ON KOSTANT'S THEOREM FOR THE LIE SUPERALGEBRA $Q(n)$

ELENA POLETAEVA AND VERA SERGANOVA

1. Introduction

A finite W-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element. Geometrically a finite W algebra is a quantization of the Poisson structure on the so-called Slodowy slice (a transversal slice to the orbit of e in the adjoint representation). In the case when $e = 0$ the finite W-algebra coincides with the universal enveloping algebra $U(\mathfrak{g})$ and in the case when e is a regular nilpotent element, the corresponding W-algebra coincides with the center of $U(\mathfrak{g})$. The latter case was studied by B. Kostant [15] who was motivated by applications to generalized Toda lattices. The general definition of a finite W-algebra was given by A. Premet in [24]. I. Losev used the machinery of Fedosov quantization to prove important results relating representations of W-algebras and primitive ideals of $U(\mathfrak{g})$ [16, 17, 18] (see also [25, 26, 27]). He used this result to prove long standing conjectures of A. Joseph and others concerning primitive ideals in $U(\mathfrak{g})$, [11].

On the other hand, affine W-algebras were first constructed by physicists [8, 9]. The role of the Slodowy slice in W-algebras in the principal case was recognized in [2]. A. De Sole and V.G. Kac in [7] established the relation between affine and finite W-algebras.

Let us mention an important discovery of physicists, [28], that for $\mathfrak{g} = \mathfrak{sl}(n)$ finite W-algebras are closely related to Yangians. This connection was further studied in [4] and [6].

It is interesting to generalize all above applications to Lie superalgebras. Finite W-algebras for Lie superalgebras have been extensively studied by C. Briot, E. Ragoucy, J. Brundan, J. Brown, S. Goodwin, W. Wang, L. Zhao and other mathematicians and physicists [3, 5, 31, 32]. Analogues of finite W-algebras for Lie superalgebras in terms of BRST cohomology were defined in by A. De Sole and V.G. Kac in [7].

In [3] C. Briot and E. Ragoucy observed that finite W-algebras associated with certain nilpotent orbits in $\mathfrak{gl}(pm|pn)$ can be realized as truncations of the super-Yangian of $\mathfrak{gl}(m|n)$, see [19] for definition.

The principal finite W-algebras for $\mathfrak{gl}(m|n)$ associated to regular (principal) nilpotent elements were described as certain truncations of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$ in [5]. It is also proven there that all irreducible modules over principal finite W-algebras are finite-dimensional for $\mathfrak{gl}(m|n)$. Furthermore, [5] contains a classification of irreducible modules using highest weight theory.

In [32] L. Zhao generalized certain results about finite W-algebras to the case of Lie superalgebras. In particular he has proved that the definition of a finite W-algebra does not depend on a choice of an isotropic subspace $\mathfrak{l}$ and a good $\mathbb{Z}$-grading. He has also proved an analogue of the Skryabin theorem establishing equivalence between the category of modules over a finite W-algebra and the category of generalized Whittaker $\mathfrak{g}$-modules. He also gave a definition of a finite W-algebra for the queer Lie superalgebra $Q(n)$.

In [22, 23] we described the finite W-algebras in the regular case for some classical and exceptional Lie superalgebras of defect one.

In this paper we are interested in the finite W-algebra associated with a regular nilpotent element $\chi \in \mathfrak{g}_{\bar{0}}^*$ for a Lie superalgebra $\mathfrak{g}$ with reductive even part $\mathfrak{g}_{\bar{0}}$. (Since not all such superalgebras admit an even invariant form, we can not identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and we use the notation W_χ instead of W_e.) We prove that for basic classical $\mathfrak{g}$ or $Q(n)$ and the regular χ the algebra W_χ satisfies the Amitsur-Levitzki identity ([1]) (Corrolary 3.6). In the proof we use some sort of reduction by constructing an injective homomorphism $\vartheta : W_\chi \to \bar{W}_\chi^{\mathfrak{s}}$, where $\mathfrak{s}$ is the reductive part of some parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, and $\bar{W}_\chi^{\mathfrak{s}}$ is an analogue of W_χ for $\mathfrak{s}$. As a corollary we obtain that all irreducible representations of W_χ are finite-dimensional (Proposition 3.7).

We study in detail the case when $\mathfrak{g} = Q(n)$ and χ is regular. In this case, $\mathfrak{p}$ is a Borel subalgebra and $\mathfrak{s}$ is a Cartan subalgebra. We obtain results about the image of ϑ in this case, which imply, in particular, that the center of W_χ coincides with the center of $U(Q(n))$ (Corollary 5.10).

Using Sergeev's construction of certain elements in the universal enveloping algebra $U(Q(n))$ ([24]), we construct generators of W_χ. Using these generators, we prove that the associated graded algebra $Gr_K W_\chi$ with respect to the Kazhdan filtration is isomorphic to $S(\mathfrak{g}^\chi)$ (the symmetric algebra of the annihilator $\mathfrak{g}^\chi$ of χ in $\mathfrak{g}$) (Conjecture 2.8 and Corollary 4.9). Furthermore, we prove that W_χ is isomorphic to a quotient of the super-Yangian of $Q(1)$ defined by M. Nazarov and A. Sergeev ([20, 21]) (Theorem 6.1). Finally, we construct n even and n odd generators in W_χ, such that all even generators commute and generate the polynomial subalgebra of rank n in W_χ, and the commutators of odd generators lie in the center of W_χ (Theorem 5.13).

Acknowledgments. The authors would like to thank J. Brundan, M. Nazarov and A. Sergeev for helpful discussions. A significant part of this work was done at the Max-Planck-Institut fur Mathematik in Bonn in the Fall of 2012. We thank the MPIM for the hospitality and support. The second author was also supported by NSF grant DMS - 1303301.

2. Finite W-algebras for Lie superalgebras

2.1. **Definitions.** Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar 0}$. Let $\chi \in \mathfrak{g}_{\bar 0}^* \subset \mathfrak{g}^*$ be an even nilpotent element in the coadjoint representation. [1] By $\mathfrak{g}^\chi$ we denote the annihilator of χ in $\mathfrak{g}$. By definition

$$\mathfrak{g}^\chi = \{x \in \mathfrak{g} \mid \chi([x, \mathfrak{g}]) = 0\}.$$

A good $\mathbb{Z}$-grading for χ is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ satisfying the following two conditions

(1) $\chi(\mathfrak{g}_j) = 0$ if $j \neq -2$;
(2) $\mathfrak{g}^\chi$ belongs to $\bigoplus_{j \geq 0} \mathfrak{g}_j$.

Note that $\chi([\cdot,\cdot]) : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}$ is a non-degenerate skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$. Let $\mathfrak{l}$ be a maximal isotropic subspace with respect to this form. We consider a nilpotent subalgebra $\mathfrak{m} = (\bigoplus_{j \leq -2} \mathfrak{g}_j) \oplus \mathfrak{l}$ of $\mathfrak{g}$. The restriction of χ to $\mathfrak{m}$

$$\chi : \mathfrak{m} \longrightarrow \mathbb{C}$$

defines a one-dimensional representation $C_\chi =< v >$ of $\mathfrak{m}$.

Definition 2.1. The induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} C_\chi \cong U(\mathfrak{g})/I_\chi,$$

where I_χ is the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$, is called *the generalized Whittaker module.*

Definition 2.2. [24]. Define *the finite W-algebra* associated to the nilpotent element χ to be

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

As in the Lie algebra case, the superalgebras W_χ are all isomorphic for different choices of good gradings and maximal isotropic subspaces $\mathfrak{l}$ [32].

If $\mathfrak{g}$ admits an even non-degenerate invariant supersymmetric bilinear form, then $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar 0}$. By the Jacobson–Morozov theorem e can be included in $\mathfrak{sl}(2) =< e, h, f >$. As in the Lie algebra case, the linear operator ad$h$ defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\}.$$

[1] Denote by $G_{\bar 0}$ the algebraic reductive group of $\mathfrak{g}_{\bar 0}$. Then χ is nilpotent if the closure of $G_{\bar 0}$-orbit in $\mathfrak{g}_{\bar 0}^*$ contains zero.

As follows from representation theory of $\mathfrak{sl}(2)$, the Dynkin $\mathbb{Z}$-grading is good. Let $\mathfrak{g}^e := \mathrm{Ker}(\mathrm{ad}e)$. Note that as in the Lie algebra case, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$ and $\mathfrak{g}^e \subseteq \bigoplus_{j\geq 0} \mathfrak{g}_j$.

Most results of this paper concern the case when $\mathfrak{g}$ admits an odd non-degenerate invariant supersymmetric bilinear form. In this case $\mathfrak{g} \simeq \Pi\mathfrak{g}^*$ and $\chi(x) = (E|x)$ for some nilpotent $E \in \mathfrak{g}_{\bar{1}}$. Among classical Lie superalgebras only $Q(n)$ or $PSQ(n)$ admit an odd non-degenerate invariant supersymmetric bilinear form. We will see that in this case there is an analogue of the Dynkin $\mathbb{Z}$-grading.

Note that by Frobenius reciprocity

$$\mathrm{End}_{U(\mathfrak{g})}(Q_\chi) = \mathrm{Hom}_{U(\mathfrak{m})}(C_\chi, Q_\chi).$$

That defines an identification of W_χ with the subspace

$$Q_\chi^{\mathfrak{m}} = \{u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in \mathfrak{m}\}.$$

In what follows we denote by $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ the natural projection. By above

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\}, \tag{2.1}$$

or, equivalently,

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \mathrm{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}. \tag{2.2}$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\mathrm{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

Definition 2.3. A $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ is called *even*, if $\mathfrak{g}_j = 0$ unless j is an even integer.

The definition of W_χ for an even good $\mathbb{Z}$-grading is simpler, since in this case $\mathfrak{g}_{-1} = 0$. Hence there is no complications of choice of a Lagrangian subspace $\mathfrak{l}$ and $\mathfrak{m} = \bigoplus_{j\geq 1} \mathfrak{g}_{-2j}$.

Let $\mathfrak{p} := \bigoplus_{j\geq 0} \mathfrak{g}_{2j}$. It follows directly from definition that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. From the PBW theorem,

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi.$$

The projection $pr : U(\mathfrak{g}) \longrightarrow U(\mathfrak{p})$ along this direct sum decomposition induces an isomorphism: $U(\mathfrak{g})/I_\chi \xrightarrow{\sim} U(\mathfrak{p})$. Thus, the algebra W_χ can be regarded as a *subalgebra* of $U(\mathfrak{p})$.

2.2. **Kazhdan filtration on W_χ.** Define the $\mathbb{Z}$-grading on $T(\mathfrak{g})$ induced by the shift by 2 of the fixed good $\mathbb{Z}$-grading. In other words, we set the degree of $X \in \mathfrak{g}_j$ to be $j+2$. It induces a filtration on $U(\mathfrak{g})$ and therefore on $U(\mathfrak{g})/I_\chi$, which is called the *Kazhdan filtration*. We will denote by Gr_K the corresponding graded algebras. Recall that by (2.1) $W_\chi \subset U(\mathfrak{g})/I_\chi$. Hence we have the induced filtration on W_χ. It is not hard to see that $Gr_K U(\mathfrak{g})$ is supercommutative and therefore $\mathrm{Gr}_K W_\chi$ is also supercommutative. For any $X \in W_\chi$ we denote by $\mathrm{Gr}_K X$ the corresponding element in $\mathrm{Gr}_K W_\chi$. The following result is very important.

Theorem 2.4. *A. Premet* [24]. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the associated graded algebra $Gr_K W_\chi$ is isomorphic to $S(\mathfrak{g}^\chi)$.*

We believe that the above theorem holds for basic classical Lie superalgebras if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is *even.* In fact, for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and regular χ it is proven in [5]. In this paper we prove the analogous result for regular χ and $\mathfrak{g} = Q(n)$ (see Corollary 4.9).

We will prove now a weaker general result. Let $\mathfrak{l}'$ be some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties

- $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$;
- $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form $\chi([\cdot,\cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $\mathfrak{l}'$ is a maximal isotropic subspace. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $\mathfrak{l}^\perp \cap \mathfrak{l}'$ is one-dimensional and we fix $\theta \in \mathfrak{l}^\perp \cap \mathfrak{l}'$ such that $\chi([\theta,\theta]) = 2$. It is clear that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.

Let $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. By the PBW theorem, $U(\mathfrak{g})/I_\chi \simeq S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. Therefore $\mathrm{Gr}_K(U(\mathfrak{g})/I_\chi)$ is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. The good grading of $\mathfrak{g}$ induces the grading on $S(\mathfrak{p} \oplus \mathfrak{l}')$. For any $X \in S(\mathfrak{p} \oplus \mathfrak{l}')$ we denote by $\bar{X}$ the element of highest degree in this grading. Following the original Premet's proof we will prove now the following statement.

Theorem 2.5. *(a) Assume that $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. If $X \in Gr_K W_\chi$, then $\bar{X} \in S(\mathfrak{g}^\chi)$.*
(b) Assume that $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd. If $X \in Gr_K W_\chi$, then $\bar{X} \in S(\mathfrak{g}^\chi \oplus \mathbb{C}\theta)$.

Proof. We start with the following simple observation.

Lemma 2.6. *Let $x \in \mathfrak{p} \oplus \mathfrak{l}'$. Then $\chi([\mathfrak{m}, x]) = 0$ if and only if $x \in \mathfrak{g}^\chi$ for even $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ and $x \in \mathfrak{g}^\chi \oplus \mathbb{C}\theta$ for odd $\dim(\mathfrak{g}_{-1})_{\bar{1}}$.*

Proof. Note that if $x \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, then $\chi([Y,x]) \neq 0$ implies $i + j = -2$. Therefore if $x \in \mathfrak{p}$, the condition $\chi([\mathfrak{m}, x]) = 0$ implies the condition $\chi([\mathfrak{g}, x]) = 0$, and thus $x \in \mathfrak{g}^\chi$. If $x \in \mathfrak{l}'$, then the condition $\chi([\mathfrak{m}, x]) = 0$ is equivalent to the condition $\chi([\mathfrak{l}, x]) = 0$. Therefore $x \in \mathfrak{l}^\perp \cap \mathfrak{l}' = \mathbb{C}\theta$. □

Let $X \in Gr_K W_\chi$. Passing to the graded version of (2.2) we obtain that for any $Y \in \mathfrak{m}$ we have

$$\pi(\mathrm{ad}Y(X)) = 0. \tag{2.3}$$

Define $\gamma : \mathfrak{m} \otimes S(\mathfrak{p} \oplus \mathfrak{l}') \to S(\mathfrak{p} \oplus \mathfrak{l}')$ by putting

$$\gamma(Y, Z) = \pi(\mathrm{ad}Y(Z))$$

for all $Y \in \mathfrak{m}, Z \in S(\mathfrak{p} \oplus \mathfrak{l}')$. It is easy to see that if $Y \in \mathfrak{g}_{-i}$, where $i > 0$, and $Z \in S(\mathfrak{p} \oplus \mathfrak{l}')_j$, then $\gamma(Y, Z) \in S(\mathfrak{p} \oplus \mathfrak{l}')_{j-i} \oplus S(\mathfrak{p} \oplus \mathfrak{l}')_{j-i+2})$. Hence we can write $\gamma = \gamma_0 + \gamma_2$ where $\gamma_0(Y, Z)$ is the projection on $S(\mathfrak{p} \oplus \mathfrak{l}')_{j-i}$ and $\gamma_2(Y, Z)$ is the projection on $S(\mathfrak{p} \oplus \mathfrak{l}')_{j-i+2}$. The condition (2.3) implies that for any $X \in Gr_K W_\chi$

$$\gamma_2(\mathfrak{m}, \bar{X}) = 0. \tag{2.4}$$

On the other hand, $\gamma_2 : \mathfrak{m} \times S(\mathfrak{p} \oplus \mathfrak{l}') \to S(\mathfrak{p} \oplus \mathfrak{l}')$ is a derivation with respect to the second argument defined by the condition

$$\gamma_2(Y, Z) = \chi([Y, Z])$$

for any $Y \in \mathfrak{m}, Z \in \mathfrak{p} \oplus \mathfrak{l}'$. Now by induction on the polynomial degree of $\bar{X}$ in $S(\mathfrak{p} \oplus \mathfrak{l}')$, using Lemma 2.6, one can show that (2.4) implies $\bar{X} \in S(\mathfrak{g}^\chi)$ (respectively, $\bar{X} \in S(\mathfrak{g}^\chi \oplus \mathbb{C}\theta)$). □

Proposition 2.7. *Assume that* $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ *is even (respectively, odd). Let* $y_1, \dots, y_p$ *be a basis in* $\mathfrak{g}^\chi$ *homogeneous in the good* $\mathbb{Z}$*-grading. Assume that there exist* $Y_1, \dots, Y_p \in W_\chi$ *such that* $\overline{Gr_K Y_i} = y_i$ *for all* $i = 1, \dots, p$.

(a) $Y_1, \dots, Y_p$ *generate* W_χ *(respectively,* $Y_1, \dots, Y_p$ *and* $\pi(\theta)$ *generate* W_χ*) ;*

(b) $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$ *(respectively,* $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$*, where* $\mathbb{C}[\xi]$ *is the exterior algebra generated by one element* ξ*).*

Proof. We will give a proof in the case when $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. The odd case is analogous and we leave it to the reader. Let us first prove (a) by contradiction. Assume that $X \in W_\chi$ is an element of minimal Kazhdan degree such that it does not lie in the subalgebra generated by $Y_1, \dots, Y_p$. By Theorem 2.5 we have

$$\overline{\mathrm{Gr}_K X} = \sum c(a_1, \dots, a_p) y_1^{a_1} \dots y_p^{a_p}.$$

Let

$$Z = X - \sum c(a_1, \dots, a_p) Y_1^{a_1} \dots Y_p^{a_p}.$$

Then Kazhdan degree of Z is less than that of X. By minimality of degree of X we conclude that $Z = 0$. That contradicts our assumption.

To prove (b) write $\mathfrak{p} = \mathfrak{g}^\chi \oplus \mathfrak{r}$, where $\mathfrak{r}$ is some graded subspace complementary to $\mathfrak{g}^\chi$. Let $\gamma : S(\mathfrak{p} \oplus \mathfrak{l}') \to S(\mathfrak{g}^\chi)$ denote the natural projection with kernel $(\mathfrak{r} \oplus \mathfrak{l}')S(\mathfrak{p} \oplus \mathfrak{l}')$. By (a) and Theorem 2.5 the restriction $\gamma : \mathrm{Gr}_K W_\chi \to S(\mathfrak{g}^\chi)$ is an isomorphism of rings. □

Conjecture 2.8. Assume that $\mathfrak{g}$ is a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$ and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$, where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element ξ.

2.3. **Kostant's theorem and the regular case for Lie superalgebras.** A nilpotent $\chi \in \mathfrak{g}_{\bar{0}}^*$ is called *regular* if $G_{\bar{0}}$-orbit of χ has maximal dimension, i.e. the dimension of $\mathfrak{g}_{\bar{0}}^\chi$ is minimal. Let us recall that for a regular nilpotent χ and a reductive Lie algebra $\mathfrak{g}$ the algebra W_χ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, see [15].

It is not hard to see that this result of B. Kostant does not hold for Lie superalgebras. In Section 3 we will prove that for regular χ, W_χ satisfies the Amitsur–Levitzki identity and all irreducible representations of W_χ are finite-dimensional with dimension not greater than 2^{k+1}, where k is the constant depending on defect of $\mathfrak{g}$ and the parity of $\dim\mathfrak{g}_{\bar{1}}^\chi$. Recall that for a contragredient $\mathfrak{g}$ the defect of $\mathfrak{g}$ is the maximal number of mutually orthogonal linearly independent isotropic roots, [14].

2.4. **Good $\mathbb{Z}$-gradings for superalgebras in the regular case.** Good $\mathbb{Z}$-gradings for basic classical superalgebras are classified in [12]. In the case when χ is regular and $\mathfrak{g}$ is of type II (i.e. $\mathfrak{g}_{\bar{0}}$ is semisimple and $\mathfrak{g}_{\bar{1}}$ is a simple $\mathfrak{g}_{\bar{0}}$-module), the only good $\mathbb{Z}$-grading is the Dynkin $\mathbb{Z}$-grading, and it is never even. If $\mathfrak{g}$ is of type I, i.e. $\mathfrak{g}_{\bar{0}}$ has a non-trivial center, we can choose an even good $\mathbb{Z}$-grading for any χ. For the Lie superalgebra $Q(n)$ the analogue of Dynkin $\mathbb{Z}$-grading is even for any χ.

Let us concentrate on the case of basic classical or exceptional Lie superalgebras of type II and regular χ. In this case $\chi(\cdot) = (e|\cdot)$ for some principal nilpotent element $e \in \mathfrak{g}_{\bar{0}}$. We are going to describe the Dynkin $\mathbb{Z}$-grading on $\mathfrak{g}$ in terms of a specific Borel subalgebra. Let $\mathfrak{b}_{\bar{0}} \subset \mathfrak{g}_{\bar{0}}$ be the Borel subalgebra containing e. Since e is principal, this Borel subalgebra is unique. Let Π_0 denote the set of simple roots of $\mathfrak{b}_{\bar{0}}$.

Lemma 2.9. *Let $\mathfrak{g}$ be a basic classical or exceptional Lie superalgebra of type II.*

(a) There exists a Borel subalgebra $\mathfrak{b}_{\bar{0}} \subset \mathfrak{b} \subset \mathfrak{g}$ with the set of simple roots Π such that for any root $\beta \in \Pi_0$ either $\beta \in \Pi$ or $\beta = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Pi$.

(b) Let d denote the defect of $\mathfrak{g}$. Then the number of odd roots in Π equals $2d$ if $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m \geq n$, $\mathfrak{osp}(2m|2n)$ for $m \leq n$ or G_3, and the number of odd roots in Π equals $2d+1$ if $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m < n$, $\mathfrak{osp}(2m|2n)$ for $m > n$, $D(2,1;a)$ or F_4.

(c) Let e, h, f be the $\mathfrak{sl}(2)$-triple such that $h \in \mathfrak{h}$. Then $\alpha(h) = 2$ for any even $\alpha \in \Pi$ and $\alpha(h) = 1$ for any odd $\alpha \in \Pi$, i.e. the Dynkin $\mathbb{Z}$-grading is consistent.

Proof. (a) Among all Borel subalgebras containing $\mathfrak{b}_{\bar{0}}$ pick up the one that has maximal number of odd roots and contains an odd non-isotropic root if such roots exist. For ortho-symplectic superalgebra those Borel subalgebras are listed in [10].

For the exceptional superalgebras we list the simple roots using the roots description in [13]. If $\mathfrak{g} = G_3$, the set of simple roots is $\{\delta, \gamma_1 - \delta, \gamma_2\}$, where γ_1 is the short and γ_2 is the long simple root of G_2. If $\mathfrak{g} = F_4$, then the set of simple roots is

$$\{\varepsilon_1 - \varepsilon_2, \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \delta), \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \delta), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \delta)\}.$$

(b) follows by direct inspection.

(c) follows from the condition $[h, e] = 2e$ and (a). □

Corollary 2.10. *Let $\mathfrak{g}$ be a basic classical or exceptional Lie superalgebra of type II, and d be its defect. If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m \geq n$, $\mathfrak{osp}(2m|2n)$ for $m \leq n$ or G_3, then $\dim \mathfrak{g}_{-1} = (0|2d)$. If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ for $m < n$, $\mathfrak{osp}(2m|2n)$ for $m > n$, $D(2,1;a)$ or F_4, then $\dim \mathfrak{g}_{-1} = (0|2d+1)$. By Lemma 2.9(c) $\dim \mathfrak{g}_{-1}$ equals the number of irreducible $\mathfrak{sl}(2)$-components in $\mathfrak{g}_{\bar{1}}$. Therefore* $\dim(\mathfrak{g}^{\chi})_{\bar{1}} = 2d$ *or* $2d+1$.

Corollary 2.11. *Let $\mathfrak{g}$ satisfy the assumptions of Corollary 2.10.*

(a) One can choose a maximal isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{-1}$ such that $\mathfrak{l} = \mathfrak{g}_{-\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{-\alpha_d}$ for some isotropic mutually orthogonal roots $\alpha_1, \dots, \alpha_d \in \Pi$. In particular, $[\mathfrak{l}, \mathfrak{l}]$=0.

(b) There exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with Levi subalgebra $\mathfrak{s}$ such that $\mathfrak{m} \cap \mathfrak{s}$ is an even one dimensional subspace, and if $\mathfrak{n}^-$ denotes the nil radical of the opposite parabolic $\mathfrak{p}^-$, then $\mathfrak{n}^- \subset \mathfrak{m}$.

(c) If $\mathfrak{g}$ does not have non-isotropic roots (i.e. $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ or F_4), then $[\mathfrak{s}, \mathfrak{s}]$ is isomorphic to a direct sum of several copies of $\mathfrak{sl}(1|1)$ and one copy of $\mathfrak{sl}(1|2)$. If $\mathfrak{g}$ has non-isotropic roots (i.e. $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ or G_3), then $[\mathfrak{s}, \mathfrak{s}]$ is isomorphic to a direct sum of several copies of $\mathfrak{sl}(1|1)$ and one copy of $\mathfrak{osp}(1|2)$.

Proof. Let Γ denote the Dynkin diagram of Π. For any subset $C \subset \Pi$ we denote by Γ_C the corresponding subdiagram of Γ. Let Π' denote the set of all odd roots of Π, the subgraph $\Gamma_{\Pi'}$ is connected and Π' has at most one non-isotropic root. Let us choose a subset $A = \{\alpha_1, \dots \alpha_d\} \subset \Pi'$ of mutually orthogonal isotropic roots such that the subgraph $\Gamma_{\Pi' \setminus A}$ has maximal number of connected components. If Π' contains a non-isotropic root, then $\Gamma_{\Pi' \setminus A}$ is a disjoint union of single vertex diagrams. If all roots of Π' are isotropic, then $\Gamma_{\Pi' \setminus A}$ is a disjoint union of several single vertex diagrams and one diagram consisting of two connected isotropic vertices. The latter is the diagram of $\mathfrak{sl}(1|2)$.

Now we set $\mathfrak{s}$ to be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and $\mathfrak{g}_{\pm\beta}$ for all $\beta \in \Pi' \setminus A$ and let $\mathfrak{p} = \mathfrak{b} + \mathfrak{s}$. We leave to the reader to check that all requirements of the corollary are true for this choice. □

Example 2.12. Let $\mathfrak{g} = \mathfrak{osp}(3|4)$. Then Π has the Dynkin diagram

$$\otimes - \otimes \Rightarrow \bullet,$$

and A consists of one middle vertex. In this case $[\mathfrak{s}, \mathfrak{s}] \simeq \mathfrak{sl}(1|1) \oplus \mathfrak{osp}(1|2)$.

Example 2.13. Let $\mathfrak{g} = G_3$. Then Π has the Dynkin diagram

$$\bullet \Leftarrow \otimes \Lleftarrow \circ,$$

and A again coincides with the midle vertex. In this case we also have $[\mathfrak{s},\mathfrak{s}] \simeq \mathfrak{sl}(1|1)\oplus\mathfrak{osp}(1|2)$.

2.5. **The queer superalgebra $Q(n)$.** Recall that the *queer* Lie superalgebra is defined as follows

$$Q(n) := \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n\times n \text{ matrices}\}.$$

Let $\operatorname{otr}\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = \operatorname{tr} B$.

Remark 2.14. $Q(n)$ has one-dimensional center $< z >$, where $z = 1_{2n}$. Let

$$SQ(n) = \{X \in Q(n) \mid \operatorname{otr} X = 0\}.$$

The Lie superalgebra $\tilde{Q}(n) := SQ(n)/ < z >$ is simple for $n \geq 3$, see [13].

Note that $\mathfrak{g} = Q(n)$ admits an *odd* non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form

$$(x|y) := \operatorname{otr}(xy) \text{ for } x, y \in \mathfrak{g}.$$

Therefore, we identify the coadjoint module $\mathfrak{g}^*$ with $\Pi(\mathfrak{g})$, where Π is the change of parity functor.
Let $e_{i,j}$ and $f_{i,j}$ be standard bases in $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$ respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right),$$

where E_{ij} are elementary $n \times n$ matrices.
Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \operatorname{diag}(n-1, n-3, \dots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i)e_{i+1,i}.$$

Note that e is a regular nilpotent element, h defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cccc|cccc}
0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \\
\hline
0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0
\end{array}\right).$$

Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$. Since we have an isomorphism $\mathfrak{g}^* \simeq \Pi(\mathfrak{g})$, an even regular nilpotent $\chi \in \mathfrak{g}^*$ can be defined by $\chi(x) := (x|E)$ for $x \in \mathfrak{g}$. Note that the Dynkin $\mathbb{Z}$-grading is good for χ. We have that

$$\mathfrak{g}^\chi = \mathfrak{g}^E = \{z, e, e^2, \dots, e^{n-1} \mid H_0, H_1, \dots, H_{n-1}\}, \quad \dim(\mathfrak{g}^E) = (n|n), \tag{2.5}$$

where $H_k = \sum_{i=1}^{n-k}(-1)^{i+k-1} f_{i,i+k}$ for $k = 0, \dots, n-1$. Let

$$\mathfrak{m} = \bigoplus_{j=1}^{n-1} \mathfrak{g}_{-2j}.$$

Note that $\mathfrak{m}$ is generated by $e_{i+1,i}$ and $f_{i+1,i}$, where $i = 1, \dots, n-1$, and

$$\chi(e_{i+1,i}) = 1, \quad \chi(e_{i+k,i}) = 0 \text{ if } k \geq 2, \quad \chi(f_{i+k,i}) = 0 \text{ if } k \geq 1. \tag{2.6}$$

The left ideal I_χ and W_χ are defined now as usual. Moreover,

$$\mathfrak{b} := \bigoplus_{j=0}^{n-1} \mathfrak{g}_{2j}$$

is a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h} := \mathfrak{g}_0$ is a Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where

$$\mathfrak{n} := \bigoplus_{j=1}^{n-1} \mathfrak{g}_{2j}.$$

Note that the algebra W_χ can be regarded as a *subalgebra* of $U(\mathfrak{b})$.

3. Some general results

3.1. **The Harish-Chandra homomorphism.** In this section we assume that $\mathfrak{g}$ is a basic classical Lie superalgebra or $Q(n)$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra such that $\mathfrak{n}^- \subset \mathfrak{m} \subset \mathfrak{p}^-$, where $\mathfrak{n}^-$ denotes the nilradical of the opposite parabolic $\mathfrak{p}^-$. Let $\mathfrak{s}$ be the Levi subalgebra of $\mathfrak{p}$, $\mathfrak{n}$ be its nilradical and $\mathfrak{m}^\mathfrak{s} = \mathfrak{m} \cap \mathfrak{s}$. Note that $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{m}^s$. We denote by $Q_\chi^\mathfrak{s}$ the induced module $U(\mathfrak{s}) \otimes_{U(\mathfrak{m}^s)} C_\chi$, where by χ we understand the restriction of χ on $\mathfrak{s}$. Let

$$\bar{W}_\chi^\mathfrak{s} = \mathrm{End}_{U(\mathfrak{s})} Q_\chi^\mathfrak{s} = (Q_\chi^\mathfrak{s})^{\mathfrak{m}^\mathfrak{s}}.$$

Let J_χ (respectively $J_\chi^\mathfrak{s}$) be the left ideal in $U(\mathfrak{p})$ (respectively in $U(\mathfrak{s})$) generated by $a - \chi(a)$ for all $a \in \mathfrak{m}^\mathfrak{s}$.

Finally, let $\bar{\vartheta} : U(\mathfrak{p}) \to U(\mathfrak{s})$ denote the projection with the kernel $\mathfrak{n}U(\mathfrak{p})$. Note that $\bar{\vartheta}(J_\chi) = J_\chi^\mathfrak{s}$. Thus, the projection $\vartheta' : U(\mathfrak{p})/J_\chi \to U(\mathfrak{s})/J_\chi^\mathfrak{s}$ is well defined.

Note that we have an isomorphism of vector spaces $Q_\chi \simeq U(\mathfrak{p})/J_\chi$, hence W_χ can be identified with a subspace in $(U(\mathfrak{p})/J_\chi)^{\mathfrak{m}^\mathfrak{s}}$. On the other hand, $\bar{W}_\chi^\mathfrak{s}$ can be identified with the subspace $(U(\mathfrak{s})/J_\chi^s)^{\mathfrak{m}^\mathfrak{s}}$. Consider a map $\vartheta : W_\chi \to U(\mathfrak{s})/J_\chi^\mathfrak{s}$ obtained by the restriction of ϑ' to W_χ. Since $\operatorname{ad}\mathfrak{m}^{\mathfrak{s}}(\mathfrak{n}) \subset \mathfrak{n}$, ϑ maps $\operatorname{ad}\mathfrak{m}^{\mathfrak{s}}$-invariants to $\operatorname{ad}\mathfrak{m}^{\mathfrak{s}}$-invariants. In other words, $\vartheta(W_\chi) \subset \bar{W}_\chi^{\mathfrak{s}}$. Furthermore, one can easily see that $\vartheta : W_\chi \to \bar{W}_\chi^{\mathfrak{s}}$ is a homomorphism of algebras.

An important example is as follows. Assume that $\mathfrak{g}$ admits an even good grading with respect to χ. Then we can set $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i$. Then $\mathfrak{s} = \mathfrak{g}_0$, $\mathfrak{m}^{\mathfrak{s}} = 0$ and ϑ is a homomorphism $W_\chi \to U(\mathfrak{s})$.

Theorem 3.1. *The homomorphism $\vartheta : W_\chi \to \bar{W}_\chi^{\mathfrak{s}}$ is injective.*

Proof. We consider a new grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{(i)}$ such that $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_{(i)}$. Note that J_χ is a graded ideal and hence Q_χ is also a graded vector space with respect to this new grading. Note that $(Q_\chi)_{(0)} = Q_\chi^{\mathfrak{s}}$. For any $t \in \mathbb{C}\setminus\{0\}$ let ϕ_t denote the automorphism of $\mathfrak{g}$ that multiplies elements of $\mathfrak{g}_{(j)}$ by t^j. Let $X \in W_\chi = Q_\chi^{\mathfrak{m}}$. Write

$$X = \sum_{i=d}^{s} X_{(i)},$$

where $X_{(i)} \in (Q_\chi)_{(i)}$ and $X_{(d)} \neq 0$. Our goal is to show that $d = 0$. Let

$$\chi_0 = \lim_{t\to 0} \phi_t(\chi).$$

Then $\chi_0(\mathfrak{n}^-) = 0$, $\chi_0|_{\mathfrak{m}^{\mathfrak{s}}} = \chi|_{\mathfrak{m}^{\mathfrak{s}}}$. Note that

$$t^{-d}\phi_t(X) \in W_{\phi_t(\chi)}$$

and hence by the standard continuity argument $X_{(d)} \in (U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} C_{\chi_0})^{\mathfrak{m}}$. Note that

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} C_{\chi_0} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} U(\mathfrak{p}^-) \otimes_{U(\mathfrak{m})} C_{\chi_0}.$$

Furthermore, $U(\mathfrak{p}^-) \otimes_{U(\mathfrak{m})} C_{\chi_0}$ has the trivial action of $\mathfrak{n}^-$ and is isomorphic to $Q_\chi^{\mathfrak{s}}$ as an $\mathfrak{s}$-module. Thus, $X_{(d)} \in (U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} Q_\chi^{\mathfrak{s}})^{\mathfrak{m}}$.

We need now the following Lemma.

Lemma 3.2. $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} Q_\chi^{\mathfrak{s}})^{\mathfrak{n}^-} = Q_\chi^{\mathfrak{s}}$.

Proof. Let ζ be a generic central character of $U(\mathfrak{s})$ and S be a quotient of $Q_\chi^{\mathfrak{s}}$ admitting this central character. Consider the parabolically induced module $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} S$ (here we assume that $\mathfrak{n}^-$ acts trivially on S). We will prove that $M^{\mathfrak{n}^-} = S$.

Let $\gamma : Z(\mathfrak{g}) \to Z(\mathfrak{s})$ be the restriction of the Harish-Chandra projection $U(\mathfrak{g}) \to U(\mathfrak{s})$ with kernel $\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$. Note that M admits central character $\gamma^*(\zeta)$. Any simple $\mathfrak{s}$-submodule $N \subset M^{\mathfrak{n}^-}$ that admits central character ζ' generates in M a submodule admitting central character $\gamma^*(\zeta')$. Hence we have $\gamma^*(\zeta) = \gamma^*(\zeta')$.

Recall the correspondence between central characters and weights. One chooses a Borel subalgebra $\mathfrak{b}^{\mathfrak{s}}$ in $\mathfrak{s}$ and set ζ_λ to be the central character of the Verma module over $\mathfrak{s}$ with highest weight λ. Furthermore $\mathfrak{b}^{\mathfrak{s}} \oplus \mathfrak{n}^-$ is a Borel subalgebra in $\mathfrak{g}$ and we define the $U(\mathfrak{g})$-central character $\bar{\zeta}_\lambda$ to be the central character of the Verma module over $\mathfrak{g}$ with highest weight λ. Obviously, $\gamma^*(\zeta_\lambda) = \bar{\zeta}_\lambda$. Moreover, all simple $\mathfrak{s}$-subquotients of M admit central character ζ_μ for some $\mu \in \lambda + R(\mathfrak{n}^-)$ where $R(\mathfrak{n}^-)$ is the set of weights of $U(\mathfrak{n}^-)$. Recall that if λ is typical, then $\bar{\zeta}_\lambda = \bar{\zeta}_\nu$ implies that ν is obtained from λ by the shifted action of the Weyl group of $\mathfrak{g}_0$.

Let us choose a typical λ such that the intersection of the orbit of λ and $\lambda + R(\mathfrak{n}^-)$ equals λ. Suppose that there exists a simple $N \subset M^{\mathfrak{n}^-} \cap \mathfrak{n}^- M$. Then N admits $U(\mathfrak{s})$-central character ζ_μ for some $\mu \in \lambda + R(\mathfrak{n}^-)$, $\mu \neq \lambda$. But then $\bar{\zeta}_\mu \neq \bar{\zeta}_\lambda$. A contradiction.

Since S is generic, the above argument implies $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} Q^{\mathfrak{s}}_\chi)^{\mathfrak{n}^-} = Q^{\mathfrak{s}}_\chi$. □

Now we can finish the proof of the theorem. By Lemma 3.2

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} Q^{\mathfrak{s}}_\chi)^{\mathfrak{m}} = (Q^{\mathfrak{s}}_\chi)^{\mathfrak{m}^{\mathfrak{s}}} = \bar{W}^{\mathfrak{s}}_\chi.$$

That implies $d = 0$. □

3.2. **The case of a regular χ.** If χ is regular and admits an even good $\mathbb{Z}$-grading, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(m|n), \mathfrak{osp}(2|2n)$ or $Q(n)$. In this case we set $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$. If $\mathfrak{g}$ is of type II, then we define $\mathfrak{p}$ as in Corollary 2.11.

If $\mathfrak{g} = Q(n)$ we set $k = \frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd. In other cases we set $k = d$ (the defect of $\mathfrak{g}$) if $\mathfrak{g}$ is of type I or $\mathfrak{g}$ is of type II and $\dim \mathfrak{g}^\chi_{\bar{1}}$ is even. If $\mathfrak{g}$ is of type II and $\dim \mathfrak{g}^\chi_{\bar{1}}$ is odd, then we set $k = d + 1$.

Proposition 3.3. *$\bar{W}^{\mathfrak{s}}_\chi$ satisfies Amitsur–Levitzki identity, i.e. for any $u_1, \dots, u_{2^{k+1}} \in \bar{W}^{\mathfrak{s}}_\chi$*

$$\sum_{\sigma \in S_{2^{k+1}}} sgn(\sigma) u_{\sigma(1)} \dots u_{\sigma(2^{k+1})} = 0. \tag{3.1}$$

Proof. We first consider the case of even $\mathbb{Z}$-grading. Then $\bar{W}^{\mathfrak{s}}_\chi = U(\mathfrak{s})$. Let us assume first that $\mathfrak{g} = Q(n)$. Then the even good $\mathbb{Z}$-grading coincides with the Dynkin $\mathbb{Z}$-grading and $\mathfrak{s} = \mathfrak{g}_0 = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote

$$x_i = e_{i,i}, \quad \xi_i = (-1)^{i+1} f_{i,i}.$$

Then x_i lie in the center of $U(\mathfrak{h})$ and we have $[f_{i,i}, f_{i,i}] = 2x_i$. From this it is easy to see that $U(\mathfrak{h}_{\bar{0}}) = \mathbb{C}[x_1, \dots, x_n]$ coincides with the center of $U(\mathfrak{h})$.

Let F denote the algebraic closure of the field of fractions of $U(\mathfrak{h}_{\bar{0}})$ and let $U(\mathfrak{h})_F = F \otimes_{U(\mathfrak{h}_{\bar{0}})} U(\mathfrak{h})$. Then $U(\mathfrak{h})_F$ is isomorphic to the Clifford algebra associated with a non-degenerate symmetric form on an n-dimensional space. Thus, $U(\mathfrak{h})_F \simeq M_{2^k}(F)$ for even n and $U(\mathfrak{h})_F \simeq M_{2^k}(F) \times M_{2^k}(F)$ for odd n, where by $M_s(F)$ we denote the algebra of $s \times s$ matrices over F. Thus, by the Amitsur–Levitzki theorem (see [1]), $U(\mathfrak{h})_F$ satisfies (3.1). Since $U(\mathfrak{h})$ is a subalgebra of $U(\mathfrak{h})_F$, it also satisfies (3.1).

Now let $\mathfrak{g} = \mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2|2n)$. Then the even part of $\mathfrak{s}$ coincides with the Cartan subalgebra $\mathfrak{h}$, which is abelian. The basis of the odd part consists of root elements $X_1, \dots, X_k, Y_1, \dots, Y_k$ such that $[X_i, Y_j] = 0$ if $i \neq j$, $[X_i, X_j] = [Y_i, Y_j] = 0$ for all $i, j \leq k$. Thus, $\mathfrak{s}$ has a triangular decomposition $\mathfrak{s} = \mathfrak{s}^- \oplus \mathfrak{h} \oplus \mathfrak{s}^+$, with $\mathfrak{s}^+$ spanned by $X_1, \dots, X_k$ and $\mathfrak{s}^-$ spanned by $Y_1, \dots, Y_k$. Let $\lambda \in \mathfrak{h}^*$ and $M_\lambda = U(\mathfrak{s}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{s}^+)} C_\lambda$ denote the Verma module over $\mathfrak{s}$. The dimension of M_λ equals 2^k. An easy calculation shows that $\prod_{\lambda \in \mathfrak{h}^*} M_\lambda$ is a faithful $U(\mathfrak{s})$-module. Therefore $U(\mathfrak{s})$ is isomorphic to a subalgebra in $\prod_{\lambda \in \mathfrak{h}^*} \operatorname{End}_{\mathbb{C}}(M_\lambda)$. Since $\prod_{\lambda \in \mathfrak{h}^*} \operatorname{End}_{\mathbb{C}}(M_\lambda)$ satisfies the Amitsur–Levitzki identity, $U(\mathfrak{s})$ must satisfy it as well.

Finally, let us consider the case when $\mathfrak{g}$ is of type II. Here we are going to consider two subcases. We will use notations of the proof of Corollary 2.11.

First, let us assume that Π contains an odd non-isotropic root β. Then $\Pi' \setminus A = \{\beta_1, \dots, \beta_{k-1}, \beta = \beta_k\}$. Then $[\mathfrak{s}, \mathfrak{s}]$ is a direct sum of $k-1$ copies of $\mathfrak{sl}(1|1)$ generated by the root spaces $\mathfrak{g}_{\pm\beta_i}$, $i = 1, \dots, k-1$ and one copy of $\mathfrak{osp}(1|2)$ generated by $\mathfrak{g}_{\pm\beta_k}$. Furthermore, $\mathfrak{m}^{\mathfrak{s}} \subset \mathfrak{osp}(1|2)$ is generated by $\mathfrak{g}_{-2\beta}$. Let us write $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{r}$, where $\mathfrak{r} = \mathfrak{osp}(1|2)$. Then $\bar{W}^{\mathfrak{s}}_\chi = U(\mathfrak{s}') \otimes \bar{W}^{\mathfrak{r}}_\chi$, where $\bar{W}^{\mathfrak{r}}_\chi$ is the usual W-algebra for the regular χ and $\mathfrak{r} = \mathfrak{osp}(1|2)$. In the following example we give an explicit description of W-algebra for $\mathfrak{osp}(1|2)$.

Let $\mathfrak{r} = \mathfrak{osp}(1|2) =< X, Y, H \mid \theta, r >$, where

$$X = E_{23}, Y = E_{32}, H = E_{22} - E_{33}, \theta = E_{12} - E_{31}, r = E_{13} + E_{21}.$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = X, h = H, f = Y$. The element h defines a $\mathbb{Z}$-grading on $\mathfrak{r}$:

$$\mathfrak{r} = \mathfrak{r}_{-2} \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2, \text{ where}$$

$$\mathfrak{r}_{-2} =< Y >, \quad \mathfrak{r}_{-1} =< \theta >, \quad \mathfrak{r}_0 =< H >, \quad \mathfrak{r}_1 =< r >, \quad \mathfrak{r}_2 =< X > .$$

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = \frac{1}{2} str(ab)$ on $\mathfrak{r}$: $(\theta|r) = 1, \quad (X|Y) = -\frac{1}{2}, \quad (H|H) = -1$. Let $\chi(x) = (e|x)$ for $x \in \mathfrak{r}$ and let W_χ be the corresponding W-algebra. Note that $\mathfrak{g}^\chi = \mathfrak{g}^e =< X \mid r >$, $\mathfrak{m} = \mathfrak{r}_{-2}$, and $\chi(Y) = -\frac{1}{2}$. We have that $\pi(\theta) \in W^{\mathfrak{r}}_\chi$, and $\pi(\theta)^2 = \frac{1}{2}$. Let Ω be the Casimir element of $\mathfrak{r}$. Then

$$\pi(\Omega) = \pi(2X + H - H^2 + 2r\theta).$$

Let

$$R = \pi(r - H\theta).$$

Note that $\pi(\Omega)$ and R belong to W_χ.

Lemma 3.4. *a) W_χ is generated by $\pi(\Omega)$, $\pi(\theta)$ and R. The defining relations are*

$$[\pi(\Omega), R] = [\pi(\Omega), \pi(\theta)] = 0,$$
$$[R, R] = \pi(\Omega), \quad [R, \pi(\theta)] = -\tfrac{1}{2}, \quad [\pi(\theta), \pi(\theta)] = 1.$$

b) For any $c \in \mathbb{C}$, $W_\chi/(\Omega - c)$ is isomorphic to a Clifford algebra with two generators and it has a unique irreducible representation M_c of dimension 2.

Proof. Since $\overline{\mathrm{Gr}_K(\pi(\Omega))} = 2X$, $\overline{\mathrm{Gr}_K(R)} = r$, then (a) follows from Proposition 2.7 (a).

The proof of (b) is straightforward. □

We use Lemma 3.4 (b) to prove the Amitsur–Levitzki identity in the latter case. We again consider the family $M_\lambda \otimes M_c$, where M_λ is the Verma module over $\mathfrak{s}'$ and M_c is as in Lemma 3.4 (b). Then $\prod_{\lambda\in\mathfrak{h}^*, c\in\mathbb{C}} (M_\lambda \otimes M_c)$ is a faithful $\bar{W}_\chi^{\mathfrak{s}}$-module. Therefore $\bar{W}_\chi^{\mathfrak{s}}$ is isomorphic to a subalgebra in $\prod_{\lambda\in\mathfrak{h}^*, c\in\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M_\lambda \otimes M_c)$. Since $\prod_{\lambda\in\mathfrak{h}^*, c\in\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M_\lambda \otimes M_c)$ satisfies the Amitsur–Levitzki identity, $\bar{W}_\chi^{\mathfrak{s}}$ must satisfy it as well.

Finally we assume that all odd roots in Π are isotropic. Then

$$\Pi' \setminus A = \{\beta_1, \dots, \beta_{k-1}, \beta_k\}$$

with the only non-orthogonal pair β_{k-1}, β_k. In this case $[\mathfrak{s}, \mathfrak{s}]$ is a direct sum of $k-2$ copies of $\mathfrak{sl}(1|1)$ generated by the root spaces $\mathfrak{g}_{\pm\beta_i}$, $i = 1, \dots, k-2$ and one copy of $\mathfrak{sl}(1|2)$ generated by $\mathfrak{g}_{\pm\beta_{k-1}}, \mathfrak{g}_{\pm\beta_k}$. Furthermore, $\mathfrak{m}^{\mathfrak{s}} \subset \mathfrak{sl}(1|2)$ is generated by $f \in \mathfrak{g}_{-\beta_{k-1}-\beta_k}$. As in the previous case we write $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{r}$, where $\mathfrak{r} = \mathfrak{sl}(1|2)$. Then $\bar{W}_\chi^{\mathfrak{s}} = U(\mathfrak{s}') \otimes \bar{W}_\chi^{\mathfrak{r}}$, where

$$\bar{W}_\chi^{\mathfrak{r}} = (U(\mathfrak{r}) \otimes_{\mathbb{C}f} C_\chi)^f,$$

where $\chi(f) = 1$.

We realize $\mathfrak{r}$ in the standard matrix form and introduce the following notations:

$$h_1 = E_{11} + E_{33}, h_2 = E_{11} + E_{22}, f = E_{32}, e = E_{23}, e^+ = E_{13}, e^- = E_{12},$$
$$f^- = E_{31}, f^+ = E_{21}, C = h_1 + h_2.$$

Let $\pi : U(\mathfrak{r}) \to U(\mathfrak{r})/U(\mathfrak{r})(f-1)$ be the natural projection. We denote by Ω the quadratic Casimir element of $\mathfrak{r}$ and set

$$a = [f^-, e^+e^-] = h_1 e^- - e^+, \ b = [e^-, f^+f^-] = h_2 f^- - f^+.$$

The reader can easily check that $\pi(C), \pi(\Omega), \pi(e^-), \pi(f^-), \pi(a)$ and $\pi(b)$ belong to $\bar{W}_\chi^{\mathfrak{r}}$.

Lemma 3.5. *a) $\bar{W}_\chi^{\mathfrak{r}}$ is generated by $\pi(C), \pi(\Omega), \pi(e^-), \pi(f^-), \pi(a)$ and $\pi(b)$. It is clear that $\pi(\Omega)$ lies in the center of $\bar{W}_\chi^{\mathfrak{r}}$. The other defining relations are*

$$[\pi(C), \pi(e^-)] = \pi(e^-), [\pi(C), \pi(a)] = \pi(a),$$
$$[\pi(C), \pi(f^-)] = -\pi(f^-), [\pi(C), \pi(b)] = -\pi(b),$$
$$[\pi(e^-), \pi(f^-)] = 1, [\pi(a), \pi(b)] = \pi(\Omega),$$

and the commutators of all other odd generators are zero.

b) Let $c, d \in \mathbb{C}$, $c \neq 0$, U be the subalgebra in $\bar{W}^{\mathfrak{r}}_{\chi}$ generated by $\pi(C), \pi(\Omega), \pi(a)$ and $\pi(e^-)$, $U_{c,d} = U/(\pi(\Omega) - c, \pi(C) - d, \pi(a), \pi(e^-))$ be the one-dimensional U-module. The induced module $M_{c,d} = \bar{W}^{\mathfrak{r}}_{\chi} \otimes_U U_{c,d}$ is simple and has dimension 4. The product $\prod_{c,d\in\mathbb{C}} M_{c,d}$ is a faithful $\bar{W}^{\mathfrak{r}}_{\chi}$-module.

Proof. We leave the proof to the reader. For assertion (a) one should use a suitable modification of Proposition 2.7 (a). □

We also leave to the reader the proof of Proposition 3.3 in the last case since it is completely similar to the previous case. □

In what follows we denote by $\mathcal{A}$ the image $\vartheta(W_\chi)$ of W_χ in $\bar{W}^{\mathfrak{s}}_{\chi}$.

Corollary 3.6. *W_χ satisfies (3.1).*

Proof. By Proposition 3.1, $\mathcal{A} \simeq W_\chi$. By Proposition 3.3, $\mathcal{A}$ satisfies (3.1). □

Proposition 3.7. *Let M be a simple W_χ-module. Then $\dim M \leq 2^{k+1}$.*

Proof. Consider M as a module over the associative algebra W_χ, forgetting the $\mathbb{Z}_2$-grading. Then either M is simple or M is a direct sum of two non-homogeneous simple submodules: $M = M_1 \oplus M_2$.

In the former case we claim that $\dim M \leq 2^k$. Indeed, assume $\dim M > 2^k$. Let V be a subspace of dimension $2^k + 1$. By the density theorem for any $X_1, \dots, X_{2^{k+1}} \in \mathrm{End}_{\mathbb{C}}(V)$ one can find $u_1, \dots, u_{2^{k+1}}$ in W_χ such that $(u_i)_{|V} = X_i$ for all $i = 1, \dots, 2^{k+1}$. Since $\mathrm{End}_{\mathbb{C}}(V)$ does not satisfy (3.1), we obtain contradiction with Corollary 3.6.

In the latter case, we can prove in the same way that $\dim M_1 \leq 2^k$ and $\dim M_2 \leq 2^k$. Therefore $\dim M \leq 2^{k+1}$.

□

Conjecture 3.8. Every irreducible representation of $\mathcal{A} \simeq W_\chi$ is isomorphic to a subquotient of some irreducible representation of $\bar{W}^{\mathfrak{s}}_{\chi}$ restricted to $\mathcal{A}$.

4. Generators of W_χ for the queer Lie superalgebra $Q(n)$

In the rest of the paper we study in detail the case when χ is regular and $\mathfrak{g} = Q(n)$.

In this section we construct some generators of W_χ. In particular, we will prove that W_χ is finitely generated. We use the elements $e^{(m)}_{i,j}$ and $f^{(m)}_{i,j}$ of $U(Q(n))$ defined in [29] recursively:

$$e^{(m)}_{i,j} = \sum_{k=1}^{n} e_{i,k} e^{(m-1)}_{k,j} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} f^{(m-1)}_{k,j}, \tag{4.1}$$

$$f^{(m)}_{i,j} = \sum_{k=1}^{n} e_{i,k} f^{(m-1)}_{k,j} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} e^{(m-1)}_{k,j}.$$

Then

$$[e_{i,j}, e^{(m)}_{k,l}] = \delta_{j,k} e^{(m)}_{i,l} - \delta_{i,l} e^{(m)}_{k,j}, \quad [e_{i,j}, f^{(m)}_{k,l}] = \delta_{j,k} f^{(m)}_{i,l} - \delta_{i,l} f^{(m)}_{kj},$$
$$[f_{i,j}, e^{(m)}_{k,l}] = (-1)^{m+1}\delta_{j,k} f^{(m)}_{i,l} - \delta_{i,l} f^{(m)}_{k,j},$$
$$[f_{i,j}, f^{(m)}_{k,l}] = (-1)^{m+1}\delta_{j,k} e^{(m)}_{i,l} + \delta_{i,l} e^{(m)}_{k,j}. \tag{4.2}$$

Proposition 4.1. *A. Sergeev* [29].
The elements $\sum_{i=1}^{n} e^{(2m+1)}_{i,i}$ *generate* $Z(Q(n))$.

Remark 4.2. In contrast with the Lie algebra case the center $Z(Q(n))$ is not Noetherian, in particular, it is not finitely generated.

Lemma 4.3. $\pi(e^{(m)}_{n,1})$ *and* $\pi(f^{(m)}_{n,1})$ *belong to* W_χ.

Proof. By (4.2) we have that

$$[e_{i,j}, e^{(m)}_{n,1}] = [f_{i,j}, e^{(m)}_{n,1}] = [e_{i,j}, f^{(m)}_{n,1}] = [f_{i,j}, f^{(m)}_{n,1}] = 0$$

for all $i > j$. In other words, $e^{(m)}_{n,1}, f^{(m)}_{n,1} \in U(\mathfrak{g})^{ad\mathfrak{m}}$. Hence $\pi(e^{(m)}_{n,1}), \pi(f^{(m)}_{n,1}) \in W_\chi$. □

Lemma 4.4. *Let* $1 \le l \le n-1$. *Then*

$$\pi(e^{(l)}_{m,1}) = \begin{cases} 1 & if \quad m = l+1, \\ 0 & if \quad l+2 \le m \le n, \end{cases} \qquad \pi(f^{(l)}_{m,1}) = 0, \ if\, l+1 \le m \le n. \tag{4.3}$$

Proof. We will prove the statement by induction in l. For $l = 1$ we have that

$$\pi(e^{(1)}_{m,1}) = \pi(e_{m,1}), \quad \pi(f^{(1)}_{m,1}) = \pi(f_{m,1}). \tag{4.4}$$

Then (4.3) follows from (2.6). Assume that (4.3) holds for l. From (4.1) we have that

$$e^{(l+1)}_{m,1} = \sum_{k=1}^{n} e_{m,k} e^{(l)}_{k,1} + (-1)^l \sum_{k=1}^{n} f_{m,k} f^{(l)}_{k,1},$$
$$f^{(l+1)}_{m,1} = \sum_{k=1}^{n} e_{m,k} f^{(l)}_{k,1} + (-1)^l \sum_{k=1}^{n} f_{m,k} e^{(l)}_{k,1}. \tag{4.5}$$

Note that

$$[e_{m,k}, e^{(l)}_{k,1}] = e^{(l)}_{m,1}, \quad [e_{m,k}, f^{(l)}_{k,1}] = f^{(l)}_{m,1},$$
$$[f_{m,k}, e^{(l)}_{k,1}] = (-1)^{l+1} f^{(l)}_{m,1}, \quad [f_{m,k}, f^{(l)}_{k,1}] = (-1)^{l+1} e^{(l)}_{m,1}. \tag{4.6}$$

Hence

$$e^{(l+1)}_{m,1} = \sum_{k=1}^{m-1} (e^{(l)}_{k,1} e_{m,k} + e^{(l)}_{m,1}) + \sum_{k=m}^{n} e_{m,k} e^{(l)}_{k,1} + (-1)^l \Big(\sum_{k=1}^{m-1} (-f^{(l)}_{k,1} f_{m,k} + (-1)^{l+1} e^{(l)}_{m,1}) + \sum_{k=m}^{n} f_{m,k} f^{(l)}_{k,1} \Big),$$

$$f^{(l+1)}_{m,1} = \sum_{k=1}^{m-1} (f^{(l)}_{k,1} e_{m,k} + f^{(l)}_{m,1}) + \sum_{k=m}^{n} e_{m,k} f^{(l)}_{k,1} + (-1)^l \Big(\sum_{k=1}^{m-1} (e^{(l)}_{k,1} f_{m,k} + (-1)^{l+1} f^{(l)}_{m,1}) + \sum_{k=m}^{n} f_{m,k} e^{(l)}_{k,1} \Big).$$

Then

$$\pi(e_{m,1}^{(l+1)}) = \sum_{k=1}^{m-1} \pi(e_{k,1}^{(l)})\pi(e_{m,k}) + \sum_{k=m}^{n}(\pi(e_{m,k})\pi(e_{k,1}^{(l)}) + (-1)^l\pi(f_{m,k})\pi(f_{k,1}^{(l)}))+$$
$$(-1)^{l+1}\Big(\sum_{k=1}^{m-1}\pi(f_{k,1}^{(l)})\pi(f_{m,k})\Big),$$
$$\pi(f_{m,1}^{(l+1)}) = \sum_{k=1}^{m-1} \pi(f_{k,1}^{(l)})\pi(e_{m,k}) + \sum_{k=m}^{n}(\pi(e_{m,k})\pi(f_{k,1}^{(l)}) + (-1)^l\pi(f_{m,k})\pi(e_{k,1}^{(l)}))+$$
$$(-1)^{l}\Big(\sum_{k=1}^{m-1}\pi(e_{k,1}^{(l)})\pi(f_{m,k})\Big).$$

Then by (2.6)

$$\pi(e_{m,1}^{(l+1)}) = \pi(e_{m-1,1}^{(l)}) + \sum_{k=m}^{n}\Big(\pi(e_{m,k})\pi(e_{k,1}^{(l)}) + (-1)^l\pi(f_{m,k})\pi(f_{k,1}^{(l)})\Big), \tag{4.7}$$

$$\pi(f_{m,1}^{(l+1)}) = \pi(f_{m-1,1}^{(l)}) + \sum_{k=m}^{n}\Big(\pi(e_{m,k})\pi(f_{k,1}^{(l)}) + (-1)^l\pi(f_{m,k})\pi(e_{k,1}^{(l)})\Big). \tag{4.8}$$

Let $m \geq l+2$. Then by induction hypothesis,

$$\pi(e_{k,1}^{(l)}) = \pi(f_{k,1}^{(l)}) = 0 \text{ for } k = m, \dots, n. \tag{4.9}$$

If $m = l+2$, then $\pi(e_{m,1}^{(l+1)}) = \pi(e_{l+1,1}^{(l)}) = 1$, and if $m \geq l+3$, then $\pi(e_{m,1}^{(l+1)}) = \pi(e_{m-1,1}^{(l)}) = 0$. Also, if $m \geq l+2$, then $\pi(f_{m,1}^{(l+1)}) = \pi(f_{m-1,1}^{(l)}) = 0$. Hence (4.3) holds for $l+1$. □

Corollary 4.5.

$$\pi(e_{n,1}^{(m)}) = 0 \textit{ for } m \leq n-2, \quad \pi(e_{n,1}^{(n-1)}) = 1; \quad \pi(f_{n,1}^{(m)}) = 0 \textit{ for } m \leq n-1, \tag{4.10}$$

Lemma 4.6.

$$\pi(e_{n,1}^{(n)}) = \pi(z), \quad \pi(f_{n,1}^{(n)}) = \pi(H_0).$$

Proof. Let $1 \leq m \leq n$. We will show that

$$\pi(e_{m,1}^{(m)}) = \sum_{k=1}^{m}\pi(e_{k,k}), \quad \pi(f_{m,1}^{(m)}) = \sum_{k=1}^{m}(-1)^{k-1}\pi(f_{k,k}). \tag{4.11}$$

Again we proceed by induction on m. If $m=1$, then (4.11) obviously holds by (4.4). Assume that (4.11) holds for m. From (4.1) and (4.2) we have that

$$e^{(m+1)}_{m+1,1} = \sum_{k=1}^{n} e_{m+1,k}e^{(m)}_{k,1} + (-1)^m \sum_{k=1}^{n} f_{m+1,k}f^{(m)}_{k,1} = \sum_{k=1}^{m}\left(e^{(m)}_{k,1}e_{m+1,k} + e^{(m)}_{m+1,1}\right) + \sum_{k=m+1}^{n} e_{m+1,k}e^{(m)}_{k,1} +$$
$$(-1)^m\Big(\sum_{k=1}^{m}(-f^{(m)}_{k,1}f_{m+1,k} + (-1)^{m+1}e^{(m)}_{m+1,1}) + \sum_{k=m+1}^{n} f_{m+1,k}f^{(m)}_{k,1}\Big),$$
$$f^{(m+1)}_{m+1,1} = \sum_{k=1}^{n} e_{m+1,k}f^{(m)}_{k,1} + (-1)^m \sum_{k=1}^{n} f_{m+1,k}e^{(m)}_{k,1} = \sum_{k=1}^{m}\left(f^{(m)}_{k,1}e_{m+1,k} + f^{(m)}_{m+1,1}\right) + \sum_{k=m+1}^{n} e_{m+1,k}f^{(m)}_{k,1} +$$
$$(-1)^m\Big(\sum_{k=1}^{m}(e^{(m)}_{k,1}f_{m+1,k} + (-1)^{m+1}f^{(m)}_{m+1,1}) + \sum_{k=m+1}^{n} f_{m+1,k}e^{(m)}_{k,1}\Big).$$

Using (2.6) and (4.3) we obtain

$$\pi(e^{(m+1)}_{m+1,1}) = \pi(e^{(m)}_{m,1}) + \pi(e_{m+1,m+1}),$$
$$\pi(f^{(m+1)}_{m+1,1}) = \pi(f^{(m)}_{m,1}) + (-1)^m\pi(f_{m+1,m+1}).$$

By induction hypothesis we have

$$\pi(e^{(m+1)}_{m+1,1}) = \sum_{k=1}^{m}\pi(e_{k,k}) + \pi(e_{m+1,m+1}) = \sum_{k=1}^{m+1}\pi(e_{k,k}),$$
$$\pi(f^{(m+1)}_{m+1,1}) = \sum_{k=1}^{m}(-1)^{k-1}\pi(f_{k,k}) + (-1)^m\pi(f_{m+1,m+1}) = \sum_{k=1}^{m+1}(-1)^{k-1}\pi(f_{k,k}).$$

Hence $\pi(e^{(n)}_{n,1}) = \pi(z)$ and $\pi(f^{(n)}_{n,1}) = \pi(H_0)$. □

Consider the Kazhdan filtration on $U(\mathfrak{b})$. By definition, the graded algebra $Gr_K U(\mathfrak{b})$ is isomorphic to $S(\mathfrak{b})$. Moreover, $Gr_K U(\mathfrak{b}) \simeq S(\mathfrak{b})$ is a commutative graded ring, where the grading is induced from the Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$. For any $X \in U(\mathfrak{b})$ let $Gr_K(X)$ denote the corresponding element in $Gr_K U(\mathfrak{b})$ and $P(X)$ denote the highest weight component of $Gr_K(X)$ in the Dynkin $\mathbb{Z}$-grading. For $X \in U(\mathfrak{b})$, we denote by $\deg P(X)$ the Kazhdan degree of $Gr_K(X)$ and by $\mathrm{wt} P(X)$ the weight of the highest weight component of $Gr_K(X)$.

Lemma 4.7. $P(\pi(e^{(n)}_{n,1})) = z$,

$$P(\pi(e^{(n-1+k)}_{n,1})) = e^{k-1}, k = 2, \ldots, n, \tag{4.12}$$
$$P(\pi(f^{(n-1+k)}_{n,1})) = H_{k-1}, k = 1, \ldots, n.$$

Proof. We will prove a more general statement. We claim that for $0 \leq l \leq n-1$ and $1 \leq p \leq n$

$$P(\pi(e_{p,1}^{(p+l)})) = \sum_{i=1}^{r} e_{i,i+l},$$
$$P(\pi(f_{p,1}^{(p+l)})) = \sum_{i=1}^{r} (-1)^{l+1-i} f_{i,i+l},\ r = \min\{p, n-l\}. \tag{4.13}$$

In particular,

$$\deg P(\pi(e_{p,1}^{(p+l)})) = \deg P(\pi(f_{p,1}^{(p+l)})) = 2l+2,$$
$$\mathrm{wt} P(\pi(e_{p,1}^{(p+l)})) = \mathrm{wt} P(\pi(f_{p,1}^{(p+l)})) = 2l.$$

We proceed to the proof of (4.13) by induction on l and p. Note that if $l=0$, then (4.13) holds for any $1 \leq p \leq n$ by (4.11). Assume that if $l \leq k-1$, then (4.13) holds for any $1 \leq p \leq n$. Let $l = k$. Show that (4.13) holds for $p=1$. Note that

$$e_{1,1}^{(1+k)} = \Big(\sum_{i=1}^{n} e_{1,i} e_{i,1}^{(k)}\Big) + (-1)^k \Big(\sum_{i=1}^{n} f_{1,i} f_{i,1}^{(k)}\Big).$$

Let $X = \pi(e_{1,i} e_{i,1}^{(k)})$, $Y = \pi(f_{1,i} f_{i,1}^{(k)})$ where $i = 1, \ldots, k$. Note that

$$\deg P(\pi(e_{1,i})) = \deg P(\pi(f_{1,i})) = 2i,$$
$$\mathrm{wt} P(\pi(e_{1,i})) = \mathrm{wt} P(\pi(f_{1,i})) = 2i-2.$$

By induction hypothesis,

$$\deg P(\pi(e_{1,i}^{(k)})) = \deg P(\pi(f_{1,i}^{(k)})) = 2(k-i)+2,$$
$$\mathrm{wt} P(\pi(e_{1,i}^{(k)})) = \mathrm{wt} P(\pi(f_{1,i}^{(k)})) = 2(k-i).$$

Then

$$\deg P(X) = 2k+2, \quad \mathrm{wt} P(X) = 2k-2,$$
$$\deg P(Y) = 2k+2, \quad \mathrm{wt} P(Y) = 2k-2.$$

Let $X = \pi(e_{1,k+1} e_{k+1,1}^{(k)})$. Then by (4.3) $X = \pi(e_{1,k+1})$. Hence

$$\deg P(X) = 2k+2, \quad \mathrm{wt} P(X) = 2k.$$

Finally, by (4.3)

$$\pi(e_{k+i,1}^{(k)}) = 0 \text{ for } i = 2, \ldots, n-k, \quad \pi(f_{k+i,1}^{(k)}) = 0 \text{ for } i = 1, \ldots, n-k.$$

Hence

$$P(\pi(e_{1,1}^{(1+k)})) = e_{1,k+1}.$$

Let $l=k$ and assume that (4.13) holds for $p\le m$. Show that it holds for $p=m+1$. Note that

$$e^{(m+1+k)}_{m+1,1}=\Big(\sum_{i=1}^{n}e_{m+1,i}e^{(m+k)}_{i,1}\Big)+(-1)^{m+k}\Big(\sum_{i=1}^{n}f_{m+1,i}f^{(m+k)}_{i,1}\Big).$$

Thus

$$\pi(e^{(m+1+k)}_{m+1,1})=\Big(\sum_{i=1}^{m-1}\pi(e_{m+1,i}e^{(m+k)}_{i,1})\Big)+\pi(e_{m+1,m}e^{(m+k)}_{m,1})+$$
$$\sum_{i=1}^{k}\pi(e_{m+1,m+i}e^{(m+k)}_{m+i,1})+\pi(e_{m+1,m+k+1}e^{(m+k)}_{m+k+1,1})+\sum_{i=2}^{n-m-k}\pi(e_{m+1,m+k+i}e^{(m+k)}_{m+k+i,1})+$$
$$(-1)^{m+k}\Big(\sum_{i=1}^{m}\pi(f_{m+1,i}f^{(m+k)}_{i,1})+\sum_{i=1}^{k}\pi(f_{m+1,m+i}f^{(m+k)}_{m+i,1})+$$
$$\sum_{i=1}^{n-m-k}\pi(f_{m+1,m+k+i}f^{(m+k)}_{m+k+i,1})\Big).$$

Let $X=\pi(e_{m+1,i}e^{(m+k)}_{i,1})$, where $i=1,\dots,m-1$, and $Y=\pi(f_{m+1,i}f^{(m+k)}_{i,1})$, where $i=1,\dots,m$. Then by (4.2) and (2.6)

$$\begin{aligned}X&=\pi(e^{(m+k)}_{i,1}e_{m+1,i}+e^{(m+k)}_{m+1,1})=\pi(e^{(m+k)}_{m+1,1}),\\ Y&=\pi(-f^{(m+k)}_{i,1}f_{m+1,i}+(-1)^{m+k+1}e^{(m+k)}_{m+1,1})=\pi((-1)^{m+k+1}e^{(m+k)}_{m+1,1}).\end{aligned}$$

By induction hypothesis

$$\begin{aligned}&\deg P(X)=\deg P(Y)=2k,\\ &\mathrm{wt}P(X)=\mathrm{wt}P(Y)=2k-2.\end{aligned}\tag{4.14}$$

Let $X=\pi(e_{m+1,m}e^{(m+k)}_{m,1})$. Then by (4.2) and (2.6)

$$X=\pi(e^{(m+k)}_{m,1}e_{m+1,m}+e^{(m+k)}_{m+1,1})=\pi(e^{(m+k)}_{m,1}+e^{(m+k)}_{m+1,1}).$$

By induction hypothesis

$$\begin{aligned}&\deg P(\pi(e^{(m+k)}_{m+1,1}))=2k,\\ &\mathrm{wt}P(\pi(e^{(m+k)}_{m+1,1}))=2k-2,\end{aligned}\tag{4.15}$$

$$\begin{aligned}&\deg P(\pi(e^{(m+k)}_{m,1}))=2k+2,\\ &\mathrm{wt}P(\pi(e^{(m+k)}_{m,1}))=2k.\end{aligned}\tag{4.16}$$

Let $X = \pi(e_{m+1,m+i}e^{(m+k)}_{m+i,1})$, $Y = \pi(f_{m+1,m+i}f^{(m+k)}_{m+i,1})$ for $i = 1,\dots,k$. Then by induction hypothesis

$$\begin{aligned}\deg P(X) = \deg P(Y) = 2k+2,\\ \mathrm{wt}P(X) = \mathrm{wt}P(Y) = 2k-2.\end{aligned} \tag{4.17}$$

Let $X = \pi(e_{m+1,m+k+1}e^{(m+k)}_{m+k+1,1})$. Hence by (4.3) $X = \pi(e_{m+1,m+k+1})$. Then

$$\begin{aligned}\deg P(X) = 2k+2,\\ \mathrm{wt}P(X) = 2k.\end{aligned} \tag{4.18}$$

Finally, by (4.3) $\pi(e_{m+1,m+k+i}e^{(m+k)}_{m+k+i,1}) = 0$ for $i = 2,\dots,n-m-k$ and $\pi(f_{m+1,m+k+i}f^{(m+k)}_{m+k+i,1}) = 0$ for $i = 1,\dots,n-m-k$. From (4.14)-(4.18) one can see that the highest degree component in $\pi(e^{(m+1+k)}_{m+1,1})$ has degree $2k+2$, and its highest weight component has weight $2k$. In fact, if $m \geq n-k$, then by (4.16) this component is $P(\pi(e^{(m+k)}_{m,1}))$. By induction hypothesis $P(\pi(e^{(m+k)}_{m,1})) = \sum_{i=1}^{n-k} e_{i,i+k}$. If $m < n-k$, then $P(\pi(e^{(m+k)}_{m,1})) = \sum_{i=1}^{m} e_{i,i+k}$. Note that in this case $\pi(e^{(m+1+k)}_{m+1,1})$ has an additional element $\pi(e_{m+1,m+k+1})$ of degree $2k+2$ and weight $2k$ according to (4.18). Clearly, $P(\pi(e_{m+1,m+k+1})) = e_{m+1,m+k+1}$ and $P(\pi(e^{(m+k)}_{m,1})) + P(\pi(e_{m+1,m+k+1})) \neq 0$. Hence

$$P(\pi(e^{(m+1+k)}_{m+1,1})) = P(\pi(e^{(m+k)}_{m,1})) + P(\pi(e_{m+1,m+k+1})) = \sum_{i=1}^{m+1} e_{i,i+k}.$$

Then in either case,

$$P(\pi(e^{(m+1+k)}_{m+1,1})) = \sum_{i=1}^{r} e_{i,i+k}, \text{ where } r = \min\{m+1, n-k\}.$$

Thus if $0 \leq l \leq n-1$ and $1 \leq p \leq n$, then

$$P(\pi(e^{(p+l)}_{p,1})) = \sum_{i=1}^{r} e_{i,i+l}, \text{ where } r = \min\{p, n-l\}.$$

Similarly, one can prove that

$$P(\pi(f^{(p+l)}_{p,1})) = \sum_{i=1}^{r} (-1)^{l+1-i} f_{i,i+l}, \ r = \min\{p, n-l\}.$$

In particular, if $p = n$ and $l = k$, where $k = 0,\dots,n-1$, we have

$$P(\pi(e^{(n+k)}_{n,1})) = \sum_{i=1}^{n-k} e_{i,i+k} = e^k,$$

$$P(\pi(f^{(n+k)}_{n,1})) = \sum_{i=1}^{n-k} (-1)^{k+1-i} f_{i,i+k} = H_k.$$

□

Proposition 4.8. *$\pi(e^{(m)}_{n,1})$ and $\pi(f^{(m)}_{n,1})$ for $m = n, \dots, 2n-1$ generate W_χ.*

Proof. The statement follows from Lemma 4.7 and Proposition 2.7 (a). □

Corollary 4.9. *Lemma 4.7 and Proposition 2.7 (b) imply that Conjecture 2.8 is true for $\mathfrak{g} = Q(n)$ and regular χ.*

Corollary 4.10. *The natural homomorphism $U(\mathfrak{g})^{ad\mathfrak{m}} \to W_\chi$ is surjective.*

Proof. Since $e^{(m)}_{n,1}, f^{(m)}_{n,1} \in U(\mathfrak{g})^{ad\mathfrak{m}}$, the statement follows from Proposition 4.8. □

5. Further results about the structure of W_χ for $\mathfrak{g} = Q(n)$

5.1. **The Harish-Chandra homomorphism for $Q(n)$.** Recall that for $\mathfrak{g} = Q(n)$ and regular χ we have $\mathfrak{p} = \mathfrak{b}$. We study in detail the restriction of the Harish-Chandra homomorphism $\vartheta : U(\mathfrak{b}) \longrightarrow U(\mathfrak{h})$ to W_χ. We start with calculating the images of the generators.

Proposition 5.1.

$$\vartheta(\pi(e^{(n+k-1)}_{n,1})) = [\sum_{i_1 \geq i_2 \geq \dots \geq i_k} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even}, \tag{5.1}$$

$$\vartheta(\pi(f^{(n+k-1)}_{n,1})) = [\sum_{i_1 \geq i_2 \geq \dots \geq i_k} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

Proof. We will prove by induction on l and p that if $0 \leq l \leq n-1$ and $1 \leq p \leq n$ then

$$\vartheta(\pi(e^{(p+l)}_{p,1})) + \vartheta(\pi(f^{(p+l)}_{p,1})) = \sum_{p \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} (x_{i_1} + (-1)^l \xi_{i_1}) \dots (x_{i_l} - \xi_{i_l})(x_{i_{l+1}} + \xi_{i_{l+1}}). \tag{5.2}$$

Note that if $l = 0$, then (5.2) holds for any $1 \leq p \leq n$ since by (4.11)

$$\vartheta(\pi(e^{(p)}_{p,1})) + \vartheta(\pi(f^{(p)}_{p,1})) = \vartheta(\sum_{i=1}^{p} \pi(e_{i,i})) + \vartheta(\sum_{i=1}^{p} (-1)^{i-1}\pi(f_{i,i})) = \sum_{i=1}^{p} (x_i + \xi_i).$$

Assume that if $l \leq k-1$, then (5.2) holds for any $1 \leq p \leq n$. Let $l = k$, show that (5.2) holds for $p = 1$. We have

$$\vartheta(\pi(e_{1,1}^{(1+k)})) + \vartheta(\pi(f_{1,1}^{(1+k)})) =$$
$$\vartheta(\pi(e_{1,1})\pi(e_{1,1}^{(k)}) + (-1)^k\pi(f_{1,1})\pi(f_{1,1}^{(k)})) + \vartheta(\pi(e_{1,1})\pi(f_{1,1}^{(k)}) + (-1)^k\pi(f_{1,1})\pi(e_{1,1}^{(k)})) =$$
$$(e_{1,1} + (-1)^k f_{1,1})(\vartheta(\pi(e_{1,1}^{(k)}) + \vartheta(\pi(f_{1,1}^{(k)})) =$$
$$(x_1 + (-1)^k\xi_1)\sum_{i_1=i_2=\ldots=i_k=1}(x_{i_1} + (-1)^{k-1}\xi_{i_1})\ldots(x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) =$$
$$\sum_{i_1=i_2=\ldots=i_{k+1}=1}(x_{i_1} + (-1)^k\xi_{i_1})\ldots(x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}}).$$

Let $l = k$ and assume that (5.2) holds for $p \leq m$. Show that it holds for $p = m+1$. By induction hypothesis we have

$$\vartheta(\pi(e_{m+1,1}^{(m+1+k)})) + \vartheta(\pi(f_{m+1,1}^{(m+1+k)})) = \vartheta(\pi(e_{m,1}^{(m+k)})) + \vartheta(\pi(f_{m,1}^{(m+k)}))+$$
$$(e_{m+1,m+1} + (-1)^{m+k}f_{m+1,m+1})\vartheta(\pi(e_{m+1,1}^{(m+k)})) + \vartheta(\pi(f_{m+1,1}^{(m+k)})) =$$
$$\sum_{m\geq i_1\geq i_2\geq\ldots\geq i_{k+1}\geq 1}(x_{i_1} + (-1)^k\xi_{i_1})\ldots(x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}})+$$
$$(x_{m+1} + (-1)^k\xi_{m+1})\sum_{m+1\geq i_1\geq i_2\geq\ldots\geq i_k\geq 1}(x_{i_1} + (-1)^{k-1}\xi_{i_1})\ldots(x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) =$$
$$\sum_{m+1\geq i_1\geq i_2\geq\ldots\geq i_{k+1}\geq 1}(x_{i_1} + (-1)^k\xi_{i_1})\ldots(x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}}).$$

Thus (5.2) is proven. In particular, if $p = n$ we obtain (5.1). □

Proposition 5.2.

$$\pi(e_{n,1}^{(n+1)}) = \pi(\frac{1}{2}\sum_{i=1}^{n} e_{i,i}^2 + \sum_{i=1}^{n-1} e_{i,i+1} + \sum_{i<j}(-1)^{i-j}f_{i,i}f_{j,j} + \frac{1}{2}z^2 - z), \tag{5.3}$$

and

$$\vartheta(\pi(e_{n,1}^{(n+1)})) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \sum_{i<j}\xi_i\xi_j + \frac{1}{2}z^2 - z. \tag{5.4}$$

Proof. We will prove by induction on m that for $1 \leq m \leq n$

(5.5)

$$\pi(e_{m,1}^{(m+1)}) = \pi(\frac{1}{2}\sum_{i=1}^{m} e_{i,i}^2 + \sum_{i=1}^{min(m,n-1)} e_{i,i+1} + \sum_{1\leq i<j\leq m}(-1)^{i-j}f_{i,i}f_{j,j} + \frac{1}{2}(\sum_{i=1}^{m} e_{i,i})^2 - \sum_{i=1}^{m} e_{i,i}).$$

If $m = 1$, then

$$\pi(e_{1,1}^{(2)}) = \pi(e_{1,1}^2 + e_{1,2} - f_{1,1}^2) = \pi(e_{1,1}^2 + e_{1,2} - e_{1,1}).$$

Assume that (5.5) holds for m. By (4.1)

$$e^{(m+2)}_{m+1,1} = \sum_{i=1}^{n} e_{m+1,i} e^{(m+1)}_{i,1} - (-1)^m \sum_{k=i}^{n} f_{m+1,i} f^{(m+1)}_{i,1}.$$

Then by (2.6) and (4.3)

$$\pi(e^{(m+2)}_{m+1,1}) = \pi(e^{(m+1)}_{m,1}) + \pi(e_{m+1,m+1})\pi(e^{(m+1)}_{m+1,1}) + \pi(e_{m+1,m+2})\pi(e^{(m+1)}_{m+2,1}) - (-1)^m \pi(f_{m+1,m+1})\pi(f^{(m+1)}_{m+1,1}).$$

By induction hypothesis and using (4.11) we have

$$\pi(e^{(m+2)}_{m+1,1}) = \pi\Big(\frac{1}{2}\sum_{i=1}^{m} e^2_{i,i} + \sum_{i=1}^{min(m,n-1)} e_{i,i+1} + \sum_{1\leq i<j\leq m} (-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2}(\sum_{i=1}^{m} e_{i,i})^2 - \sum_{i=1}^{m} e_{i,i}\Big) +$$

$$\pi\Big(e_{m+1,m+1}(\sum_{i=1}^{m+1} e_{i,i}) + e_{m+1,m+2} - (-1)^m f_{m+1,m+1}(\sum_{i=1}^{m+1} (-1)^{i-1} f_{i,i})\Big) =$$

$$\pi\Big(\frac{1}{2}\sum_{i=1}^{m+1} e^2_{i,i} + \sum_{i=1}^{min(m+1,n-1)} e_{i,i+1} + \sum_{1\leq i<j\leq m+1} (-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2}(\sum_{i=1}^{m+1} e_{i,i})^2 - \sum_{i=1}^{m+1} e_{i,i}\Big).$$

Thus (5.5) is proven. In particular, if $m = n$ we obtain (5.3). Finally, applying ϑ to (5.3) we obtain (5.4). □

5.2. **On the center of W_χ.** Recall that we denote by $\mathcal{A}$ the image $\vartheta(W_\chi)$ of W_χ in $U(\mathfrak{h})$. Set $\mathcal{A}^0 = \mathcal{A} \cap U(\mathfrak{h}_{\bar{0}})$.

Lemma 5.3. *Define odd elements $\Phi_0, \dots, \Phi_{n-1}$ of W_χ as follows:*

$$\Phi_0 = \pi(f^{(n)}_{n,1}) = \pi(H_0),$$
$$\Phi_k = \Big(\frac{1}{2}\,\mathrm{ad}(\pi(e^{(n+1)}_{n,1}))\Big)^k(\Phi_0), \quad k = 1, \dots, n-1.$$

Then (a) $P(\Phi_k) = H_k$,

(b)

$$[\Phi_m, \Phi_p] = 0, \text{ if } m + p \text{ is odd },$$

(c) there exist $z_0, z_2, \dots \in \pi(Z(Q(n)))$ such that

$$[\Phi_m, \Phi_p] = (-1)^m z_{m+p} \text{ if } m + p \text{ is even.}$$

Proof. Let $X, Y \in W_\chi$. To prove (a) observe that if $P(X), P(Y) \in \mathfrak{g}^\chi$ and $[P(X), P(Y)] \neq 0$, then $P([X, Y]) = [P(X), P(Y)]$. Since $P(\pi(e^{(n+1)}_{n,1})) = e$ and $P(\Phi_0) = H_0$, the statement follows from the relation

$$H_k = (\frac{1}{2}\,\mathrm{ad}(e))^k(H_0).$$

To prove (b) and (c) we use ϑ. We first notice that (5.4) implies

$$\vartheta(\Phi_k) = \sum_{j=1}^{n} \phi_j^{(k)} \xi_j$$

for some polynomial $\phi_j^{(k)} \in \mathbb{C}[x_1, \dots, x_n]$ of degree k. Hence $[\vartheta(\Phi_m), \vartheta(\Phi_p)] \in \mathbb{C}[x_1, \dots, x_n]$. Since x_i lie in the center of $\mathfrak{h}$, we get

$$[\vartheta(\Phi_{m+1}), \vartheta(\Phi_p)] = \frac{1}{2}[[\vartheta(\pi(e_{n,1}^{(n+1)})), \vartheta(\Phi_m)], \theta(\Phi_p)] =$$
$$-\frac{1}{2}[\vartheta(\Phi_m), [\vartheta(\pi(e_{n,1}^{(n+1)})), \vartheta(\Phi_p)]] = -[\vartheta(\Phi_m), \vartheta(\Phi_{p+1})].$$

Since ϑ is injective, that implies

$$[\Phi_p, \Phi_q] = (-1)^{r-p}[\Phi_r, \Phi_s],$$

if $p + q = r + s$.

In particular, if $p + q$ is odd we have

$$[\Phi_p, \Phi_q] = (-1)^{q-p}[\Phi_q, \Phi_p] = 0.$$

This implies (b).

To prove (c) we set

$$z_i = [\Phi_0, \Phi_i] \qquad \text{for even } i, \quad 0 \le i \le n-1.$$

Since $\vartheta(z_i) \in \mathcal{A}^0$, (c) follows from Lemma 5.4. □

Lemma 5.4. $\mathcal{A}^0 = \vartheta(\pi(Z(\mathfrak{g})))$.

Proof. It is not hard to see that the restriction of ϑ on $Z(\mathfrak{g})$ coincides with the standard Harish-Chandra homomorphism. Thus, from Sergeev's result, [30], we know that $\vartheta(Z(\mathfrak{g}))$ coincides with the space of symmetric polynomials p in $x_1, \dots, x_n$ satisfying the additional condition

$$\frac{\partial p}{\partial x_i} - \frac{\partial p}{\partial x_j} \in (x_i + x_j)U(\mathfrak{h}_{\bar{0}}) \tag{5.6}$$

for all $i < j \le n$. In view of Proposition 3.1, it is sufficient to prove that if $p \in \mathcal{A}^0$, then p is symmetric and satisfies (5.6).

First, we will prove the last assertion in the case when $n = 2$. It follows from Lemma 5.3 and Theorem 5.1 that $\mathcal{A}$ is generated by $z_0 = 2x_1 + 2x_2$, $\phi_0 = \xi_1 + \xi_2$, $\phi_1 = x_2\xi_1 - x_1\xi_2$ and $z_1 = -\vartheta(\pi(e_{2,1}^3)) + \frac{1}{4}z_0^2 - \frac{1}{2}z_0 = x_1x_2 - \xi_1\xi_2$. By direct calculation we can check that

$$\phi_0^2 = \frac{1}{2}z_0, \phi_0\phi_1 = -\frac{1}{2}z_0\xi_1\xi_2, \phi_1^2 = \frac{1}{2}z_0x_1x_2, [z_1, \phi_0] = -2\phi_1, [z_1, \phi_1] = 2x_1x_2\phi_0.$$

Let $\mathcal{A}_0$ denote the even part of $\mathcal{A}$. The above relations imply that $\mathcal{A}_0$ is a subring in $\mathbb{C}[z_0, x_1x_2] \oplus \mathbb{C}[z_0, x_1x_2]\xi_1\xi_2$. Moreover, $\mathcal{A}_0/(z_0\mathcal{A}_0) = \mathbb{C}[z_1]$. Therefore $\mathcal{A}^0 = \mathbb{C} \oplus z_0\mathbb{C}[z_0, x_1x_2]$, i.e. $\mathcal{A}^0$ consists of symmetric polynomials satisfying (5.6).

Let $p = \vartheta(u)$ for some $u \in W_\chi \subset U(\mathfrak{b})$. Then we have

$$\pi(\operatorname{ad} e_{i+1,i}(u)) = 0 \tag{5.7}$$

and

$$\pi(\operatorname{ad} f_{i+1,i}(u)) = 0 \tag{5.8}$$

for all $i = 1, \dots, n-1$.

Let $\mathfrak{s}_i$ be the subalgebra in $\mathfrak{g}$ generated by $e_{i,i+1}, e_{i,i}, e_{i+1,i}, e_{i+1,i+1}, f_{i,i+1}, f_{i,i}, f_{i+1,i}, f_{i+1,i+1}$. Clearly, $\mathfrak{s}_i$ is isomorphic to $Q(2)$. Note that the orthogonal compliment $\mathfrak{s}_i^\perp$ (with respect to the invariant form) is $\operatorname{ad}\mathfrak{s}_i$-invariant, $\mathfrak{b} \cap \mathfrak{s}_i^\perp$ is a Lie subalgebra and, moreover,

$$\pi(\operatorname{ad} e_{i+1,i}(u)) = 0, \quad \pi(\operatorname{ad} f_{i+1,i}(u)) = 0$$

whenever $u \in U(\mathfrak{b} \cap \mathfrak{s}_i^\perp)$.

Therefore any $u \in W_\chi$ satisfying (5.7) and (5.8) for a given i can be written in the form $u = \sum u_j v_j$ for some $u_j \in U(\mathfrak{s}_i \cap \mathfrak{b})$ satisfying (5.7) and (5.8) and arbitrary $v_j \in U(\mathfrak{s}_i^\perp \cap \mathfrak{b})$.

Thus, (5.7) and (5.8) can be checked locally for $\mathfrak{s}_i$. Indeed, if $\vartheta(u) \in U(\mathfrak{h}_{\bar{0}})$, then

$$\vartheta(u) = \sum \vartheta(u_j)\vartheta(v_j),$$

where $\vartheta(v_j) \in U(\mathfrak{h}_{\bar{0}} \cap \mathfrak{s}_i^\perp) = \mathbb{C}[x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n]$ and $\vartheta(u_j) \in U(\mathfrak{h}_{\bar{0}} \cap \mathfrak{s}_i) = \mathbb{C}[x_i, x_{i+1}]$. Since we already know the result for $Q(2)$, we obtain $\vartheta(u_j)(x_{i+1}, x_i) = \vartheta(u_j)(x_i, x_{i+1})$ and $\frac{\partial\vartheta(u_j)}{\partial x_i} - \frac{\partial\vartheta(u_j)}{\partial x_{i+1}} \in (x_i + x_{i+1})U(\mathfrak{h}_{\bar{0}} \cap \mathfrak{s}_i)$. Therefore $\vartheta(u)$ is invariant under all adjacent transpositions and therefore is symmetric. Moreover,

$$\frac{\partial\vartheta(u)}{\partial x_i} - \frac{\partial\vartheta(u)}{\partial x_{i+1}} \in (x_i + x_{i+1})U(\mathfrak{h}_{\bar{0}}).$$

Since $\vartheta(u)$ is symmetric, the last condition implies (5.6) for $\vartheta(u)$. □

Let $\phi_k := \vartheta(\Phi_k)$. Consider $U(\mathfrak{h})$ as a free $U(\mathfrak{h}_{\bar{0}})$-module and let V denote the free submodule generated by $\xi_1, \dots, \xi_n$. Then V is equipped with $U(\mathfrak{h}_{\bar{0}})$-valued bilinear symmetric form $B(x, y) = [x, y]$. If $\omega = \vartheta(\pi(\frac{1}{2}e_{n,1}^{(n+1)}))$, then $T = \operatorname{ad}\omega$ is an $U(\mathfrak{h}_{\bar{0}})$-linear operator. As we have seen in the proof of Lemma 5.3, T is skew-symmetric with respect to the form B, i.e.

$$B(Tv, w) + B(v, Tw) = 0.$$

Furthermore, in these terms $\phi_k = T^k(\phi_0)$. The matrix of T in the standard basis $\xi_1, \dots, \xi_n$ has 0 on the diagonal and

$$t_{ij} = \begin{cases} x_j & if \quad i < j, \\ -x_j & if \quad i > j. \end{cases} \tag{5.9}$$

Lemma 5.5. *The characteristic polynomial* $\det(\lambda \operatorname{Id} - T)$ *of* T *equals*

$$\lambda^n + \sigma_2 \lambda^{n-2} + \cdots + \sigma_{2k} \lambda^{n-2k},$$

where $\sigma_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \dots x_{i_r}$ *are the elementary symmetric functions.*

Proof. Let

$$p_n(x_1, \dots, x_n; \lambda) = \det(\lambda \operatorname{Id} - T) = \lambda^n + \sum_{i=1}^{n} f_{n,i}(x_1, \dots, x_n) \lambda^{n-i}.$$

Note that $f_{n,i}(x_1, \dots, x_n)$ is a symmetric polynomial, since the substitutions $x_i \mapsto x_j, x_j \mapsto x_i$ preserves the determinant of $\lambda \operatorname{Id} - T$. It is also easy to calculate that $\det T = x_1 \dots x_n$ if n is even. If n is odd, then $\det T = 0$, since T is skew-symmetric with respect to B. Finally, if $x_n = 0$ we have a relation

$$p_n(x_1, \dots, x_{n-1}, 0; \lambda) = \lambda p_{n-1}(x_1, \dots, x_{n-1}; \lambda).$$

That implies

$$f_{n,i}(x_1, \dots, x_{n-1}, 0) = f_{n-1,i}(x_1, \dots, x_{n-1}),$$

for $i \leq n-1$. Since it is also easy to show that the degree of $f_{n,i}$ is i, we can finish the proof by induction in n. $\square$

Corollary 5.6. *There exists* $s = (s_1, \dots, s_n) \in \mathbb{R}^n_{>0}$ *such that the specialization of* $\det(\lambda \operatorname{Id} - T)$ *at the point* $x_1 = s_1, \dots, x_n = s_n$ *has distinct eigenvalues.*

Proof. Assume that $n = 2k$ is even. Let Pol^{ev}_n denote the set of monic even polynomials in $\mathbb{C}[\lambda]$ of degree n and $\mathrm{Pol}^{ev,+}_n$ denote the subset of polynomials with real positive coefficients. Let $\varphi : \mathbb{R}^n_{>0} \to \mathrm{Pol}^{ev,+}_n$ be the specialization map, i.e. $\varphi(s)$ is the specialization of $\det(\lambda \operatorname{Id} - T)$ at $s \in \mathbb{R}^n_{>0}$. From the above Lemma, $d\varphi(s)$ is surjective for generic $s \in \mathbb{R}^n_{>0}$. Therefore $\mathrm{Im}\varphi$ contains a non-empty open subset in $\mathbb{R}^n_{>0}$.

Define the map $\rho : \mathbb{C}^n \to \mathrm{Pol}^{ev}_n$ by the formula

$$\rho(t_1, \dots, t_k) = \prod_{i=1}^{k} (\lambda^2 - t_i^2).$$

Obviously, ρ is surjective. Set

$$\mathcal{U} = \{(t_1, \dots, t_k) \in \mathbb{C}^k \mid t_i \neq \pm t_j \text{ for all } i \neq j\}.$$

Then $\rho(\mathcal{U})$ is Zariski open in Pol^{ev}_n. Therefore the intersection $\mathrm{Pol}^{ev,+}_n \cap \rho(\mathcal{U})$ is a non-empty Zariski open subset in $\mathrm{Pol}^{ev,+}_n$. Hence $\mathrm{Pol}^{ev,+}_n \cap \rho(\mathcal{U})$ is dense in $\mathrm{Pol}^{ev,+}_n$ in the usual topology and the intersection $\mathrm{Im}\varphi \cap \rho(\mathcal{U})$ is not empty. This implies the statement for even n.

For odd n the proof is similar and we leave it to the reader. $\square$

Lemma 5.7. $\phi_0, \dots \phi_{n-1}$ *are linearly independent over* $U(\mathfrak{h}_{\bar{0}})$.

Proof. For each $s=(s_1,\dots,s_n)\in\mathbb{C}^n$ consider the ideal $I_s=(x_1-s_1,\dots,x_n-s_n)\in U(\mathfrak{h}_{\bar 0})$. Let $V_s=V/I_sV$ and T_s, B_s and $(\phi_i)_s$ denote the corresponding operator, form and vector in V_s. It suffices to show that $(\phi_0)_s,\dots(\phi_{n-1})_s$ are linearly independent for some s. By Corollary 5.6 we can find $s\in\mathbb{R}^n_{>0}$ such that all eigenvalues of T_s are distinct. Let $v_1,\dots,v_n$ denote an eigenbasis for T_s, and let H_s denote the Hermitian form such that $H_s(\xi_i,\xi_j)=B_s(\xi_i,\xi_j)$ for all $i,j=1,\dots,n$. Then H_s is positive definite and T_s is skew-hermitian with respect to H_s. Hence all eigenvalues of T_s are purely imaginary and $H_s(v_i,v_j)=0$ if $i\neq j$. Let $(\phi_0)_s=\sum_{i=1}^n a_iv_i$. Since all eigenvalues of H_s are distinct and $(\phi_l)_s=T^l(\phi_0)_s$, linear independence of $(\phi_0)_s,\dots(\phi_{n-1})_s$ is equivalent to the fact that a_i are not zero for all $i=1,\dots n$. Assume that some $a_i=0$. Since $a_i=\frac{H_s(v_i,(\phi_0)_s)}{H_s(v_i,v_i)}$, that implies $H_s(v_i,(\phi_0)_s)=0$. Let $v_i=t_1\xi_1+\dots+t_n\xi_n$. Then the last condition implies $\sum_{i=1}^n s_it_i=0$. But then the first coordinate of T_sv_i equals $s_2t_2+\dots+s_nt_n=-s_1t_1$. Since $T_sv_i=av_i$ for some purely imaginary a, we obtain $t_1=0$. Repeating this argument we can prove by induction that all t_i are zero and obtain a contradiction. □

Problem. Calculate $\vartheta(\Phi_i)$ and $\vartheta(z_i)$.

Lemma 5.8. *The centralizer of $\mathcal{A}$ in $U(\mathfrak{h})$ coincides with $U(\mathfrak{h}_{\bar 0})$.*

Proof. Suppose that u lies in the centralizer of $\mathcal{A}$. Recall that F denotes the field of fractions of $U(\mathfrak{h}_{\bar 0})$. Then since $U(\mathfrak{h})$ is a free $U(\mathfrak{h}_{\bar 0})$-module, $U(\mathfrak{h})\subset U(\mathfrak{h})_F$. By Lemma 5.7, $\mathcal{A}_F$ contains $\xi_1,\dots,\xi_n$. Hence we have $[\xi_i,u]=0$ for all $i=1,\dots,n$. Therefore u lies in the center of $U(\mathfrak{h})$, which coincides with $U(\mathfrak{h}_{\bar 0})$. □

Corollary 5.9. *The center of $\mathcal{A}$ coincides with $\mathcal{A}^0$.*

Proposition 3.1, Lemma 5.4 and Corollary 5.9 imply.

Corollary 5.10. *The center of W_χ coincides with $\pi(Z(Q(n)))$.*

5.3. **New generators and relations.** We will need the following realization of $Q(n)$ given by M. Nazarov and S. Sergeev in [21]. Let the indices i,j run through $-n,\dots,-1,1,\dots,n$. Put $p(i)=0$ if $i>0$ and $p(i)=1$ if $i<0$. As a vector space $Q(n)$ is spanned by the elements

$$F_{ij}=E_{ij}+E_{-i,-j}.$$

Note that $F_{-i,-j}=F_{ij}$. The elements F_{ij} with $i>0$ form a basis of $Q(n)$.

For any indices $n\geq 1$ and $i,j=\pm1,\dots,\pm n$, we denote by $F^{(m)}_{ij}$ the following element of $U(Q(n))$:

$$F^{(m)}_{ij}=\sum_{k_1,\dots,k_{m-1}}(-1)^{p(k_1)+\dots+p(k_{m-1})}F_{ik_1}F_{k_1k_2}\dots F_{k_{m-2}k_{m-1}}F_{k_{m-1}j}. \tag{5.10}$$

Note that

$$F^{(m)}_{ij}=(-1)^{m-1}F^{(m)}_{-i,-j}, \tag{5.11}$$

$$F_{ij}^{(m)} = e_{ij}^{(m)}, \quad \text{for } i,j>0, \tag{5.12}$$

$$F_{ij}^{(m)} = (-1)^{m+1} f_{-i,j}^{(m)}, \quad \text{for } i<0, j>0. \tag{5.13}$$

Proposition 5.11. *For odd k and m we have*

$$[\pi(e_{n,1}^{(n+k)}), \pi(e_{n,1}^{(n+m)})] = 0. \tag{5.14}$$

Proof. We prove the statement by induction on $l = k+m$. Obviously, if $l=2$, then (5.14) is true. Assume that the statement is true for odd k and m such that $k+m \le l-2$. According to [21]

$$[F_{n,1}^{(m)}, F_{n,1}^{(k)}] = \sum_{r=1}^{m-1}[F_{n,1}^{(k+r-1)}, F_{n,1}^{(m-r)}] + \sum_{r=1}^{m-1}(-1)^r(F_{-n,1}^{(k+r-1)}F_{-n,1}^{(m-r)} - F_{n,-1}^{(m-r)}F_{n,-1}^{(k+r-1)}).$$

Thus from (5.11), (5.12), (5.13) we have

$$[e_{n,1}^{(m)}, e_{n,1}^{(k)}] = \sum_{r=1}^{m-1}[e_{n,1}^{(k+r-1)}, e_{n,1}^{(m-r)}] + \sum_{r=1}^{m-1}(-1)^{r+1}((-1)^{k+m}f_{n,1}^{(k+r-1)}f_{n,1}^{(m-r)} + f_{n,1}^{(m-r)}f_{n,1}^{(k+r-1)}). \tag{5.15}$$

Furthermore, from [21]

$$[F_{-n,1}^{(m)}, F_{-n,1}^{(k)}] = \sum_{r=1}^{m-1}(F_{-n,1}^{(k+r-1)}F_{-n,1}^{(m-r)} - F_{-n,1}^{(m-r)}F_{-n,1}^{(k+r-1)})$$
$$+ \sum_{r=1}^{m-1}(-1)^{r+1}(F_{n,1}^{(k+r-1)}F_{n,1}^{(m-r)} - F_{-n,-1}^{(m-r)}F_{-n,-1}^{(k+r-1)}).$$

Thus from (5.11), (5.12), (5.13) we have

$$[f_{n,1}^{(m)}, f_{n,1}^{(k)}] = -\sum_{r=1}^{m-1}(f_{n,1}^{(k+r-1)}f_{n,1}^{(m-r)} - f_{n,1}^{(m-r)}f_{n,1}^{(k+r-1)}) \tag{5.16}$$
$$+ \sum_{r=1}^{m-1}(-1)^{r+1}((-1)^{k+m}e_{n,1}^{(k+r-1)}e_{n,1}^{(m-r)} + e_{n,1}^{(m-r)}e_{n,1}^{(k+r-1)}).$$

Lemma 5.12. *For odd m we have*

$$[\pi(f_{n,1}^{(n)}), \pi(f_{n,1}^{(n+m)})] = 0.$$

Proof. From (5.16)

$$[\pi(f_{n,1}^{(n)}), \pi(f_{n,1}^{(n+m)})] = -\sum_{r=1}^{m-1}\Big(\pi(f_{n,1}^{(n+m+r-1)})\pi(f_{n,1}^{(n-r)}) - \pi(f_{n,1}^{(n-r)})\pi(f_{n,1}^{(n+m+r-1)})\Big)$$
$$+ \sum_{r=1}^{m-1}(-1)^{r+1}\Big(-\pi(e_{n,1}^{(n+m+r-1)})\pi(e_{n,1}^{(n-r)}) + \pi(e_{n,1}^{(n-r)})\pi(e_{n,1}^{(n+m+r-1)})\Big).$$

Note that the first sum is zero, since $\pi(f_{n,1}^{(n-r)})=0$ for $r\geq 1$ by (4.10), and the second sum is also zero, since $\pi(e_{n,1}^{(n-1)})=1$ and $\pi(e_{n,1}^{(n-r)})=0$ for $r\geq 2$ by (4.10). □

Let

(5.17)
$$[e_{n,1}^{(n+k)},e_{n,1}^{(n+m)}]^e=[e_{n,1}^{(n+m)},e_{n,1}^{(n+k-1)}]+[e_{n,1}^{(n+m+1)},e_{n,1}^{(n+k-2)}]+\ldots+[e_{n,1}^{(n+m+k-2)},e_{n,1}^{(n+1)}],$$

(5.18)
$$[e_{n,1}^{(n+k)},e_{n,1}^{(n+m)}]^f=[f_{n,1}^{(n+m)},f_{n,1}^{(n+k-1)}]-[f_{n,1}^{(n+m+1)},f_{n,1}^{(n+k-2)}]+\ldots-[f_{n,1}^{(n+m+k-2)},f_{n,1}^{(n+1)}].$$

Then
$$[\pi(e_{n,1}^{(n+k)}),\pi(e_{n,1}^{(n+m)})]=\pi([e_{n,1}^{(n+k)},e_{n,1}^{(n+m)}]^e)+\pi([e_{n,1}^{(n+k)},e_{n,1}^{(n+m)}]^f),$$
since by (4.10)
$$\pi(e_{n,1}^{(n)})=\pi(z),\quad \pi(e_{n,1}^{(n-1)})=1,\quad \pi(e_{n,1}^{(n-r)})=0 \text{ for } r\geq 2,\quad \pi(f_{n,1}^{(n-r)})=0 \text{ for } r\geq 1,$$
and by Lemma 5.12
$$[\pi(f_{n,1}^{(n+m+k-1)}),\pi(f_{n,1}^{(n)})]=0,$$
since $m+k-1$ is odd.

Note that each of the sums in (5.17) and (5.18) has $k-1$ terms, where $k-1$ is even.

Denote by $[e_{n,1}^{(m)},e_{n,1}^{(k)}]_e$ and by $[e_{n,1}^{(m)},e_{n,1}^{(k)}]_f$ the first and the second sum in (5.15), respectively. Thus
$$[e_{n,1}^{(m)},e_{n,1}^{(k)}]=[e_{n,1}^{(m)},e_{n,1}^{(k)}]_e+[e_{n,1}^{(m)},e_{n,1}^{(k)}]_f.$$
Also, denote by $[f_{n,1}^{(m)},f_{n,1}^{(k)}]_f$ and by $[f_{n,1}^{(m)},f_{n,1}^{(k)}]_e$ the first and the second sum in (5.16), respectively. Thus
$$[f_{n,1}^{(m)},f_{n,1}^{(k)}]=[f_{n,1}^{(m)},f_{n,1}^{(k)}]_f+[f_{n,1}^{(m)},f_{n,1}^{(k)}]_e.$$

Let
$$A^m=\pi([e_{n,1}^{(n+m)},e_{n,1}^{(n+k-1)}]+[e_{n,1}^{(n+m+1)},e_{n,1}^{(n+k-2)}]+[f_{n,1}^{(n+m)},f_{n,1}^{(n+k-1)}]-[f_{n,1}^{(n+m+1)},f_{n,1}^{(n+k-2)}]).$$
We claim that $A^m=0$. Note that $A^m=A^m_e+A^m_f$, where
$$A^m_e=\pi([e_{n,1}^{(n+m)},e_{n,1}^{(n+k-1)}]_e+[e_{n,1}^{(n+m+1)},e_{n,1}^{(n+k-2)}]_e+[f_{n,1}^{(n+m)},f_{n,1}^{(n+k-1)}]_e-[f_{n,1}^{(n+m+1)},f_{n,1}^{(n+k-2)}]_e),$$
$$A^m_f=\pi([e_{n,1}^{(n+m)},e_{n,1}^{(n+k-1)}]_f+[e_{n,1}^{(n+m+1)},e_{n,1}^{(n+k-2)}]_f+[f_{n,1}^{(n+m)},f_{n,1}^{(n+k-1)}]_f-[f_{n,1}^{(n+m+1)},f_{n,1}^{(n+k-2)}]_f).$$
Let us show that $A^m_e=0$. Note that
$$[e_{n,1}^{(n+m)},e_{n,1}^{(n+k-1)}]_e=[e_{n,1}^{(n+k-1)},e_{n,1}^{(n+m-1)}]+[e_{n,1}^{(n+k)},e_{n,1}^{(n+m-2)}]+$$
$$[e_{n,1}^{(n+k+1)},e_{n,1}^{(n+m-3)}]+[e_{n,1}^{(n+k+2)},e_{n,1}^{(n+m-4)}]+\ldots,$$
$$[f_{n,1}^{(n+m)},f_{n,1}^{(n+k-1)}]_e=-[e_{n,1}^{(n+k-1)},e_{n,1}^{(n+m-1)}]+[e_{n,1}^{(n+k)},e_{n,1}^{(n+m-2)}]$$
$$-[e_{n,1}^{(n+k+1)},e_{n,1}^{(n+m-3)}]+[e_{n,1}^{(n+k+2)},e_{n,1}^{(n+m-4)}]+\ldots.$$

Thus

$[e_{n,1}^{(n+m)}, e_{n,1}^{(n+k-1)}]_e + [f_{n,1}^{(n+m)}, f_{n,1}^{(n+k-1)}]_e = 2[e_{n,1}^{(n+k)}, e_{n,1}^{(n+m-2)}] + 2[e_{n,1}^{(n+k+2)}, e_{n,1}^{(n+m-4)}] + \ldots.$

By induction hypothesis

$$[\pi(e_{n,1}^{(n+k)}), \pi(e_{n,1}^{(n+m-2)})] = [\pi(e_{n,1}^{(n+k+2)}), \pi(e_{n,1}^{(n+m-4)})] = \ldots = 0$$

for positive $k, m-2, m-4, \ldots$ Note that (5.14) also holds for odd k, m such that $k \leq -1$ or $m \leq -1$ by (4.10). Similarly

$[e_{n,1}^{(n+m+1)}, e_{n,1}^{(n+k-2)}]_e - [f_{n,1}^{(n+m+1)}, f_{n,1}^{(n+k-2)}]_e = 2[e_{n,1}^{(n+k-2)}, e_{n,1}^{(n+m)}] + 2[e_{n,1}^{(n+k)}, e_{n,1}^{(n+m-2)}] + \ldots = 0.$

By induction hypothesis

$$[\pi(e_{n,1}^{(n+k-2)}), \pi(e_{n,1}^{(n+m)})] = [\pi(e_{n,1}^{(n+k)}), \pi(e_{n,1}^{(n+m-2)})] = \ldots = 0.$$

Hence $A_e^m = 0$. Let us show that $A_f^m = 0$. Note that

$[e_{n,1}^{(n+m)}, e_{n,1}^{(n+k-1)}]_f = -[e_{n,1}^{(n+k-1)}, e_{n,1}^{(n+m)}]_f =$
$(f_{n,1}^{(n+m)} f_{n,1}^{(n+k-2)} - f_{n,1}^{(n+k-2)} f_{n,1}^{(n+m)}) - (f_{n,1}^{(n+m+1)} f_{n,1}^{(n+k-3)} - f_{n,1}^{(n+k-3)} f_{n,1}^{(n+m+1)}) + \ldots,$
$[e_{n,1}^{(n+m+1)}, e_{n,1}^{(n+k-2)}]_f = -[e_{n,1}^{(n+k-2)}, e_{n,1}^{(n+m+1)}]_f =$
$(f_{n,1}^{(n+m+1)} f_{n,1}^{(n+k-3)} - f_{n,1}^{(n+k-3)} f_{n,1}^{(n+m+1)}) - (f_{n,1}^{(n+m+2)} f_{n,1}^{(n+k-4)} - f_{n,1}^{(n+k-4)} f_{n,1}^{(n+m+2)}) + \ldots,$
$[f_{n,1}^{(n+m)}, f_{n,1}^{(n+k-1)}]_f = [f_{n,1}^{(n+k-1)}, f_{n,1}^{(n+m)}]_f =$
$-(f_{n,1}^{(n+m)} f_{n,1}^{(n+k-2)} - f_{n,1}^{(n+k-2)} f_{n,1}^{(n+m)}) - (f_{n,1}^{(n+m+1)} f_{n,1}^{(n+k-3)} - f_{n,1}^{(n+k-3)} f_{n,1}^{(n+m+1)}) - \ldots,$
$-[f_{n,1}^{(n+m+1)}, f_{n,1}^{(n+k-2)}]_f = -[f_{n,1}^{(n+k-2)}, f_{n,1}^{(n+m+1)}]_f =$
$(f_{n,1}^{(n+m+1)} f_{n,1}^{(n+k-3)} - f_{n,1}^{(n+k-3)} f_{n,1}^{(n+m+1)}) + (f_{n,1}^{(n+m+2)} f_{n,1}^{(n+k-4)} - f_{n,1}^{(n+k-4)} f_{n,1}^{(n+m+2)}) + \ldots.$

In the sum of the right-hand sides of these equations all terms cancel out. Hence $A_f^m = 0$. Then $A^m = 0$. Similarly,

$$A^{m+2} = A^{m+4} = \ldots = A^{m+k-3} = 0.$$

Then

$$[\pi(e_{n,1}^{(n+k)}), \pi(e_{n,1}^{(n+m)})] = \sum_{i=0}^{\frac{1}{2}(k-3)} A^{m+2i} = 0.$$

□

We set

$$z_i = \pi(e_{n,1}^{(n+i)}) \quad \text{for odd } i, \quad 1 \leq i \leq n-1.$$

Theorem 5.13. *Elements $z_0, \ldots, z_{n-1}$ are algebraically independent in W_χ. Together with $\Phi_0, \ldots, \Phi_{n-1}$ they form a complete set of generators in W_χ.*

Proof. By Lemma 5.3, we have $P(\Phi_i) = H_i$ for $i \leq n-1$, $P(z_i) = e^i$ for even $0 < i \leq n-1$ and $P(z_0) = z$. By Lemma 4.7, $P(z_i) = e^i$ for odd $i \leq n-1$. Therefore the second assertion follows from Proposition 2.7. The algebraic independence of $z_0, \dots, z_{n-1}$ follows from algebraic independence of the corresponding elements in $S(\mathfrak{g}^\chi)$. □

Conjecture 5.14. Let $\mathfrak{g}$ be a basic classical Lie superalgebra and χ is regular. Then it is possible to find a set of generators of W_χ such that even generators commute, and the commutators of odd generators are in $\pi(Z(\mathfrak{g}))$.

6. Super-Yangian of $\mathbf{Q(n)}$

Super-Yangian $Y(Q(n))$ was studied by M. Nazarov and A. Sergeev [21]. Recall that $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{ij}^{(m)} \text{ where } m = 1, 2, \dots \text{ and } i, j = \pm 1, \pm 2, \dots, \pm n.$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{ij}^{(m)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0, \text{ and } p(i) = 1 \text{ if } i < 0.$$

To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots. \tag{6.1}$$

Then for all possible indices i, j, k, l we have the relations

$$\begin{aligned}(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\ = (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\ - (u-v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)},\end{aligned} \tag{6.2}$$

where v is a formal parameter independent of u, so that (6.2) is an equality in the algebra of formal Laurent series in u^{-1}, v^{-1} with coefficients in $Y(Q(n))$.
For all indices i, j we also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u). \tag{6.3}$$

Note that the relations (6.2) and (6.3) are equivalent to the following defining relations:

$$\begin{aligned}&([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = \\ &T_{k,j}^{(m)} T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)} T_{i,l}^{(r)} - T_{k,j}^{(r-1)} T_{i,l}^{(m)} - T_{k,j}^{(r)} T_{i,l}^{(m-1)} \\ &+ (-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)} T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)} T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)} T_{i,-l}^{(m)} - T_{k,-j}^{(r)} T_{i,-l}^{(m-1)}),\end{aligned} \tag{6.4}$$

$$T^{(m)}_{-i,-j} = (-1)^m T^{(m)}_{i,j}, \tag{6.5}$$

where $m, r = 1, \dots$ and $T^{(0)}_{ij} = \delta_{ij}$.

Theorem 6.1. *There exists a surjective homomorphism:*

$$\varphi : Y(Q(1)) \longrightarrow W_\chi$$

defined as follows:

$$\varphi(T^{(k)}_{1,1}) = (-1)^k \pi(e^{(n+k-1)}_{n,1}), \quad \varphi(T^{(k)}_{-1,1}) = (-1)^k \pi(f^{(n+k-1)}_{n,1}), \text{ for } k = 1, 2, \dots.$$

Proof. Note that even and odd generators of $Y(Q(1))$ are $T^{(m)}_{1,1}$ and $T^{(m)}_{-1,1}$, respectively, where $m = 1, 2, \dots$. We are going to check that the relations (6.4) for generators of $Y(Q(1))$ are preserved by φ. We separate this checking in the following three cases.
Case 1: Even generators. We want first to check that φ preserves the relation

(6.6)
$$[T^{(m)}_{1,1}, T^{(p)}_{1,1}] - [T^{(m-2)}_{1,1}, T^{(p+2)}_{1,1}] =$$
$$T^{(m-1)}_{1,1} T^{(p)}_{1,1} + T^{(m-2)}_{1,1} T^{(p+1)}_{1,1} - T^{(p)}_{1,1} T^{(m-1)}_{1,1} - T^{(p+1)}_{1,1} T^{(m-2)}_{1,1} +$$
$$- T^{(m-1)}_{-1,1} T^{(p)}_{-1,1} + T^{(m-2)}_{-1,1} T^{(p+1)}_{-1,1} + (-1)^{m+p-1} T^{(p)}_{-1,1} T^{(m-1)}_{-1,1} - (-1)^{m+p-1} T^{(p+1)}_{-1,1} T^{(m-2)}_{-1,1}.$$

First, we will prove the relation

(6.7)
$$(-1)^{m+p}\Big([e^{(m+n-1)}_{n,1}, e^{(p+n-1)}_{n,1}] - [e^{(m+n-3)}_{n,1}, e^{(p+n+1)}_{n,1}]\Big) =$$
$$(-1)^{m+p-1}\Big(e^{(m+n-2)}_{n,1} e^{(p+n-1)}_{n,1} + e^{(m+n-3)}_{n,1} e^{(p+n)}_{n,1} - e^{(p+n-1)}_{n,1} e^{(m+n-2)}_{n,1} - e^{(p+n)}_{n,1} e^{(m+n-3)}_{n,1}\Big) +$$
$$(-1)^{m+p-1}\Big(- f^{(m+n-2)}_{n,1} f^{(p+n-1)}_{n,1} + f^{(m+n-3)}_{n,1} f^{(p+n)}_{n,1}\Big) +$$
$$f^{(p+n-1)}_{n,1} f^{(m+n-2)}_{n,1} - f^{(p+n)}_{n,1} f^{(m+n-3)}_{n,1}.$$

Note that

$$[e^{(m+n-1)}_{n,1}, e^{(p+n-1)}_{n,1}]_e - [e^{(m+n-3)}_{n,1}, e^{(p+n+1)}_{n,1}]_e = \sum_{r=1}^{2} [e^{(p+n+r-2)}_{n,1}, e^{(m+n-r-1)}_{n,1}], \tag{6.8}$$

$$\begin{aligned}&[e^{(m+n-1)}_{n,1}, e^{(p+n-1)}_{n,1}]_f - [e^{(m+n-3)}_{n,1}, e^{(p+n+1)}_{n,1}]_f = \\ &\sum_{r=1}^{2} (-1)^{r+1} ((-1)^{m+p} f^{(p+n+r-2)}_{n,1} f^{(m+n-r-1)}_{n,1} + f^{(m+n-r-1)}_{n,1} f^{(p+n+r-2)}_{n,1}).\end{aligned} \tag{6.9}$$

Multiplying the sum of equations (6.8) and (6.9) by $(-1)^{m+p}$, we obtain (6.7).

By Lemma 4.4, the application of π to (6.7) implies that φ preserves the relation (6.6).

Case 2: Odd generators. Next we will check that φ preserves the relation

$$\begin{aligned}&-([T^{(m)}_{-1,1}, T^{(p)}_{-1,1}] - [T^{(m-2)}_{-1,1}, T^{(p+2)}_{-1,1}]) =\\ &T^{(m-1)}_{-1,1}T^{(p)}_{-1,1} + T^{(m-2)}_{-1,1}T^{(p+1)}_{-1,1} - T^{(p)}_{-1,1}T^{(m-1)}_{-1,1} - T^{(p+1)}_{-1,1}T^{(m-2)}_{-1,1}+\\ &T^{(m-1)}_{1,1}T^{(p)}_{1,1} - T^{(m-2)}_{1,1}T^{(p+1)}_{1,1} + (-1)^{m+p}T^{(p)}_{1,1}T^{(m-1)}_{1,1} - (-1)^{m+p}T^{(p+1)}_{1,1}T^{(m-2)}_{1,1}.\end{aligned}$$

We claim that the following relation holds

(6.10)

$$\begin{aligned}&(-1)^{m+p-1}\Big([f^{(m+n-1)}_{n,1}, f^{(p+n-1)}_{n,1}] - [f^{(m+n-3)}_{n,1}, f^{(p+n+1)}_{n,1}]\Big) =\\ &(-1)^{m+p-1}\Big(f^{(m+n-2)}_{n,1}f^{(p+n-1)}_{n,1} + f^{(m+n-3)}_{n,1}f^{(p+n)}_{n,1} - f^{(p+n-1)}_{n,1}f^{(m+n-2)}_{n,1} - f^{(p+n)}_{n,1}f^{(m+n-3)}_{n,1}\Big)+\\ &(-1)^{m+p-1}\Big(e^{(m+n-2)}_{n,1}e^{(p+n-1)}_{n,1} - e^{(m+n-3)}_{n,1}e^{(p+n)}_{n,1}\Big)+\\ &-e^{(p+n-1)}_{n,1}e^{(m+n-2)}_{n,1} + e^{(p+n)}_{n,1}e^{(m+n-3)}_{n,1}.\end{aligned}$$

Indeed, use

$$\begin{aligned}&[f^{(m+n-1)}_{n,1}, f^{(p+n-1)}_{n,1}]_f - [f^{(m+n-3)}_{n,1}, f^{(p+n+1)}_{n,1}]_f =\\ &-\sum_{r=1}^{2}(f^{(p+n+r-2)}_{n,1}f^{(m+n-r-1)}_{n,1} - f^{(m+n-r-1)}_{n,1}f^{(p+n+r-2)}_{n,1}),\end{aligned} \tag{6.11}$$

$$\begin{aligned}&[f^{(m+n-1)}_{n,1}, f^{(p+n-1)}_{n,1}]_e - [f^{(m+n-3)}_{n,1}, f^{(p+n+1)}_{n,1}]_e =\\ &\sum_{r=1}^{2}(-1)^{r+1}((-1)^{m+p}e^{(p+n+r-2)}_{n,1}e^{(m+n-r-1)}_{n,1} + e^{(m+n-r-1)}_{n,1}e^{(p+n+r-2)}_{n,1}).\end{aligned} \tag{6.12}$$

Multiplying the sum of equations (6.11) and (6.12) by $(-1)^{m+p-1}$, we obtain (6.10). The end of the proof is as in the previous case.

Case 3: Even and odd generators. Finally, we will check that φ preserves the relation

$$\begin{aligned}&[T^{(m)}_{1,1}, T^{(p)}_{-1,1}] - [T^{(m-2)}_{1,1}, T^{(p+2)}_{-1,1}] =\\ &T^{(m-1)}_{-1,1}T^{(p)}_{1,1} + T^{(m-2)}_{-1,1}T^{(p+1)}_{1,1} - T^{(p)}_{-1,1}T^{(m-1)}_{1,1} - T^{(p+1)}_{-1,1}T^{(m-2)}_{1,1}+\\ &T^{(m-1)}_{1,1}T^{(p)}_{-1,1} - T^{(m-2)}_{1,1}T^{(p+1)}_{-1,1} + (-1)^{m+p}T^{(p)}_{1,1}T^{(m-1)}_{-1,1} - (-1)^{m+p}T^{(p+1)}_{1,1}T^{(m-2)}_{-1,1}.\end{aligned}$$

We claim that the following relation holds

(6.13)
$$(-1)^{m+p}\Big([e_{n,1}^{(m+n-1)}, f_{n,1}^{(p+n-1)}] - [e_{n,1}^{(m+n-3)}, f_{n,1}^{(p+n+1)}]\Big) =$$
$$(-1)^{m+p-1}\Big(f_{n,1}^{(m+n-2)}e_{n,1}^{(p+n-1)} + f_{n,1}^{(m+n-3)}e_{n,1}^{(p+n)} - f_{n,1}^{(p+n-1)}e_{n,1}^{(m+n-2)} - f_{n,1}^{(p+n)}e_{n,1}^{(m+n-3)}\Big)+$$
$$(-1)^{m+p-1}\Big(e_{n,1}^{(m+n-2)}f_{n,1}^{(p+n-1)} - e_{n,1}^{(m+n-3)}f_{n,1}^{(p+n)}\Big)$$
$$- e_{n,1}^{(p+n-1)}f_{n,1}^{(m+n-2)} + e_{n,1}^{(p+n)}f_{n,1}^{(m+n-3)}.$$

According to [21]
$$[F_{n,1}^{(m)}, F_{-n,1}^{(k)}] = \sum_{r=1}^{m-1}(F_{n,1}^{(k+r-1)}F_{-n,1}^{(m-r)} - F_{n,1}^{(m-r)}F_{-n,1}^{(k+r-1)})+$$
$$\sum_{r=1}^{m-1}(-1)^{r+1}(F_{-n,1}^{(k+r-1)}F_{n,1}^{(m-r)} + (-1)^{m+k}F_{-n,1}^{(m-r)}F_{n,1}^{(k+r-1)}).$$

Thus from (5.11), (5.12), (5.13) we have
$$[e_{n,1}^{(m)}, f_{n,1}^{(k)}] = \sum_{r=1}^{m-1}(-1)^r((-1)^{m+k}e_{n,1}^{(k+r-1)}f_{n,1}^{(m-r)} + e_{n,1}^{(m-r)}f_{n,1}^{(k+r-1)})+ \tag{6.14}$$
$$\sum_{r=1}^{m-1}(f_{n,1}^{(k+r-1)}e_{n,1}^{(m-r)} - f_{n,1}^{(m-r)}e_{n,1}^{(k+r-1)}).$$

We denote by $[e_{n,1}^{(m)}, f_{n,1}^{(k)}]_{ef}$ and by $[e_{n,1}^{(m)}, f_{n,1}^{(k)}]_{fe}$ the first and the second sum in (6.14), respectively, then
$$[e_{n,1}^{(m)}, f_{n,1}^{(k)}] = [e_{n,1}^{(m)}, f_{n,1}^{(k)}]_{ef} + [e_{n,1}^{(m)}, f_{n,1}^{(k)}]_{fe}.$$

Note that
$$[e_{n,1}^{(m+n-1)}, f_{n,1}^{(p+n-1)}]_{ef} - [e_{n,1}^{(m+n-3)}, f_{n,1}^{(p+n+1)}]_{ef} = \tag{6.15}$$
$$\sum_{r=1}^{2}((-1)^{m+p}e_{n,1}^{(p+n+r-2)}f_{n,1}^{(m+n-r-1)} + e_{n,1}^{(m+n-r-1)}f_{n,1}^{(p+n+r-2)}),$$

$$[e_{n,1}^{(m+n-1)}, f_{n,1}^{(p+n-1)}]_{fe} - [e_{n,1}^{(m+n-3)}, f_{n,1}^{(p+n+1)}]_{fe} = \tag{6.16}$$
$$\sum_{r=1}^{2}(f_{n,1}^{(p+n+r-2)}e_{n,1}^{(m+n-r-1)} - f_{n,1}^{(m+n-r-1)}e_{n,1}^{(p+n+r-2)}).$$

Multiplying the sum of equations (6.15) and (6.16) by $(-1)^{m+p}$, we obtain (6.13). The proof can be finished by the same argument as in two previous cases.

□

References

1. A. S. Amitsur, J. Levitzki, Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449–463.
2. J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, Toda theory and W-algebra from a gauged WZNW point of view, *Ann. Physics* **203** (1990), 76–136.
3. C. Briot, E. Ragoucy, W-superalgebras as truncations of super-Yangians, *J. Phys. A* **36** (2003), no. 4, 1057–1081.
4. J. Brown, Twisted Yangians and finite W-algebras, *Transform. Groups* **14** (2009), 87–114.
5. J. Brown, J. Brundan, S. Goodwin, Principal W-algebras for $GL(m|n)$, Algebra Numb. Theory 7 (2013), 1849–1882.
6. J. Brundan, A. Kleshchev, Shifted Yangians and finite W-algebras, *Adv. Math.* **200** (2006), 136-195,
7. A. De Sole and V. Kac, Finite vs affine W-algebras, *Jpn. J. Math.* **1** (2006) 137–261.
8. L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, Generalized Toda theories and W-algebras associated with integral gradings, *Ann. Physics* **213** (1992) 1–20.
9. L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories, *Phys. Rep.* **222** (1992) 1–64.
10. C. Gruson, V. Serganova, Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras, Proc. Lond. Math. Soc. (3) 101 (2010), no. 3, 852892.
11. A.Joseph, Kostant's problem, Goldie rank and the Gelfand–Kirillov conjecture, *Invent. Math.* **56** (1980), 191–213.
12. C. Hoyt, Good gradings of basic Lie superalgebras, *Israel J. Math.* **192** (2012) 251–280.
13. V. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977) 8–96.
14. V. G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory. Lie theory and geometry, 415456, Progr. Math., 123, Birkhuser Boston, Boston, MA, 1994.
15. B. Kostant, On Wittaker vectors and representation theory, *Invent. Math.* **48** (1978) 101–184.
16. I. Losev, Finite W-algebras, *Proceedings of the International Congress of Mathematicians.* Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010. arXiv:1003.5811v1.
17. I. Losev, Quantized symplectic actions and W-algebras, *J. Amer. Math. Soc.* **23** (2010) 35–59.
18. I. Losev, Finite-dimensional representations of W-algebras, *Duke Math. J.* **159** (2011), 99–143.
19. A. Molev, Yangians and classical Lie algebras, Mathematical Surveys and Monographs, **143**, *Amer. Math. Soc.*, Providence, RI, 2007.
20. M. Nazarov, Yangian of the queer Lie superalgebra, *Comm. Math. Phys.* **208** (1999) 195–223.
21. M. Nazarov, A. Sergeev, Centralizer construction of the Yangian of the queer Lie superalgebra, Studies in Lie Theory, 417–441, *Progr. Math.* **243**, Birkhäuser Boston, Boston, MA, 2006.
22. E. Poletaeva, V. Serganova, On finite W-algebras for Lie superalgebras in the regular case, In: *Lie Theory and Its Applications in Physics*, V. Dobrev (ed.), IX International Workshop. 20-26 June 2011, Varna, Bulgaria. Springer Proceedings in Mathematics and Statistics, Vol. **36** (2013) 487–497.
23. E. Poletaeva, On Kostant's Theorem for Lie superalgebras, in M. Gorelik, P. Papi (eds.) *Advances in Lie Superalgebras*, Springer INdAM Series, Vol. **7** (2014) 167-180.
24. A. Premet, Special transverse slices and their enveloping algebras, *Adv. Math.* **170** (2002) 1–55.
25. A. Premet, Enveloping algebras of Slodowy slices and the Joseph ideal, *J. Eur. Math. Soc.* **9** (2007) 487–543.
26. A. Premet, Primitive ideals, non-restricted representations and finite W-algebras, *Mosc. Math. J.* **7** (2007) 743–762.

27. A. Premet, Enveloping algebras of Slodowy slices and Goldie rank, *Trans. groups*, **16** (2011) 857–888.
28. E. Ragoucy and P. Sorba, Yangian realizations from finite W-algebras, *Comm. Math. Phys.* **203** (1999) 551–572.
29. A. Sergeev, The centre of enveloping algebra for Lie superalgebra $Q(n, \mathbb{C})$, *Lett. Math. Phys.* 7 (1983) 177–179.
30. A. Sergeev, The invariant polynomials on simple Lie superalgebras, *Represent. Theory* 3 (1999) 250–280 (electronic).
31. W. Wang, Nilpotent orbits and finite W-algebras, Geometric representation theory and extended affine Lie algebras, 71–105, *Fields Inst. Commun.* **59**, Amer. Math. Soc., Providence, RI, 2011; arXiv:0912.0689v2.
32. L. Zhao, Finite W-superalgebras for queer Lie superalgebras. arXiv:1012.2326v2.

Dept. of Mathematics, University of Texas-Pan American, Edinburg, TX 78539
E-mail address: `elenap@utpa.edu`

Dept. of Mathematics, University of California at Berkeley, Berkeley, CA 94720
E-mail address: `serganov@math.berkeley.edu`