NORMAL LATTICE OF CERTAIN METABELIAN p-GROUPS G WITH $G/G' \simeq (p,p)$

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Abstract. Let p be an odd prime. The lattice of all normal subgroups and the terms of the lower and upper central series are determined for all metabelian p-groups with generator rank $d = 2$ having abelianization of type (p, p) and minimal defect of commutativity $k = 0$. It is shown that many of these groups are realized as Galois groups of second Hilbert p-class fields of an extensive set of quadratic fields which are characterized by principalization types of p-classes.

1. Introduction

Let $p \geq 3$ be an odd prime number, and $G = \langle x, y\rangle$ be a two-generated metabelian p-group having an elementary bicyclic derived quotient G/G' of type (p, p).

Assume further that G is of order $|G| = p^n$ with $n \geq 2$, and of nilpotency class $\mathrm{cl}(G) = m - 1$ with $m \geq 2$. Then G is of coclass $\mathrm{cc}(G) = n - m + 1 = e - 1$ with $e \geq 2$. Denote by

$$G = \gamma_1(G) > \gamma_2(G) = G' > \ldots > \gamma_{m-1}(G) > \gamma_m(G) = 1$$

the (descending) lower central series of G, where $\gamma_j(G) = [\gamma_{j-1}(G), G]$ for $j \geq 2$, and by

$$1 = \zeta_0(G) < \zeta_1(G) < \ldots < G' = \zeta_{m-2}(G) < \zeta_{m-1}(G) = G$$

the (ascending) upper central series of G, where $\zeta_j(G)/\zeta_{j-1}(G) = \mathrm{Centre}(G/\zeta_{j-1}(G))$ for $j \geq 1$.

Let $s_2 = t_2 = [y, x]$ denote the main commutator of G, such that $\gamma_2(G) = \langle s_2, \gamma_3(G)\rangle$. By means of the two series $s_j = [s_{j-1}, x]$ for $j \geq 3$ and $t_\ell = [t_{\ell-1}, y]$ for $\ell \geq 3$ of higher commutators and the subgroups $\Sigma_j = \langle s_j, \ldots, s_{m-1}\rangle$ with $j \geq 3$ and $T_\ell = \langle t_\ell, \ldots, t_{e+1}\rangle$ with $\ell \geq 3$, we obtain the following fundamental distinction of cases.

(1) The *uniserial* case of a CF group (*cyclic factors*) of coclass $\mathrm{cc}(G) = 1$ (maximal class), where $t_3 \in \Sigma_3$, $\gamma_3(G) = \langle s_3, \gamma_4(G)\rangle$, $e = 2$, and $m = n$. There are two subcases:
(1.1) $t_3 = 1 \in \gamma_m(G)$, where G contains an abelian maximal subgroup and $k = 0$,
(1.2) $1 \neq t_3 \in \gamma_{m-k}(G)$, $1 \leq k \leq m - 4$, where all maximal subgroups are non-abelian.

(2) The *biserial* case of a non-CF or BCF group (*bicyclic or cyclic factors*) of coclass $\mathrm{cc}(G) \geq 2$, where $t_3 \notin \Sigma_3$, $\gamma_3(G) = \langle s_3, t_3, \gamma_4(G)\rangle$, $e \geq 3$, and $m < n$. Again there exist two subcases, characterized by the *defect of commutativity* k of G:
(2.1) $t_{e+1} = 1 \in \gamma_m(G)$, where $\Sigma_3 \cap T_3 = 1$ and $k = 0$,
(2.2) $1 \neq t_{e+1} \in \gamma_{m-k}(G)$, for some $k \geq 1$, where $\Sigma_3 \cap T_3 \leq \gamma_{m-k}(G)$.

In this article, we are interested in two-generator metabelian p-groups $G = \langle x, y\rangle$ of coclass $\mathrm{cc}(G) \geq 2$ having the convenient property $\Sigma_3 \cap T_3 = 1$, resp. $k = 0$, where the product $\Sigma_3 \times T_3$ is direct and coincides with the major part of the *normal lattice* of G, as shown in Figure 1.

Definition 1.1. A pair (U, V) of normal subgroups of a p-group G, such that $V < U \leq G$ and $(U : V) = p^2$, is called a *diamond* if the quotient U/V is abelian of type (p, p).

If (U, V) is a diamond and $U = \langle u_1, u_2, V\rangle$, then the $p+1$ intermediate subgroups of G between U and V are given by $\langle u_2, V\rangle$ and $\langle u_1 u_2^{i-2}, V\rangle$ with $2 \leq i \leq p + 1$.

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2. The normal lattice

In this section, let $G = \langle x, y\rangle$ be a metabelian p-group with two generators x, y, having abelianization G/G' of type (p, p) and satisfying the independence condition $\Sigma_3 \cap T_3 = 1$, that is, G is a metabelian p-group with defect of commutativity $k = 0$ [14, § 3.1.1, p. 412, and § 3.3.2, p. 429]. We assume that G is of coclass $\mathrm{cc}(G) \geq 2$, since the normal lattice of p-groups of maximal class has been determined by Blackburn [5].

Theorem 2.1. *The complete normal lattice of G contains the heading diamond (G, G') and the rectangle $\left((P_{j,\ell}, P_{j+1,\ell+1})\right)_{3\leq j\leq m-1,\ 3\leq \ell\leq e}$ of trailing diamonds, where $P_{j,\ell} = \Sigma_j \times T_\ell$ for $3 \leq j \leq m$ and $3 \leq \ell \leq e + 1$. The structure of the normal lattice is visualized in Figure 1.*

Note that $P_{j,\ell} = \langle s_j, \dots, s_{m-1}\rangle \times \langle t_\ell, \dots, t_e\rangle = \langle s_j, t_\ell, P_{j+1,\ell+1}\rangle$ for $3 \leq j \leq m-1$, $3 \leq \ell \leq e$.

Conjecture 2.1. The complete normal lattice of G consists exactly of the normal subgroups given in Theorem 2.1.

Corollary 2.1. *The total number of normal subgroups of G is given by*

$$me - (m + 2e) + 6 + [me - (2m + 3e) + 7] \cdot (p - 1),$$

in particular, for $p = 3$ it is given by

$$3me - (5m + 8e) + 20.$$

Corollary 2.2. *Blackburn's two-step centralizers of G [5] are given by*

$$\chi_j(G) = \begin{cases} G' \text{ for } 1 \leq j \leq e-1, \\ \langle y, G'\rangle \text{ for } e \leq j \leq m-2, \\ G \text{ for } j \geq m-1, \end{cases}$$

in particular, none of the maximal subgroups of G occurs as a two-step centralizer, when $e = m-1$.

(1) *The factors of the lower central series of G are given by*

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (p, p) \text{ for } j = 1 \text{ and } 3 \leq j \leq e, \\ (p) \text{ for } j = 2 \text{ and } e + 1 \leq j \leq m-1. \end{cases}$$

(2) *The terms of the lower central series of G are given by*

$$\gamma_j(G) = \begin{cases} \langle x, y, G'\rangle \text{ for } j = 1, \\ \langle s_2, \gamma_3(G)\rangle \text{ for } j = 2, \\ P_{j,j} \text{ for } 3 \leq j \leq e, \\ \Sigma_j \text{ for } e + 1 \leq j \leq m-1. \end{cases}$$

(3) *The factors of the upper central series of G are given by*

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (p, p) \text{ for } 1 \leq j \leq e-2 \text{ and } j = m-1, \\ (p) \text{ for } e-1 \leq j \leq m-2. \end{cases}$$

(4) *The terms of the upper central series of G are given by*

$$\zeta_j(G) = \begin{cases} P_{m-j,e+1-j} \text{ for } 1 \leq j \leq e-2, \\ P_{m-j,3} \text{ for } e-1 \leq j \leq m-3, \\ \langle s_2, \zeta_{m-3}(G)\rangle \text{ for } j = m-2, \\ \langle x, y, \zeta_{m-2}(G)\rangle \text{ for } j = m-1. \end{cases}$$

Proof. We prove the invariance of all claimed normal subgroups under inner automorphisms of $G = \langle x, y\rangle$.

It is well known that the subgroups in the heading diamond are normal, since they contain the commutator subgroup $G' = \gamma_2(G)$.

We start the proof with the tops of trailing diamonds. For $g \in P_{j,\ell}$ and $s \in G'$ we have $s^{-1}gs = s^{-1}sg = g$, since $P_{j,\ell} < G'$, for $j \geq 3$, $\ell \geq 3$, and G was assumed to be metabelian. Now, $P_{j,\ell}$ is the direct product of Σ_j and T_ℓ, since we suppose that $\Sigma_3 \cap T_3 = 1$. So it suffices to show invariance of Σ_j and T_ℓ under conjugation with the generators x and y of G. We have $x^{-1}s_jx = s_j[s_j,x] = s_js_{j+1} \in \Sigma_j$ and $y^{-1}s_jy = s_j[s_j,y] = s_j \in \Sigma_j$ for $j \geq 3$. And similarly we have $x^{-1}t_\ell x = t_\ell[t_\ell,x] = t_\ell \in T_\ell$ and $y^{-1}t_\ell y = t_\ell[t_\ell,y] = t_\ell t_{\ell+1} \in T_\ell$ for $\ell \geq 3$.

Next we prove invariance of intermediate groups between top and bottom of trailing diamonds. They are of the shape $\langle t_\ell, P_{j+1,\ell+1}\rangle$ or $\langle s_jt_\ell^i, P_{j+1,\ell+1}\rangle$ with $0 \leq i \leq p-1$. For t_ℓ, invariance has been shown above. So we investigate $s_jt_\ell^i$. We have $x^{-1}s_jt_\ell^ix = x^{-1}s_jx(x^{-1}t_\ell x)^i = s_js_{j+1}t_\ell^i$, where $s_{j+1} \in P_{j+1,\ell+1}$, and $y^{-1}s_jt_\ell^iy = y^{-1}s_jy(y^{-1}t_\ell y)^i = s_jt_\ell^it_{\ell+1}^i$, where $t_{\ell+1}^i \in P_{j+1,\ell+1}$. (Here we probably are tacitly using power conditions like $s_j^p \in \Sigma_{j+1}$ for $j \geq 3$ and $t_\ell^p \in T_{\ell+1}$ for $\ell \geq 3$.)

Thus we have proved the invariance of all claimed normal subgroups under inner automorphisms.

The number of all (heading and trailing) diamonds of the normal lattice is $1+(m-1-2)\cdot(e-2) = 1+(m-3)\cdot(e-2) = 1+me-2m-3e+6 = me-(2m+3e)+7$.

There are $p-1$ inner vertices of valence 2 in each diamond, which gives a total of $(me-[2m+3e]+7)\cdot(p-1)$ inner vertices.

The remaining (outer) vertices form the heading square and the trailing rectangle with $4+(m-1+1-2)\cdot(e+1-2) = 4+(m-2)\cdot(e-1) = 4+me-m-2e+2 = me-(m+2e)+6$ vertices.

Outer and inner vertices together form a lattice of $me-(m+2e)+6+(me-[2m+3e]+7)\cdot(p-1)$ normal subgroups.

For $p = 3$, this formula yields $me-m-2e+6+2me-4m-6e+14 = 3me-(5m+8e)+20$.

For each $j \geq 2$, Blackburn's two-step centralizer $\chi_j(G)$ is defined as the biggest intermediate group between G and $G' = \gamma_2(G)$ such that $[\gamma_j(G), \chi_j(G)] \leq \gamma_{j+2}(G)$. Since $[\gamma_j(G), \gamma_2(G)] \leq \gamma_{j+2}(G)$, for any $j \geq 2$, $\chi_j(G)$ certainly contains $\gamma_2(G)$. Since $[s_j,x] = s_{j+1} \notin \gamma_{j+2}(G)$ for $2 \leq j \leq m-2$, $[t_\ell,y] = t_{\ell+1} \notin \gamma_{\ell+2}(G)$ for $2 \leq \ell \leq e-1$, and $e \leq m-1$, neither x nor y can be an element of $\chi_j(G)$ for $2 \leq j \leq e-1$. However, since $[t_e,y] = t_{e+1} = 1 \in \gamma_{e+2}(G)$ and $[s_e,y] = 1 \in \gamma_{e+2}(G)$, we have $\chi_j(G) = \langle y, \gamma_2(G)\rangle$ for $e \leq j \leq m-2$, provided that $e \leq m-2$. Finally, since $[s_{m-1},x] = s_m = 1 \in \gamma_m(G) = \gamma_{m+1}(G) = 1$, the two-step centralizers $\chi_j(G)$ with $j \geq m-1$ coincide with the entire group G.

The members of the lower central series can be constructed recursively by $\gamma_j(G) = [\gamma_{j-1}(G), G]$. There is a unique ramification generating the series Σ_3 and T_3 for $j = 3$, since $\gamma_3(G) = [\gamma_2(G), G] = [\langle s_2, \gamma_3(G)\rangle, G] = \langle [s_2,x], [s_2,y], \gamma_4(G)\rangle = \langle s_3, t_3, \gamma_4(G)\rangle$. Otherwise the series Σ_3 and T_3 do not mix and we have $\gamma_j(G) = [\gamma_{j-1}(G), G] = [\langle s_{j-1}, t_{j-1}, \gamma_j(G)\rangle, G]$
$= \langle [s_{j-1},x], [s_{j-1},y], [t_{j-1},x], [t_{j-1},y], \gamma_{j+1}(G)\rangle = \langle s_j, t_j, \gamma_{j+1}(G)\rangle$, since $[s_{j-1},y] = [t_{j-1},x] = 1$ for $j \geq 4$. For $j = e+1$ the bicyclic factors stop, since $t_{e+1} = [t_e,y] = 1$, and γ_{e+1} is simply given by Σ_{e+1}.

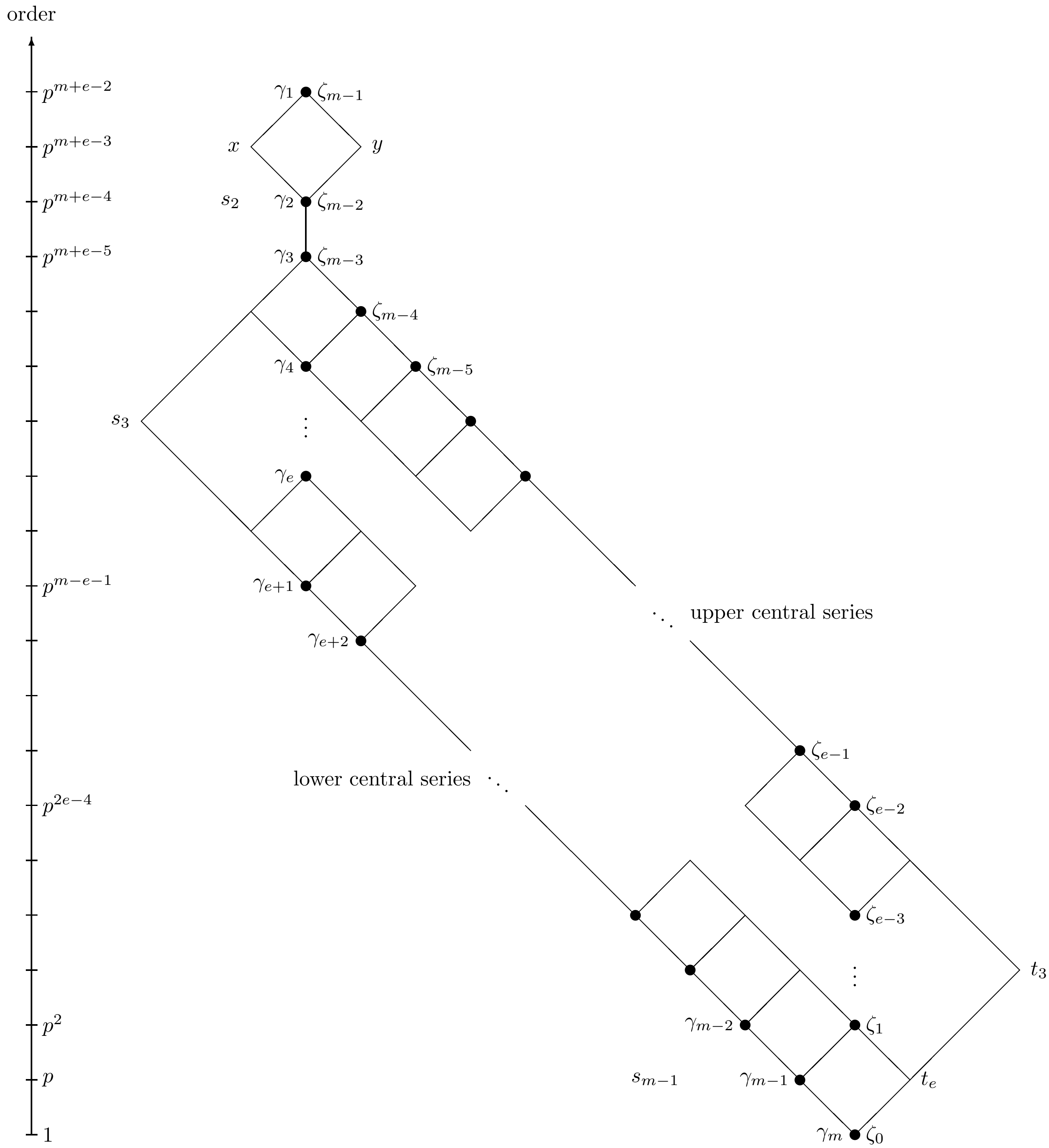
The members of the upper central series can be constructed recursively by $\zeta_j(G)/\zeta_{j-1}(G) = \text{Centre}(G/\zeta_{j-1}(G))$. All groups G with the assigned properties have a bicyclic centre $\zeta_1(G) = \langle s_{m-1}, t_e\rangle$, since $[s_{m-1},x] = [t_e,y] = 1$.

Generally, the equations $[s_{m-j},x] = s_{m-(j-1)}$, $[s_{m-j},y] = 1$, $[t_{e+1-j},x] = 1$, $[t_{e+1-j},y] = t_{e+1-(j-1)}$, whose right sides are elements of $\zeta_{j-1}(G)$, show that s_{m-j} and t_{e+1-j} commute with all elements of G modulo $\zeta_{j-1}(G)$. Therefore, we have $\zeta_j(G) = P_{m-j,e+1-j}$.

However, for $j = e-1$ the bicyclic factors stop, since $[t_{e+1-j},x] = [t_2,x] = [s_2,x] = s_3$, which is not contained in $\zeta_{e-2}(G)$, except for $e = m-1$. Consequently, $\zeta_j(G) = P_{m-j,3}$ for $j \geq e-1$, since it cannot contain $t_2 = s_2$.

□

FIGURE 1. Full normal lattice, including lower and upper central series, of a p-group G with $G/G' \simeq (p,p)$, $\mathrm{cl}(G) = m-1$, $\mathrm{cc}(G) = e-1$, $\mathrm{dl}(G) = 2$, $k(G) = 0$.



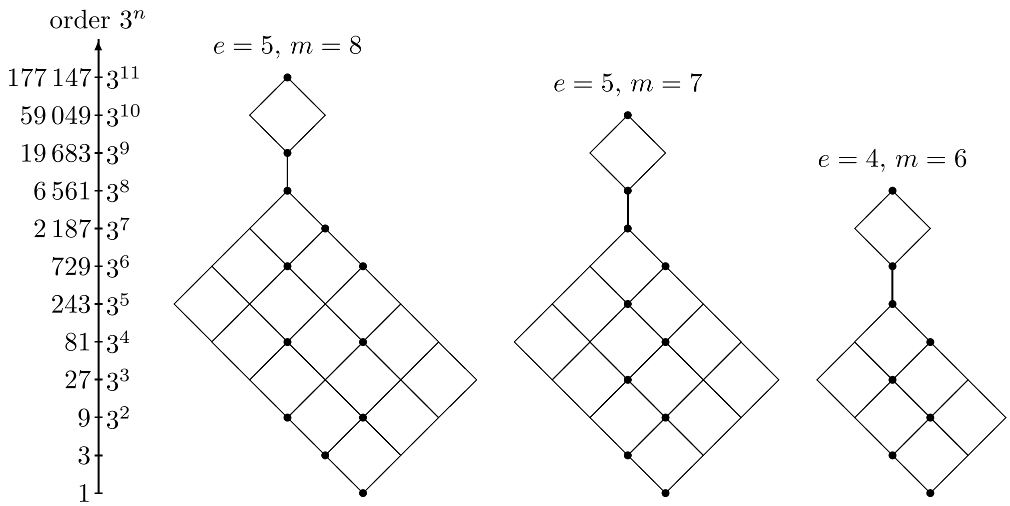
3. Applications in Algebraic Number Theory

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic number field with discriminant D and denote by $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ the Galois group of the second Hilbert p-class field $\mathrm{F}_p^2(K)$ of K, that is, the maximal metabelian unramified p-extension of K. We recall that coclass and class of G are given by the equations $\mathrm{cc}(G) = r = e - 1$ and $\mathrm{cl}(G) = m - 1$ in terms of the invariants e and m. Due to our extensive computations for the papers [12, 14], we are able to underpin the present theory of normal lattices by numerical data concerning the 2 020 complex and the 2 576 real quadratic fields with 3-class group of type $(3, 3)$ and discriminant in the range $-10^6 < D < 10^7$.

Figure 2 shows several examples of normal lattices of 3-groups G with *bicyclic and cyclic factors* of the central series. They are located on coclass trees of coclass graphs $\mathcal{G}(3, r)$ [15, p. 189 ff].

Here, the length of the rectangle of trailing diamonds is bigger than the width, $m - 1 > e$, the upper central series is different from the lower central series, and the last lower central $\gamma_{m-1}(G)$ is cyclic, whence the parent $\pi(G) = G/\gamma_{m-1}(G)$ is of the same coclass. Such groups were called *core groups* in [14]. Concerning the principalization type $\varkappa(K)$ of K which coincides with the transfer kernel type (TKT) $\varkappa(G)$ of G, see [13, 14]. Different TKTs can give rise to equal normal lattices.

Figure 2. 3-groups $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ with bicyclic and cyclic factors.



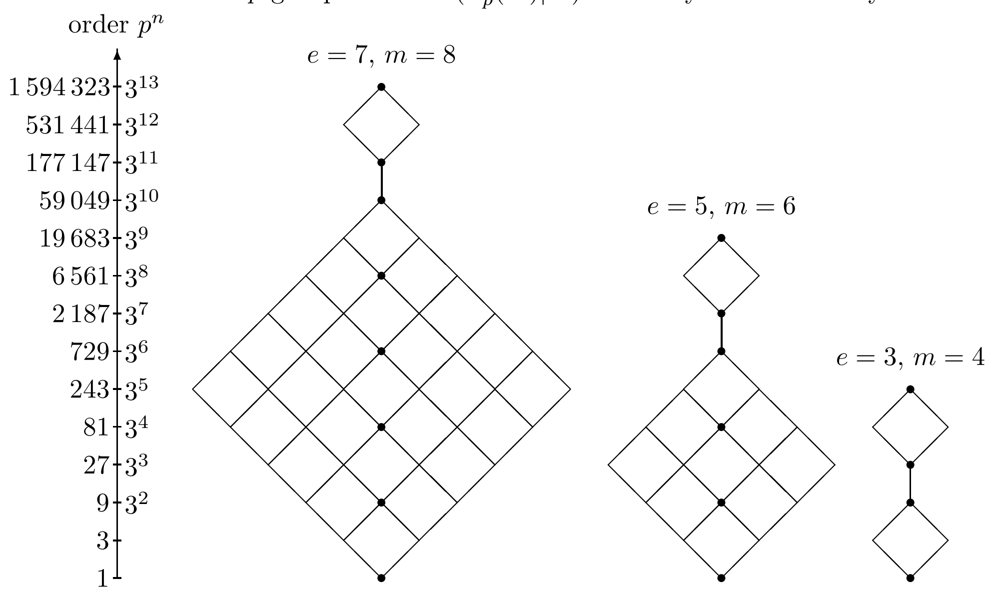
Example 3.1. 3-groups G of coclass $3 \le \mathrm{cc}(G) \le 4$.

- Coclass $\mathrm{cc}(G) = 4$, class $\mathrm{cl}(G) = 7$:
  a total of 14 complex quadratic fields, e. g.,
  $D = -159\,208$ with principalization type F.13,
  $D = -249\,371$ with principalization type F.12,
  $D = -469\,787$ with principalization type F.11,
  $D = -469\,816$ with principalization type F.7,
  and a single real quadratic field of discriminant
  $D = 8\,127\,208$ with principalization type F.13,
  branch groups of depth 1, visualized by Figure 2, $e = 5$, $m = 8$.
- Coclass $\mathrm{cc}(G) = 4$, class $\mathrm{cl}(G) = 6$:
  a single real quadratic field of discriminant
  $D = 8\,491\,713$ with principalization type d*.25,
  mainline group, visualized by Figure 2, $e = 5$, $m = 7$.
- Coclass $\mathrm{cc}(G) = 3$, class $\mathrm{cl}(G) = 5$:
  two real quadratic fields of discriminant
  $D = 1\,535\,117$ with principalization type d.23,
  $D = 2\,328\,721$ with principalization type d.19,
  branch groups of depth 1, visualized by Figure 2, $e = 4$, $m = 6$.

In Figure 3 we display numerous examples of normal lattices of p-groups G with *bicyclic factors* of the central series, except the bottle neck $\gamma_2(G)/\gamma_3(G)$. They are located as vertices on the sporadic part $\mathcal{G}_0(p,r)$ of coclass graphs $\mathcal{G}(p,r)$, outside of coclass trees, [14, Fig. 3.5, p. 439].

Here, the rectangle of trailing diamonds degenerates to a square with $e = m - 1$, the upper central series is the reverse lower central series, and thus the last lower central $\gamma_{m-1}(G)$ is bicyclic, whence the (generalized) parent $\tilde{\pi}(G) = G/\gamma_{m-1}(G)$ is of lower coclass. Such groups were called *interface groups* in [14].

FIGURE 3. p-groups $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ with bicyclic factors only.

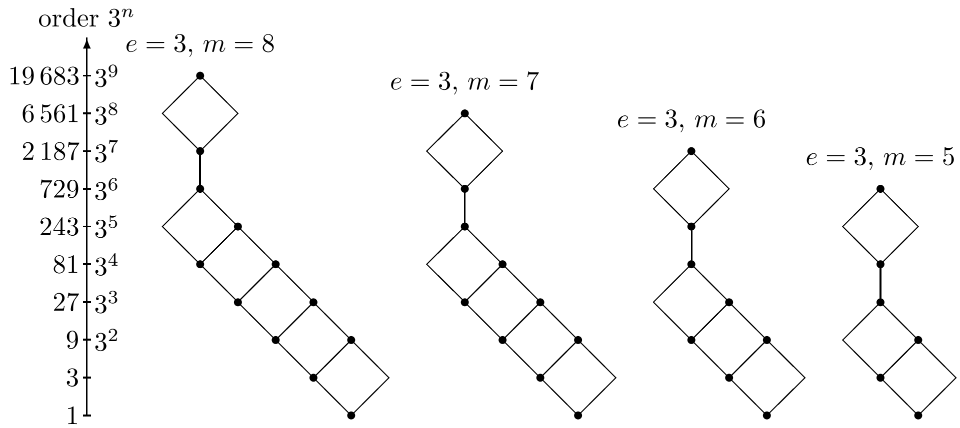


Example 3.2. p-groups G with $p \in \{3,5,7\}$.

- $p = 3$, coclass $\mathrm{cc}(G) = 6$, class $\mathrm{cl}(G) = 7$:
  a single complex quadratic field of discriminant
  $D = -423\,640$ with principalization type F.12,
  sporadic group, visualized by Figure 3, $e = 7$, $m = 8$.
- $p = 3$, coclass $\mathrm{cc}(G) = 4$, class $\mathrm{cl}(G) = 5$:
  a total of 78 complex quadratic fields, e. g.,
  $D = -27\,156$ with principalization type F.11,
  $D = -31\,908$ with principalization type F.12,
  $D = -67\,480$ with principalization type F.13,
  $D = -124\,363$ with principalization type F.7,
  and a single real quadratic field of discriminant
  $D = 8\,321\,505$ with principalization type F.13,
  sporadic groups, visualized by Figure 3, $e = 5$, $m = 6$.
- $p = 3$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 3$:
  a total of 936 complex quadratic fields, e. g.,
  $D = -4\,027$ with principalization type D.10,
  $D = -12\,131$ with principalization type D.5,
  and a total of 140 real quadratic fields, e. g.,
  $D = 422\,573$ with principalization type D.10,
  $D = 631\,769$ with principalization type D.5,
  sporadic groups, visualized by Figure 3, $e = 3$, $m = 4$.
- $p = 5$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 3$: see [14, Tbl. 3.13, p. 450].
- $p = 7$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 3$: see [14, Tbl. 3.14, p. 450].

Figure 4 shows many examples of normal lattices of "small" p-groups G with *bicyclic and cyclic factors* of the central series. They are located on coclass trees of coclass graphs $\mathcal{G}(p,r)$ [14, Fig. 3.6–3.7, pp. 442–443].

FIGURE 4. Small p-groups $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ with bicyclic and cyclic factors.



Example 3.3. Small p-groups G with $p \in \{3,5,7\}$.

- $p = 3$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 7$:
  a total of 28 complex quadratic fields, e. g.,
  $D = -262\,744$ with principalization type E.14,
  $D = -268\,040$ with principalization type E.6,
  $D = -297\,079$ with principalization type E.9,
  $D = -370\,740$ with principalization type E.8,
  branch groups of depth 1, visualized by Figure 4, $e = 3$, $m = 8$.
- $p = 3$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 6$:
  two real quadratic fields, e. g.,
  $D = 1\,001\,957$ with principalization type c.21,
  mainline groups, visualized by Figure 4, $e = 3$, $m = 7$.
- $p = 3$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 5$:
  a total of 383 complex quadratic fields, e. g.,
  $D = -9\,748$ with principalization type E.9,
  $D = -15\,544$ with principalization type E.6,
  $D = -16\,627$ with principalization type E.14,
  $D = -34\,867$ with principalization type E.8,
  and a total of 21 real quadratic fields, e. g.,
  $D = 342\,664$ with principalization type E.9,
  $D = 3\,918\,837$ with principalization type E.14,
  $D = 5\,264\,069$ with principalization type E.6,
  $D = 6\,098\,360$ with principalization type E.8,
  branch groups of depth 1, visualized by Figure 4, $e = 3$, $m = 6$.
- $p = 3$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 4$:
  a total of 54 real quadratic fields, e. g.,
  $D = 534\,824$ with principalization type c.18,
  $D = 540\,365$ with principalization type c.21,
  mainline groups, visualized by Figure 4, $e = 3$, $m = 5$.
- $p = 5$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 5$: see [14, Tbl. 3.13, p. 450].
- $p = 7$, coclass $\mathrm{cc}(G) = 2$, class $\mathrm{cl}(G) = 5$: see [14, Tbl. 3.14, p. 450].

4. Final Remarks

- Among the 2 020 complex quadratic fields with 3-class group of type $(3,3)$ and discriminant in the range $-10^6 < D < 0$, the dominating part of 1 440, that is 71.29 %, has a second 3-class group with minimal defect of commutativity $k = 0$. The remaining 28.71 % have $k = 1$ and TKTs G.16, G.19 and H.4.
- Among the 2 576 real quadratic fields with 3-class group of type $(3,3)$ and discriminant in the range $0 < D < 10^7$, a modest part of 273, i. e. 10.6 %, has a second 3-class group of coclass at least 2. A dominating part of 222 among these 273 second 3-class groups, that is 81.3 %, has minimal defect of commutativity $k = 0$, whereas 18.7 % have $k = 1$ and TKTs b.10, G.16, G.19 and H.4.
- It should be pointed out that the power-commutator presentations which we used for proving Theorem 2.1 and its Corollaries are rudimentary, since in fact they consist of commutator relations only. Thus they define an isoclinism family of p-groups of fixed order, rather than a single isomorphism class of p-groups.

  On the other hand, experience shows that the transfer kernel type (TKT) of a p-group mainly depends on the power relations. This explains why different TKTs frequently give rise to equal normal lattices.

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