arXiv:1403.4021v2 [math.GT] 20 Jul 2014

MARKOV TRACES ON THE BIRMAN-WENZL-MURAKAMI ALGEBRAS

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Abstract. We classify the Markov traces factoring through the Birman-Wenzl-Murakami (BMW) algebras. For this purpose, we define a common 'cover' for the two variations of the BMW-algebra originating from the quantum orthogonal/symplectic duality, which are responsible for the so-called 'Dubrovnik' variation of the Kauffman polynomial. For generic values of the defining parameters of the BMW algebra, this cover is isomorphic to the BMW algebra itself, and this fact provides a shorter defining relation for it, in the generic case. For a certain 1-dimensional family of special values however, it is a non-trivial central extension of the BMW-algebra which induces a central extension of the Temperley-Lieb algebra. Inside this 1-dimensional family, exactly two values provide possibly additional $\mathbb{Q}$-valued Markov traces. We describe both of these potential traces on the (extended) Temperley-Lieb subalgebra. While we only conjecture the existence of one of them, we prove the existence of the other by introducing a central extension of the Iwahori-Hecke algebra at $q=-1$ for an *arbitrary* Coxeter system, and by proving that this extension indeed admits an exotic Markov trace. These constructions provide several natural non-vanishing Hochschild cohomology classes on these classical algebras.

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Date: July 17, 2014.

1. Introduction

1.1. **Context.** When the Jones and Homfly polynomial appeared, they were first described as Markov traces on the group algebra of the braid group, and therefore as invariants of oriented links produced by the virtue of Markov theorem. Indeed this theorem says that an invariant of oriented links is the same thing as a Markov trace on the tower of the group algebra of the braid group. This approach has had a great descent, on the existence of quantum invariants originating from a quantum trace, in the study of Markov traces on generalized Hecke algebras, and the Khovanov homology in more recent years has a description in terms of Soergel bimodules that can be seen as part of the same thread of thinking.

On the other hand, the construction of its cousin the Kauffman polynomial has a very different story. Indeed this polynomial was first described as an invariant of regular isotopy by way of *un*oriented skein relations, and then turned into an invariant of (oriented) links by twisting the invariant of regular isotopy of the underlying unoriented link by the writhe of the link. This approach was shown to be also applicable to the Jones polynomial (and later on used in the definition of Khovanov homology) through the Kauffman bracket, whose description involves unoriented skein relations.

Conversely, there has been very soon an attempt to describe the Kauffman polynomial in terms of Markov traces. This lead to the definition, independently by Murakami and Birman-Wenzl, of the BMW-algebra. This algebra however, in spite of the quantum moto describing both the Hecke and BMW algebra as ‘quantizations’ of the symmetric group and of the Brauer algebra respectively, and therefore as quantizations of the centralizer algebras of the tensor powers of the standard modules of the most classical Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{so}_n/\mathfrak{sp}_n$, never reached the level of recognition of the Hecke algebra. As a sample test for a not so nice behavior, a completely satisfactory 'categorification' is still lacking, and the known generalizations of this algebra to other Coxeter or complex reflection groups have flatness issues (as modules over the natural ring attached to this context). The most natural description of the BMW-algebra is probably as an algebra of unoriented tangle diagrams, following the original work of Kauffman. Then, BMW_n can be described as an homomorphic image of the group algebra of the braid group, by converting a given braid into a non-oriented tangle. Describing the kernel of this map as a finite set of meaningful relations presenting BMW_n as a quotient is the next natural goal. Following the insight given by the tangle picture, relations on 2 strands are easily found to be given by a cubic relation $(s_i - a)(s_i - b)(s_i - c)$ on the Artin generators, hence BMW_n belongs to the family of cubic quotients of the group algebra of the braid group B_n. In the sequel we will denote $x = b + c$ and $y = bc$ since most definitions involving the BMW-algebras are symmetric in b, c (but not in a, b, c !). We will also assume $x \neq 0$ which is necessary in order to express the elementary tangles e_i as linear combinations of braids. As opposed to the quadratic quotients, for which the Hecke algebra provides a universal finite-dimensional model, the 'cubic Hecke algebra' does not enjoy universal finiteness properties, because the factor group B_n/s_i^3 is infinite for $n \geq 6$. There are thus additional relations on 3 strands, and it turns out that these relations on 3 strands are enough to define the BMW_n algebras for every single n.

1.2. **Presentation.** The origin of this work was the apparently innocent question to determine whether the 'Kauffman trace', that is the Markov trace affording the Kauffman polynomial, was indeed the only Markov trace factoring through the BMW-algebra (in addition to the one factoring to the smaller Hecke algebra quotient). When we tried to answer this question, we faced another problem. First of all, it is well-known that there are two variations of the Kauffman polynomial, one of them having been given the name of the city of Dubrovnik. These two variations can be seen as Markov traces, after specialization of the parameters a, b, c, subject to a polynomial relation $a = bc$ or $a = -bc$. But they are not Markov trace on the same BMW-algebra ! More precisely, although the two relevant BMW-algebras are isomorphic as algebras, they are a priori not the same quotient of the group algebra of the braid group. Indeed, the additional relation on 3 strands involves a, b, c and depends on whether $a = bc$ or $a = -bc$. Viewing the BMW-algebras as centralizer algebras for quantum groups actions, these two specializations correspond to the difference between the orthogonal and symplectic groups (see the end of §3).

Because of this, we tried to dig deeper into the defining relations of the BMW-algebras. When translated into braid words, the usual additional relation (in both cases), relating $e_i s_j e_i$ and e_i for $|i - j| = 1$, involves 12 terms. We use another relation, that relates $s_j^{-1} e_i s_j^{-1}$ and $s_i e_j s_i$, involving only 6 terms when expanded into braid words, which holds inside both BMW-algebras, and which is enough to ensure the finiteness of the dimension. Because of this, we can define a finite-dimensional 'cover' of the two BMW-algebras involved in the computation of the Kauffman polynomial, as well as the ortho-symplectic quantum invariants, that we call $\widetilde{BMW}_n$.

From this we get a satisfying algebraic setup to explore the possible Markov traces for every single value of the parameters. We prove that, for generic values of the parameters, this cover is actually isomorphic to the usual BMW-algebras, thus providing in these cases an even simpler definition (as a quotient of the group algebra of B_n) of the BMW-algebras (see propositions 4.4 and 4.5). However, for generic values satifying $a^2 = y$, we find that $\widetilde{BMW}_3$ has one dimension more than expected.

Concerning Markov traces we first get (proposition 5.5) that, when $a^2 \neq y^2$ and $a^2 \neq y$ (and actually : also when $a^2 = y$ and $a^2 \neq y^2$, see proposition 5.7), the only Markov trace factoring through $\widetilde{BMW}_n$ is the Ocneanu trace, defined on the Hecke algebra quotient.

We specialize to the situation $a^2 = y^2$. Then, we have in addition the Kauffman trace. When $y \neq 1$, we prove that the only Markov traces factoring through our algebra lead either to the Kauffman polynomial or to the Homfly polynomial, and that this algebra is actually isomorphic to the usual BMW-algebra. When $y = 1$, that is when $a^2 = y^2$ and $a^2 = y$, we have first to exclude a very degenerate case, $x = -2a$, for which there is an infinite number of Markov traces, namely the ones factoring through the group algebra of the symmetric group (see proposition 5.10) ; it is well-known that these ones detect only the number of components of the links. In the general case $y = 1$, we get an additional Markov trace $t_n^{\dagger\dagger}$ that basically provides the parity of the number of components of the link. These 3 Markov traces exhaust all possible traces, and are linearly independent one from the other, for generic x. The special values for which this does not hold, besides $x = -2a$, are $x = a$ and $x = 2a$.

In order to understand what happens in these two special cases, we provide a description of the algebra $\widetilde{BMW}_n$ when specialized at $a^2 = y = 1$. For this we define by generators and relations an algebra over $\mathbb{Q}[a, x, x^{-1}]/(a^2 - 1)$ that we denote F_n. It is a free module of rank 1 more than the dimension of BMW_n (corollary 6.6) and it can be viewed as a central extension of BMW_n by a 1-dimensional ideal spanned by some element that we call C. This element squares to 0, and therefore the extension cannot split, precisely when $x = a$ and $x = 2a$. We prove that it is indeed the specialization we want of $\widetilde{BMW}_n$, as soon as $x \neq -2a$ (theorem 6.8), while the specialization of $\widetilde{BMW}_n$ for the case $x = -2a$ provides a larger algebra, of which we provide a partly conjectural description (see section 6.8). Finally, we use the structure of F_n to check that the space of Markov traces factoring through F_n has dimension at most 3 : there is at most one way to find an additional Markov trace in the special cases $x = a$ and $x = 2a$.

In passing, we deduce from the existence of F_n a similar central extension $\widetilde{TL}_n$ of the Temperley-Lieb algebra, which is a subalgebra of F_n. We find natural (diagrammatic) interpretations of the two (potential) additional Markovs trace when restricted to this subalgebra. Finally, we manage to construct the expected additional Markov trace in the case $x = 2a$ by constructing a central extension of the classical Hecke algebra.

More generally, we prove that, for an arbitrary Coxeter system (W, S), the usual Iwahori-Hecke algebra at $q = -1$ has a natural non-split central extension, of dimension $1 + \# W$ if W is finite, and that there existes a Markov trace on this algebra when $W = \mathfrak{S}_n$ (see theorems 6.22 and 6.26). This Markov trace provides what we need. The case $x = a$ remains conjectural, although we are confident that the corresponding invariant exists. We guess that an algebraic proof of the existence of the Kauffman trace similar to the one that Jones provided for the Ocneanu trace should be easy to generalize to our central extension. However it appears that no one provided such a proof yet, and finding such a proof seems to us to be quite more tricky than the Hecke algebra case.

1.3. **Organisation of the paper.** The plan of the paper is as follows. In §2 we compile a few results on the 'cubic Hecke algebra' on 3 strands, namely the quotient of the group algebra of B_3 by a generic cubic relations $(s_i - a)(s_i - b)(s_i - c) = 0$. The finite-dimensionality as well as the symmetric algebra structure of this algebra is a crucial tool in the sequel. In §3 and §4 we explore the algebraic structure of the BMW-algebra, and define a suitable cover of its two avatars appearing in the construction of the Kauffman polynomial. This provides a suitable setting for studying the Markov traces, and we do this in §5. Inside §5, we rediscover the classical Markov traces, and describe an additional one when $y = a^2 = 1$. We prove there that these exhaust all possible traces, except when $x = a$ or $x = 2a$. In §6 we introduce our central extensions of the BMW-algebra, Hecke algebras and Temperley-Lieb algebras. We define two additional traces on these extended Temperley-Lieb algebra, and one on the extended Hecke algebra. Finally, §7 is devoted to the exploration of the link invariants obtained in this way. We show that the additional trace for $y = a^2 = 1$ simply counts the parity of the number of components of the link, and we tabulate the two special ones, for which an interpretation is lacking.

Because of the large number of specializations that we are using, we provide here a table of the various rings involved in the paper, as a common place for reference. The second table provide a list of the main algebras used in the paper, together with the rings involved in their definition.

$R = \mathbb{Q}[a,b,c,(abc)^{-1}]$	$\overline{R} = R/(a^2-y^2)$	$R_{\pm} = R/(a \mp y)$
$S = R[(b+c)^{-1}]$	$\overline{S} = S/(a^2-y^2)$	$S_{\pm} = S/(a \mp y)$
$\overline{S}' = \overline{S}[(bc-1)^{-1}]$	$\overline{S}'_{\pm} = \overline{S}'/(a \mp y)$	$S^{\dagger} = S/(a^2-y)$
$S^{\dagger\dagger} = S^{\dagger}/(a^2-1)$	$S^{\dagger\dagger}_{\pm} = S^{\dagger}/(a \mp 1)$	

Algebra	Ring
H_n	R
$\widetilde{BMW}_n$	R
$BMW^{\pm}_n$	$S_{\pm}$
$\overline{BMW}_n$	$\overline{R}$

Acknowledgements. We thank S. Bouc, F. Digne and A. Zimmermann for discussions and references.

2. Preliminaries on the cubic Hecke algebras on 3 strands

In order to insure the coherence of notations with the forthcoming sections, we let

$$R = \mathbb{Q}[a, a^{-1}, b, b^{-1}, c, c^{-1}] = \mathbb{Q}[a, b, c, (abc)^{-1}],$$

although all the results of the present section are actually already valid with $R = \mathbb{Z}[a, b, c, (abc)^{-1}]$.

We let H_n denote the R-algebra defined as the quotient of the group algebra RB_n of the braid group on n strands by the relations $(s_i - a)(s_i - b)(s_i - c) = 0$ for $1 \le i \le n-1$ or, equivalently – since each s_i is conjugated to s_1 – by the relation $(s_1 - a)(s_1 - b)(s_1 - c) = 0$. It is known that H_n is a free R-module of finite rank for $n \le 5$ (see [16]). More precisely, for $n = 3$, one may excerpt from [16] the following result (see also [4, 7, 18] for related statements).

Proposition 2.1.

(i) *The algebra H_3 is a free H_2-module of rank 8, with basis the elements $1, s_2, s_2^{-1}$, $s_1^{\alpha} s_2^{\beta}$ for $\alpha, \beta \in \{1, -1\}$, $s_2 s_1^{-1} s_2$.*

(ii) *The algebra H_3 is a free R-module of rank 24, with basis the elements*

$$\begin{aligned} \mathcal{B}_1 \;=\; & (1, s_1, s_1^{-1}, s_2, s_2^{-1}, s_1 s_2, s_1 s_2^{-1}, s_1^{-1} s_2, s_1^{-1} s_2^{-1}, s_1 s_2 s_1, s_1 s_2 s_1^{-1}, s_1^{-1} s_2 s_1, s_1^{-1} s_2 s_1^{-1}, \\ & s_1 s_2^{-1} s_1, s_1^{-1} s_2^{-1} s_1, s_2 s_1, s_2^{-1} s_1, s_2 s_1^{-1}, s_2^{-1} s_1^{-1}, s_1 s_2^{-1} s_1^{-1}, s_1^{-1} s_2^{-1} s_1^{-1}, s_2 s_1^{-1} s_2, \\ & s_1 s_2 s_1^{-1} s_2, s_1^{-1} s_2 s_1^{-1} s_2). \end{aligned}$$

Proof. From [16] theorem 3.2 we know that H_3 is generated as a H_2-module by the 8 elements on the first statement. Since H_2 is spanned by $1, s_1, s_1^{-1}$ it follows that H_3 is generated as a

H_3-module by the 24 elements of the second statement. Since Γ_3 has 24 elements and by an a argument of [5] (see also [17], proposition 2.4 (1)) it follows that these 24 elements are a basis over R of H_3. It readily follows that the 8 original elements provide a basis of H_3 as a H_2-module. □

A consequence is that H_3 is a free deformation of the group algebra $R\Gamma_3$, where Γ_n denotes the quotient of the braid group by the relations $s_i^3 = 1$, and H_3 becomes isomorphic to it after extension of scalars to the algebraic closure $\overline{K}$ of the field of fractions K of R. Actually, one has the stronger result $H_3 \otimes_R K \simeq K\Gamma_3$, because the irreducible representations of KH_3 are absolutely irreducible.

We will use the following explicit matrix models for the representations, which are basically the same which were obtained in [4], §5B. We endow $\{a, b, c\}$ with the total order $a < b < c$. We denote

(i) S_α for $\alpha \in \{a, b, c\}$ the 1-dimensional representation $s_1, s_2 \mapsto \alpha$

(ii) $U_{\alpha,\beta}$ for $\alpha, \beta \in \{a, b, c\}$ with $\alpha < \beta$ the 2-dimensional representation

$$U_{\alpha,\beta} : s_1 \mapsto \begin{pmatrix} \alpha & 0 \\ -\alpha & \beta \end{pmatrix} s_2 \mapsto \begin{pmatrix} \beta & \beta \\ 0 & \alpha \end{pmatrix}$$

(iii) V the 3-dimensional irreducible representation

$$s_1 \mapsto \begin{pmatrix} c & 0 & 0 \\ ac + b^2 & b & 0 \\ b & 1 & a \end{pmatrix} s_2 \mapsto \begin{pmatrix} a & -1 & b \\ 0 & b & -ac - b^2 \\ 0 & 0 & c \end{pmatrix}$$

We note the important feature that these representations are actually defined over R. As a consequence, these formulas provide an explicit embedding

$$\Phi_{H_3} : H_3 \hookrightarrow R^3 \oplus M_2(R)^3 \oplus M_3(R)$$

and the RHS is easy to identify with R^{24} as a R-module.

For an algebra A, we let $[A, A]$ denote the submodule spanned by the $ab - ba$ for $a, b \in A$.

Proposition 2.2. *$H_3/[H_3, H_3]$ is freely generated over R by the 7 elements*

$$1, s_1, s_1^{-1}, s_1 s_2, s_1 s_2^{-1}, s_1^{-1} s_2^{-1}, s_1^{-1} s_2 s_1^{-1} s_2.$$

Proof. From the basis above it is readily proved (note for instance that the element $s_1 s_2 s_1$ conjugates s_1 and s_2) that $H_3/[H_3, H_3]$ is generated by these 7 elements. The freeness is a consequence of the fact that, if $\tilde{H}_3 = H_3 \otimes_R K$, then $\tilde{H}_3/[\tilde{H}_3, \tilde{H}_3] \simeq Z(\tilde{H}_3) \simeq Z(K\Gamma_3)$ has dimension 7. Since these 7 elements also generate $\tilde{H}_3/[\tilde{H}_3, \tilde{H}_3]$ they are a basis of the 7-dimensional K-vector space $\tilde{H}_3/[\tilde{H}_3, \tilde{H}_3]$. If a R-linear combination of these elements belonged to $[H_3, H_3]$, this would yield a contradiction since $H_3 \subset \tilde{H}_3$. □

Proposition 2.3.

(i) *The family $\mathcal{B}_0$ below provides a basis of H_3 as a R-module.*

$$\mathcal{B}_0 = \begin{array}{l} 1, s_1, s_1^2, s_2, s_2^2, s_1 s_2, s_1 s_2^2, s_1^2 s_2, s_1^2 s_2^2, s_1 s_2 s_1, s_1 s_2 s_1^2, s_1^2 s_2 s_1, s_1^2 s_2 s_1^2, s_1 s_2^2 s_1, s_1^2 s_2^2 s_1, s_2 s_1, \\ s_2^2 s_1, s_2 s_1^2, s_2^2 s_1^2, s_1 s_2^2 s_1^2, s_1^2 s_2^2 s_1^2, s_2 s_1^2 s_2, s_1 s_2 s_1^2 s_2, s_1^2 s_2 s_1^2 s_2 \end{array}$$

(ii) *The linear form $t_0 : H_3 \to R$ defined by $t_0(1) = 1$ and $t_0(g) = 0$ for all $g \in \mathcal{B}_0 \setminus \{1\}$ is a (nondegenerate) symmetrizing form for H_3.*

Proof. It was checked in [4] §4B that the linear span of $\mathcal{B}_0$ (actually of the image of $\mathcal{B}_0$ under the anti-automorphism $s_i \mapsto s_i$, see [16]) was stable under left multiplication by s_1, s_2, hence generates H_3, and thus provides a basis of H_3. It appears however that the computations in [4] might have (incorrectly) been made inside $H_3 \otimes K$ instead of H_3, so we need to provide another argument. Since we proved that $\mathcal{B}_1$ is a basis of H_3, it is sufficient to show that every element of $\mathcal{B}_1$ can be expressed as a linear combination of the elements of $\mathcal{B}_0$. This readily follows from the expression of s_i^{-1} as a linear combination of $1, s_i$ and s_i^2. This proves (1). Using the explicit injective morphism $\Phi : H_3 \to R^3 \times M_2(R)^3 \times M_3(R)$ above, calculations inside H_3 are easy, and we can explicitely check that $(x, y) \mapsto t_0(xy)$ is indeed a symmetrizing trace, more precisely that the matrix $t_0(b_i b_j)$ for $b_i \in \mathcal{B}_0$ (resp. $\mathcal{B}_1$) is a symmetric matrix with determinant $-(abc)^{54}$ (resp. $-(abc)^2$), which belongs to $R^\times$. $\square$

Because of the proposition, it is possible to apply the theory of Geck's 'Schur elements' (see e.g. [8]) to H_3, that is to determine elements $p_\chi \in R$ attached to each of the irreducible representations χ of $H_3 \otimes K$ such that $t_0 = \sum_\chi \frac{1}{p_\chi} tr_\chi$ where tr_χ denotes the matrix trace attached to the irreductible representation χ of $H_3 \otimes K$. Once convenient matrix models as well as an explicit description of t_0 are known, it is a simple matter to determine them. These elements were already determined in [14].

The one attached to S_a is

$$p_{S_a} = \frac{(a-c)\left(a^2 - ac + c^2\right)(a-b)\left(a^2 - ab + b^2\right)\left(bc + a^2\right)}{b^4 c^4}$$

the one to $U_{b,c}$ is

$$p_{U_{b,c}} = -\frac{\left(b^2 + c^2 - bc\right)(a-c)(a-b)\left(bc + a^2\right)}{a^4 bc}$$

and the one to V is

$$p_V = \frac{\left(bc + a^2\right)\left(ab + c^2\right)\left(ac + b^2\right)}{a^2 c^2 b^2}$$

By this theory of Schur elements (see [8] theorem 7.2.6) we have that, for each morphism $\varphi : R \to k$ for k a field, $H_3 \otimes_\varphi k$ is semisimple if and only if $\varphi(p_\chi) \neq 0$ for all irreducible representation χ of H_3. Another related computation that can be found in [18] is that the discriminant of the trace form of the regular representation of H_3, that is to say of the action of H_3 on itself by left multiplication, is

$$2^{12} 3^9 a^6 b^6 c^6 (c-b)^{10} (a-c)^{10} (b-a)^{10}$$
$$(c^2 - cb + b^2)^6 (a^2 - ac + c^2)^6 (b^2 - ba + a^2)^6 (a^2 + bc)^{14} (b^2 + ac)^{14} (c^2 + ab)^{14}.$$

From this computation of Schur elements we get the following lemma.

Lemma 2.4. *Let R_1 be a domain, and $\varphi : R \to R_1$ a morphism of rings such that $\varphi(p_\chi) \neq 0$ for all the irreducible representation χ of H_3 . Then the induced map*

$$\Phi_{H_3} \otimes R_1 : H_3 \otimes_\varphi R_1 \to R_1^3 \oplus M_2(R_1)^3 \oplus M_3(R_1)$$

is injective.

Proof. Let k denote an algebraic closure of the fraction field of R_1. By the remarks above we get the semisimplicity of $H_3 \otimes_\varphi k$ and, by Tits deformation theorem, that $H_3 \otimes_\varphi k$ is isomorphic to $k^3 \oplus M_2(k)^3 \oplus M_3(k)$. Moreover (see [8] theorem 7.4.6), the 'decomposition map' between $H_3 \otimes K$ and $H_3 \otimes_\varphi k$ induces an isomorphism between simple modules, which implies that the morphism that we consider $\Phi_{H_3 \otimes_\varphi} : H_3 \otimes_\varphi k \to k^3 \oplus M_2(k)^3 \oplus M_3(k)$ is a morphism from $H_3 \otimes_\varphi k$ to the sum of the matrix algebras associated to its simple modules. Because $H_3 \otimes_\varphi k$ is semisimple, this morphism is indeed an isomorphism. Since H_3 is free over R_1 the conclusion follows. $\square$

Let M be a R-module. Since the natural map $H_2 \to H_3$ is injective, we can identify H_2 with a R-subalgebra of H_3.

For M a R-module, we let $MT_n(M)$ be the R-module of R-linear maps $t : H_n \to M$ such that $t(xy) = t(yx)$ for all $x, y \in H_n$, and such that $t(xs_{n-1}) = t(xs_{n-1}^{-1})$, for all x in the image of the natural morphism $H_{n-1} \to H_n$.

Proposition 2.5. *Let M be a R-module. Then $MT_3(M)$ is isomorphic to $\mathrm{Hom}_R(R^4, M)$ under $t \mapsto t(1)e_1^* + t(s_1)e_2^* + t(s_1s_2)e_3^* + t(s_1s_2^{-1}s_1s_2^{-1})e_4^*$, where $(e_1, \dots, e_4)$ is the canonical basis of R^4 and the $e_i^* \in \mathrm{Hom}(R^4, M)$ are the obvious dual maps.*

Proof. Let $t \in MT_3(M)$. Since s_2 is conjugated to s_1 and t vanishes on $s_2 - s_2^{-1}$, it also vanishes on $s_1 - s_1^{-1}$, hence $t(s_1^{-1}) = t(s_1)$. Since t also vanishes on $s_1s_2 - s_1s_2^{-1}$, and because

$$\begin{array}{rcl} s_1^{-1}s_2^{-1} - s_1s_2 & = & (s_1^{-1}s_2^{-1} - s_1^{-1}s_2) + (s_1^{-1}s_2 - s_1s_2) \\ & = & (s_1^{-1}s_2^{-1} - s_1^{-1}s_2) + (s_1s_2s_1)(s_2^{-1}s_1 - s_2s_1)(s_1s_2s_1)^{-1} \end{array}$$

we get that $t(s_1s_2) = t(s_1s_2^{-1}) = t(s_1^{-1}s_2^{-1})$. By proposition 2.2 it follows that the map $MT_3(M) \to \mathrm{Hom}_R(R^4, M)$ described in the statement is injective (note that $t(s_1s_2^{-1}s_1s_2^{-1}) = t(s_2s_1^{-1}s_2s_1^{-1}) = t(s_1^{-1}s_2s_1^{-1}s_2)$ using $s_1s_2s_1$-conjugation and the invariance of traces under cyclic rotation). Conversely, because by proposition 2.2 $H_3/[H_3, H_3]$ is a free module, it is possible to associate to each element in $\mathrm{Hom}_R(R^4, M)$ a trace t on H_3 satisfying $t(s_1^{-1}) = t(s_1)$, $t(s_1s_2) = t(s_1s_2^{-1}) = t(s_1^{-1}s_2^{-1})$. Now, since H_2 is spanned by $1, s_1, s_1^{-1}$, the R-module spanned by the $xs_2 - xs_2^{-1}$ for $x \in H_2$ is spanned by $s_2 - s_2^{-1}$, $s_1s_2 - s_1s_2^{-1}$, $s_1^{-1}s_2 - s_1^{-1}s_2^{-1}$, hence t clearly vanishes on it. $\square$

3. Two BMW algebras as quotients of the braid groups

We still denote $R = \mathbb{Q}[a, b, c, (abc)^{-1}]$ and let $S = R[(b+c)^{-1}]$, $S_\pm = S/(a \mp y)$. There are two variants of the Birman-Wenzl-Murakami algebras, one which can be defined over S_+, the other one over S_-. Usually, they are defined as a algebras over $\mathbb{Q}[\alpha, \alpha^{-1}, q, q^{-1}, (q \pm q^{-1})^{-1}]$, by generators $\sigma_1, \dots, \sigma_{n-1}$, braid relations between the σ_i's, and three series of relations involving the additional elements

$$e_i = \frac{\sigma_i^{-1} \pm \sigma_i}{q \pm q^{-1}} \mp 1,$$

namely

(i) $\sigma_i e_i = \alpha^{-1} e_i$
(ii) $e_i \sigma_{i+1} e_i = \alpha e_i$
(iii) $e_i \sigma_{i+1}^{-1} e_i = \alpha e_i$

That these relations are enough to present the algebra originally introduced in [1] was shown in [21]. A classical remark is that, using conjugating properties inside the braid group, these three *series of relations* are equivalent to the three *relations*

(i) $\sigma_1 e_1 = \alpha^{-1} e_1$
(ii) $e_1 \sigma_2 e_1 = \alpha e_1$
(iii) $e_1 \sigma_2^{-1} e_1 = \alpha^{-1} e_1$

A slightly more convenient presentation for our purposes is to replace the generators σ_i by $s_i = \alpha^{-1}\sigma_i$. The formulas above become

$$e_i = \frac{\alpha^{-1}s_i^{-1} \pm \alpha s_i}{q \pm q^{-1}} \mp 1 = \frac{\alpha^{-2}s_i^{-1} \pm s_i}{\alpha^{-1}q \pm \alpha^{-1}q^{-1}} \mp 1,$$

and

(i) $s_1 e_1 = \alpha^{-2} e_1$
(ii) $e_1 s_2 e_1 = e_1$
(iii) $e_1 s_2^{-1} e_1 = e_1$

A classical consequence of the first relation is that s_1 and, therefore, all the s_i's, satisfy a cubic relation, and more precisely

$$(s_i - \alpha^{-2})(s_i - \alpha^{-1}q)(s_i \mp \alpha^{-1}q^{-1}) = 0$$

It follows that this algebra is actually defined over

$$\mathbb{Q}[\alpha^2, \alpha^{-2}, \alpha^{-1}q, \alpha^{-1}q^{-1}, (\alpha^{-1}q \pm \alpha^{-1}q^{-1})^{-1}]$$

which is isomorphic to $S_\pm = S/(a \mp bc)$ under $a \mapsto \alpha^{-2}$, $b \mapsto \alpha^{-1}q$, $c \mapsto \pm\alpha^{-1}q^{-1}$. Using this isomorphism, we get that

$$e_i = \frac{a s_i^{-1} \pm s_i}{b + c} \mp 1,$$

and the defining relations become, in addition of the braid relations,

(i) $(s_1 - a)(s_1 - b)(s_1 - c) = 0$
(ii) $e_1 s_2 e_1 = e_1$
(iii) $e_1 s_2^{-1} e_1 = e_1$

and thus $BMW_n^\pm$ appears as the quotient of $H_n \otimes_S S_\pm$ by the ideal generated by two elements $\mathcal{S}_\pm$ and $\mathcal{S}'_\pm$, namely $e_1 s_2 e_1 - e_1$ and $e_1 s_2^{-1} e_1 - e_1$. Notice that each of these elements, expressed in the s_i's, is a linear combination of 12 terms originating from the braid group. We call these the two 12-terms defining relations of the BMW algebras. In this setting, the classical definition of the BMW algebras can be formulated as follows.

Definition 3.1. $BMW_n^\pm = (H_n \otimes_R S_\pm)/(\mathcal{S}_\pm, \mathcal{S}'_\pm)$.

We let $x = b + c$, $y = bc$. We recall the following easy consequences of the cubic relation and of the definition of e_i :

- $e_i^2 = \delta e_i$ with $\delta = \frac{1 \pm a \mp x}{x}$
- $e_i s_i = a e_i = e_i s_i$

We first prove that, if $x - a$ is made invertible, then these two 12-terms relations are equivalent, in other words that the defining ideal is generated by either one of these two relations.

Proposition 3.2.

$$BMW_n^\pm \otimes_{S_\pm} S_\pm[(x-a)^{-1}] = \left(H_n \otimes_R S_\pm[(x-a)^{-1}]\right)/(\mathcal{S}_\pm)$$

$$BMW_n^\pm \otimes_{S_\pm} S_\pm[(x-a)^{-1}] = \left(H_n \otimes_R S_\pm[(x-a)^{-1}]\right)/(\mathcal{S}'_\pm)$$

Proof. We need to prove that the two relations $e_1 s_2^{-1} e_1 = e_1$ and $e_1 s_2 e_1 = e_1$ are implied one by the other, inside $H_3 \otimes_R S_\pm[(x-a)^{-1}]$. We assume that $e_1 s_2 e_1 = e_1$, and show $e_1 s_2^{-1} e_1 = e_1$, the proof of the converse implication being similar.

We have $X = (e_1 s_2 e_1) s_2^{-1} e_1 = e_1 s_2^{-1} e_1$, and also $X = e_1 (s_2 e_1 s_2^{-1}) e_1 = e_1 s_1^{-1} e_2 s_1 e_1$, because the braid relations imply $s_2 s_1^u s_2^{-1} = s_1^{-1} s_2^u s_1$ for all $u \in \mathbb{Z}$ and by expressing e_i as a linear combination of $1, s_i$ and s_i^{-1}. Now the cubic relation implies $s_1 e_1 = a e_1$ and $e_1 s_1^{-1} = a^{-1} e_1$, hence $X = (e_1 s_1^{-1}) e_2 (s_1 e_1) = e_1 e_2 e_1$. By definition of e_2, this is

$$X = e_1 \left(\frac{a s_2^{-1} \pm s_2}{x} \mp 1 \right) e_1 = \frac{a e_1 s_2^{-1} e_1 \pm e_1 s_2 e_1}{x} \mp e_1^2 = \frac{a e_1 s_2^{-1} e_1 \pm e_1}{x} \mp \delta e_1.$$

Altogether, this yields

$$\left(1 - \frac{a}{x}\right) e_1 s_2^{-1} e_1 = \mp \left(\delta - \frac{1}{x}\right) e_1 = \mp \left(\frac{1 \pm a \mp x - 1}{x} \right) e_1 = \left(\frac{-a + x}{x} \right) e_1 = \left(1 - \frac{a}{x}\right) e_1$$

whence the conclusion. ☐

Recall that, inside $BMW_n^\pm$, we have $a = \pm y$.

Proposition 3.3. *We have $s_2^{-1} e_1 s_2^{-1} = a^{-2} s_1 e_2 s_1 = y^{-2} s_1 e_2 s_1$, and*

$$\frac{1}{x} s_2^{-1} s_1 s_2^{-1} - \frac{1}{xy^2} s_1 s_2 s_1 - s_2^{-1} + s_1^2 = \frac{1}{xy} s_1 s_2^{-1} s_1 - \frac{y}{x} s_2^{-1} s_1^{-1} s_2^{-1}.$$

Proof. We have $e_1 = \frac{a s_1^{-1} \pm s_1}{x} \mp 1$, that is $a s_1^{-1} \pm s_1 = x e_1 \pm x$ hence $s_1 = \pm x e_1 + x \mp a s_1^{-1}$. It follows that $s_1 e_2 s_1 = \pm x e_1 e_2 s_1 + x e_2 s_1 \mp a s_1^{-1} e_2 s_1$. By the braid relations we have $s_1^{-1} e_2 s_1 = s_2 e_1 s_2^{-1}$, and we have $e_1 e_2 s_1 = a e_1 s_1^{-1} e_2 s_1 = a e_1 s_2 e_1 s_2^{-1} = a e_1 s_2^{-1}$. Thus

$$s_1 e_2 s_1 = \pm x a e_1 s_2^{-1} + x e_2 s_1 \mp a s_2 e_1 s_2^{-1}.$$

Similarly, $s_2^{-1} = \mp a^{-1} s_2 + a^{-1} x e_2 \pm x a^{-1}$, hence $s_2^{-1} e_1 s_2^{-1} = \mp a^{-1} s_2 e_1 s_2^{-1} + a^{-1} x e_2 e_1 s_2^{-1} \pm x a^{-1} e_1 s_2^{-1}$. We have $e_2 e_1 s_2^{-1} = a^{-1} (e_2 s_2) e_1 s_2^{-1} = a^{-1} e_2 (s_2 e_1 s_2^{-1}) = a^{-1} (e_2 s_1^{-1} e_2) s_1 = a^{-1} e_2 s_1$ and

$$\begin{aligned} s_2^{-1} e_1 s_2^{-1} &= \mp a^{-1} s_2 e_1 s_2^{-1} + a^{-2} x e_2 s_1 \pm x a^{-1} e_1 s_2^{-1} \\ &= a^{-2} (\pm x a e_1 s_2^{-1} + x e_2 s_1 \mp a s_2 e_1 s_2^{-1}) \\ &= a^{-2} s_1 e_2 s_1. \end{aligned}$$

Using again $e_i = \frac{a s_i^{-1} \pm s_i}{x} \mp 1$ and $a = \pm y$ on both sides one gets the conclusion by straightforward computation. ☐

Finally, following the method of Birman and Wenzl in [1], we define Markov traces $t_n^\pm : BMW_n^\pm \to S_\pm$ by their images on words in the $\sigma_i' s$, by closing the braid corresponding to it and applying the Kauffman invariant of links, in its original or Dubrovnik variation (see [12]). We recall that this is done by applying the skein relations of Figure 1, starting from the additional conventional choice that the trivial knot diagram is mapped to 1, and then multiply by α^r where r is the writhe of the diagram, namely the number of crossings of the original braid (in other terms, the image of the abelianization morphism $\ell : B_n \to \mathbb{Z}$ which maps $\sigma_i \mapsto 1$). In particular, we have $t_n^\pm(s_1 \dots s_{n-1}) = 1$, and

$$t_2^\pm(1) = \delta_K^\pm = \frac{y \mp x + 1}{x}.$$

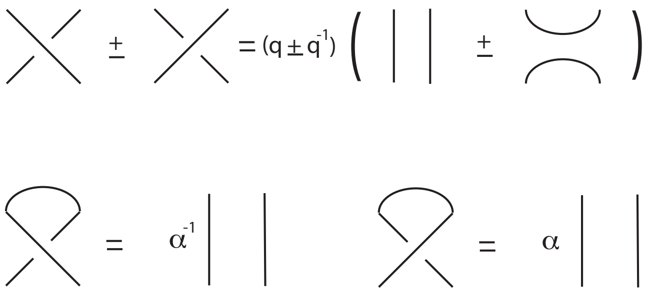

FIGURE 1. Skein relations for the Kauffman polynomial of unoriented links

The fact that the value of such a trace on braids lies inside the $S_\pm$ is a consequence of the following probably classical lemma.

Lemma 3.4. *For all* $\beta \in B_n$, $t_n^\pm(\beta) \in \mathbb{Q}[a, a^{-1}, x, x^{-1}] \subset S_\pm$

Proof. Let L be the closure of the braid β, and $\overrightarrow{D}$ an oriented link diagram representing it. We prove more generally that, for an oriented link diagram $\overrightarrow{D}$, the value of the Kauffman invariant $\overrightarrow{K}(\overrightarrow{D})$ lies in $\mathbb{Q}[a, a^{-1}, x, x^{-1}]$. We do this by a double induction, first on the number of crossings, and then, the number of crossings being fixed, on the minimal number of crossings needed to be changed in order to get a diagram representing a trivial link. If $\overrightarrow{D}$ represents a trivial link with r components, we have $\overrightarrow{K}(\overrightarrow{D}) = t_n^\pm(1) = \delta_K^{\pm r} \in S_0$. Otherwise, by choosing a suitable crossing we can apply the first relation of Figure 1 to the corresponding unoriented diagram D. Letting D' the other diagram with the same number of crossings, and D_0, D_∞ the two other ones, and $K(D)$,$K(D')$, etc. the Kauffman polynomial for unoriented links associated to them, we get $K(D) = \mp K(D') \mp \varepsilon(q \pm q^{-1})(K(D_0) + K(D_\infty))$ for some $\varepsilon \in \{-1, 1\}$. There exists oriented diagrams $\overrightarrow{D'}, \overrightarrow{D_0}, \overrightarrow{D_\infty}$ whose underlying unoriented diagrams are D', D_0, D_∞. Then $\overrightarrow{K}(\overrightarrow{D})$ is equal to

$$\mp \overrightarrow{K}(\overrightarrow{D}')\alpha^{w(\overrightarrow{D})-w(\overrightarrow{D'})} \mp \varepsilon(q \pm q^{-1})\alpha\left(\alpha^{w(\overrightarrow{D})-w(\overrightarrow{D_0})-1}\overrightarrow{K}(\overrightarrow{D}_0) \pm \alpha^{w(\overrightarrow{D}_0)-w(\overrightarrow{D_\infty})-1}\overrightarrow{K}(\overrightarrow{D}_\infty)\right)$$

Since the parity of the writhe only depends on the number of crossings the conclusion follows by induction. $\square$

More precisely, this lemma shows that the value of $t_n^\pm$ on such a braid belongs to the subalgebra of $S_\pm$ generated by $\alpha^{-2} = a$ and $\alpha^{-1}(q \pm q^{-1}) = b + c = x$ as well as their inverses. If $z = q \pm q^{-1}$, this subalgebra may also be seen as the fixed subalgebra of $\mathbb{Q}[\alpha, \alpha^{-1}, z, z^{-1}]$ fixed by the involutive automorphism $\alpha \mapsto -\alpha$, $z \mapsto -z$.

Using the skein relations one gets in particular the following formula (which is the value of the Kauffman polynomial on the figure-eight knot 4_1).

$$t_3^\pm(s_1 s_2^{-1} s_1 s_2^{-1}) = x^3(a^2 \pm a) + x^2(a^2 + 2a \pm 1) - x(1 \pm a) - (\pm 1 \pm a + a^{-1})$$

In terms of quantum groups, these two variants of the BMW algebras have their origin in the disctinction between the symplectic and orthogonal groups acting on their standard

$$\begin{tikzcd}
 & & V(\varpi_1) \arrow[dll] \arrow[d] \arrow[drr] & & \\
V(2\varpi_1) \arrow[dd] \arrow[drr] \arrow[ddrr] & & \mathbb{1} \arrow[d] & & V(\varpi_2) \arrow[dll] \arrow[ddll] \arrow[dd] \\
 & & V(\varpi_1) & & \\
V(3\varpi_1) & & V(\varpi_1+\varpi_2) & & V(\varpi_3)
\end{tikzcd}$$

FIGURE 2. Decomposition of $V \otimes V$ and $V \otimes V \otimes V$

	$\mathbb{1}$ $\tilde{a}$	$2\varpi_1$ $\tilde{b}$ or $\tilde{c}$	ϖ_2 $\tilde{b}$ or $\tilde{c}$
$SO(V)$	q^{1-m}	q	$-q^{-1}$
$SP(V)$	$-q^{1-m}$	q^{-1}	$-q$

TABLE 1. Eigenvalues of the braid action

module. Indeed, let V denote the finite dimensional complex vector space acted upon by the isometry group G of some non-degenerate bilinear form. We assume G is split and fix a Cartan subalgebra of its Lie algebra $\mathfrak{g}$, and use Bourbaki conventions and notations for the weights (see [2]). Then V is a fundamental module of highest weight ϖ_1. Then, Figure 2 represents the graph corresponding to the relation $x \to y$ meaning 'y appears as a constituent in $x \otimes V$' (which turns out to be a symmetric relation because V is selfdual), and therefore is also the Bratteli diagram of the tower of centralizers algebra, which are well-known to be the algebras of Brauer diagrams. The difference between the orthogonal and symplectic case is that $S^2V = \mathbb{1} + V(2\varpi_1)$ in the former case, while $\Lambda^2 V = \mathbb{1} + V(\varpi_2)$ in the latter – where $V(\lambda)$ denotes the highest weight module corresponding to λ, and $\mathbb{1} = V(0)$ is the trivial representation.Therefore, the quantum representation of the braid group obtained by monodromy of the KZ 1-form

$$\frac{h}{\mathrm{i}\pi} \sum_{i<j} \Omega_{ij} \mathrm{dlog}(z_i - z_j)$$

is such that the spectrum $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ of the Artin generators is given in table 1 with $q = e^{h/2(m-2)}$, $m = \dim V$ in the orthogonal case, $m = -\dim V$ in the symplectic case (recall that Ω_{ij} is the action on the tensor factors in position (i,j) of $V^{\otimes n}$ of $\sum_k f_k \otimes f_k \in \mathfrak{g} \otimes \mathfrak{g}$ where $f_1, f_2, \dots$ is an orthonormal basis of $\mathfrak{g}$ w.r.t. the Killing form). Renormalizing the eigenvalues by the formula $x = \tilde{x} q^{1-m}$, we get that $a = -bc$ in the orthogonal case, while $a = bc$ in the symplectic case. Therefore the algebras BMW_n^+ and BMW_n^- corresponds to the symplectic and orthogonal groups, respectively. Moreover, a is specialized to $\mp q^{2(1-m)}$.

4. A universal cover for the two BMW algebras, and a shorter defining relation

We still denote $R = \mathbb{Q}[a,b,c,(abc)^{-1}]$, $S = R[(b+c)^{-1}]$. In all what follows, $\mathbb{Q}$ could be replaced by $\mathbb{Z}[\frac{1}{2}]$ without damage ; however the invertibility of 2 is crucial at several steps. Indeed, the BMW-algebras in characteristic 2 present very specific features (see [6] for results in this direction). For $n \geq 3$, we let I_n denote the ideal of H_n generated by the elements

$$\mathcal{R}_i = -bcs_is_{i+1}^{-1}s_i + (bc)^2s_{i+1}^{-1}\ s_is_{i+1}^{-1} - s_is_{i+1}s_i - (b+c)b^2c^2s_{i+1}^{-2} + (b+c)s_i^2 + (bc)^3s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}$$

for $1 \leq i \leq n-2$. Note that $\mathcal{R}_i = -bcs_is_{i+1}^{-1}s_i + (bc)^2s_{i+1}^{-1}\ s_is_{i+1}^{-1} - s_is_{i+1}s_i - (b+c)b^2c^2s_{i+1}^{-2} + (b+c)s_i^2 + (bc)^3s_i^{-1}s_{i+1}^{-1}s_i^{-1}$, and that each $\mathcal{R}_i$ is conjugated to $\mathcal{R}_1$ inside H_n. As a consequence I_n is generated as an ideal by $\mathcal{R}_1$.

Definition 4.1. *We let $\widetilde{BMW}_n = H_n/I_n$.*

We also note that $\mathcal{R}_i$ actually has coefficients in $R_0 = \mathbb{Q}[b,c,(bc)^{-1}] \subset R$. Let ε be the involutive automorphism of the $\mathbb{Q}$-algebra R_0 defined by $b \mapsto c$, $c \mapsto b$. We have $R_0^\varepsilon = \mathbb{Q}[x,y^{\pm 1}]$ with $y = bc$, $x = b+c$, and $\mathcal{R}_i$ has coefficients in R_0^ε. The relation $\mathcal{R}_i$ can thus be written

$$0 \equiv -ys_is_{i+1}^{-1}s_i + y^2s_{i+1}^{-1}\ s_is_{i+1}^{-1} - s_is_{i+1}s_i - xy^2s_{i+1}^{-2} + xs_i^2 + y^3s_i^{-1}s_{i+1}^{-1}s_i^{-1}$$

or (see Figure 3)

$$s_{i+1}^{-1}\ s_is_{i+1}^{-1} \equiv \frac{1}{y}s_is_{i+1}^{-1}s_i + \frac{1}{y^2}s_is_{i+1}s_i + xs_{i+1}^{-2} - \frac{x}{y^2}s_i^2 - ys_i^{-1}s_{i+1}^{-1}s_i^{-1}.$$

The cubic relation $(s_i-a)(s_i^2-xs_i+y) = s_i^3-(a+x)s_i^2+(y+ax)s_i-ay = 0$ implies $s_i^2 = (a+x)s_i - (y+ax) + ays_i^{-1}$, hence that $s_{i+1}s_i^{-1}s_{i+1} = s_i^{-1}(s_is_{i+1}s_i^{-1})s_{i+1} = s_i^{-1}s_{i+1}^{-1}s_is_{i+1}s_{i+1} = s_i^{-1}s_{i+1}^{-1}s_is_{i+1}^2 = (a+x)s_i^{-1}(s_{i+1}^{-1}s_is_{i+1}) - (y+ax)s_i^{-1}s_{i+1}^{-1}s_i + ays_i^{-1}s_{i+1}^{-1}s_is_{i+1}^{-1} = (a+x)s_{i+1}s_i^{-1} - (y+ax)s_i^{-1}s_{i+1}^{-1}s_i + ays_i^{-1}s_{i+1}^{-1}s_is_{i+1}^{-1}$. Thus, $\mathcal{R}_i$ implies the following relation $\mathcal{R}'_i$:

$$\begin{array}{rcl} s_{i+1}s_i^{-1}s_{i+1} & \equiv & (a+x)s_{i+1}s_i^{-1} - (y+ax)s_i^{-1}s_{i+1}^{-1}s_i + as_{i+1}^{-1}s_i + \frac{1}{y}as_{i+1}s_i \\ & & +xays_i^{-1}s_{i+1}^{-2} - \frac{x}{y}as_i - ay^2s_i^{-2}s_{i+1}^{-1}s_i^{-1}. \end{array}$$

We let $\mathcal{H}_n(b,c)$ denote the usual Hecke algebra $RB_n/(s_1-b)(s_1-c)$. The natural projection $RB_n \twoheadrightarrow \mathcal{H}_n(b,c)$ obviously factorizes through H_n.

Proposition 4.2.

(i) *The R-algebra morphism $H_n \twoheadrightarrow \mathcal{H}_n(b,c)$ factorizes through $\widetilde{BMW}_n$.*

(ii) *If we abuse notations by letting $\widetilde{BMW}_n$ also denote the image of $\widetilde{BMW}_n$ inside $\widetilde{BMW}_{n+1}$ under the natural morphism $s_i \mapsto s_i$, we have*

$$\widetilde{BMW}_{n+1} = \widetilde{BMW}_n + \widetilde{BMW}_ns_n\widetilde{BMW}_n + \widetilde{BMW}_ns_n^{-1}\widetilde{BMW}_n.$$

(iii) *For all n, $\widetilde{BMW}_n$ is a finitely generated R-module.*

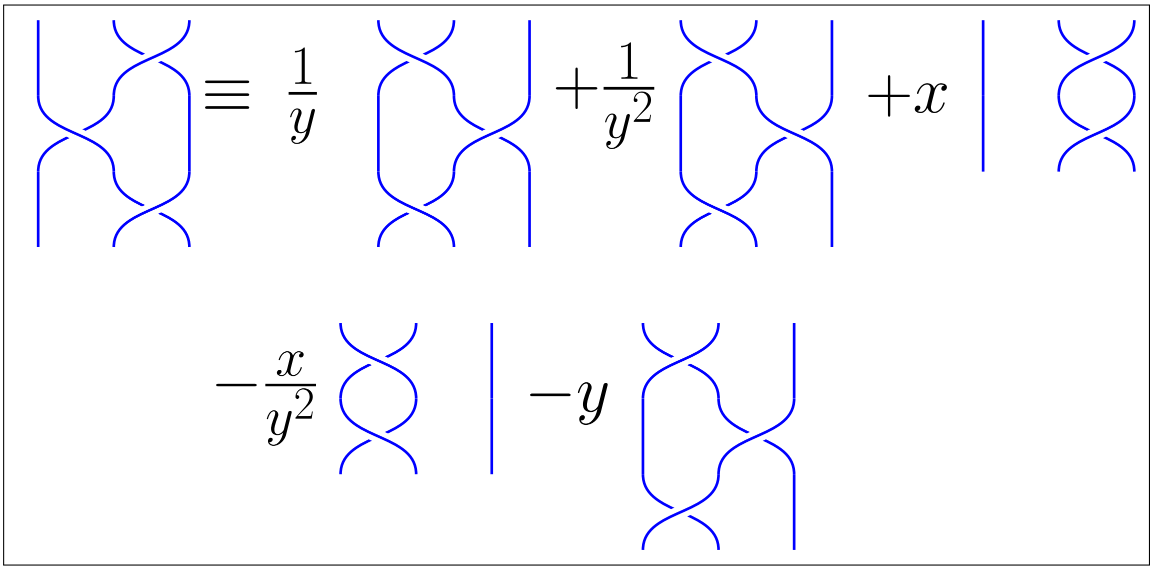

FIGURE 3. 6-terms defining relation for $\widetilde{BMW}_3$

Proof. The defining relation of $\mathcal{H}_n(b,c)$ is $s_i^2 = xs_i - y$, and $\mathcal{R}_i = -ys_is_{i+1}^{-1}s_i + y^2s_{i+1}^{-1}\,s_is_{i+1}^{-1} - s_is_{i+1}s_i - xy^2s_{i+1}^{-2} + xs_i^2 + y^3s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}$. Inside $\mathcal{H}_n(b,c)$, $ys_{i+1}^{-1} = x - s_{i+1}$ hence $-ys_is_{i+1}^{-1}s_i = -xs_i^2 + s_is_{i+1}s_i$ and $\mathcal{R}_i = y^2s_{i+1}^{-1}\,s_is_{i+1}^{-1} - xy^2s_{i+1}^{-2} + y^3s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}$. From $ys_i^{-1} = x - s_i$ we get $y^3s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1} = xy^2s_{i+1}^{-2} - y^2s_{i+1}^{-1}s_is_{i+1}^{-1}$ hence $\mathcal{R}_i = 0$ in $\mathcal{H}_n(b,c)$, which proves (i). We prove (ii) by induction, with $\widetilde{BMW}_1 = R$, the case $n = 1$ being trivially true, as $\widetilde{BMW}_2$ is spanned over R by $1, s_1, s_1^{-1}$. The method is now similar to the one used in [6], proposition 4.2. Let $U = \widetilde{BMW}_n + \widetilde{BMW}_n s_n \widetilde{BMW}_n + \widetilde{BMW}_n s_n^{-1}\widetilde{BMW}_n \subset \widetilde{BMW}_{n+1}$. It is a $\widetilde{BMW}_n$-submodule of $\widetilde{BMW}_{n+1}$ containing 1, so we only need to prove $s_nU \subset U$. We have $s_n\widetilde{BMW}_n \subset U$, so we need to prove $s_n\widetilde{BMW}_n s_n^{\pm 1}\widetilde{BMW}_n \subset U$, and actually only $s_n\widetilde{BMW}_n s_n^{\pm 1} \subset U$ is needed. By induction we know $\widetilde{BMW}_n = \widetilde{BMW}_{n-1} + \widetilde{BMW}_{n-1}s_{n-1}\widetilde{BMW}_{n-1} + \widetilde{BMW}_{n-1}s_{n-1}^{-1}\widetilde{BMW}_{n-1}$ hence $s_n\widetilde{BMW}_n s_n^{\pm 1}$ is equal to

$$\begin{aligned} & s_n\widetilde{BMW}_{n-1}s_n^{\pm 1} + s_n\widetilde{BMW}_{n-1}s_{n-1}\widetilde{BMW}_{n-1}s_n^{\pm 1} + s_n\widetilde{BMW}_{n-1}s_{n-1}^{-1}\widetilde{BMW}_{n-1}s_n^{\pm 1} \\ = \; & \widetilde{BMW}_{n-1}s_ns_n^{\pm 1} + \widetilde{BMW}_{n-1}s_ns_{n-1}s_n^{\pm 1}\widetilde{BMW}_{n-1} + \widetilde{BMW}_{n-1}s_ns_{n-1}^{-1}s_n^{\pm 1}\widetilde{BMW}_{n-1}. \end{aligned}$$

It is thus sufficient to prove that $s_ns_{n-1}^{-1}s_n \in U$, since $s_ns_{n-1}s_n^{\pm 1} = s_{n-1}^{\pm 1}s_ns_{n-1} \in U$ and $s_ns_{n-1}^{-1}s_n^{-1} = s_{n-1}^{-1}s_ns_{n-1} \in U$. This follows from relation $\mathcal{R}'_n$, and proves (ii) by induction. (iii) is a trivial consequence of (ii). □

Inside $\widetilde{BMW}_n \otimes_R S$ and S we introduce the following elements

$$e_i = \frac{a}{y}\left(\frac{ys_i^{-1} + s_i}{x} - 1\right), \qquad \tilde{\delta} = \frac{a^2 - ax + y}{y}.$$

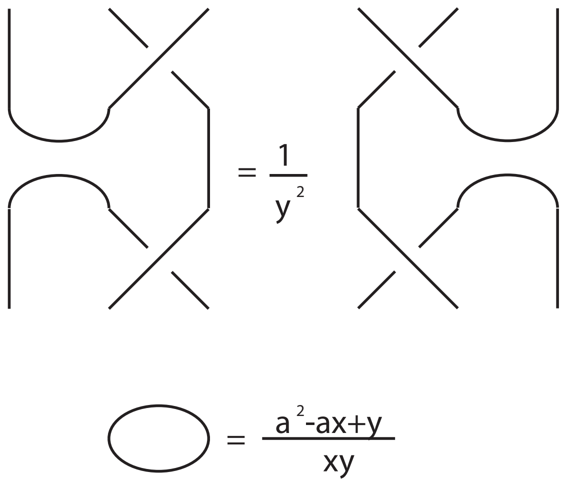

FIGURE 4. Diagrammatic relations for $\widetilde{BMW}_n \otimes_R S$

Proposition 4.3. *We have* $e_i s_i = a e_i$ *and* $s_{i+1}^{-1} e_i s_{i+1}^{-1} = y^{-2} s_i e_{i+1} s_i$. *Moreover, these two relations together with* $e_i = \frac{a}{y}\left(\frac{y s_i^{-1} + s_i}{x} - 1\right)$ *provide a presentation of* $\widetilde{BMW}_n \otimes_R S$ *with generators* $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$, *and we have* $e_i^2 = \tilde{\delta} x^{-1} e_i$.

Proof. The fact that $e_i^2 = \tilde{\delta} x^{-1} e_i$ is a straightforward computation using the cubic relation and the definition of e_i. Using the definition of e_i, $s_{i+1}^{-1} e_i s_{i+1}^{-1} = y^{-2} s_i e_{i+1} s_i$ is a rewriting of the defining relation $\mathcal{R}_i$, which proves the first claim. The fact that these relations provide a presentation of $\widetilde{BMW}_n$ follows from the fact that $e_i s_i = a e_i$, $e_i^2 = \delta e_i$ (with $\delta = \tilde{\delta}/x$) together with the relation between e_i and s_i implies the cubic relation on s_i, and then again $s_{i+1}^{-1} e_i s_{i+1}^{-1} = y^{-2} s_i e_{i+1} s_i$. □

It is thus possible to depict elements of $\widetilde{BMW}_n \otimes_R S$ as diagrams in a similar flavour as the elements of the usual BMW algebras (see Figure 4).

For the sequel we need to compute the image of $\mathcal{R}_1$ inside the various representations of H_3. We get that $S_b, S_c, U_{b,c}, V$ all map $\mathcal{R}_1$ to 0, while

$$S_a : \mathcal{R}_1 \mapsto -\frac{(-c+a)(a-b)\left(a^2 - bc\right)\left(a^2 + bc\right)}{a^3}$$

$$U_{a,c} : \mathcal{R}_1 \mapsto \begin{pmatrix} (a-b)\left(ca+b^2\right) & -\frac{c(a-b)\left(ca+b^2\right)}{a} \\ -\frac{c(a-b)\left(ca+b^2\right)}{a} & \frac{c^2(a-b)\left(ca+b^2\right)}{a^2} \end{pmatrix}$$

$$U_{a,b} : \mathcal{R}_1 \mapsto \begin{pmatrix} (-c+a)\left(ab+c^2\right) & -\frac{b(-c+a)\left(ab+c^2\right)}{a} \\ -\frac{b(-c+a)\left(ab+c^2\right)}{a} & \frac{b^2(-c+a)\left(ab+c^2\right)}{a^2} \end{pmatrix}$$

Recall that the *numerical invariant* of a (absolutely) semisimple k-algebra A is the tuple $(n_1 \leq \cdots \leq n_r)$ such that $A \otimes \overline{k} \simeq M_{n_1}(\overline{k}) \times \cdots \times M_{n_r}(\overline{k})$, where $\overline{k}$ denotes any algebraically closed field containing k.

Proposition 4.4. *For all* n, $\widetilde{BMW}_n \otimes K$ *is a semisimple algebra whose numerical invariant is the same as* BMW_n^+*. In particular it has dimension* $1.3.5.\dots.(2n+1)$.

Proof. Let $\psi : R \to S_+[\lambda^{\pm 1}]$ be the algebra morphism defined by $a \mapsto \lambda bc$, $b \mapsto \lambda b$, $c \mapsto \lambda c$. We define a surjective morphism $S_+[\lambda^{\pm 1}]B_n \to BMW_n^+ \otimes_{S_+} S_+[\lambda^{\pm 1}]$ which maps $\sigma_i \mapsto \lambda s_i$, where σ_i is the i-th Artin generator of the braid group B_n. It is straightforward to check that this morphism factorizes according to the following diagram, where the map $S_+[\lambda^{\pm 1}]B_n \to H_n \otimes_\psi S_+[\lambda^{\pm 1}]$ is $\sigma_i \mapsto s_i$, and $H_n \otimes_\psi S_+[\lambda^{\pm 1}] \to \widetilde{BMW}_n \otimes_\psi S_+[\lambda^{\pm 1}]$ is naturally induced by the canonical map $H_n \twoheadrightarrow \widetilde{BMW}_n$.

$$
\begin{tikzcd}
S_+[\lambda^{\pm 1}]B_n \arrow[r] \arrow[d] & BMW_n^+ \otimes_{S_+} S_+[\lambda^{\pm 1}] \\
H_n \otimes_\psi S_+[\lambda^{\pm 1}] \arrow[ur] \arrow[dr] & \\
& \widetilde{BMW}_n \otimes_\psi S_+[\lambda^{\pm 1}] \arrow[uu]
\end{tikzcd}
$$

Let $\tilde{K}$ the field of fractions of $S_+[\lambda^{\pm 1}]$. From this factorisation we get surjective morphisms $H_n \otimes_\psi \tilde{K} \to \widetilde{BMW}_n \otimes_\psi \tilde{K} \to BMW_n^+ \otimes_{S_+} \tilde{K}$. The kernel F_n of $H_n \otimes_\psi \tilde{K} \to BMW_n^+ \otimes_{S_+} \tilde{K}$ is the ideal generated by the image of F_3 under the natural morphism $H_3 \otimes_\psi \tilde{K} \to H_n \otimes_\psi \tilde{K}$, and similarly the kernel G_n of $H_n \otimes_\psi \tilde{K} \to \widetilde{BMW}_n \otimes_\psi \tilde{K}$ is generated by the image of G_3 under the natural morphism $H_3 \otimes_\psi \tilde{K} \to H_n \otimes_\psi \tilde{K}$. In order to prove that the morphism $\widetilde{BMW}_n \otimes_\psi \tilde{K} \to BMW_n^+ \otimes_{S_+} \tilde{K}$ is an isomorphism, it is thus sufficient to check that $BMW_3^+ \otimes_{S_+} \tilde{K}$ and $\widetilde{BMW}_3 \otimes_\psi \tilde{K}$ have the same dimension. Since BMW_3^+ is free of rank 15 over S_+ we need to show that $\widetilde{BMW}_3 \otimes_\psi \tilde{K}$ has dimension 15. For example because of the computation of the Schur elements, we know by section 2 that $H_3 \otimes_\psi \tilde{K}$ is semisimple, and isomorphic to a direct sum of matrix algebras. Because of this, the ideal generated by an element is uniquely determined by the collection of simple modules on which it vanishes. Because of the calculations above we know that this element is nonzero exactly on $U_{a,c}, U_{b,c}, S_a$. It follows that the ideal has dimension $1 + 2 \times 2^2 = 9$ hence the dimension of $\widetilde{BMW}_3 \otimes_\psi \tilde{K}$ is $24 - 9 = 15$, as required.

We thus proved that $\widetilde{BMW}_n \otimes_\psi \tilde{K}$ is semisimple, because $BMW_n^+ \otimes_{S_+} \tilde{K}$ is so, and that these two $\tilde{K}$-algebras have the same numerical invariant. Now notice that $\tilde{K} = \mathbb{Q}(\lambda, b, c)$ is isomorphic to $K = \mathbb{Q}(a, b, c)$ under the isomorphisms

$$
\begin{array}{lll}
\tau^{-1} : & K \to \tilde{K} & a \mapsto \lambda bc, b \mapsto \lambda b, c \mapsto \lambda c \\
\tau : & \tilde{K} \to K & \lambda \mapsto \frac{bc}{a}, b \mapsto \frac{a}{c}, c \mapsto \frac{a}{b}
\end{array}
$$

and we have $\widetilde{BMW}_n \otimes_R K \simeq (\widetilde{BMW}_n \otimes_\psi \tilde{K}) \otimes_\tau K$ as K-algebras. This isomorphism τ between $\tilde{K}$ and K can be extended to an isomorphism between their algebraic closures k, $\tilde{k}$.

If $\widetilde{BMW}_n \otimes_\psi \tilde{k} \simeq M_{n_1}(\tilde{k}) \oplus \cdots \oplus M_{n_r}(\tilde{k})$, then $\widetilde{BMW}_n \otimes_R k \simeq M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$ and the conclusion follows. $\square$

We let $\overline{R} = R/(a^2-(bc)^2) = R/(a^2-y^2)$, $R_\pm = R/(a \mp y)$, and $\overline{S} = S/(a^2-(bc)^2) = S/(a^2-y^2)$, $S_\pm = S/(a \mp y)$. By the Chinese Reminder Theorem we have $\overline{R} \simeq R_+ \oplus R_-$, $\overline{S} \simeq S_+ \oplus S_-$. Let $\overline{BMW}_n = \widetilde{BMW}_n \otimes_R \overline{R}$. From the preceding section we know that we have natural $\overline{R}$-algebra morphisms $\overline{BMW}_n \to BMW_n^\pm$, hence a $\overline{R}$-algebra morphism $\overline{BMW}_n \to BMW_n^+ \oplus BMW_n^-$.

Proposition 4.5. *The natural morphism $\overline{BMW}_n \to BMW_n^+ \oplus BMW_n^-$ becomes an isomorphism after tensorisation by $\overline{S}' = \overline{S}[(bc-1)^{-1}]$.*

In other words, if $(bc-1)$ is assumed to be invertible, then the 6 terms relation that we introduce here imply the defining 12 terms relations of the BMW algebras, and thus we can use it to get a simpler definition of the BMW algebras as quotients of the group algebra of the braid group.

Recall that the standard (12 terms) generators of the defining ideal of BMW_n^+ are

$$\begin{aligned}\mathcal{S}_+ \ := \ & x^{-2}s_1s_2s_1 - yx^{-1}s_1^{-1}s_2 - yx^{-1}s_2s_1^{-1} + s_2 - x^{-1}s_1s_2 - x^{-1}s_2s_1 + yx^{-2}s_1s_2s_1^{-1} \\ & + yx^{-2}s_1^{-1}s_2s_1 + y^2x^{-2}s_1^{-1}s_2s_1^{-1} - x^{-1}s_1 - yx^{-1}s_1^{-1} + 1\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}'_+ \ := \ & x^{-2}s_1s_2^{-1}s_1 - yx^{-1}s_1^{-1}s_2^{-1} - yx^{-1}s_2^{-1}s_1^{-1} + s_2^{-1} - x^{-1}s_1s_2^{-1} - x^{-1}s_2^{-1}s_1 + yx^{-2}s_1s_2^{-1}s_1^{-1} \\ & + yx^{-2}s_1^{-1}s_2^{-1}s_1 + y^2x^{-2}s_1^{-1}s_2^{-1}s_1^{-1} - x^{-1}s_1 - yx^{-1}s_1^{-1} + 1\end{aligned}$$

and that the corresponding generators of the defining ideal of BMW_n^- are

$$\begin{aligned}\mathcal{S}_- \ := \ & x^{-2}s_1s_2s_1 - yx^{-1}s_1^{-1}s_2 - yx^{-1}s_2s_1^{-1} + s_2 - x^{-1}s_1s_2 - x^{-1}s_2s_1 + yx^{-2}s_1s_2s_1^{-1} \\ & + yx^{-2}s_1^{-1}s_2s_1 + y^2x^{-2}s_1^{-1}s_2s_1^{-1} + x^{-1}s_1 + yx^{-1}s_1^{-1} - 1\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}'_- \ := \ & x^{-2}s_1s_2^{-1}s_1 - yx^{-1}s_1^{-1}s_2^{-1} - yx^{-1}s_2^{-1}s_1^{-1} + s_2^{-1} - x^{-1}s_1s_2^{-1} - x^{-1}s_2^{-1}s_1 + yx^{-2}s_1s_2^{-1}s_1^{-1} \\ & + yx^{-2}s_1^{-1}s_2^{-1}s_1 + y^2x^{-2}s_1^{-1}s_2^{-1}s_1^{-1} + x^{-1}s_1 + yx^{-1}s_1^{-1} - 1\end{aligned}$$

Proof. We first notice that, since $\overline{R} \simeq R_+ \oplus R_-$, we have $\overline{BMW}_n \otimes_{\overline{R}} \overline{S}' = \widetilde{BMW}_n \otimes_R S'_+ \oplus \widetilde{BMW}_n \otimes_R S'_-$, where $S'_\pm = \overline{S}'/(a \mp bc)$, so we only need to prove that the natural maps $\widetilde{BMW}_n \otimes_R S'_\pm \to BMW_n^\pm \otimes_{R_\pm} S'_\pm$ are isomorphisms of $S'_\pm$-algebras. For $n=2$ this is clearly true, so we can assume $n \geq 3$.

By definition, $BMW_n^+ \otimes_{S_+} S'_+$ is the quotient of $H_n \otimes_S S'_+$ by the ideal generated by $\mathcal{S}$. The claim thus amounts to saying that $\mathcal{S}$ is contained inside the twosided ideal of $H_n \otimes_S S'_+$ generated by $\mathcal{R}_1$. This claim is substantiated by the next lemma, which concludes the proof of the proposition.

Lemma 4.6. *Inside $H_3 \otimes_R R_+$, we have*

$$x^2(y-1)\mathcal{S}_+ = \left(\frac{x+1}{y} - \frac{1}{y}s_1 - s_2^{-1}\right)\mathcal{R}_1 s_2$$

and

$$x^2(y-1)\mathcal{S}'_+ = \left(\frac{x+y}{y^2} + \frac{y-1}{y}s_1^{-1} - \frac{1}{y^2}s_1 - s_2^{-1}\right)\mathcal{R}_1 s_2.$$

Inside $H_3 \otimes_R R_-$, we have

$$x^2(y-1)\mathcal{S}_- = \left(\frac{1-x}{y} - \frac{1}{y}s_1 - s_2^{-1}\right)\mathcal{R}_1 s_2$$

and

$$x^2(y-1)\mathcal{S}'_- = \left(\frac{y-x}{y^2} + \frac{y-1}{y}s_1^{-1} - \frac{1}{y^2}s_1 - s_2^{-1}\right)\mathcal{R}_1 s_2.$$

Proof. In order to check equalities in $H_3 \otimes_R R_\pm$, we use the map $H_3 \otimes_R R_\pm \hookrightarrow R_\pm^3 \times M_2(R_\pm)^3 \times M_3(R_\pm)$ described in section 2, which is injective by lemma 2.4, and check by direct computation that both sides are equal. □

□

We add a comment on how we got the equalities of lemma 4.6, inside the algebra $R_\pm^3 \times M_2(R_\pm)^3 \times M_3(R_\pm)$. First of all, the images of both $\mathcal{S}_\pm$ and $\mathcal{R}$ are 0 on two of the 1-dimensional factors, on the factor $M_3(R_\pm)$, and on two of the factors $M_2(R_\pm)$. In order to get an expression of $\mathcal{S}_\pm$ inside the ideal generated by $\mathcal{R}_1$, we only need to consider the map $H_3 \otimes_R R_\pm \to \mathrm{End}(S_{\pm bc}) \oplus \mathrm{End}(U_{\pm bc,c}) \oplus \mathrm{End}(U_{\pm bc,b})$. Moreover, because $\mathcal{R}_1$ and $\mathcal{S}$ have coefficients in R^ε, we can restrict ourselves to consider the map $H_3 \otimes_R R_\pm \to \mathrm{End}(S_{\pm bc}) \oplus \mathrm{End}(U_{\pm bc,c})$ and look for a linear combination with coefficients in $R_\pm^\varepsilon$ of terms of the form $g\mathcal{R}_1 g'$ with g, g' elements of the braid group. Now, a direct computation shows that the image of $\mathcal{S}$ as well as of $\mathcal{R}_1 s_2$ inside $\mathrm{End}(S_{\pm bc,b})$ are matrices of the form $\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$. Taking into account their images in $S_{\pm bc}$, which belong to $\mathbb{Q}(x,y) = \mathbb{Q}(b,c)^\varepsilon$, and because $\mathbb{Q}(b,c)^2 \simeq \mathbb{Q}(x,y)^4$ as a $\mathbb{Q}(x,y)$-vector space, we get that $\mathcal{S}_\pm$ and the elements of $H_3\mathcal{R}_1 s_2$ are uniquely determined by their image into a 5-dimensional vector space over $\mathbb{Q}(x,y)$.

By multiplying $\mathcal{R}_1 s_2$ on the left by suitable elements of H_3 one readily gets a basis of this vector space, namely $\mathcal{R}_1 s_2$, $s_1^{-1}\mathcal{R}_1 s_2$, $s_1\mathcal{R}_1 s_2$, $s_2^{-1}\mathcal{R}_1 s_2$, $s_2\mathcal{R}_1 s_2$. Expressing the image of $\mathcal{S}_\pm$ in this basis we get the linear combinations of the lemma.

Since $\bar{S}/(y-1) = \bar{S}/(a^2-y) = S^\dagger/(a^2-y^2)$ with $S^\dagger = S/(a^2-y)$, the special case $a^2 = y = 1$ is a consequence of the study of the algebra $BMW_n^\dagger = \widetilde{BMW}_n \otimes_R S^\dagger$. We let $K^\dagger$ denote the fraction field of $S^\dagger$.

Proposition 4.7. *The dimension of $BMW_3^\dagger \otimes_{S^\dagger} K^\dagger$ is 16.*

Proof. By the computations above we now that the representation $S_a : H_3 \to R$ factorizes through $BMW_3^\dagger$ when tensored with $S^\dagger$, and that it is not the case for the representations $U_{a,c}$ and $U_{a,b}$. Since $H_3 \otimes_R K^\dagger$ is semisimple, the conclusion follows, as $1^2+1^2+1^2+2^2+3^3 = 16$. □

Corollary 4.8. *$\widetilde{BMW}_3$ is not free over S.*

By the same method, using the freeness of H_4, H_5 proved in [16], and modulo the still conjectural existence of convenient symmetrizing forms assumed in [15], we can check from the Schur elements computed in [15] that $H_4 \otimes K^\dagger$ and $H_5 \otimes K^\dagger$ are semisimple, and compute with the same method the dimension of $BMW_n^\dagger \otimes K^\dagger$: the irreducible representations corresponding to the ideal I_n are the ones whose restriction to $BMW_3^\dagger$ contains a constituent isomorphic to $U_{b,c}$ or $U_{a,c}$. By this method we get that, if these symmetrizing forms exist, then

$$\dim_{K^\dagger} BMW_n^\dagger \otimes K^\dagger = 1 + \dim_K \widetilde{BMW}_n \otimes K = 1 + \dim BMW_n^+$$

for $n = 3, 4, 5$. Note however that this formula is not valid for $n = 2$. We leave the general case open.

Finally, we notice the following fact. Let $\eta \in \mathrm{Aut}(R)$ be defined by $a \mapsto -a$, $b \mapsto -b$, $c \mapsto -c$. The fixed subring of R is denoted R^η. It is straightforward to check that there is an involutive automorphism of R^η-algebra $\hat{E}$ of H_n mapping $s_i \mapsto -s_i$ and acting as η on elements of R. One checks easily that $\hat{E}(\mathcal{R}_i) = -\mathcal{R}_i$, $\hat{E}(\mathcal{S}_+) = -\mathcal{S}_-$, $\hat{E}(\mathcal{S}_-) = -\mathcal{S}_+$. As a consequence we get the following.

Proposition 4.9. *There is an automorphism E of the R^η-algebra $\widetilde{BMW}_n$ defined by $r \mapsto \eta(r)$ for $r \in R$ and $s_i \mapsto -s_i$. It induces an automorphism $\overline{E}$ of $\overline{BMW_n} \otimes_{\overline{R}} \overline{S}$ exchanging BMW_n^+ and BMW_n^-.*

The automorphism of proposition 4.9 relates the traces t_n^+ and t_n^-, as we will see in corollary 5.4 below.

5. Traces on $\widetilde{BMW}_n$

By definition, a Markov trace on the tower of algebras $(\widetilde{BMW}_n)_{n\geq 1}$ with values in a fixed R-module M is a sequence $(t_n)_{n\geq 1}$ of R-linear maps $t_n = \widetilde{BMW}_n \to M$ which are traces on each $\widetilde{BMW}_n$ (that is, $t_n(xy) = t_n(yx)$ for $x, y \in \widetilde{BMW}_n$) and which satisfy the Markov conditions $t_{n+1}(\iota_n(x)s_n^{\pm 1}) = t_n(x)$ for each $x \in \widetilde{BMW}_n$, where ι_n denotes the natural map $\iota_n : \widetilde{BMW}_n \to \widetilde{BMW}_{n+1}$. We let (t_n^H) denote the Markov trace induced by the Ocneanu trace on the Hecke algebra $\mathcal{H}_n(b, c)$ through the factorization of proposition 4.2 (i), normalized by $t_1^H(1) = 1$.

5.1. General inductive properties.

The following proposition is a corollary of proposition 4.2 (ii).

Proposition 5.1. *For all $n \geq 2$, t_{n+1} is uniquely determined by t_n and $t_{n+1}(1)$.*

Proof. In the sequel, we abuse notations by letting $\widetilde{BMW}_r$ denote the image of $\widetilde{BMW}_r$ inside $\widetilde{BMW}_{n+1}$. We prove that, for all $r \leq n+1$, t_{n+1} is uniquely determined by t_n and by its value on $\widetilde{BMW}_r$. The case $r = 1$ is the statement of the proposition, since $\widetilde{BMW}_1 = R$. We prove this by descending induction on r, the case $r = n+1$ being trivial. Because of proposition 4.2 (ii) we know that $\widetilde{BMW}_{r+1} = \widetilde{BMW}_r + \widetilde{BMW}_r s_r \widetilde{BMW}_r + \widetilde{BMW}_r s_r^{-1} \widetilde{BMW}_r$. If $x, y \in \widetilde{BMW}_r$, we have $t_{n+1}(xs_r^{\pm 1}y) = t_{n+1}(yxs_r^{\pm 1})$. Conjugating by $(s_1 s_2 \dots s_n)^{n-r}$ maps $s_r^{\pm 1}$ to s_n and x, y to elements x', y' of $\widetilde{BMW}_n$. This yields $t_{n+1}(yxs_r^{\pm 1}) = t_{n+1}(y'x's_n^{\pm 1}) = t_n(y'x')$. This proves that t_{n+1} is determined by t_n and by its restriction to $\widetilde{BMW}_r$. The conclusion follows by induction. □

Proposition 5.2. *Let t'_n be the composition of t_n with $M \to M \otimes_R R[(a^2 - bc)^{-1}, (b + c)^{-1}]$. Then (t'_n) is uniquely determined by $t'_1(1)$ and $t'_2(1)$.*

Proof. In view of the previous proposition, it is sufficient to prove that, for all $n \geq 2$, $t'_{n+1}(1)$ is determined by t'_n. We have that $\mathcal{R}_{n-1}$ belongs to

$$H_n s_n H_n + H_n s_n^{-1} H_n - (b + c)b^2c^2 s_n^{-2} + (b + c)s_{n-1}^2 + (bc)^2 s_n^{-1} s_{n-1} s_n^{-1}.$$

Since $s_n^{-1} s_{n-1} s_n^{-1}$ is conjugated to $s_{n-1}^{-1} s_n s_{n-1}^{-1}$, the Markov property implies that $(b+c)t_{n+1}(s_{n-1}^2 - (bc)^2 s_n^{-2})$ is determined by t_n. It is straightforward to check that, because of the cubic relation,

$s_{n-1}^2 - (bc)^2 s_n^{-2}$ is equal to $\frac{(b+c)}{a}(a^2 - bc).1$ plus a linear combination of s_{n-1}, s_{n-1}^{-1}, s_n, s_n^{-1}, on which the value of t_{n+1} is clearly determined by t_n. It follows that $(b+c)^2(a^2-bc)t_{n+1}(1)$ is uniquely determined by t_n, and the conclusion follows.

□

We may compare this statement with the stronger assertion one has on BMW_n^+.

Proposition 5.3. *If $(T_n : BMW_n^+ \to S_+)$ is a Markov trace, then it is uniquely determined by $T_1(1)$ and $T_2(1)$.*

Proof. As before it is sufficient to prove that, for all $n \geq 2$, $T_{n+1}(1)$ is determined by T_n. We have that the 12 terms relation can be written as

$$\begin{aligned} e_{n-1}s_ne_{n-1} - e_{n-1} \quad = \quad & x^{-2}s_{n-1}s_ns_{n-1} - yx^{-1}s_{n-1}^{-1}s_n - yx^{-1}s_ns_{n-1}^{-1} + s_n - x^{-1}s_{n-1}s_n \\ & -x^{-1}s_ns_{n-1} + yx^{-2}s_{n-1}s_ns_{n-1}^{-1} + yx^{-2}s_{n-1}^{-1}s_ns_{n-1} \\ & +y^2x^{-2}s_{n-1}^{-1}s_ns_{n-1}^{-1} - x^{-1}s_{n-1} - yx^{-1}s_{n-1}^{-1} + 1 \end{aligned}$$

hence belongs to $BMW_n^+s_nBMW_n^+ + BMW_n^+s_n^{-1}BMW_n^+ - xs_{n-1} - yx^{-1}s_{n-1}^{-1} + 1$, hence, since $T_{n+1}(s_{n-1}^{\pm 1}) = T_{n+1}(s_n^{\pm 1}) = T_n(1)$, we get that $T_{n+1}(1)$ is indeed determined by T_n and the conclusion follows. □

Corollary 5.4. *Over BMW_n^+, we have $t_n^+ = (-1)^{n-1}\eta \circ t_n^- \circ \overline{E}$.*

Proof. We first prove that the RHS defines a Markov trace on BMW_n^+, with values in S_+. We denote T_n this RHS. We clearly have $T_n(\lambda\beta) = \lambda T_n(\beta)$, $T_n(\alpha+\beta) = T_n(\alpha) + T_n(\beta)$ and $T_n(\alpha\beta) = T_n(\beta\alpha)$ for all braids $\alpha, \beta \in BMW_n^+$ and scalar $\lambda \in S_+$, and also $T_{n+1}(\beta s_n^{\pm 1}) = (-1)^n t_{n+1}^-(\overline{E}(\beta s_n^{\pm 1})) = (-1)^n t_{n+1}^-(\overline{E}(\beta)\overline{E}(s_n^{\pm 1}))) = (-1)^{n-1}t_{n+1}^-(\overline{E}(\beta)s_n^{\pm 1})) = (-1)^{n-1}t_n^-(\overline{E}(\beta))) = T_n(\beta)$. By the proposition above we first need to prove that $T_2(1) = t_2^+(1)$ and $T_2(s_1) = t_2^+(s_1)$. Recall that $t_2^{\pm}(1) = \delta_K^{\pm}$, $t_2^{\pm}(s_1) = 1$ with

$$\delta_K^{\pm} = \frac{y \mp x + 1}{x}.$$

Since $\eta(\delta_K^-) = -\delta_K^+$ we get $T_2(1) = \delta_K^+ = t_2^+(1)$ and $T_2(s_1) = 1 = t_2^+(s_1)$, and the conclusion follows.

□

This corollary proves that the 'natural' diagram below is commutative only up to a sign depending on n.

$$\begin{tikzcd} & \overline{BMW}_n \otimes_{\overline{R}} \overline{S} \arrow[dl] \arrow[dr] & \\ BMW_n^+ \arrow[rr, "\overline{E}"] \arrow[d, "t_n^+"'] & & BMW_n^- \arrow[d, "t_n^-"] \\ S_+ \arrow[dr] & & S_- \arrow[ll, "\eta"] \arrow[dl] \\ & S & \end{tikzcd}$$

5.2. **Restrictions to 3 strands, and the Ocneanu trace.** We recall that the notation $MT_3(M)$ was defined in section 2. We note that

$$t_3^H(1) = \left(\frac{y+1}{x}\right)^2 \quad t_3^H(s_1) = \frac{y+1}{x} \quad t_3^H(s_1s_2) = 1$$

and define additional elements t_3^S, t_3^K of $MT_3(M)$ by

$$t_3^S(1) = a^3, \quad t_3^S(s_1) = t_3^S(s_1s_2) = 0, \quad t_3^S((s_1s_2^{-1})^2) = x^2(2a+x),$$

$$t_3^K(1) = \delta_K^2, \quad t_3^K(s_1) = \delta_K, \quad t_3^K(s_1s_2) = 1, \quad \delta_K = \frac{y^2-ax+y}{xy},$$

$$t_3^K((s_1s_2^{-1})^2) = x^3y(y+1) + x^2(y^2+2a+\frac{a}{y}) - x(1+y) - (y+a+\frac{a}{y}).$$

Proposition 5.5. *Let $t \in MT_3(M)$ factorizing through $\widetilde{BMW}_3$.*

(i) *The trace t is uniquely determined by $t(1)$, $t(s_1) = t(s_2)$ and $t(s_1s_2)$.*
(ii) *If $x(a^2-y^2)$ is invertible in M then t is uniquely determined by $t(1)$ and $t(s_1s_2)$.*
(iii) *If $x^2(a^2-y)$ is invertible in M then t is uniquely determined by $t(s_1)$ and $t(s_1s_2)$.*
(iv) *If $x(a^2-y^2)$ and $x^2(a^2-y)$ are invertible in M then t is uniquely determined by $t(s_1s_2)$.*

Proof. Let $t \in MT_3(M)$ factorizing though $\widetilde{BMW}_3$. With the notations of proposition 2.5 we write $t = \alpha e_1^* + \beta e_2^* + \gamma e_3^* + \delta e_4^*$. By direct calculation, we check that the equation $t(\mathcal{R}_1 s_1) = 0$ means

$$0 = -x^2\left(a^2+ax+y\right)\alpha + \frac{x\left(2\,(y+1)\,a^3 + 2\,x\,(y+1)\,a^2 + a\left(x^2+y\,(y+1)\right) + yx\right)}{a}\beta$$
$$+\frac{-\,(y+1)^2\,a^3 - x\,(y+1)^2\,a^2 + a\left(-x^2 - 2\,x^2y + y\left(y^2+y+1\right)\right) - yx\,(y+1)}{a}\gamma + y^2\delta$$

and, since y is invertible, this proves that t is determined by α, β and γ, meaning that t is uniquely determined by $t(1)$, $t(s_1)$ and $t(s_1s_2)$, which proves (i). Similarly, we get that $t(s_1^{-1}\mathcal{R}_1) = 0$ reads

$$(a^2-y^2)\left(xt(s_1) - (y+1)t(s_1s_2)\right) = 0$$

and $t(\mathcal{R}_1) = 0$ means

$$x^2(a^2-y)t(1) = \frac{(a^2-y)(y+1) - x(y-1)a}{a}\left(xt(s_1) - (y+1)t(s_1s_2)\right) + \frac{x}{a}(y+1)(a^2-y)t(s_1)$$

which easily implies (ii)-(iv). □

Corollary 5.6. *Let (t_n) be a Markov trace factoring through $(\widetilde{BMW}_n)$ with values in M. If $x(a^2-y^2)$ and $x^2(a^2-y)$ are invertible in M then (t_n) is a multiple of (t_n^H) composed with some morphism $R \to M$.*

Proof. Immediate consequence of (iv) together with proposition 5.2. □

Recall that $S^\dagger = S/(a^2-y) = R[x^{-1}]/(a^2-y)$ and $BMW_n^\dagger = \widetilde{BMW}_n \otimes_R S^\dagger$.

Proposition 5.7. *Let t_n' be the composite of t_n with $M \to M \otimes_R S^\dagger[(y-1)^{-1}]$. Then (t_n') is a scalar multiple of the composite of t_n^H with some morphism $R \to M$ and with $M \to M \otimes_R S^\dagger[(y-1)^{-1}]$.*

Proof. Inside $S^{\dagger}[(y-1)^{-1}]$ we have $a^2 - y^2 = y(1-y)$, hence by proposition 5.5 (ii) we have that t'_3 is uniquely determined by $t'_3(1)$ and $t'_3(s_1s_2)$, and more precisely the equation $t'_3(s_1^{-1}\mathcal{R}_1) = 0$ means $t'(s_1) = \frac{y+1}{x}t'(s_1s_2)$. Let $t_n^\circ = t'_n - t'_3(s_1s_2)t_n^H$. Then $t_3^\circ(s_1s_2) = 0$ hence $t_3^\circ(s_1) = 0$ and therefore $t_3^\circ = a^{-3}t_3^\circ(1)t_3^S$. Let $\tau = t_4^\circ \circ \iota_3$, where $\iota_3 : \widetilde{BMW}_3 \to \widetilde{BMW}_4$ is the natural map. Again by 5.5 (ii) we know that τ is uniquely determined by $\tau(1)$ and $\tau(s_1s_2)$. Moreover $\tau(s_1s_2) = t_4^\circ(s_1s_2) = t_4^\circ(s_2s_3) = t_3^\circ(s_2) = t_3^S(s_2) = 0$. It follows that $\tau(s_1) = 0$, since $\tau(s_1) = \frac{y+1}{x}\tau(s_1s_2)$. But $\tau(s_1) = t_4^\circ(s_1) = t_4^\circ(s_3) = t_3^\circ(1)$, hence $t_3(1) = 0$ and $t_3^\circ = 0$. We prove by induction on n that $t_n^\circ(1) = 0$. Assume we know that $t_r^\circ(1) = 0$ for all $r \leq n$, and consider the composite τ of t_{n+1}° with the natural map $\widetilde{BMW}_3 \to \widetilde{BMW}_{n+1}$. We have $\tau(s_1s_2) = t_{n+1}^\circ(s_1s_2) = t_{n+1}^\circ(s_{n-1}s_n) = t_{n-1}^\circ(1) = 0$. It follows as before that $\tau(s_1) = 0$, hence $t_n^0(1) = t_{n+1}^0(s_n) = t_{n+1}^0(s_1) = \tau(s_1) = 0$. It follows that $t_n^\circ(1) = 0$, and the claim follows by induction because of proposition 5.1. $\square$

5.3. **The Kauffman trace.** Recall that we defined in §3 two Markov traces $t_n^{\pm} : BMW_n^{\pm} \to S_{\pm}$. Using the Chinese Remainder Theorem isomorphism $\overline{S} \simeq S_+ \oplus S_-$ they can be patched together into a Markov trace $\overline{BMW}_n \otimes_{\overline{R}} \overline{S} \to \overline{S}$, which extends to a Markov trace $t_n^K : \widetilde{BMW}_n \to \overline{S}$.

$$\begin{tikzcd}
 & \overline{BMW}_n \otimes_{\overline{R}} S_+ \arrow[r] & BMW_n^+ \arrow[r, "t_n^+"] & S_+ \arrow[dr] & \\
\overline{BMW}_n \otimes_{\overline{R}} \overline{S} \arrow[ur] \arrow[dr] \arrow[rrrr, dotted, "t_n^K"] & & & & \overline{S} \\
 & \overline{BMW}_n \otimes_{\overline{R}} S_- \arrow[r] & BMW_n^- \arrow[r, "t_n^-"'] & S_- \arrow[ur] &
\end{tikzcd}$$

Note that the value of $t_3^K((s_1s_2^{-1})^2)$ define before indeed matches this new definition, because of the computation of the Kauffman invariant of the figure-eight knot we did in section 3.

Proposition 5.8. *Let t'_n be the composite of t_n with $M \to M \otimes_R \overline{S}[(a^2-y)^{-1}]$. Then (t'_n) is a linear combination of the composites of t_n^H and t_n^K with some morphism $\overline{S} \to M \otimes_R \overline{S}[(a^2-y)^{-1}]$.*

Proof. By proposition 5.2 we know that t'_n is uniquely determined by $t'_1(1) = t'_3(s_1s_2)$ and $t'_2(1) = t'_3(s_1)$. It is thus sufficient to show that t_3^H and t_3^K induce a basis of $(\overline{S}[(a^2-y)^{-1}])^2$ under $t'_3 \mapsto (t'_3(s_1s_2), t'_3(s_1))$. The corresponding 2×2 matrix has for determinant

$$\begin{vmatrix} \delta_H & 1 \\ \delta_K & 1 \end{vmatrix} = \delta_H - \delta_K = \frac{a}{y} \in (\overline{S}[(a^2-y)^{-1}])^{\times}$$

with $\delta_H = \frac{y+1}{x}$, and this concludes the proof. $\square$

Remark 5.9. *It is well-known that both the Kauffman polynomial and the HOMFLYPT polynomial both specialize to the Jones polynomial (see e.g. [13] p. 180). It could have been expected that this coincidence would have appeared in the previous proposition. It clearly does not, since any specialized value of the invariants defined by our traces t_n^H and t_n^K cannot* *coincide on the 2-components unlink (we always have $\delta_H \neq \delta_K$). What happens is that, according to* [13] *proposition 16.6, the trace t_n^K provides the Jones polynomial when specialized to $\alpha = -t^{\frac{-3}{4}}$, $q = t^{\frac{1}{4}}$, that is $a = q^{-6}$, $b = -q^{-2} = -t^{\frac{-1}{2}}$, $c = -q^{-4} = -t^{-1}$ while, according to* [13] *proposition 16.5, t_n^H provides the Jones polynomial when $\{b, c\}$ is specialized to $\{-t^{\frac{1}{2}}, t^{\frac{3}{2}}\}$, and the only value of t for which these two parametrizations coincide is $t^{\frac{1}{2}} = -1$, in which case we get $b + c = 0$, which is forbidden here.*

5.4. **An additional trace when $y = a^2 = 1$.** Let $S^{\dagger\dagger} = S/(a^2 - 1, y - 1) = S^{\dagger}/(a^2 - y^2) = \overline{S}/(a^2 - y)$, and $BMW_n^{\dagger\dagger} = \widetilde{BMW}_n \otimes_R S^{\dagger\dagger}$.

We first deal with the very special case $x = -2a$.

Proposition 5.10. *Let $(x_n)_{n\geq 1}$ denote a sequence with values in some $S^{\dagger\dagger}$-module satisfying $(x + 2a)M = 0$. Then there exists a Markov trace (t_n^X) on $(\widetilde{BMW}_n)$ with values in M such that, for every braid g, $t_n^X(g) = x_{\#\hat{g}}$, where $\hat{g}$ denotes the closure of the braid g and $\#L$ denotes the number of components of the link L. Moreover, every Markov trace on $(\widetilde{BMW}_n)$ with values in M is of that form.*

Proof. We first note that the relations $(s_i - a)(s_i^2 + 2as_i + 1) = (s_i - a)(s_i + a)^2 = (s_i^2 - 1)(s_i + a) = 0$ hold true in M. Therefore, the action of $BMW_n^{\dagger\dagger}$ on M factors through the group algebra $\mathbb{Q}[a]/(a^2 - 1)\mathfrak{S}_n$ of the symmetric group. Moreover, the natural map $H_n \otimes S^{\dagger\dagger}/(x + 2a) \to \mathbb{Q}[a]/(a^2 - 1)\mathfrak{S}_n$, where H_n is the cubic Hecke algebra defined by $(s_i^2 - 1)(s_i + a) = 0$, is clearly surjective, and $\mathcal{R}_1$ is easily checked to map to 0. Therefore, the map $\widetilde{BMW}_n \otimes S^{\dagger\dagger}/(x + 2a) \to \mathbb{Q}[a]/(a^2 - 1)\mathfrak{S}_n$ is surjective.

We show that there exists a Markov trace (t_n^X) on $((\mathbb{Q}[a]/(a^2 - 1))\mathfrak{S}_n)$ fulfilling the conditions of the statement. Since the formula $t_n^X(g) = x_{\#\hat{g}}$ clearly defines an invariant of links, and therefore a Markov trace on the tower of algebras of the braid groups, it is sufficient to prove that t_n^X vanishes on the defining ideal of $\mathbb{Q}[a]/(a^2 - 1)\mathfrak{S}_n$ for any given n. This ideal is the linear span of the $As_i^2B - AB$ for A, B two arbitrary braids on n strands. Since the closures of As_i^2B and AB have the same number of components, we indeed get that (t_n^X) is a well-defined Markov trace on $(\widetilde{BMW}_n)$. Since $t_n^X(1) = x_n$, the fact that all Markov traces are obtained this way is a consequence of proposition 5.1.

□

We can now state a general statement.

Proposition 5.11.

(i) *There exists a Markov trace $(t_n^{\dagger\dagger})$ on $\widetilde{BMW}_n$ with values in $S^{\dagger\dagger}$, given by $t_n^{\dagger\dagger}(\beta) = a^n\psi_n(\beta)$, where $\psi_n : \widetilde{BMW}_n \to S^{\dagger\dagger}$ is an algebra morphism defined by $s_i \mapsto a$.*

(ii) *Let t_3' be the composite of t_3 with $M \to M \otimes_R S^{\dagger\dagger}[(a - x)^{-1}, (2a - x)^{-1}]$. Then (t_3') is a linear combination of the composites of t_3^H, t_3^K, $t_3^{\dagger\dagger}$ with coefficients inside $M \otimes_R S^{\dagger\dagger}[(a - x)^{-1}, (2a - x)^{-1}]$.*

Proof. We first check that the R-algebra morphism $\psi_n : H_n \to S^{\dagger\dagger}$ defined by $s_i \mapsto a$ indeed factorizes through $\widetilde{BMW}_n$, namely that $\psi_n(\mathcal{R}_1) = 0$, by direct calculation. Then $t_n^{\dagger\dagger}(x) = a^n\psi_n(x)$ clearly defines a trace for every n, and we need to check the Markov property, namely that $t_{n+1}^{\dagger\dagger}(xs_n^{\pm 1}) = t_n^{\dagger\dagger}(x)$ for all $x \in \widetilde{BMW}_n$. This holds because $t_{n+1}^{\dagger\dagger}(xs_n^{\pm 1}) = a^{n+1}\psi_{n+1}(xs_n^{\pm 1}) = a^{n+1}\psi_{n+1}(x)a^{\pm 1} = a^{n+1}\psi_n(x)a^{\pm 1} = a^{n+1}\psi_n(x)a = a^{n+2}\psi_n(x) = a^n\psi_n(x) =$

$t_n^{\dagger\dagger}(x)$. This proves (i). Note that $t_3^{\dagger\dagger}(1) = a^3$, $t_3^{\dagger\dagger}(s_1) = a^2 = 1$, $t_3^{\dagger\dagger}(s_1 s_2) = a$, We know that t_3' is uniquely determined by its value on $1, s_1, s_1 s_2$. It has to be a linear combination of t_3^H, t_3^K and $t_3^{\dagger\dagger}$ if and only if

$$\Delta = \begin{vmatrix} a^3 & a^2 & a \\ \delta_H^2 & \delta_H & 1 \\ \delta_K^2 & \delta_K & 1 \end{vmatrix} = a \begin{vmatrix} a^2 & a & 1 \\ \delta_H^2 & \delta_H & 1 \\ \delta_K^2 & \delta_K & 1 \end{vmatrix}$$

is invertible. Since $\delta_H = (y+1)/x = 2/x$, $\delta_K = \frac{2}{x} - a$, we get

$$\Delta = \frac{2}{x^2}(2a - x)(a - x)$$

whence the conclusion of (ii).

□

We will show below (see corollary 6.12) that part (ii) actually holds true for every n, provided that $x + 2a$ is also assumed to be invertible.

We note that, when specialized to a field, $x = a, y = 1, a^2 = 1$ imply $x \in \{-1, 1\}$, hence $\{b, c\} = \{-j, -j^2\}$ with $j = \exp(\frac{2i\pi}{3})$ if $a = 1$, and $\{b, c\} = \{j, j^2\}$ if $a = -1$; likewise, $x = 2a, y = 1, a^2 = 1$ imply $x \in \{-2, 2\}$, hence $b = c$, and $b, c \in \{-1, 1\}$, hence either $a = b = c = 1$ or $a = b = c = -1$. In these four cases, we have $a = \pm bc$, and a possible additional trace on $\widetilde{BMW}_n$ cannot factorize through $BMW_n^{\pm}$, as is immediately checked on the 12-terms relation (note that, by substracting a linear combination of the two ordinary traces, we can assume in the first two cases that this trace satisfies $t_3(1) = t_3(s_1) = 0$, $t_3(s_1 s_2) = 1$, while in the latter cases we can assume $t_3(1) = 1$, $t_3(s_1) = t_3(s_1 s_2) = 0$).

6. A central extension of BMW

6.1. **Definition.** We define an algebra F_n over $A = \mathbb{Q}[a, x, x^{-1}]/(a^2 = 1)$ by generators $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}, C$ and relations

(i) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_j$
(ii) $(s_i - a)(s_i^2 - x s_i + 1) = 0$
(iii) $e_i = a\left(\frac{s_i^{-1} + s_i}{x} - 1\right)$
(iv) $s_i e_i = a e_i$
(v) $e_i s_{i+1} e_i = e_i + C$
(vi) $e_i s_{i-1} e_i = e_i + C$
(vii) $e_i s_{i+1}^{-1} e_i = e_i + C$
(viii) $e_i s_{i-1}^{-1} e_i = e_i + C$
(ix) $s_i C = C s_i = aC$.

Letting $\tilde{\delta} = 2 - ax$, we have $e_i^2 = \tilde{\delta} x^{-1} e_i$. Immediate consequences of these relations are $s_i^{-1} C = C s_i^{-1} = aC$, $e_i C = C e_i = x^{-1} \tilde{\delta} C$, $C^2 = (x^{-2}\tilde{\delta}^2 a - x^{-1}\tilde{\delta})C = x^{-1}\tilde{\delta}(a x^{-1}\tilde{\delta} - 1)C = 2x^{-2}\tilde{\delta}(a - x)C$. Note that, in the specializations $x = a$ and $x = 2a$, we have $C^2 = 0$.

The following is easily checked

Proposition 6.1. (i) *The $S^{\dagger\dagger}$-algebra $F_n \otimes_A S^{\dagger\dagger}$ is a quotient of $H_n \otimes_R S^{\dagger\dagger}$ through $s_i \mapsto s_i$ and $A \to S^{\dagger\dagger}$ being given by $x \mapsto b + c$. This quotient factorizes through $BMW_n^{\dagger\dagger}$.*

(ii) *There is a surjective morphism of* $S_{\pm}^{\dagger\dagger}$*-algebras* $F_n \otimes_A S_{\pm}^{\dagger\dagger} \twoheadrightarrow BMW_n^{\pm} \otimes_{S_{\pm}} S_{\pm}^{\dagger\dagger}$ *satisfying* $s_i \mapsto s_i$, $e_i \mapsto e_i$, $C \mapsto 0$. *Its kernel is the linear span of* C.
(iii) *The automorphism and antiautomorphism of* A*-algebras of the group algebra* AB_n *defined by* $s_i \mapsto s_i^{-1}$ *induce an automorphism and a antiautomorphism of* F_n.
(iv) *Every Markov trace* (t_n) *factorizing through* F_n *is uniquely determined by* $t_3(1)$, $t_3(s_1)$ *and* $t_3(s_1 s_2)$.

Proof. We start with (i). Only the fact that we have a factorization through $BMW_n^{\dagger\dagger}$ requires a justification. According to proposition 4.3, it is sufficient to show that $s_1 e_2 s_1$ and $s_2^{-1} e_1 s_2^{-1}$ are mapped to the same element. Using the same computations as in the proof of proposition 3.3 we easily get that both are sent to $x e_1 s_2^{-1} + axC + x e_2 s_1 - s_2 e_1 s_2^{-1}$, and this proves the claim. (ii) is easy, because the linear span of C is clearly a two-sided ideal of F_n. (iii) is easily checked from the defining relations of F_n. We now prove (iv). From the arguments of proposition 5.3 one easily gets that such a Markov trace is uniquely determined by t_3 together with the collection of the $t_n(C)$, since $C = e_{n-1} s_n e_{n-1} - e_{n-1}$. Because $C s_n = aC$ we get $a t_{n+1}(C) = t_n(C)$, whence the Markov trace is uniquely determined by t_3, and therefore by $t_3(1)$, $t_3(s_1)$ together with $t_3(s_1 s_2)$ by proposition 5.5. □

Corollary 6.2. *If* (t_n) *is a Markov trace factoring through* F_n, *then the associated link invariant does not distinguish mirrors and does not detect non-invertible links.*

Proof. This follows from the items (iii) and (iv) of the proposition : the mirror of the closed braid $\hat{\beta}$ is the closure of the image of β under the automorphism $s_i \mapsto s_i^{-1}$, the mirror of the inverse is the closure of the image of β under the antiautomorphism $s_i \mapsto s_i^{-1}$. Since $t_3(s_1) = t_3(s_1^{-1})$, $t_3(1) = t_3(1^{-1})$, $t_3(s_1 s_2) = t_3(s_1^{-1} s_2^{-1})$, such a Markov trace coincides by (iv) with its composite with the (anti-)automorphisms defined in (iii), whence the conclusion. □

Proposition 6.3. *If* (t_n) *is a Markov trace factoring through* F_n, *then*
(i) $\forall n \geq 3 \ \ t_{n+1}(C) = a t_n(C)$
(ii) $\forall n \geq 2 \ \ t_{n+1}(C) = a t_{n+1}(1) - a x^{-1}(\tilde{\delta} + 2) t_n(1) + 2a\tilde{\delta} x^{-2} t_{n-1}(1)$
(iii) *For all* $n \geq 1$,

$$t_{n+3}(1) = \left(a + \frac{\tilde{\delta}+2}{x}\right) t_{n+2}(1) - \frac{1}{x}\left(2\frac{\tilde{\delta}}{x} + a(\tilde{\delta}+2)\right) t_{n+1}(1) + 2a\frac{\tilde{\delta}}{x^2} t_n(1)$$

Proof. (i) is trivially deduced from $s_n C = aC$ and the Markov property. From $C = e_{n-1} s_n e_{n-1} - e_{n-1}$ we get $t_{n+1}(C) = t_{n+1}(e_{n-1}^2 s_n) - t_3(e_{n-1}) = \tilde{\delta} x^{-1} t_{n+1}(e_{n-1} s_n) - t_{n+1}(e_{n-1}) = \tilde{\delta} x^{-1} t_n(e_{n-1}) - t_{n+1}(e_n)$ by the Markov property and because e_{n-1} and e_n are conjugates. Expanding $e_i = (a/x)(s_i + s_i^{-1}) - a$ and using the Markov property again we get $t_{n+1}(C) = \tilde{\delta} x^{-1}(2a/x t_{n-1}(1) - a t_n(1)) - (2a/x t_n(1) - a t_{n+1}(1)) = a t_{n+1}(1) - a x^{-1}(\tilde{\delta}+2) t_n(1) + 2a\tilde{\delta} x^{-2} t_{n-1}(1)$, namely (ii). By (i) we know $t_{n+1}(C) = a t_{n+2}(C)$ hence, by (ii), we get $t_{n+1}(C) = a t_{n+2}(C) = t_{n+2}(1) - x^{-1}(\tilde{\delta}+2) t_{n+1}(1) + 2\tilde{\delta} x^{-2} t_n(1)$ hence, again by (ii), $t_{n+2}(1) - x^{-1}(\tilde{\delta}+2) t_{n+1}(1) + 2\tilde{\delta} x^{-2} t_n(1) = a t_{n+1}(1) - a x^{-1}(\tilde{\delta}+2) t_n(1) + 2a\tilde{\delta} x^{-2} t_{n-1}(1)$ hence

$$t_{n+2}(1) = \left(a + \frac{\tilde{\delta}+2}{x}\right) t_{n+1}(1) - \frac{1}{x}\left(2\frac{\tilde{\delta}}{x} + a(\tilde{\delta}+2)\right) t_n(1) + 2a\frac{\tilde{\delta}}{x^2} t_{n-1}(1)$$

which proves (iii). □

At this stage, C could well be 0. We now prove that this is not the case.

6.2. **A genuine extension of $BMW_n^{\pm}$.** Using the abelianization morphism $B_n \twoheadrightarrow \mathbb{Z}$ we can define a 3-dimensional $H_n \otimes_R S^{\dagger\dagger}$-module by

$$s_i \mapsto \begin{pmatrix} a & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b^{-1} \end{pmatrix}$$

It is easily checked that $\mathcal{R}_1$ acts by 0 in this module, hence we get a 3-dimensional $BMW_n^{\dagger\dagger}$-module. We get that $e_i = a((s_i + s_i^{-1})/x - 1)$ is mapped to

$$\begin{bmatrix} \frac{(ab-1)(-b+a)}{b^2+1} & \frac{ab-1}{b^2+1} & \frac{b}{b^2+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while $e_i s_{i+1} e_i - e_i$ and $e_i s_{i+1}^{-1} e_i - e_i$ are both mapped to

$$2\frac{ab - b^2 - 1}{(b^2+1)^2} \begin{pmatrix} (ab-1)(a-b) & ab-1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This proves that this module induces a $F_n \otimes_A S^{\dagger\dagger}$-module, and therefore a $F_n \otimes_A S_{\pm}^{\dagger\dagger}$-module, which do not factorize through $BMW_n^{\pm}$. It follows that $BMW_n^{\dagger\dagger}$ and F_n are genuine extensions of the Birman-Wenzl-Murakami algebra. We need to prove a similar result for the specializations appearing in the previous section. However, for one of them the above argument does not work, because it corresponds to a root of $ab - b^2 - 1$. Nevertheless, we know by proposition 4.2 (iii) that $BMW_n^{\dagger\dagger}$ is finitely generated as a $S^{\dagger\dagger}$-module. Because of this, the dimension of every specialization is at least the dimension of $BMW_n^{\dagger\dagger}$ over the field of fractions of $S^{\dagger\dagger}$ (this classical fact follows for instance from Nakayama's lemma, by replacing $S^{\dagger\dagger}$ by its localization at the defining ideal of the specialization). This proves the following.

Proposition 6.4. *For every morphism $\lambda : S^{\dagger\dagger} \to \mathbb{C}$, $BMW_n^{\dagger\dagger} \otimes_\lambda \mathbb{C}$ and $(F_n \otimes_A S^{\dagger\dagger}) \otimes_\lambda \mathbb{C}$ have dimension at least $1 + \dim BMW_n^{+}$.*

Corollary 6.5. *For every morphism $\lambda : S^{\dagger\dagger} \hookrightarrow \mathbb{C}$, the image of C inside $(F_n \otimes_A S^{\dagger\dagger}) \otimes_\lambda \mathbb{C}$ is nonzero.*

Corollary 6.6. *The A-module F_n is free of rank $1 + \dim BMW_n^{\pm}$*

Proof. Let us denote $A_{\pm} = \mathbb{Q}[a, x, x^{-1}]/(a \mp 1) \simeq \mathbb{Q}[x, x^{-1}]$. By the natural A-module decomposition $A \simeq A_+ \oplus A_-$ it is enough to prove that $F_n^{\pm} = F_n \otimes_A A_{\pm}$ is a free $A_{\pm}$-module of rank $1+N$ with $N = \dim BMW_n^{\pm}$. Now notice that $BMW_n^{\pm} \otimes_{S_{\pm}} S_{\pm}^{\dagger\dagger}$ is actually defined by a presentation with coefficients in $A_{\pm}$, and that the corresponding $A_{\pm}$-form is free of rank N by [19]. Therefore, as in proposition 6.1 (ii), we get a surjective $A_{\pm}$-morphisme $F_n^{\pm} \twoheadrightarrow A_{\pm}^N$ by mapping C to 0. Letting $s_{\pm}$ denote a section of this morphism, we get a surjective morphism $u_{\pm} : A_{\pm}^{1+N} \simeq (A_{\pm}C) \oplus A_{\pm}^N \to F_n^{\pm}$ by $(\lambda C, m) \mapsto \lambda C + s_{\pm}(m)$. Letting $K_{\pm}$ denote the quotient field of $A_{\pm}$, it follows that $u_{\pm} \otimes_{A_{\pm}} K_{\pm}$ is surjective. But proposition 6.4 implies that $F_n^{\pm} \otimes_{A_{\pm}} K_{\pm}$ has dimension $1 + N$. Therefore the source and target of $u_{\pm} \otimes_{A_{\pm}} K_{\pm}$ have the same dimension hence $u_\pm \otimes K_\pm$ is injective. Since the source of $u_\pm$ is a free module this implies that $u_\pm$ is injective and this proves the claim.

□

Corollary 6.7. *The natural algebra morphism $F_n \to F_{n+1}$ is injective.*

Proof. As before, using the notations of the previous proof, it is equivalent to show that the natural maps $F_n^\pm \to F_{n+1}^\pm$ are injective. This is true because the following diagram of horizontal short exact sequences is commutative, and because its two extremal vertical arrows are known to be injective.

$$\begin{tikzcd} 0 \arrow[r] & A_\pm C \arrow[r] \arrow[d] & F_n^\pm \arrow[r] \arrow[d] & BMW_n^\pm \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_\pm C \arrow[r] & F_{n+1}^\pm \arrow[r] & BMW_{n+1}^\pm \arrow[r] & 0 \end{tikzcd}$$

□

6.3. **The algebra F_n as a specialization of $\widetilde{BMW}_n$.** The goal of this section is to prove the following theorem.

Theorem 6.8. *The morphism of $S^{\dagger\dagger}$-algebras $BMW_n^{\dagger\dagger} \to F_n \otimes_A S^{\dagger\dagger}$ induces an isomorphism after tensorisation by $S^{\dagger\dagger}[(x+2a)^{-1}]$.*

We define $\mathcal{S}_i = e_i s_{i+1} e_i - e_i$, $\mathcal{S}'_i = e_i s_{i+1}^{-1} e_i - e_i \in H_n \otimes_R S^{\dagger\dagger}$, and also $\widehat{\mathcal{R}_i} = (s_i s_{i+1} s_i)\mathcal{R}_i (s_i s_{i+1} s_i)^{-1}$, and similarly $\widehat{\mathcal{S}}_i = (s_i s_{i+1} s_i)(\mathcal{S}_i)(s_i s_{i+1} s_i)^{-1}$. The two formulas below hold inside $H_3 \otimes S^{\dagger\dagger}$ and can be checked computationally by using the morphism Φ_{H_3}, as we did for lemma 4.6 (these formulas were found by a similar procedure, too).

(6.9)
$$\mathcal{S}_1 - \mathcal{S}'_1 = \frac{1}{x}\mathcal{R}_1 s_2 - \frac{1}{x^2} s_1^{-1}\mathcal{R}_1 s_2 - \frac{1}{x^2} s_1 \mathcal{R}_1 s_2 = \frac{1}{x^2}(x - s_1^{-1} - s_1)\mathcal{R}_1 s_2 = \frac{1}{(b+c)^2}(s_1 - b)(s_1 - c)s_1^{-1}\mathcal{R}_1 s_2$$

$$2x^2\left(\mathcal{S}_1 - \widehat{\mathcal{S}}_1\right) = (x - s_1^{-1} - s_1)\mathcal{R}_1 s_2 - \mathcal{R}_1 + (-x + s_2^{-1} + s_2)\widehat{\mathcal{R}_1} s_1 + \widehat{\mathcal{R}_1} \tag{6.10}$$

We now want to show that $\mathcal{S}_1 - \mathcal{S}_2$ also belongs to the ideal generated by $\mathcal{R}_1$. For this we need to work inside $H_4 \otimes S^{\dagger\dagger}$. Since the computations become quite complicated, we specialize at $a = 1$. There is no loss of generality in doing this, as we justify it now. The natural decomposition $S^{\dagger\dagger} = S_+^{\dagger\dagger} \oplus S_-^{\dagger\dagger}$ induces a $\mathbb{Q}$-vector space decomposition $H_n \otimes_R S^{\dagger\dagger} = H_n^+ \oplus H_n^-$ with projectors p_+, p_- given by the multiplication by $(a-1)/2$ and $(a+1)/2$. It is straightforward to check that the involutive automorphism $E^\dagger$ of $H_n \otimes_R S^{\dagger\dagger}$ induced by $\hat{E}$ and η (see the notations of proposition 4.9 and before) exchanges H_n^+ and H_n^-, maps $\mathcal{S}_i \mapsto -\mathcal{S}_i$, $\mathcal{R}_i \mapsto -\mathcal{R}_i$ and intertwines p_+ and p_- up to a sign (that is: $E^\dagger \circ p_+ = -p_- \circ E^\dagger$, $E^\dagger \circ p_- = -p_+ \circ E^\dagger$). Because of this, any expression of $p_+(\mathcal{S}_1 - \mathcal{S}_2) = (\mathcal{S}_1)_+ - (\mathcal{S}_2)_+$ immediately yields an expression of $p_-(\mathcal{S}_1 - \mathcal{S}_2)$ and therefore of $\mathcal{S}_1 - \mathcal{S}_2$.

We now use the fact that H_4 is a free R-module of rank 648 in order to do explicit computations. More precisely, the computations are made as follows. First of all, we build a basis of H_4 as follows. We consider the collection $\mathcal{B}_2$ of 27 words in s_i, s_i^{-1} given in proposition 4.8 of [16]. They form a basis of H_4 as a H_3-module. Together with the list of 24 words $\mathcal{B}_1$ given by proposition 2.1 (ii), which induces a basis of H_3, we get a collection $\mathcal{B}_3 = \{g_1 g_2; (g_1, g_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}$

of $24 \times 27 = 648$ words which induces a basis of H_4. From this and the implicit rewriting rules of [16] we build an explicit regular representation $H_4 \hookrightarrow Mat_{648}(R)$ and therefore an injective map $\Phi_{H_4} : H_4 \otimes_R S_+^{\dagger\dagger} \to Mat_{648}(S_+^{\dagger\dagger})$, that we use in order to check equalities.

Letting $(\mathcal{S}_1)_\pm = \mathcal{S}_\pm$, we let $(\mathcal{S}_2)_\pm = (s_1 s_2)(\mathcal{S}_1)_\pm (s_1 s_2)^{-1}$. Inside $H_4 \otimes_R R/(a-1, y-1)$ we find that $2(x+2)^2 x^4((\mathcal{S}_1)_+ - (\mathcal{S}_2)_+)$ can be expressed as a sum of 161 terms obviously belonging to I_4, see Figures 5, 6, 7.

These terms were found as follows. For computational reasons (and the limited power of the computers we have at disposal) it is too difficult to compute the linear span of I_4 inside the field of fractions $\mathbb{Q}(x)$ or $\mathbb{Q}(b)$, so we need to circumvent this obstacle by computing inside specialisations in x. For some specific value of x we get a basis $\mathcal{B}^1$ of the linear span of the image of I_4 inside $\mathbb{Q}^{648}$. The one we get is made of terms of the form $g\mathcal{R}_1 g'$ where g, g' are products of elements s_i, s_i^{-1} (chosen inside the basis of H_4 mentionned above), or $g\mathcal{R}_2 g'$ or $g\widehat{\mathcal{R}_1} g'$ or $g\widehat{\mathcal{R}_2} g'$. The same elements form a basis of the specializations of H_4 for infinitely many values of x. We chose a number of values for which we got an expression of $\mathcal{S}_1 - \mathcal{S}_2$ as linear combination of the elements of $\mathcal{B}^1$. Assuming that these coefficients should be rational fractions in x whose denominators have low degree, we get these rational fractions by interpolation. We then check that the corresponding equality is correct by direct computation inside $H_4 \otimes \mathbb{Q}(x)$. It so happens that the choices we made in this process provide an expression of $(x-1)(x+2)^2(x^2+x-1)x^4(\mathcal{S}_1 - \mathcal{S}_2)$. Recall that, in order to deal with the odd cases of the previous section, we need to specialize at $x = 1$. For this alone, we need to start again with this specialization, and we get this time, as a linear combinations of another basis $\mathcal{B}^2$, a polynomial expression of $(x+1)(x+2)^2 x^4(\mathcal{S}_1 - \mathcal{S}_2)$. By Bezout theorem both results combined provide an expression of $2(x+2)^2 x^4((\mathcal{S}_1)_+ - (\mathcal{S}_2)_+)$ as a linear combination of $\mathcal{B}^1 \cup \mathcal{B}^2$, and this is the result that is expressed here (the cardinality of $\mathcal{B}^1 \cup \mathcal{B}^2$ is 161).

Needless to say, one could have hoped to get a nicer expression. Unfortunately we failed to find one.

The x^4 factor in this expression prevents the specialization $x = 0$, which is to be expected. The $x+2$ factor prevents the specialization $x = -2a$, which was not expected, but is explained by the proposition below. This proposition implies that the morphism under consideration does not induce an isomorphism under this specialization, since the algebras $BMW_4^\pm$ have dimension 105. Note that $(S_\pm^{\dagger\dagger})/(x+2a) \simeq \mathbb{Q}[b]/(b+b^{-1} \pm 2) \simeq \mathbb{Q}[b]/(b \pm 1)^2$.

Proposition 6.11. *The $\mathbb{Q}$-algebras $BMW_4^{\dagger\dagger} \otimes_{S^{\dagger\dagger}} \mathbb{Q}[b]/(b \pm 1)$ both have dimension 115.*

Proof. Direct computations prove that their defining ideals inside $H_4 \otimes_R R/(a \mp 1, b \pm 1, c \pm 1) \simeq \mathbb{Q}^{648}$ have dimension 533, hence the corresponding quotients have dimension $648 - 533 = 115$. □

We will elaborate a bit more on the case $x = -2a$ in section 6.8. For now, let $\pi : BMW_n^{\dagger\dagger} \otimes S^{\dagger\dagger}[(x+2a)^{-1}] \twoheadrightarrow F_n \otimes_A S^{\dagger\dagger}[(x+2a)^{-1}]$ be the obvious projection. We need to find a section f such that $f \circ \pi = \mathrm{Id}$, and for this it is enough to check that the natural projection $f : H_n \otimes S^{\dagger\dagger}[(x+2a)^{-1}] \twoheadrightarrow BMW_n^{\dagger\dagger} \otimes S^{\dagger\dagger}[(x+2a)^{-1}]$ factorizes through $F_n \otimes_A S^{\dagger\dagger}[(x+2a)^{-1}]$. Using the decomposition $S^{\dagger\dagger} = S_+^{\dagger\dagger} \oplus S_-^{\dagger\dagger}$ and the Galois automorphism mapping $a \mapsto -a$, it is sufficient to check this for $f_+ : H_n \otimes S_+^{\dagger\dagger}[(x+2)^{-1}] \twoheadrightarrow BMW_n^{\dagger\dagger} \otimes S_+^{\dagger\dagger}[(x+2)^{-1}]$. It is clear that the relators associated to $(i) - (iv)$ are mapped to 0. Now $C = e_1 s_2 e_1 - e_1$ is mapped to $\mathcal{S}_1$, and the relations 6.9, 6.10 and the fact that $\mathcal{S}_1 - \mathcal{S}_2 \in I_n$ (hence $\mathcal{S}_i - \mathcal{S}_{i+1} \in I_n$ for all i) then imply, using conjugation by elements of the braid groups, that the relators associated to

$$
\begin{aligned}
& (x+2)(x^7+4x^6-x^5-12x^4-5x^3+9x^2-1)x\mathcal{R}_1+(-x^7-4x^6+8x^4+2x^3-8x^2-2x+6)\mathcal{R}_1 s_3^{-1} \\
+\ & (2x+4)\mathcal{R}_1 s_3^{-1}s_2^{-1}s_1^{-1}+(x+2)(x^2-x-1)(x^5+3x^4-3x^3-5x^2+3x-2)x\mathcal{R}_1 s_1 \\
+\ & (x+2)(x^2-x-1)x^2\mathcal{R}_1 s_1 s_3^{-1}+(-x-2)(x^2-x-1)(x^3+3x^2-2x-3)x^2\mathcal{R}_1 s_1 s_3 \\
+\ & (-x^8-5x^7-3x^6+13x^5+17x^4-4x^3-10x^2+6x+6)\mathcal{R}_1 s_3+(2x+4)\mathcal{R}_1 s_3 s_2 s_1 \\
+\ & (-x-2)(x^3+2x^2+2x-1)x\mathcal{R}_2+(x^5+4x^4+3x^3-3x^2-2x-2)\mathcal{R}_2 s_1^{-1} \\
+\ & (x+1)(x+2)(x^2-x-1)(x^4+4x^3+2x^2-2x-4)x\mathcal{R}_2 s_1^{-1}s_3+(x^5+4x^4+3x^3-3x^2-2x-2)\mathcal{R}_2 s_1 \\
+\ & (x+1)(x+2)(x^2-x-1)(x^4+4x^3+2x^2-2x-4)x\mathcal{R}_2 s_1 s_3 \\
+\ & (-x-1)(x+2)(x^2-x-1)(x^4+3x^3-2x^2-x+2)x\mathcal{R}_2 s_2 \\
+\ & (-x-2)(x^5+2x^4-6x^3-7x^2+2x+9)x\widehat{\mathcal{R}}_1+(2x^7+9x^6+2x^5-26x^4-29x^3-x^2+12x+4)\widehat{\mathcal{R}}_1 s_3^{-1} \\
+\ & (-2x-4)\widehat{\mathcal{R}}_1 s_3^{-1}s_2^{-1}s_1^{-1}+(x^8+5x^7+4x^6-11x^5-25x^4-14x^3+9x^2+14x+4)\widehat{\mathcal{R}}_1 s_3 \\
+\ & (-2x-4)\widehat{\mathcal{R}}_1 s_3 s_2 s_1+(-x-1)(x+2)(x^6+3x^5-4x^4-9x^3+2x^2+9x+1)x\widehat{\mathcal{R}}_2 \\
+\ & (x^9+7x^8+14x^7-4x^6-47x^5-49x^4+5x^3+34x^2+10x-4)\widehat{\mathcal{R}}_2 s_1^{-1} \\
+\ & (x^9+7x^8+14x^7-3x^6-44x^5-52x^4+32x^2+12x-4)\widehat{\mathcal{R}}_2 s_1 \\
+\ & (-x^8-5x^7-2x^6+17x^5+25x^4+3x^3-24x^2-20x-2s_3^{-1})\mathcal{R}_1+(2x+4)(x^2+x+2)s_3^{-1}\mathcal{R}_1 s_3^{-1} \\
+\ & (-x^6-3x^5-4x^4-2x^3+7x^2+10x+4)s_3^{-1}\mathcal{R}_1 s_1^{-1} \\
+\ & (-x^8-5x^7-x^6+14x^5+9x^4-11x^3-10x^2+4x+4)s_3^{-1}\mathcal{R}_1 s_1 \\
+\ & (x-1)(x+2)(x^2-x-1)(x^2+4x+2)xs_3^{-1}\mathcal{R}_1 s_1 s_3 \\
+\ & (-x^8-4x^7+x^6+12x^5+12x^4+5x^3-4x^2-12x-10)s_3^{-1}\mathcal{R}_1 s_2 \\
+\ & (x+1)(x^5+3x^4-4x^3-9x^2+2x+10)xs_3^{-1}\mathcal{R}_1 s_3+(2x+4)s_3^{-1}\mathcal{R}_1 s_3 s_2^{-1} \\
+\ & (-x^8-5x^7-3x^6+13x^5+25x^4+15x^3-5x^2-12x-4)s_3^{-1}\widehat{\mathcal{R}}_1 \\
+\ & (-x-2)(x^5+3x^4-3x^3-7x^2-4x+2)s_3^{-1}\widehat{\mathcal{R}}_1 s_3^{-1}+(x+2)(x^4+x^3+2x^2-3)xs_3^{-1}\widehat{\mathcal{R}}_1 s_2^{-1} \\
+\ & (x^7+4x^6-2x^5-14x^4-15x^3-2x^2+14x+8)s_3^{-1}\widehat{\mathcal{R}}_1 s_1^{-1} \\
+\ & (x^5-4x^4-12x^3-4x^2+10x+6)s_3^{-1}\widehat{\mathcal{R}}_1 s_1 \\
+\ & (-x^7-4x^6+10x^4+12x^3+3x^2-4x-6)s_3^{-1}\widehat{\mathcal{R}}_1 s_3+(-x-1)(x+2)(x^2+x-1)xs_3^{-1}\widehat{\mathcal{R}}_2 \\
+\ & (x^5+4x^4+3x^3-3x^2-2x-2)s_3^{-1}\widehat{\mathcal{R}}_2 s_1^{-1}+(x^5+4x^4+3x^3-3x^2-2x-2)s_3^{-1}\widehat{\mathcal{R}}_2 s_1 \\
+\ & (x-1)(x^2+4x+2)s_3^{-1}s_2^{-1}\widehat{\mathcal{R}}_1 s_3+(2x+4)s_3^{-1}s_2^{-1}s_1\mathcal{R}_2+(-2x-4)s_3^{-1}s_2^{-1}s_1\widehat{\mathcal{R}}_2 \\
+\ & (-x-2)(x^5+3x^4-3x^3-7x^2-4x+2)s_3^{-1}s_2\mathcal{R}_1 s_3^{-1}+(-x^5-3x^4+4x^3+10x^2+4x-4)s_3^{-1}s_2\mathcal{R}_1 s_3 \\
+\ & (x+1)(x^5+3x^4-4x^3-8x^2+6x+11)x^2s_2^{-1}\mathcal{R}_2+(-x^5-4x^4-5x^3-x^2+2x-2)s_2^{-1}\mathcal{R}_2 s_1^{-1} \\
+\ & (-x^7-4x^6-x^5+6x^4+6x^3-x^2-2)s_2^{-1}\mathcal{R}_2 s_1+(x+2)(x^4+3x^3-6x+1)xs_2^{-1}\widehat{\mathcal{R}}_1 \\
+\ & (-x^4-3x^3+3x+2)xs_2^{-1}\widehat{\mathcal{R}}_1 s_3^{-1}+(-x^4-4x^3-3x^2+5x+4)xs_2^{-1}\widehat{\mathcal{R}}_1 s_3 \\
+\ & (x+1)(x^5+2x^4-6x^3-x^2+10x+6)xs_2^{-1}\widehat{\mathcal{R}}_2 \\
+\ & (-x^8-5x^7-4x^6+10x^5+18x^4+6x^3-7x^2-4x-2)s_2^{-1}\widehat{\mathcal{R}}_2 s_1^{-1} \\
+\ & (-x^8-5x^7-4x^6+10x^5+18x^4+6x^3-7x^2-4x-2)s_2^{-1}\widehat{\mathcal{R}}_2 s_1+(-2x-4)s_2^{-1}s_1\mathcal{R}_2 s_1^{-1} \\
+\ & (2x+4)s_2^{-1}s_3\widehat{\mathcal{R}}_1 s_3^{-1}+(-x^7-4x^6+8x^4-7x^2-x-4)xs_1^{-1}\mathcal{R}_1 \\
+\ & (-x^8-3x^7+5x^6+13x^5-9x^3+2x^2+4x-2)s_1^{-1}\mathcal{R}_1 s_3^{-1}+(x^7+4x^6-9x^4-3x^3+4x^2-2)s_1^{-1}\mathcal{R}_1 s_3 \\
+\ & (-x^7-3x^6-2x^5+2x^4+7x^3+2x^2-4x+2)s_1^{-1}\mathcal{R}_2 \\
+\ & (x^7+4x^6-2x^5-10x^4+x^3+2x^2-7x-6)xs_1^{-1}\mathcal{R}_2 s_2^{-1} \\
+\ & (x^8+4x^7-2x^6-10x^5+x^4+2x^3-7x^2-4x+4)xs_1^{-1}\mathcal{R}_2 s_2 \\
+\ & (-x^7-4x^6+2x^5+10x^4-x^3-2x^2+7x+6)xs_1^{-1}\mathcal{R}_2 s_3 \\
+\ & (-x+1)(x+1)(x^4+2x^3-5x^2+2x+14)xs_1^{-1}\widehat{\mathcal{R}}_1 \\
+\ & (-x^7-4x^6+11x^4+15x^3+5x^2-2x-6)s_1^{-1}\widehat{\mathcal{R}}_1 s_3^{-1} \\
+\ & (-x-1)(x^6+2x^5-5x^4-3x^3-5x^2+2x+6)s_1^{-1}\widehat{\mathcal{R}}_1 s_3 \\
+\ & \ldots
\end{aligned}
$$

FIGURE 5. For $a=1$, $2(x+2)^2x^4(\mathcal{S}_1-\mathcal{S}_2)$ to be continued.

$$
\begin{array}{ll}
 & \ldots \\
+ & (-x^7-4x^6+2x^5+10x^4-x^3+15x+14)xs_1^{-1}\widehat{\mathcal{R}}_2 \\
+ & (-x^7-3x^6-2x^5+2x^4+7x^3+2x^2-4x+2)s_1^{-1}\widehat{\mathcal{R}}_2 s_3^{-1} \\
+ & (2x^6+7x^5-6x^4-11x^3+2x^2-10x-4)xs_1^{-1}\widehat{\mathcal{R}}_2 s_2^{-1} \\
+ & (-x+1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1^{-1}\widehat{\mathcal{R}}_2 s_1^{-1} \\
+ & (-x^7-4x^6+10x^4+11x^3-2x^2-6x-4)s_1^{-1}\widehat{\mathcal{R}}_2 s_1+(-2x-4)s_1^{-1}\widehat{\mathcal{R}}_2 s_1 s_2^{-1} \\
+ & (2x^8+8x^7-3x^6-24x^5-10x^4+12x^3+6x^2-2x-2)s_1^{-1}\widehat{\mathcal{R}}_2 s_2 \\
+ & (x^7+4x^6-x^5-13x^4-10x^3+x^2-6)s_1^{-1}s_3^{-1}\mathcal{R}_1 s_3^{-1}+(-x^3-5x^2-2x-2s_1^{-1})s_3^{-1}\mathcal{R}_1 s_3 \\
+ & (x^6+4x^5-10x^3-13x^2-4x-2)s_1^{-1}s_3^{-1}\widehat{\mathcal{R}}_1 s_3^{-1} \\
+ & (x^6+3x^5-3x^4-7x^3-6x^2-2x+4)s_1^{-1}s_3^{-1}\widehat{\mathcal{R}}_1 s_3 \\
+ & (-x^7-4x^6+x^5+13x^4+10x^3-x^2+6)s_1^{-1}s_2^{-1}\mathcal{R}_2 s_1^{-1} \\
+ & (-x^7-3x^6+5x^5+13x^4-12x^2+8)s_1^{-1}s_2^{-1}\mathcal{R}_2 s_1+(2x+4)s_1^{-1}s_2^{-1}s_3^{-1}\mathcal{R}_1+(-2x-4)s_1^{-1}s_2^{-1}s_3^{-1}\widehat{\mathcal{R}}_1 \\
+ & (2x+4)s_1^{-1}s_2^{-1}s_3\mathcal{R}_1+(-2x-4)s_1^{-1}s_2^{-1}s_3\widehat{\mathcal{R}}_1 \\
+ & (-x-2)(x^2-x-1)(x^5+3x^4-4x^3-2x^2+3x-4)xs_1^{-1}s_2\mathcal{R}_2 \\
+ & (x+1)(x^5+3x^4-4x^3-10x^2-4x+4)xs_1^{-1}s_3\mathcal{R}_1 s_3^{-1}+(-x-1)(x+2)(x^2+2x-2)s_1^{-1}s_3\mathcal{R}_1 s_3 \\
+ & (-x+1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1^{-1}s_3\mathcal{R}_2 s_1^{-1} \\
+ & (-x^7-4x^6+10x^4+11x^3-2x^2-6x-4)s_1^{-1}s_3\mathcal{R}_2 s_1 \\
+ & (x+1)(x^6+4x^5-x^4-13x^3-9x^2+2x+6)s_1^{-1}s_3\widehat{\mathcal{R}}_1 s_3^{-1} \\
+ & (x+2)(x^6+3x^5-4x^4-9x^3-x^2+2x+6)s_1^{-1}s_3\widehat{\mathcal{R}}_1 s_3 \\
+ & (-x^8-4x^7+2x^6+17x^5+5x^4-22x^3-18x^2+2x+10)xs_1\mathcal{R}_1 \\
+ & (-x^7-4x^6-x^5+10x^4+18x^3+7x^2-2x-6)s_1\mathcal{R}_1 s_3^{-1} \\
+ & (x^8+3x^7-5x^6-13x^5+5x^4+21x^3+9x^2-8x-6)s_1\mathcal{R}_1 s_3 \\
+ & (-x^6-2x^5-x^4-4x^3-8x^2-8x+2)s_1\mathcal{R}_2 \\
+ & (x^5+3x^4+4x^3-3x^2-9x-2)xs_1\mathcal{R}_2 s_2^{-1}+(2x+4)s_1\mathcal{R}_2 s_1^{-1}s_2 \\
+ & (x^7+5x^6+2x^5-9x^4-11x^3-4x^2-2)xs_1\mathcal{R}_2 s_2 \\
+ & (x^7+4x^6-2x^5-16x^4-12x^3+10x^2+17x+4)xs_1\mathcal{R}_2 s_3 \\
+ & (x^8+5x^7+4x^6-10x^5-23x^4-20x^3+3x^2+19x+12)xs_1\widehat{\mathcal{R}}_1 \\
+ & (-x^7-6x^6-8x^5+11x^4+34x^3+22x^2-5x-10)xs_1\widehat{\mathcal{R}}_1 s_3^{-1} \\
+ & (-2x^7-9x^6-3x^5+24x^4+33x^3+7x^2-15x-8)xs_1\widehat{\mathcal{R}}_1 s_3 \\
+ & (x^7+4x^6-2x^5-16x^4-12x^3+10x^2+17x+4)xs_1\widehat{\mathcal{R}}_2 \\
+ & (-x^6-2x^5-x^4-2x^3+2)s_1\widehat{\mathcal{R}}_2 s_3^{-1} \\
+ & (-x^6-3x^5+6x^4+13x^3+3x^2-14x-10)xs_1\widehat{\mathcal{R}}_2 s_2^{-1} \\
+ & (-x^8-5x^7-3x^6+15x^5+24x^4-2x^3-20x^2-8x+8)s_1\widehat{\mathcal{R}}_2 s_1^{-1} \\
+ & (-x^8-5x^7-3x^6+14x^5+21x^4+x^3-15x^2-6x+6)s_1\widehat{\mathcal{R}}_2 s_1 \\
+ & (x^9+5x^8+3x^7-14x^6-22x^5+3x^4+20x^3+4x^2-8x-2)s_1\widehat{\mathcal{R}}_2 s_2 \\
+ & (x+2)(x^2-x-1)(x^5+5x^4+3x^3-2x^2-2x-2)s_1 s_3^{-1}\mathcal{R}_1 \\
+ & (x^6+4x^5-10x^3-13x^2-4x-2)s_1 s_3^{-1}\mathcal{R}_1 s_3^{-1} \\
+ & (-x^7-3x^6+5x^5+12x^4-5x^3-16x^2-6x+4)s_1 s_3^{-1}\mathcal{R}_1 s_3 \\
+ & (x^7+5x^6+4x^5-10x^4-21x^3-11x^2+2x+6)s_1 s_3^{-1}\widehat{\mathcal{R}}_1 s_3^{-1} \\
+ & (2x+2)(x^2+4x+2)(x^4-x^3-3/2x^2+1)s_1 s_3^{-1}\widehat{\mathcal{R}}_1 s_3 \\
+ & \ldots
\end{array}
$$

FIGURE 6. For $a=1$, $2(x+2)^2x^4(\mathcal{S}_1-\mathcal{S}_2)$, continued.

$$
\begin{array}{ll}
+ & (-x-1)(x^5+3x^4-3x^3-7x^2-6x+2)s_1s_2^{-1}\mathcal{R}_2s_1^{-1}+(2x+4)xs_1s_2^{-1}\mathcal{R}_2s_1 \\
+ & (x-1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2^{-1}\widehat{\mathcal{R}}_2s_1^{-1} \\
+ & (x-1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2^{-1}\widehat{\mathcal{R}}_2s_1 \\
+ & (-x-2)(x^2-x-1)(x^4+5x^3+3x^2-x-4)xs_1s_2\mathcal{R}_2 \\
+ & (x-1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2\mathcal{R}_2s_1^{-1} \\
+ & (x-1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2\mathcal{R}_2s_1 \\
+ & (-x+1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2\widehat{\mathcal{R}}_2s_1^{-1} \\
+ & (-x+1)(x+1)^2(x^2-x-1)(x^2+4x+2)s_1s_2\widehat{\mathcal{R}}_2s_1+(2x+4)s_1s_2s_3^{-1}\mathcal{R}_1 \\
+ & (-2x-4)s_1s_2s_3^{-1}\widehat{\mathcal{R}}_1+(2x+4)s_1s_2s_3\mathcal{R}_1+(-2x-4)s_1s_2s_3\widehat{\mathcal{R}}_1 \\
+ & (x+2)(x^2-x-1)(x^4+x^3-4x^2-2)s_1s_3\mathcal{R}_1 \\
+ & (x+1)(x^6+4x^5-x^4-13x^3-9x^2+2x+6)s_1s_3\mathcal{R}_1s_3^{-1} \\
+ & (x+2)(x^5+2x^4-6x^3-5x^2+6)s_1s_3\mathcal{R}_1s_3 \\
+ & (-x^8-5x^7-3x^6+15x^5+24x^4-2x^3-20x^2-8x+8)s_1s_3\mathcal{R}_2s_1^{-1} \\
+ & (-x^8-5x^7-3x^6+14x^5+21x^4+x^3-15x^2-6x+6)s_1s_3\mathcal{R}_2s_1 \\
+ & (x+1)^2(x^2+4x+2)(x^4-x^3-x^2-x+1)s_1s_3\widehat{\mathcal{R}}_1s_3^{-1} \\
+ & (x+2)(x^6+4x^5-x^4-13x^3-9x^2+4x+8)xs_1s_3\widehat{\mathcal{R}}_1s_3 \\
+ & (-x-2)(x^5+2x^4-6x^3-7x^2+4x+5)xs_2\mathcal{R}_1+(x^3-2x-2)(x^4+5x^3+5x^2-x-1)s_2\mathcal{R}_1s_3^{-1} \\
+ & (x^6+3x^5-4x^4-13x^3-6x^2+4x+2)s_2\mathcal{R}_1s_3 \\
+ & (x+1)(x^7+4x^6-x^5-12x^4-5x^3+8x^2+6x+2)xs_2\mathcal{R}_2 \\
+ & (-x^8-5x^7-4x^6+10x^5+18x^4+6x^3-7x^2-4x-2)s_2\mathcal{R}_2s_1^{-1} \\
+ & (-x^8-5x^7-4x^6+10x^5+18x^4+6x^3-7x^2-4x-2)s_2\mathcal{R}_2s_1 \\
+ & (x^6+4x^5-x^4-12x^3-4x^2+13x+11)x^2s_2\widehat{\mathcal{R}}_2 \\
+ & (-x^9-6x^8-9x^7+7x^6+34x^5+31x^4-9x^3-26x^2-6x+8)s_2\widehat{\mathcal{R}}_2s_1^{-1} \\
+ & (-x^9-6x^8-9x^7+7x^6+34x^5+31x^4-9x^3-26x^2-6x+8)s_2\widehat{\mathcal{R}}_2s_1+(-2x-4)s_2s_3^{-1}\mathcal{R}_1s_3 \\
+ & (2x+4)s_2s_1^{-1}\widehat{\mathcal{R}}_2s_1+(-x^7-3x^6+4x^5+14x^4+6x^3-10x^2-12x-2)s_3\mathcal{R}_1 \\
+ & (-x-1)(x+1)(x^5+2x^4-6x^3-4x^2+4)s_3\mathcal{R}_1s_3^{-1}+(-x-1)(x^5+2x^4-x^3-x^2-4x-4)s_3\mathcal{R}_1s_1^{-1} \\
+ & (-x^7-x^6+7x^5+2x^4-13x^3-10x^2+4x+4)s_3\mathcal{R}_1s_1 \\
+ & (x-1)(x+2)(x^2-x-1)(x^2+4x+2)xs_3\mathcal{R}_1s_1s_3 \\
+ & (-x+1)(x^8+6x^7+9x^6-6x^5-29x^4-28x^3-16x^2-14x-10)s_3\mathcal{R}_1s_2 \\
+ & (x+2)(x^3+3x^2-6)s_3\mathcal{R}_1s_3 \\
+ & (-x-1)(x+2)(x^6+3x^5-4x^4-9x^3+2x^2+7x+1)xs_3\mathcal{R}_2 \\
+ & (x^8+6x^7+7x^6-13x^5-27x^4-9x^3+10x^2+2x-4)s_3\mathcal{R}_2s_1^{-1} \\
+ & (x^8+6x^7+8x^6-10x^5-30x^4-14x^3+8x^2+4x-4s_3)\mathcal{R}_2s_1 \\
+ & (-x^7-3x^6+3x^5+11x^4+13x^3+7x^2-2x-4)s_3\widehat{\mathcal{R}}_1 \\
+ & (-x-1)(x^7+4x^6-10x^4-12x^3-3x^2+4x+6)s_3\widehat{\mathcal{R}}_1s_3^{-1}+(-2x-4)s_3\widehat{\mathcal{R}}_1s_3^{-1}s_2 \\
+ & (x^5+3x^4+x^3-x-8)xs_3\widehat{\mathcal{R}}_1s_2^{-1}+(x^6+x^5-7x^4-5x^3-4x^2+10x+8)s_3\widehat{\mathcal{R}}_1s_1^{-1} \\
+ & (-x^6-2x^5+2x^4-3x^3-4x^2+6x+6)s_3\widehat{\mathcal{R}}_1s_1 \\
+ & (-x-1)(x+2)(x^6+3x^5-4x^4-9x^3+4x+4)s_3\widehat{\mathcal{R}}_1s_3 \\
+ & (x-1)(x+1)(x^2+4x+2)s_3s_2^{-1}\widehat{\mathcal{R}}_1s_3^{-1}+(x-1)(x+2)(x^2+4x+2)s_3s_2^{-1}\widehat{\mathcal{R}}_1s_3 \\
+ & (-x-1)(x^5+3x^4-4x^3-10x^2-4x+4)s_3s_2\mathcal{R}_1s_3^{-1}+(x+1)(x+2)(x^2+2x-2)s_3s_2\mathcal{R}_1s_3 \\
+ & (2x+4)s_3s_2s_1^{-1}\mathcal{R}_2+(-2x-4)s_3s_2s_1^{-1}\widehat{\mathcal{R}}_2.
\end{array}
$$

FIGURE 7. For $a=1$, $2(x+2)^2x^4(\mathcal{S}_1-\mathcal{S}_2)$, last part.

(v) and (viii) are also mapped to 0. Since $s_1e_1 = e_1s_1 = ae_1$ it is clear that $s_1\mathcal{S}_1 = \mathcal{S}_1s_1 = as_1$ and this implies $s_iC - aC \mapsto 0, Cs_i - aC \mapsto 0$ for all i, which justifies that the 9th type of relator is also mapped to 0. This completes the proof of the theorem.

Corollary 6.12. *Let (t_n) be a Markov trace on $(BMW_n^{\dagger\dagger})$ with value in some $S^{\dagger\dagger}$-module M, and t'_n its composite with $M \to M \otimes S^{\dagger\dagger}[(a-x)^{-1},(2a-x)^{-1},(2a+x)^{-1}]$. Then t'_n is a linear combination t_n^H, t_n^K , $t_n^{\dagger\dagger}$ with coefficients in $M \otimes_R S^{\dagger\dagger}[(a-x)^{-1},(2a-x)^{-1},(2a+x)^{-1}]$.*

Proof. We know from proposition 5.11 (ii) that t'_3 is a linear combination of t_3^H, t_3^K, $t_3^{\dagger\dagger}$. The statement is then a consequence of theorem 6.8 together with proposition 6.1 (iv). □

6.4. **Computations inside F_n.** Following the arguments of Wenzl in [21] p. 400 we can derive additional useful relations inside F_n.

Lemma 6.13. *Assume $|j-i| = 1$. Then we have*

(i) $s_js_ie_j = e_is_js_i$
(ii) $s_ie_je_i = as_j^{-1}e_i + C$, $e_ie_js_i = ae_is_j^{-1} + C$
(iii) $e_ie_je_i = e_i + \frac{2a}{x}C$
(iv) $e_is_js_i = ae_ie_j - C$

Proof. By the braid relations, we have $s_js_is_j^{\alpha} = s_i^{\alpha}s_js_i$ whenever $\alpha \in \{-1,0,1\}$. Since $e_k = a((s_k^{-1}+s_k)/x - 1)$ we get (i). Using the defining relations of F_n and (i) we get $s_ie_je_i = s_j^{-1}(s_js_ie_j)e_i = s_j^{-1}e_is_j(s_ie_i) = as_j^{-1}(e_is_je_i) = as_j^{-1}(C+e_i) = as_j^{-1}C + as_j^{-1}e_i = C + as_j^{-1}e_i$ hence the first part of (ii). The second part is similar. Now $e_ie_je_i$ is equal to

$$ae_i\left(\frac{s_j+s_j^{-1}}{x} - 1\right)e_i = a\left(\frac{1}{x}e_is_je_i + \frac{1}{x}e_is_j^{-1}e_i - e_i^2\right) = a\left(\frac{1}{x}(e_i+C) + \frac{1}{x}(e_i+C) - \frac{\tilde{\delta}}{x}e_i\right)$$

hence $e_ie_je_i = a\left(\frac{ax}{x}e_i + \frac{2}{x}C\right)$ which proves (iii). In order to prove (iv), we use that, because of (iii), $e_is_js_i = (e_ie_je_i - \frac{2a}{x}C)s_js_i = e_i(e_je_is_j)s_i - \frac{2a}{x}Cs_js_i = e_i(ae_js_i^{-1} + C)s_i - \frac{2a}{x}C = ae_ie_js_i^{-1}s_i + e_iCs_i - \frac{2a}{x}C = ae_ie_j + \frac{\tilde{\delta}}{x}aC - \frac{2a}{x}C = ae_ie_j + \frac{2-ax-2}{x}aC$ which proves (iv).

□

6.5. **A central extension of the Temperley-Lieb algebra.**

Definition 6.14. *We define a unital algebra $\widetilde{TL}_n$ over $A = \mathbb{Q}[a,x,x^{-1}]/(a^2=1)$ by generators $e_1,\ldots,e_{n-1},C$ and relations*

(i) $e_i^2 = \tilde{\delta}x^{-1}e_i$
(ii) $e_iC = Ce_i = \tilde{\delta}x^{-1}C$
(iii) $C^2 = 2x^{-2}\tilde{\delta}(a-x)C$
(iv) $e_ie_j = e_je_i$ *if* $|j-i| \geq 2$
(v) $e_ie_je_i = e_i + \frac{2a}{x}C$ *if* $|j-i| = 1$

We have a natural morphism $\widetilde{TL}_n \to F_n$ of unital A-algebras. The next proposition shows that $\widetilde{TL}_n$ can be identified with a subalgebra of F_n, and is a genuine extension of the ordinary Temperley-Lieb algebra TL_n defined as the quotient of $\widetilde{TL}_n$ by the two-sided ideal generated by C.

Proposition 6.15. *The natural morphism $\widetilde{TL}_n \otimes_A S^{\dagger\dagger} \to F_n \otimes_A S^{\dagger\dagger}$ is injective. The $S^{\dagger\dagger}$-module $\widetilde{TL}_n \otimes_A S^{\dagger\dagger}$ is free of rank $1 + Cat_n$, where Cat_n denotes the n-th Catalan number.*

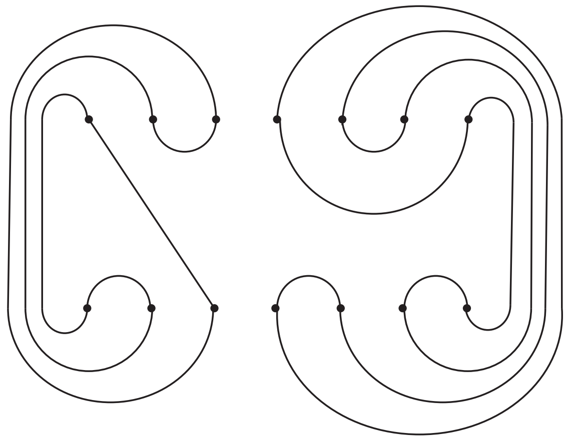

FIGURE 8. Closure of a Temperley-Lieb diagram with 2 components.

Proof. Let $\mathcal{B}$ the set of words in the e_i's that provides the usual basis of the Temperley-Lieb algebra, namely 'increasing products of increasing strings', see [10] p. 27. Their image inside $\overline{BMW}_n \otimes_{\overline{R}} S^{\dagger\dagger}$ through $\widetilde{TL}_n \otimes_A S^{\dagger\dagger} \to F_n \otimes_A S^{\dagger\dagger} \to \overline{BMW}_n \otimes_{\overline{R}} S^{\dagger\dagger}$ is a linearly independent subset of Kauffman's tangle algebra, see [19]. It follows that $\mathcal{B} \sqcup \{C\}$ is linearly independent in $\widetilde{TL}_n \otimes_A S^{\dagger\dagger}$. For, if such a linear combination $\sum_{b\in\mathcal{B}} \lambda_b b + \lambda_C C$ was 0, then its image inside $\overline{BMW}_n \otimes_{\overline{R}} S^{\dagger\dagger}$ would also be 0. But this image is equal to the image of $\sum_{b\in\mathcal{B}} \lambda_b b$, which is zero only if $\lambda_b = 0$ for all $b \in \mathcal{B}$. But then $\lambda_C C$ is mapped to $\lambda_C C \in F_n \otimes_A S^{\dagger\dagger}$, and we know that this is zero only if $\lambda_C = 0$. The remaining assertions are then obvious. $\square$

We let $A_1 = A/(x-a)$, and $\widetilde{TL}_n(1) = \widetilde{TL}_n \otimes_A A_1$. Note that, inside $\widetilde{TL}_n(1)$, we have $e_i^2 = ae_i$. We let $\bar{e}_i$ denote the image of $e_i \in \widetilde{TL}_n$ under the natural projection $\widetilde{TL}_n \to TL_n$.

Proposition 6.16. *There exists a family of traces $t_n : \widetilde{TL}_n(1) \to A_1$ satisfying $t_n(C) = -a^{n+1}$, and*

$$t_n(e_{i_1} \dots e_{i_k}) = a^{k+n} \left(\mathcal{N}(\bar{e}_{i_1} \dots \bar{e}_{i_k}) - k\right)$$

where $\mathcal{N}(\bar{e}_{i_1} \dots \bar{e}_{i_k})$ denotes the number of connected components of the diagrammatic closure of $\bar{e}_{i_1} \dots \bar{e}_{i_k} \in TL_n$ (see Figure 8)

Proof. We fix n, and we prove that this formula indeed provides a trace on $\widetilde{TL}_n(1)$. We first note that the formula $t_n(e_{i_1} \dots e_{i_k}) = a^{k+n} \left(\mathcal{N}(\bar{e}_{i_1} \dots \bar{e}_{i_k}) - k\right)$ provides a linear form on the free A_1 algebra on $e_1, \dots, e_{n-1}$, and that it is indeed a trace on this algebra, because $\mathcal{N}(e_{i_1} \dots e_{i_k})$ is invariant under cyclic permutation of the $e'_{i_r}s$ as is easily seen, for instance by representing the vertices of the diagram on a circle. It is easily checked that the formula $t_n(C) = -a^{n+1}$ extends this trace to the sum of this algebra with the (non-unital) 1-dimensional algebra spanned by C, defined by $C^2 = 0$ and $Ce_i = e_iC = aC$ for all i.

It then remains to check that the defining relations of $\widetilde{TL}_n(1)$ as a quotient module are mapped to 0 under t_n. Let x, y be words in the e_i's, and assume xe_iy has length k. Then, by definition of t_n, we have

$$t_n(xe_i^2y) = a^{k+n+1}(\mathcal{N}(\bar{x}\bar{e}_i^2\bar{y}) - k - 1) = a^{k+n+1}(\mathcal{N}(\bar{x}\bar{e}_i\bar{y}) + 1 - k - 1) = at_n(xe_iy)$$

hence the ideal generated by $e_i^2 - ae_i$ is mapped to 0. Similarly, let i,j such that $|j-i|=1$. Then

$$\begin{array}{rclcl} t_n(xe_ie_je_iy) & = & a^{k+2+n}(\mathcal{N}(\bar{x}\bar{e}_i\bar{e}_j\bar{e}_i\bar{y}) - k - 2) & = & a^{k+n}(\mathcal{N}(\bar{x}\bar{e}_i\bar{y}) - k - 2) \\ & = & t_n(xe_iy) - 2a^{k+n} & = & t_n(xe_iy) + 2t_n(xCy) \end{array}$$

and this is equal to $t_n(x(e_i+2C)y)$ hence the ideal generated by $e_ie_je_i - (e_i+2C)$ is mapped to 0. Assume now $|j-i| \geq 2$. Since $\bar{e}_i\bar{e}_j = \bar{e}_j\bar{e}_i$ it is clear that t_n vanishes on the ideal generated by the $e_ie_j - e_je_i$. The conclusion follows.

□

We let $A_0 = A/(x-2a)$, and $\widetilde{TL}_n(0) = \widetilde{TL}_n \otimes_A A_0$. Note that, inside $\widetilde{TL}_n(0)$, we have $\tilde{\delta} = 0$ whence $e_i^2 = Ce_i = e_iC = 0$, $e_ie_je_i = e_i + C$ whenever $|j-i| = 1$.

Proposition 6.17. *Let $n \geq 3$ and $u_n, v_n \in A_0$. There exists a trace on $\widetilde{TL}_n(0)$ defined by $t_n(1) = v_n$, $t_n(C) = -u_n$, $t_n(e_i) = u_n$ for all $i \in [1, n-1]$ and*

$$t_n(e_{i_1} \dots e_{i_k}) = 0 \text{ if } k \geq 2.$$

Proof. Similar to the previous proof, only easier. □

Note that such a trace is never symmetrizing, because $t_n((e_1 - e_2)x) = 0$ for every $x \in \widetilde{TL}_n(0)$.

We finally notice that the extension is not split in one of the two special cases we are interested in, therefore providing a non-zero Hochschild cohomology 2-class inside $\mathrm{HH}^2(TL_n(1), \mathbb{Q}[a]/(a^2-1))$.

Proposition 6.18. *Let k be a field of characteristic 0 and $\varphi : A \twoheadrightarrow k$ be a morphism of $\mathbb{Q}$-algebras. Assume $n \geq 3$. The natural short exact sequences $0 \to kC \to \widetilde{TL}_n \otimes_\varphi k \to TL_n \otimes_\varphi k \to 0$ splits if and only if $\varphi(x-a) \neq 0$. In that case, the splitting is given by the map*

$$\bar{e}_i \mapsto e_i - \left(\frac{\varphi(x)}{2\varphi(a-x)}\right) C.$$

Moreover, this splitting is unique.

Proof. For the proof, we identify x, a with their value under φ, and $\widetilde{TL}_n$, TL_n with their specialization. Let $\check{e}_i$ denote the image of $\bar{e}_i$ under such a splitting. We have $\check{e}_i = e_i + \lambda_i C$ for some $\lambda_i \in k$. For a given i, the relation $\check{e}_i^2 = \frac{\tilde{\delta}}{x}\check{e}_i$ is equivalent to the equation

$$(*) \qquad \frac{\lambda_i\tilde{\delta}}{x}\left(1 + \frac{2\lambda_i}{x}(a-x)\right) = 0.$$

If $\lambda_i = 0$ for some i, then, choosing j with $|j-i| = 1$ we get that $\check{e}_i\check{e}_j\check{e}_i = e_i\check{e}_je_i$ is equal to $\check{e}_i$ iff $2a + \lambda_j\tilde{\delta}^2/x = 0$, hence $\tilde{\delta} \neq 0$ and $\lambda_j = -2ax/\tilde{\delta}^2$. But then the equation $\check{e}_j^2 = (\tilde{\delta}/x)\check{e}_j$ implies $\lambda_j x/\tilde{\delta} = 0$, a contradiction. Therefore $\lambda_i = 0$ for every i.

If $\tilde{\delta} = 0$, that is $x = 2a$, for every i,j with $|j-i| = 1$ we have $\check{e}_i\check{e}_j\check{e}_i = e_ie_je_i = e_i + C$, therefore $\lambda_i = 1 = \frac{-x}{2(a-x)}$, and the formula $e_i \mapsto e_i + C$ is easily checked to provide a splitting in this case.

We can thus assume $\tilde{\delta} \neq 0$, and $\lambda_i \neq 0$ for all i. Then $(*)$ implies $x \neq a$ and $\lambda_i = \frac{-x}{2(a-x)}$. The fact that this formula provides a splitting is again checked by direct computation, and this proves the claim. □

6.6. **Splittings.** In this section we show that our extension of the BMW-algebra is not split exactly in the two cases we are interested in.

Proposition 6.19. *Let k be a field of characteristic 0 and $\varphi : S_{\pm}^{\dagger\dagger} \to k$ a morphism of $\mathbb{Q}$-algebras. Then, for $n \geq 3$, the natural short exact sequence $0 \to kC \to (F_n \otimes_A S_{\pm}^{\dagger\dagger}) \otimes_{\varphi} k \to (BMW_n^{\pm}) \otimes S_{\pm}^{\dagger\dagger}) \otimes_{\varphi} k \to 0$ splits iff $\varphi(Q)$ has a root in k, where*

$$Q(\lambda) = x^4(1 + u\tilde{\delta} + u^2\tilde{\delta}) \in S_{\pm}^{\dagger\dagger}[\lambda].$$

and $u = 2\lambda a(a-x)x^{-2}$. If this is the case, each one of the roots λ provides a splitting $s_i \mapsto s_i + \lambda C$, and these are the only two possible splittings. In particular, if k is algebraically closed, then the short exact sequence splits iff $\varphi((x-a)(x-2a)) \neq 0$, and it admits exactly one splitting iff $\varphi(x+2a) = 0$.

Proof. In the proof we work inside the specializations, and identify $x, a, \dots$ with their images under φ. Let us assume that there is a splitting, given by $e_i \mapsto \check{e}_i$, $s_i \mapsto \check{s}_i$. This splitting provides a splitting of the extension of the Temperley-Lieb subalgebra, and therefore, by proposition 6.18, one needs to have $x \neq a$, and $\check{e}_i = e_i - \frac{x}{2(a-x)}C$. We have $\check{s}_i = s_i + \lambda_i C$ for some $\lambda_i \in k$. Moreover, the equation $(s_i - a)(s_i^2 - xs_i + 1) = 0$ implies that $s_i^{-1} = as_i^2 - (ax+1)s_i + (x+a)$. Since this equation also holds for $\check{s}_i$, we get $\check{s}_i^{-1} = a\check{s}_i^2 - (ax+1)\check{s}_i + (x+a)$. Expanding $\check{s}_i = s_i + \lambda_i C$ we get $\check{s}_i^{-1} = s_i^{-1} + \lambda_i a(a-x)(1 + 2\lambda_i\tilde{\delta}/x^2)C$. From this we then get that the equation $\check{e}_i = a\left(\frac{\check{s}_i + \check{s}_i^{-1}}{x} - 1\right)$ imposes $Q(\lambda_i) = 0$. We now consider the braid relation $\check{s}_i\check{s}_j\check{s}_i = \check{s}_j\check{s}_i\check{s}_j = 0$ when $|j-i| = 1$. We get $\check{s}_i\check{s}_j\check{s}_i = s_is_js_i + (2\lambda_i + \lambda_j)C + (\lambda_i^2 + 2\lambda_i\lambda_j)C^2 + \lambda_i^2\lambda_jC^3$ and therefore $\check{s}_i\check{s}_j\check{s}_i - \check{s}_j\check{s}_i\check{s}_j$ is equal to

$$(\lambda_i - \lambda_j)\left(2\frac{\tilde{\delta}}{x^2}(a-x)\lambda_i + a\right)\left(2\frac{\tilde{\delta}}{x^2}(a-x)\lambda_j + a\right)C.$$

Therefore, either $\lambda_i = \lambda_j$, or $\tilde{\delta} \neq 0$ and one of the two values λ_i or λ_j is equal to $\lambda_0 = -ax^2/(2\tilde{\delta}(a-x))$. Since $Q(\lambda_0) = x^4/\tilde{\delta}$ this is excluded hence $\lambda_i = \lambda_j$. This proves that λ_i is independent of i, hence the splitting has the form $\check{s}_i = s_i + \lambda C$ with λ independent of i. It then remains to prove that this formula, with λ a root of Q, provides a splitting. The relation $\check{s}_i\check{s}_j = \check{s}_j\check{s}_i$ when $|j-i| \geq 2$ is clear, and therefore the only two relations that remain to be checked are $\check{e}_i\check{s}_j^{\varepsilon}\check{e}_i = \check{e}_i$ for $|j-i| = 1$ and $\varepsilon \in \{-1, 1\}$. For this we first check that $\check{e}_iC = 0$ by direct computation. Then, $\check{e}_i\check{s}_j\check{e}_i = \check{e}_i(s_j + \lambda C)\check{e}_i = \check{e}_is_j\check{e}_i + \lambda\check{e}_iC\check{e}_i = \check{e}_is_j\check{e}_i + 0$ and, expanding $\check{e}_i = e_i - \frac{x}{2(a-x)}C$, we get $\check{e}_is_j\check{e}_i = \check{e}_i$. The case $\varepsilon = -1$ is similar, and this concludes the proof.

□

Remark 6.20. *In cohomological terms, the non splitting in the cases $x = a$ and $x = 2a$ provides a non-zero cohomology 2-class in the Hochschild cohomology of these specializations of the BMW algebras with values in the one-dimensional bimodule given by $s_i \mapsto a$ (which factorizes through $BMW_n^{\pm}$ in these cases). If $x \notin \{a, 2a, -2a\}$ and the sequence splits, necessarily in two different ways, then the two splittings afford to distinct $BMW_n^{\pm}$-bimodule structures on k, namely $s_iC = Cs_i = bC$ and $s_iC = Cs_i = cC$.*

Remark 6.21. *Another natural question is for which specializations $\varphi : A \to k$ (with k a field of characteristic 0) the natural morphism $\widetilde{TL}_n \otimes_{\varphi} k \to F_n \otimes_{\varphi} k$ admits a retraction. A* *straightforward computation shows that this holds if and only if* $\varphi(x+2a)=0$ *and that, in this case, there is exactly one retraction. It is given by* $e_i \mapsto e_i$, $s_i \mapsto -e_i - a$, $s_i^{-1} \mapsto -e_i - a$, $C \mapsto C$.

6.7. **A central extension of the** (-1)**-Hecke algebras.** We introduce the two-sided ideal F_n^+ of F_n generated by $e_1,\dots,e_{n-1},C$, and we let F_n^{++} denote the ideal $(F_n^+)^2$.

Inside F_n/F_n^{++} we have $e_i = -C$ for all i, and therefore $\tilde{\delta}x^{-1}C = e_iC = -C^2 = -2\tilde{\delta}(a-x)C$, that is $\left(\frac{\tilde{\delta}}{x}\right)^2 aC = 0$. We thus let $\overline{F}_n$ denote the quotient of $F_n(0) = F_n \otimes_A A_0$ by the ideal $(A_0F_n^+)^2$. It is spanned over A_0 by elements $E_w, w \in \mathfrak{S}_n$, and C. Indeed, one can easily check the following formula, when we also denote $s_1,\dots,s_{i-1}$ the Coxeter generator of the symmetric group : if $\ell(s_iw) = \ell(w)+1$ then $s_i.E_w = E_{s_iw}$, otherwise $s_i.E_w = -2a^{\ell(w)}C + 2aE_w - E_{s_iw}$; moreover $C^2 = 0$ and $C.E_w = a^{\ell(w)}C$. Actually, a similar algebra can be associated to every Coxeter system, as we show now.

Theorem 6.22. *Let* (W,S) *be a Coxeter system, and* k *a field of characteristic* $\neq 2$. *The formulas*

$$\left\{\begin{array}{lcll} s.E_w & = & E_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ & = & -2a^{\ell(w)}C + 2aE_w - E_{sw} & \text{otherwise} \\ s.C & = & aC & \end{array}\right.$$

for all $s \in S$, $w \in W$, *define a representation of the Artin-Tits group* B *associated to* (W,S) *on the free module over* $k[a]/(a^2-1)$ *spanned by* C *and the* $E_w, w \in W$. *When* W *is finite, the image of the group algebra of* B *inside this representation is a free module of rank* $1+|W|$. *In all cases, this image projects onto the Iwahori-Hecke algebra of* (W,S) *defined by the relation* $(s-a)^2 = 0$ *for all* $s \in S$, *with kernel the linear span of* $\tilde{C} = -(s-a)^2$ *for an arbitrary choice of* $s \in S$. *When* W *admits a single conjugacy class of reflections, this algebra is the quotient of the group algebra of* B *by the relations* $(t-a)(s-a)^2 = (s-a)^2(t-a) = 0$ *for all* $s,t \in S$.

Proof. For every $s \in S$ we introduce the endomorphism R_s defined by

$$\left\{\begin{array}{lcll} R_s(E_w) & = & E_{ws} & \text{if } \ell(ws) = \ell(w)+1 \\ & = & -2a^{\ell(w)}C + 2aE_w - E_{ws} & \text{otherwise} \end{array}\right.$$

and $R_s(C) = aC$. We want to prove that the action of $\underbrace{stst\dots}_{m_{st}}$ on each E_w coincides with the action of $\underbrace{tsts\dots}_{m_{st}}$. As in the classical proof for the usual Hecke algebras (see [2], ex. 23 a) in ch. IV §2) we check that R_s, R_t commute with the actions of s and t by a straightforward computation, only using that the two conditions $\ell(swt) = \ell(w)$ and $\ell(sw) = \ell(wt)$, when met at the same time, imply $sw = wt$. From this and the obvious fact that

$$\underbrace{stst\dots}_{m_{st}}.E_1 = E_{\underbrace{stst\dots}_{m_{st}}} = E_{\underbrace{tsts\dots}_{m_{st}}} = \underbrace{tsts\dots}_{m_{st}}.E_1$$

we deduce that, writing any w as a reduced expression $t_{i_1}\dots t_{i_r}$, and letting $R_{\underline{w}} = R_{t_1}\dots R_{t_r}$, we get

$$\underbrace{stst\dots}_{m_{st}}.E_w = \underbrace{stst\dots}_{m_{st}}.R_{\underline{w}}.E_1 = R_{\underline{w}}\underbrace{stst\dots}_{m_{st}}.E_1 = R_{\underline{w}}\underbrace{tsts\dots}_{m_{st}}.E_1 = \underbrace{tsts\dots}_{m_{st}}.R_{\underline{w}}E_1 = \underbrace{tsts\dots}_{m_{st}}.E_w$$

and this proves the first claim. Let $\tilde{H}$ denote the image of the group algebra of B in this representation. By the same argument, we get that the morphism $g \mapsto g.E_1$ induces an injective module morphism between $\tilde{H}$ and the linear span $\mathcal{E}$ of the E_w and C. Letting $\overline{\mathcal{E}}$ denote the quotient of $\mathcal{E}$ by the linear span of C, we get an action of $\tilde{H}$ on $\overline{\mathcal{E}}$ which factorises through the regular representation of the usual Hecke algebra H of (W,S). Letting $\overline{E}_w \in \overline{\mathcal{E}}$ denote the image of $E_w \in \mathcal{E}$, we get therefore a surjective map $x \mapsto x.\overline{E}_1$ from $\tilde{H}$ onto a free module with basis the $\overline{E}_w, w \in W$. If W is finite, we deduce from this that the rank of $\tilde{H}$ is $1+|W|$. Since $\tilde{H} \to \mathcal{E}$ is injective we know that the kernel of $\tilde{H} \to \overline{E}$ is the linear span of C, and this kernel coincides with the kernel of $\tilde{H} \to H$ by the faithfulness of the regular representation. If W is finite, we deduce from this that the rank of $\tilde{H}$ is $1+|W|$. We have $(s-a).E_1 = E_s - aE_1$ and $(s-a)^2.E_1 = -2aC + 2aE_s - E_1 - aE_s - aE_s + E_1 = -2aC$. Letting $\tilde{C} \in \tilde{H}$ denote the action of $-(s-a)^2$, we get $\tilde{C}.E_1 = 2aC$, $\tilde{C}.C = 0$ hence $\tilde{C}^2 = 0$. Also note that, since $\tilde{C}.E_1 = -2aC$ does not depend on the choice of $s \in S$, the definition of $\tilde{C}$ does not depend on the choice of s either. Since $(t-a).C = 0$ for all $t \in S$ we get $(t-a)\tilde{C}.E_1 = 0$. Moreover $\tilde{C}.E_t = \tilde{C}R_t(E_1) = R_t(\tilde{C}.E_1) = 2aR_t(C) = 2C$. Therefore $\tilde{C}.(t-a).E_1 = \tilde{C}.E_t - a\tilde{C}.E_1 = 2C - 2C = 0$. Let now $\hat{H}$ denote the quotient of the group algebra $k[a]/(a^2-1)B$ of B by the relations $(s-a)^2(t-a) = (t-a)(s-a)^2 = 0$ for all $s,t \in S$. We proved that the natural surjective morphism $k[a]/(a^2-1)B \to \tilde{H}$ factors through $\hat{H}$. For $s \in S$, let $\hat{C}_s$ denote the image of $-(s-a)^2$ inside $\hat{H}$. Since $(t-a)\hat{C}_s = \hat{C}_s(t-a) = 0$ we get $t\hat{C}_s = a\hat{C}_s = \hat{C}_s t$ hence $t\hat{C}_s t^{-1} = \hat{C}_s$ for all $t \in S$ hence $b\hat{C}_s b^{-1} = \hat{C}_s$ for all $b \in B$. If W has a single conjugacy class of reflections this implies that $\hat{C}_s$ does not depend of $s \in S$, because in that case all the elements of S are conjugated one to the other inside B, and $b\hat{C}_s b^{-1} = \hat{C}_{bsb^{-1}}$ whenever $bsb^{-1} \in S$. Therefore we note $\hat{C} = \hat{C}_s$. By the above we know that its linear span is a two-sided ideal of $\hat{C}$, and it is clear that the composite map $\hat{H} \to \tilde{H} \to H$ factors through $\hat{H}/\langle \hat{C} \rangle \to H$. But $\hat{H}/\langle \hat{C} \rangle$ is the quotient of $k[a]/(a^2-1)B$ by the relations $(s-a)^2 = 0$ for all $s \in S$ hence this map is an isomorphism. By the short five-lemma this implies that $\hat{H} \to \tilde{H}$ is also an isomorphism.

□

Remark 6.23. *These extensions of the* (-1)*-Hecke algebras are non-split ; indeed, a splitting would provide elements* $\hat{s} = s + \lambda C$ *for some* $\lambda \in k$ *such that* $(\hat{s}-a)^2 = 0$. *But an easy computation shows that* $(\hat{s}-a)^2 = (s-a)^2 \neq 0$. *Therefore, these extensions provide natural non-zero Hochschild 2-cohomology classes in the cohomology of these Hecke algebras with values in the trivial bimodule afforded by the obvious augmentation map. When* $W = \mathfrak{S}_n$ *and* k *has characteristic* 0*, it is proved in* [3] *(theorem 1.1) that the corresponding (Hochschild) cohomology group has dimension* 1. *A natural question is then whether our extension spans this 2nd cohomology group whenever* W *has a single class of reflections.*

Remark 6.24. *When* W *has several classes of reflections, then the quotient of* $k[a]^2/(a^2-1)B$ *by the relations* $(s-a)^2(t-a) = (t-a)(s^2-a) = 0$ *defines a larger algebra. This algebra projects onto the usual Hecke algebra and the kernel of the projection is a two-sided nilpotent ideal of rank the number* r *of conjugacy classes of reflections. The action of* $k[a]/(a^2-1)B$ *on this algebra admits a similar description : a basis is given by elements* $E_w, w \in W$ *together with elements* $C_s, s \in S$ *such that* $C_s = C_{s'}$ *whenever* s *is a conjugate of* s'*, and the action itself is given by the formulas* $s.E_w = E_{sw}$ *is* $\ell(sw) = \ell(w)+1$ *and* $s.E_w = -2a^{\ell(w)}C_s + 2aE_w - E_{sw}$ *otherwise. The proof is similar and left to the reader.*

When there is a single conjugacy class of reflections, one may wonder if we could reduce the number of relations by asking for e.g. $(t-a)(s-a)^2 = 0$ for all $s,t \in S$, but *not* for $(s-a)^2(t-a) = 0$ for all $s,t \in S$. The answer is positive, as we show now.

Proposition 6.25. *If (W,S) is an irreducible Coxeter system with a single conjugacy class of reflections and char.k $=0$, then the algebra $\tilde{H}(W,S)$ is the quotient of the group algebra of $k[a]/(a^2-1)B$ by the relations $(t-a)(s-a)^2=0$ for $s,t\in S$. The corresponding ideal is also generated by the relations $(s-a)^2(t-a)=0$ for $s,t\in S$. If (W,S) is simply laced, these statements remain valid under the weaker assumption char.k $\neq 2$.*

Proof. We first give two proofs of this statement in the important simply-laced case. These proofs are straightforward and explicit, in contrast with the proof of the general case that we provide after that.

In this simply-laced case, the statement can be reduced to the special case where W has type A_2. If $char.k=0$, by using $k[a]/(a^2-1)\simeq k[a]/(a-1)\oplus k[a]/(a+1)\simeq k\oplus k$ and the fact that $\tilde{H}$ is in this case a quotient of the Hecke algebra a specialization of H_3, we can compute the dimension of the ideal generated by these relations, and concludes in this way that the ideal coincides with the ideal generated by the relations $(t-a)(s-a)^2=(s-a)^2(t-a)=0$ for $a\in\{-1,1\}$. By this method one may actually get explicit expressions over $\mathbb{Q}$ whose denominators are powers of 2, thus getting the conclusion for every field of characteristic $\neq 2$. We provide an alternative, à la Coxeter argument. We compute inside the quotient A of kB_n by the relations $(t-a)(s-a)^2=0$ for all $s,t\in S$. Again for all $s,t\in S$, since $(s-a)^3=0$ we get $s^{-1}=as^2-3s+3a$ and therefore the identity $s(t-a)^2=a(t-a)^2$ implies $s^{-1}(t-a)^2=a(t-a)^2$. We assume that (W,S) has type A_2 and we let $S=\{s,t\}$. Then, $sts=tst$ implies $(t-a)^2=(ts)^{-1}(s-a)^2ts=s^{-1}t^{-1}(s-a)^2ts=(s-a)^2ts$. Let then $X=(s-a)^2t-2a(t-a)^2+(t-a)^2s$. We get $X.s=(s-a)^2ts+(t-a)^2(s^2-2as)=(t-a)^2(s^2-2as+1)=(t-a)(t-a)(s-a)^2=0$. Since s is invertible this implies $X=0$, that is $(s-a)^2t=2a(t-a)^2-(t-a)^2s$. Symmetrically we get $(t-a)^2s=2a(s-a)^2-(s-a)^2t$ hence $2a(s-a)^2=(t-a)^2s+(s-a)^2t=(t-a)^2s+2a(t-a)^2-(t-a)^2s=2a(t-a)^2$. Thus $(s-a)^2=(t-a)^2$ hence $(s-a)^2t=(t-a)^2t=t(t-a)^2=a(t-a)^2=a(s-a)^2$ and symmetrically $(t-a)^2s=a(t-a)^2$. The conclusion follows.

We now provide an argument for the general case in characteristic 0. First of all, we may as in our very first argument assume that H and $\tilde{H}$ are defined over k and $a\in\{-1,1\}\subset k$. Secondly, we can assume $a=1$, for the map $s\mapsto -s$ defines an isomorphism between the two variations of $\tilde{H}$ that permutes consistently the ideals under consideration. Thus, H is the quotient of kB by the relations $(s-1)^2=0$ for $s\in S$, while $\tilde{H}$ is the quotient of kB by the relations $(t-1)(s-1)^2=(s-1)^2(t-1)=0$ for $s,t\in S$. We let $\hat{H}$ denote the quotient of kB by the relations $(t-1)(s-1)^2=0$ for $s,t\in W$. For $s\in S$ we let $C_s=(s-1)^2\in\hat{H}$.

We want to show that $\hat{H}=\tilde{H}$ and for this we can assume that $W=\langle s,t\rangle$ is of type $I_2(m)$ with m odd. Indeed, recall that our assumption that there is only one conjugacy class of reflections implies that every two elements $s,t\in S$ are connected by a chain $s=s_1,s_2,\dots,s_r=t$ such that $\langle s_i,s_{i+1}\rangle$ is a finite dihedral group of odd type. Therefore, if we can prove our statement for each $\langle s_i,s_{i+1}\rangle$, we get $C_{s_i}s_{i+1}=C_{s_i}=C_{s_i}s_i$; since s_i and s_{i+1} are conjugates inside $\langle s_i,s_{i+1}\rangle$ this proves $C_{s_i}=C_{s_{i+1}}$, and by induction $C_s=C_t$. But then $C_s(t-1)=C_t(t-1)=0$, which proves the claim. We thus assume from now on that $W=\langle s,t\rangle$ is of type $I_2(m)$ with m odd.

Then $\hat{H}=\tilde{H}$ is equivalent to saying that $\hat{H}$ acts on kC_s by right multiplication through its (unique) 1-dimensional representation $s,t\mapsto 1$. Indeed, if this is the case we have $(s-1)^2(t-1)=0$ and this implies $(t-1)^2(s-1)=0$ since these two expressions are conjugated by an element of B. This is what we prove now.

First notice that this action factorizes through H. Indeed, whenever f is the image in $\hat{H}$ of an element of B, we have $g(s-1)^2 = (s-1)^2$ and $g(t-1)^2 = (t-1)^2$, and therefore

$$C_s g s^2 = C_s g(s-1)^2 + 2C_s g s - C_s g = C_s(s-1)^2 + 2C_s g s - C_s g = 2C_s g s - C_s g$$

and similarly $C_s g t^2 = 2C_s g t - C_s g$ since $C_s(t-1)^2 = (s-1)(s-1)(t-1)^2 = 0$. Since $\hat{H}$ is spanned by the image of B, this proves that the right action on $C_s\hat{H}$ factorizes through H. Also note that k can be assumed to be algebraically closed. We now need to make a few remarks on H.

Recall from e.g. [8] that the generic Hecke algebra admits two 1-dimensional irreducible representations and $(m-1)/2$ two-dimensional ones, that can be defined by explicit formulas. It is straightforward to check that the specializations at $q = -1$ of the 2-dimensional ones

$$s \mapsto \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad t \mapsto \begin{pmatrix} -1 & c_j \\ 0 & -1 \end{pmatrix}$$

with $c_j = -2 - (\zeta^j + \zeta^{-j})$ where ζ is a primitive m-th root of 1 and $1 \leq j \leq (m-1)/2$ are pairwise non-isomorphic irreducible representations of H. Note in passing that st is mapped to a matrix of trace $c_j + 2$ and determinant 1, and therefore is conjugated to $\mathrm{diag}(-\zeta^j, -\zeta^{-j})$. Since the two 1-dimensional irreducible representations become one, it follows that the Jacobson radical $J(H)$ has dimension 1.

We claim that $J(H)$ is spanned by $X = \sum_{w \in W} (-1)^{\ell(w)} T_w \in H$. For this, we first note that $X \mapsto 0$ under the 1-dimensional representation $s, t \mapsto 1$. Then, letting $X_s = T_1 - T_t + T_{st} - T_{tst} + \ldots$ we have $X = X_s - X_s.T_s$ and therefore $X.T_s = X_s.T_s - X_s.T_s^2 = X_s.T_s - 2X_s.T_s + X_s = X$. Similarly, with obvious notations, $X = X_t - X_t.T_t$ and therefore $X.T_t = X$. It follows that kX is a 1-dimensional two-sided ideal of H whose image under $H/J(H) \simeq k \oplus M_2(k)^{(m-1)/2}$ cannot be the two-sided ideal k. Since all the other proper 2-sided ideals of $H/J(H)$ have dimension at least 4, this proves that its image inside $H/J(H)$ is 0 hence $kX = J(H)$.

From this we deduce that the right action of H on $C_s\hat{H}$ factorizes through $H/J(H)$. Indeed, letting $Y_s = T_1 - T_t + T_{ts} - T_{tst} + \ldots$ we have $X = Y_s - T_s.Y_s$ hence $C_s.X = C_s.Y_s - C_s.s.Y_s = C_s Y_s - C_s Y_s = 0$, since $C_s.s = (s-1)^2(s-1) + (s-1)^2 = (s-1)^2 = C_s$. Finally, since $(st)^m = (ts)^m$ is central, we have $C_s(st)^m = (st)^m C_s = C_s$ hence the two-sided ideal of H generated by $(st)^m - 1$ acts by 0. But the image of $(st)^m$ inside each 2-dimensional irreducible representation of H is $\mathrm{diag}(-\zeta^j, -\zeta^{-j})^m = -1$ hence, because $-2 \neq 0$ in k, the ideal generated by the image of $(st)^m - 1$ inside $H/J(H) \simeq k \oplus M_2(k)^{(m-1)/2}$ is $M_2(k)^{(m-1)/2}$. This proves that the right action of H on C_s factorizes through k, and this proves the claim.

The second statement about the relations $(s-a)^2(t-a) = 0$ is obviously similar. □

Thus $\overline{F}_n = F_n(0)/(A_0 F_n^+)^2$ is the extension of the theorem corresponding to $W = \mathfrak{S}_n$. Notice that the natural map $\overline{F}_n \to \overline{F}_{n+1}$ is into for all $n \geq 1$. We now prove that there is indeed a Markov trace on F_n factorizing through $\overline{F}_n$. Our proof is essentially an adaptation of Jones's proof of existence for the Ocneanu trace (see [9] theorem 5.1).

Theorem 6.26. *There exists a unique family of traces $t_n : \overline{F}_n \to A_0$ satisfying $t_{n+1}(x s_n^{\pm 1}) = t_n(x)$ for all $x \in \overline{F}_{n-1}$ and $t_2(C) = 1$.*

Proof. Because $t_{n+1}(C) = a t_{n+1}(C s_n)$ and $C \in \overline{F}_{n-1}$, the condition $t_2(C) = 1$ implies $t_n(C) = a^n$ for all $n \geq 2$. We recall that every element of $\mathfrak{S}_{n+1}$ admits a reduced expression of the form $w y_k$ with w a reduced expression of some element in $\mathfrak{S}_n$ and $y_k = s_n s_{n-1} \ldots s_k$ with $1 \leq k \leq n+1$, with the convention $y_{n+1} = 1$. We assume that t_n is uniquely defined with a trace satisfying the Markov property, and we show from this that t_{n+1} is also uniquely defined. We let $\hat{y}_k = s_{n-1} \dots s_k$ for $1 \le k \le n$.

First of all, we note that $1 = \frac{a}{2}(s_i + s_i^{-1}) - ae_i = \frac{a}{2}(s_i + s_i^{-1}) + aC$, hence the Markov property imposes that, for all $x \in \overline{F}_n$, we have

$$t_{n+1}(x) = at_n(x) + at_n(xC) = at_n(x) + a^{1+\ell(w)+n}$$

if x is given by the reduced expression w. Now $\overline{F}_{n+1} = \overline{F}_n \oplus \bigoplus_{k<n+1} \overline{F}_n y_k$. Therefore, t_{n+1} is uniquely defined by its value on $\overline{F}_n$, that we already defined, and on the $\overline{F}_n y_k$ for $k < n+1$. But the Markov property imposes $t_{n+1}([w]y_k) = t_{n+1}([w]s_n\hat{y}_k) = t_n([w]\hat{y}_k)$, so we take this as a definition.

We need to prove that t_{n+1} is a trace and that is satisfies the Markov condition. We start by the latter property, and actually we prove first that $t_{n+1}(xs_n^{\pm 1}y) = t_n(xy)$ for all $x, y \in \overline{F}_n$. First of all we note that $t_{n+1}(xs_n\hat{y}_k) = t_n(x\hat{y}_k)$ for all $x \in \overline{F}_n$, since it holds for $x = C$ as well as all the $x = [w]$ for w a reduced expression in $\mathfrak{S}_n$. We can then restrict ourselves to proving that $t_{n+1}(xs_ny) = t_n(xy)$ for all y of the form $[w]\hat{y}_k$ for w a reduced expression in $\mathfrak{S}_{n-1}$ and $1 \le k \le n$. Then $t_{n+1}(xs_ny) = t_{n+1}(xs_n[w]\hat{y}_k) = t_{n+1}(x[w]s_n\hat{y}_k) = t_{n+1}(x[w]y_k) = t_n(x[w]\hat{y}_k) = t_n(xy)$. We now prove that $t_{n+1}(xs_n^{-1}y) = t_n(xy)$ under the same assumptions on x, y. We can assume $x = [w]$ and $y = [m]$ for w, m reduced expression. We then notice that $s_n^{-1} = 2a + 2e_n - s_n = 2a - 2C - s_n$ and therefore $t_{n+1}(xs_n^{-1}y) = 2at_{n+1}(xy) - 2a^{\ell(w)+\ell(m)+n} - t_{n+1}(xs_ny) = 2at_{n+1}(xy) - 2a^{\ell(w)+\ell(m)+n} - t_{n+1}(xs_ny) = 2t_n(xy) + 2a^{n+\ell(w)+\ell(m)} - 2a^{\ell(w)+\ell(m)+n} - t_n(xy) = t_n(xy)$.

We now prove that t_{n+1} is a trace. We need to prove $t_{n+1}(s_ix) = t_{n+1}(xs_i)$ for all $i \le n$. We first assume $i < n$. If $x \in \overline{F}_n$ this is an immediate consequence of the relation between $t_{n+1}(x)$ and $t_n(x)$. If not, we can assume $x = [w]s_n\hat{y}_k$. Then $t_{n+1}(s_ix) = t_{n+1}(s_i[w]s_n\hat{y}_k) = t_n(s_i[w]\hat{y}_k)$ by the Markov property. Since t_n is a trace this is equal to $t_n(s_i[w]\hat{y}_k) = t_n([w]\hat{y}_ks_i) = t_{n+1}([w]s_n\hat{y}_ks_i) = t_{n+1}(xs_i)$.

We now let $i = n$. If $x \in \overline{F}_n$ this is a consequence of the Markov property : $t_{n+1}(s_nx) = t_n(x) = t_{n+1}(xs_n)$. If not, we can assume $x = us_nv$ with $u, v \in \overline{F}_n$. Then $t_{n+1}(s_nx) = t_{n+1}(s_nus_nv)$.

- If $u, v \in \overline{F}_{n-1}$ this is equal to $t_{n+1}(s_nuvs_n) = t_{n+1}(us_nvs_n) = t_{n+1}(xs_n)$.
- If $u \in \overline{F}_{n-1}$ and $v \notin \overline{F}_{n-1}$ this is equal to $t_{n+1}(us_n^2v) = -2at_{n+1}(uCv) - t_{n+1}(uv) + 2at_{n+1}(us_nv)$ since $s_n^2 = -2aC - 1 + 2as_n$, and therefore to $-2a^{\ell(u)+\ell(v)+n+1} - t_{n+1}(uv) + 2at_{n+1}(us_nv) = -2a^{\ell(u)+\ell(v)+n+1} - t_{n+1}(uv) + 2at_n(uv) = -2at_{n+1}(uCvs_n) - t_{n+1}(uv) + 2at_{n+1}(uvs_n)$. On the other hand we can write $v = [w]s_{n-1}[w']$ with w, w' reduced expressions in $\mathfrak{S}_{n-1}$. Then $t_{n+1}(us_nvs_n) = t_{n+1}(us_n[w]s_{n-1}[w']s_n) = t_{n+1}(u[w]s_ns_{n-1}s_n[w']) = t_{n+1}(u[w]s_{n-1}s_ns_{n-1}[w']) = t_n(u[w]s_{n-1}^2[w']) = -2at_n(u[w]C[w']) - t_n(u[w][w']) + 2at_n(u[w]s_{n-1}[w']) = -2at_n(u[w]C[w']) - t_n(u[w][w']) + 2at_n(uv) = -2at_{n+1}(uCvs_n) - t_{n+1}(uv) + 2at_{n+1}(uvs_n)$ since $t_n(u[w][w']) = at_{n-1}(u[w][w']) + a^{n+\ell(u)+\ell(w)+\ell(w')} = at_n(u[w]s_{n-1}[w']) + a^{n+\ell(u)+\ell(w)+\ell(w')} = t_{n+1}(uv)$.
- the case $u \notin \overline{F}_{n-1}$ and $v \in \overline{F}_{n-1}$ is similar and left to the reader.
- We now assume $u = [w]s_{n-1}[w']$ and $v = [m]s_{n-1}[m']$. Then

$$\begin{array}{rcl} t_{n+1}(s_nus_nv) & = & t_{n+1}(s_n[w]s_{n-1}[w']s_n[m]s_{n-1}[m']) \\ & = & t_{n+1}([w]s_ns_{n-1}s_n[w'][m]s_{n-1}[m']) \\ & = & t_{n+1}([w]s_{n-1}s_ns_{n-1}[w'][m]s_{n-1}[m']) \\ & = & t_n([w]s_{n-1}^2[w'][m]s_{n-1}[m']) \end{array}$$

by the Markov property. Similarly,

$$t_{n+1}(us_nvs_n) = t_n([w]s_{n-1}[w'][m]s_{n-1}^2[m']).$$

Expanding s_{n-1}^2 we get $t_n([w]s_{n-1}^2[w'][m]s_{n-1}[m']) = -2at_n([w]C[w'][m]s_{n-1}[m']) - t_n([w][w'][m]s_{n-1}[m'])+2at_n([w]s_{n-1}[w'][m]s_{n-1}[m'])$ while $t_n([w]s_{n-1}[w'][m]s_{n-1}^2[m']) = -2at_n([w]s_{n-1}[w'][m]C[m'])-t_n([w]s_{n-1}[w'][m][m'])+2at_n([w]s_{n-1}[w'][m]s_{n-1}[m'])$. We have $t_n([w]C[w'][m]s_{n-1}[m']) = t_n([w]s_{n-1}[w'][m]C[m']) = a^{n+1+\ell(w)+\ell(w')+\ell(m)+\ell(m')}$ $t_n([w][w'][m]s_{n-1}[m']) = t_{n-1}([w][w'][m][m']) = t_n([w]s_{n-1}[w'][m][m'])$ by the Markov property, whence $t_{n+1}(s_nus_nv) = t_{n+1}(us_nvs_n)$ and the conclusion.

$\square$

6.8. **The case** $x = -2a$**.** Let $\mathcal{B}_n$ denote the $\mathbb{Q}[a]/(a^2-1)$-algebra $BMW_4^{\dagger\dagger}\otimes S^{\dagger\dagger}/(x+2a)$ and $\mathcal{B}_n^{\pm} = \mathcal{B}_n \otimes \mathbb{Q}[a]/(a\mp 1)$. Inside $\mathcal{B}_n$ we have $x=-2a$, $e_i^2 = 2ae_i$, $\tilde{\delta} = 4$ and $(s_i-a)(s_i+a)^2 = (s_i+a)(s_i^2-1) = 0$. We specialize H_4 accordingly, and let $U^{\pm}$, $V^{\pm}$ denote the kernels of its projection onto $\mathcal{B}_4^{\pm}$ and $BMW_4^{\pm}$, respectively. We denote $\mathcal{I}_n^{\pm} = \mathrm{Ker}(\mathcal{B}_n^{\pm} \twoheadrightarrow BMW_n^{\pm})$ and we identify $\mathcal{I}_4^{\pm}$ with the vector space $U^{\pm}/V^{\pm}$. By proposition 6.11 we know that they have dimension $115-105 = 10$. As we noticed in the proof of proposition 5.10, we have natural morphisms $\mathcal{B}_n^{\pm} \twoheadrightarrow \mathbb{Q}\mathfrak{S}_n$.

We let $C_i^{\pm}$ denote the image of $\mathcal{S}_i$ inside $\mathcal{B}_n^{\pm}$. We have $(C_i^{\pm})^2 = 6aC_i^{\pm} = \pm 6C_i^{\pm}$. $\dim \mathcal{B}_n^{\pm} = 115$. The quotient $\mathcal{F}_n^{\pm}$ of $\mathcal{B}_n^{\pm}$ by the ideal generated by $C_1^{\pm}-C_2^{\pm}$ is $F_n\otimes A/(a\mp 1, x+2a)$. We have $\mathcal{B}_n^{\pm} \twoheadrightarrow \mathcal{F}_n^{\pm} \twoheadrightarrow BMW_n^{\pm}$, and $\mathcal{B}_3 \simeq \mathcal{F}_3$. Recall from proposition 6.19 that the morphism $\mathcal{F}_n^{\pm} \to BMW_n^{\pm}$ admits exactly one splitting, given by $s_i \mapsto s_i - (a/3)C$.

By explicit computations inside $U^{\pm}/V^{\pm}$, we get the following.

(i) The bimodule action of H_4 on $\mathcal{I}_4^{\pm}$ factorizes through $(\mathbb{Q}\mathfrak{S}_4)\otimes(\mathbb{Q}\mathfrak{S}_4)^{\mathrm{op}}$ (that is, the left and right actions of the s_i^2 are trivial). The corresponding representations are $\chi_4\otimes\chi_4 + \chi_{31}\otimes\chi_{31}$ if $a=1$, $\chi_{1^4}\otimes\chi_{1^4}+\chi_{211}\otimes\chi_{211}$ if $a=-1$, where χ_λ denotes the irreducible representation of $\mathfrak{S}_n$ associated to the partition λ of n, with the convention that $\chi_{[n]}$ is the trivial representation.

(ii) The subalgebra of $\mathcal{I}_4^{\pm}$ generated by $C_1^{\pm}$ and $C_2^{\pm}$ is 5-dimensional, and defined by the relations $(C_i^{\pm})^2 = 6aC_i^{\pm}$, $C_1^{\pm}C_2^{\pm}C_1^{\pm} - C_2^{\pm}C_1^{\pm}C_2^{\pm} = 4(C_1^{\pm}-C_2^{\pm})$, one possible basis being $C_1^{\pm}, C_2^{\pm}, C_1^{\pm}C_2^{\pm}, C_2^{\pm}C_1^{\pm}, C_1^{\pm}C_2^{\pm}C_1^{\pm}$.

(iii) By direct computation, we check that the ideal $\mathcal{I}_4^{\pm}$ is generated by $C_1^{\pm}$, $C_2^{\pm}$ and the e_i's. Since we have $\mathcal{B}_3^{\pm} = \mathcal{F}_3^{\pm}$, by lemma 6.13 (iii) we know that the $C_i^{\pm}$'s belong to the subalgebra generated by the e_i's. Therefore, the extension $\mathcal{B}_4^{\pm}$ of $BMW_4^{\pm}$ is basically determined by the induced extension of the Temperley-Lieb subalgebra of $BMW_n^{\pm}$, at least when $n=4$. We suspect it is the case in general.

(iv) As an algebra, using known algorithms used by GAP4, we check that $\mathcal{I}_4^{\pm}$ can be split into a direct sum of two unital $\mathbb{Q}$-algebras, one of them being 1-dimensional, the other one being 9-dimensional. We check that the latter is central and simple, but not a division ring. Therefore, we have $\mathcal{I}_4^{\pm} \simeq \mathbb{Q}\oplus Mat_3(\mathbb{Q})$ as a $\mathbb{Q}$-algebra.

One problem we face to extend these properties further is that we need to know whether $C_3^{\pm}$ and $C_1^{\pm}$ do commute (or to what extent they do not). This should be doable by computing inside H_5, which is still finite-dimensional. However its dimension (155520) is a lot larger than the dimension of H_4, and there is no software capable of dealing with it yet.

These computations are however sufficient to guess a plausible conjecture. For $n\geq 3$, let $TL_n^{\pm}$ denote the quotient of the group algebra $\mathbb{Q}\mathfrak{S}_n$ of the symmetric group by the ideal $\mathcal{J}_n^{\pm}$

generated by $\mathcal{T} = s_1s_2s_1 + as_1s_2 + as_2s_1 + s_1 + s_2 + a$ with $a = \pm 1$. It is a specialization of the Temperley-Lieb algebra, and has dimension the Catalan number Cat_n. Now recall from the proof of proposition 5.10 that we have a surjective morphism $\mathcal{B}_n^{\pm} \twoheadrightarrow \mathbb{Q}\mathfrak{S}_n$. By direct computation we get that $\mathcal{S}_1$ is mapped onto $\mathcal{T}$. Therefore, we have a commutative diagram of horizontal short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_n^{\pm} \arrow[r] \arrow[d, "\pi_n^{\pm}"'] & \mathcal{B}_n^{\pm} \arrow[r] \arrow[d] & BMW_n^{\pm} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}_n^{\pm} \arrow[r] & \mathbb{Q}\mathfrak{S}_n \arrow[r] & TL_n^{\pm} \arrow[r] & 0
\end{tikzcd}$$

The computations above together with the identification of $\mathcal{B}_3^{\pm}$ with a specialization of F_3 shows that, for $n = 3$ and $n = 4$, the leftmost vertical map $\pi_n^{\pm}$ is an isomorphism. We conjecture that it is the case in general.

Conjecture 6.27. *$\pi_n^{\pm}$ is an isomorphism for all $n \geq 3$.*

If this conjecture holds true, then the algebra structure of $\mathcal{B}_n^{\pm}$ would be completely determine by an explicit linear splitting of $\mathcal{B}_n^{\pm} \to BMW_n^{\pm}$ together with the corresponding Hochschild 2-cocycle. Indeed, the bimodule action of $BMW_n^{\pm}$ on $\mathcal{I}_n^{\pm}$ would be easily determined, since the action of every braid on $\mathcal{I}_n^{\pm}$ would be identified under $\pi_n^{\pm}$ with the action of the corresponding permutation on the ideal $\mathcal{J}_n^{\pm}$. Moreover, we would have for the dimension of $\mathcal{B}_n^{\pm}$ the conjectural formula

$$\begin{aligned} \dim \mathcal{B}_n^{\pm} &= \dim BMW_n^{\pm} + \dim \mathbb{Q}\mathfrak{S}_n - \dim TL_n^{\pm} \\ &= 1.3.\dots.(2n-1) + n! - Cat_n \end{aligned}$$

Finally, we suspect that the special retraction pointed out in remark 6.21 echoes some special phenomenon in $\mathcal{B}_n^{\pm}$ that needs to be understood further.

7. Knot invariants

7.1. **Number of connected components, special trace and change of variables.** If β is a braid on n strands, then its closure L is an oriented link. As before, we denote $\ell : B_n \to \mathbb{Z}$ the abelianization morphism which maps σ_i to 1. We will use the following classical fact :

Lemma 7.1. *If $\beta \in B_n$ and L is the closure of β, then $n + \ell(\beta) \equiv (\#L) \mod 2$, where $\#L$ denote the number of components of the link L.*

Proof. Decomposing the projection $\overline{\beta} \in \mathfrak{S}_n$ of β under $B_n \to \mathfrak{S}_n$ into a product of disjoint cycles, we get r cycles of even lengths $2a_1, \dots, 2a_r$ and s cycles of odd lengths $b_1, \dots, b_s$. Then $\#L = r + s$, $n = 2\sum a_i + \sum b_i \equiv s \mod 2$ and $\ell(\beta) \equiv r \mod 2$. This proves the claim. □

This provides a simple interpretation of the trace $t_n^{\dagger\dagger}$, with values in $S^{\dagger\dagger}$. Recall that $a^2 = 1$ inside $S^{\dagger\dagger}$.

Proposition 7.2. *If $\beta \in B_n$ and L is the closure of β, then $t_n^{\dagger\dagger}(\beta) = a^{\#L}$.*

Proof. By construction, we have $t_n^{\dagger\dagger}(\beta) = a^n a^{\ell(\beta)} = a^{n+\ell(\beta)}$. Since $a^2 = 1$ the conclusion follows from lemma 7.1. □

There is a well-known connection between 'two versions' of the Kauffman polynomial, one being connected to the other through a sign depending on the number of connected components of the link. In our formalism this is seen as follows, using the automorphism $\overline{E}$ of $\overline{BMW}_n \otimes_{\overline{R}} \overline{S} = BMW_n^+ \oplus BMW_n^-$. First recall that the values of t_n^+ and t_n^- on braids belong to a submodule $S_0 = \mathbb{Q}[a^{\pm 1}, x^{\pm 1}]$ of S_+ and S_-, respectively. Also recall that S_0 can be considered as a submodule of $\mathbb{Q}(\alpha, q)$ under $a \mapsto \alpha^{-2}$, $x \mapsto \alpha^{-1}(q \pm q^{-1})$. We let $\tau_n^\pm$ denote the Markov traces $t_n^\pm$ viewed as functions with values in the same ring S_0.

We can now show how this well-known connection between the two versions fits into our setting. This connection can be stated as follows (see e.g. [13], p. 177).

Proposition 7.3. *If β is a braid on n strands whose closure is the link L, then*

$$\tau_n^-(\beta)\Big|_{\substack{a \mapsto -a \\ x \mapsto -x}} = (-1)^{\#L-1} \tau_n^+(\beta)$$

where we identified β with its image in BMW_n^+ on the LHS, and its image in BMW_n^- in the RHS, and $\#L$ denotes the number of components of L.

Proof. The LHS can be viewed inside S_+ as $\eta \circ t_n^- \circ \overline{E}(\overline{E}(\beta))$, which is equal to $(-1)^{n-1} t_n^+(\overline{E}(\beta))$ by corollary 5.4. Now $(-1)^{n-1} t_n^+(\overline{E}(\beta)) = (-1)^{n-1}(-1)^{\ell(\beta)} t_n^+(\beta)$ where $\ell : B_n \to \mathbb{Z}$ denotes as before the abelianization morphism. The conclusion then follows from the identity $n + \ell(\beta) \equiv (\#L) \mod 2$, proved in lemma 7.1. $\square$

7.2. **The special case $a = \pm 1$, $b = \mp j$, $c = \mp j^2$.** Recall that j denotes a primitive 3-root of 1. We have $y = 1$, $x = a$. In this case, the link invariant afforded by the Kauffman trace t_n^+ is constant equal to 1 (that is, it takes the value 1 on every link), see [13] p. 186, table 16.3 row A, and this is reproved by the observation that, in this case, $\delta_K = a$ hence $t_n^K = t_n^{\dagger\dagger}$. For $y = 1$ and $x = a = -1$, according to [13] p. 186, table 16.3 row D, the link invariant associated to t_n^H maps a link L to $(\mathrm{i}\sqrt{2})^{d_2(T(L))}$, where $T(L)$ is the 3-fold cyclic cover of S^3 branched over L, and $d_2(T(L)) = \dim_{\mathbb{F}_2} H_1(T(L), \mathbb{F}_2)$. When $y = 1$, $x = a = 1$, we consider the automorphism φ of the group algebra of the braid group defined by $s_i \mapsto s_i^{-1}$, and show that, as in the proposition 7.3 and corollary 5.4, the formula $T_n(b) = (-1)^{n-1} t_n(\varphi(b))$ induces a bijection between the traces factorizing through the two Hecke algebras with quadratic conditions $s^2 + s + 1 = 0$ and $s^2 - s + 1 = 0$, hence the two invariants are related by multiplication by $(-1)^{\#L-1}$. As a consequence we get the general formula $t_n^H(\beta) = a^{\#\hat{\beta}-1}(\mathrm{i}\sqrt{2})^{d_2(T(L))}$.

We are looking for a Markov trace t_n^0 on $\widetilde{BMW}_n \otimes_R R/(a = \pm 1, b = \mp j, c = \mp j^2)$ such that $t_3^0(1) = t_3^0(s_1) = 0$ and $t_3^0(s_1 s_2) = 1$. This implies $t_3^0(s_1^\alpha s_2^\beta) = 1$ for all $\alpha, \beta \in \{-1, 1\}$.

We checked by computer that there is a (necessarily unique) extension of t_3^0 to F_4 satisfying the Markov property. If there is an obstruction to these traces to genuine Markov traces one thus needs to look for it on at least 5 strands.

Conjecture 7.4. *The trace t_3^0 can be extended to a Markov trace on the tower (F_n).*

Assuming that conjecture, we get that the corresponding link invariant would take the value $a^n \times (4.2^n - 4 - 2n)$ on the $(n+3)$-components unlink.

7.3. **The special case $a = b = c = \pm 1$.** We have $y = 1$, $x = 2a = \pm 2$. In this case, $\delta_H = a$, hence t_n^H coincides with $t_n^{\dagger\dagger}$ (see also [13] table 6.3, row A), and, for $a = y = 1$, the Kauffman invariant maps a link L to $(\det L)^2$, where $\det L = \Delta_L(-1) = \nabla_L(0)$ (see [13] table 6.3, row B). Since this invariant is afforded by t_n^+, and because of proposition 7.3, this implies that $t_n^K(\beta) = a^{\#\hat{\beta}-1}(\det \hat{\beta})^2$ for every braid $\beta \in B_n$.

We are looking for a Markov trace t_n^0 on $\widetilde{BMW}_n \otimes_R R/(a = b = c = \pm 1)$ such that $t_3^0(1) = 1$, $t_3^0(s_1) = t_3^0(s_1 s_2) = 0$. This implies $t_3^0(s_1^\alpha s_2^\beta) = 0$ for all $\alpha, \beta \in \{-1, 1\}$. By theorem 6.26 we know that such a Markov trace exists, factoring through $\overline{F}_n$. Moreover, by induction we easily get that $t_{n+3}^0(1) = a^n \times (n+1)$, and therefore the corresponding link invariant takes the value $a^n \times (n+1)$ on the $(n+3)$-components unlink. Finally, we can easily identify its restriction to the subalgebra $\widetilde{TL}_n(0)$, as follows.

Proposition 7.5. *There exists a family of traces* $t_n : \widetilde{TL}_n(0) \to A_0$ *satisfying* $t_n(C) = a^n$, $t_1(1) = 0$, $t_n(1) = (n-2)a^{n+1}$ *if* $n \geq 2$, *that would coincide with the restriction of* t_n^0 *to* $\widetilde{TL}_n(0)$ *if* (t_n^0) *is well-defined.*

Proof. Letting $u_n = -a^n$,$v_1 = 0$, $v_n = (n-2)a^{n+1}$ in proposition 6.17 provides a family of traces t_n. We know prove that t_n necessarily coincides with the restriction of the putative Markov trace t_n^0. We have $t_n(1) = t_n^0(1)$ for $n \leq 3$ and, by proposition 6.3 (iii), $t_{n+3}^0(1) = 2at_{n+2}^0(1) - t_{n+1}^0(1)$ since, we know that t_n^0 factors through (F_n) by theorem 6.8. Because of this, we have $t_n^0(1) = t_n(1)$ for all n. By proposition 6.3 we also get $t_n(C) = t_n^0(C)$ for all n. From the relation $e_i = (s_i + s_i^{-1})/2 - a$ and the Markov property we get $t_n^0(e_i) = t_n(e_i)$ for all i. It is well-known that the Temperley-Lieb algebra admits a basis made of words in the e_i which has the property that the e_i of maximal index appears exactly once. Let us consider such a basis element of $\widetilde{TL}_n$ of the form Ae_iB with A, B words in the e_j for $j < i$. Expanding e_i as before, we get from the Markov property that $t_{n+1}^0(Ae_iB) = t_n^0(AB) - at_{n+1}^0(AB)$. By induction on the length of the words in the e_i's, we get from this formula that $t_{n+1}^0(Ae_iB) = 0$ as soon as A or B has length ≥ 1, and therefore t_n^0 coincides on t_n on every basis element of $\widetilde{TL}_n(0)$, which proves the claim.

□

7.4. **Tables.** We gather here some computations that we made of these two invariants, one of them (for $x = a$) being still conjectural.

Knot	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$
0_1			3_1	4	0	4_1	10	16
5_1	1	0	5_2	13	48	6_1	7	80
6_2	13	96	6_3	25	144	7_1	1	0
7_2	-2	112	7_3	10	160	7_4	7	224
7_5	25	288	7_6	25	336	7_7	31	416

Knot	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$
8_1	-2	160	8_2	1	240	8_3	13	288
8_4	-2	352	8_5	-8	384	8_6	25	528
8_7	13	480	8_8	13	624	8_9	25	576
8_{10}	16	672	8_{11}	28	720	8_{12}	37	816
8_{13}	22	832	8_{14}	37	960	8_{15}	40	1104
8_{16}	25	1152	8_{17}	49	1296	8_{18}	64	1936
8_{19}	-8	-48	8_{20}	-8	96	8_{21}	16	240

Knot	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$
9_1	4	0	9_2	7	224	9_3	1	336
9_4	7	416	9_5	1	528	9_6	4	720
9_7	13	816	9_8	1	960	9_9	13	960
9_{10}	31	1088	9_{11}	7	1040	9_{12}	22	1216
9_{13}	22	1360	9_{14}	10	1360	9_{15}	31	1520
9_{16}	16	1536	9_{17}	7	1472	9_{18}	37	1680
9_{19}	25	1680	9_{20}	13	1632	9_{21}	34	1840
9_{22}	10	1792	9_{23}	28	2016	9_{24}	40	1968
9_{25}	34	2224	9_{26}	37	2160	9_{27}	37	2352
9_{28}	40	2544	9_{29}	4	2544	9_{30}	46	2752
9_{31}	49	2976	9_{32}	49	3408	9_{33}	61	3648
9_{34}	67	4688	9_{35}	-14	720	9_{36}	-2	1312
9_{37}	34	2016	9_{38}	52	3264	9_{39}	46	3040
9_{40}	70	5536	9_{41}	-2	2416	9_{42}	-14	64
9_{43}	-14	112	9_{44}	-2	304	9_{45}	10	544
9_{46}	-11	80	9_{47}	13	656	9_{48}	37	704
9_{49}	34	640						

Knot	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$	*Knot*	$x=a$	$x=2a$
$3_1\#3_1$	16	64	$3_1\#3_1\#3_1$	10	704	$3_1\#3_1\#3_1\#3_1$	40	6528
$3_1\#4_1$	22	208	$4_1\#4_1$	28	608			

For knots of at most 10 crossings as well as small links we use the notations in Rolfsen's book, see [20]. For knots of 11 crossings we use the notation of the KnotScape software, and give at the same time a braid description in order not to avoid possible ambiguities. It can be checked that the knots 8_2 and 11_{373}, the latter one being the braid closure of $\bar{1}2\bar{1}2\bar{3}\bar{3}\bar{4}223\bar{4}\bar{4}$, have the same Alexander polynomial, but both invariants distinguish them (and so do the Homfly polynomial). The knots 10_{41} and 10_{94} have the same Jones polynomial, but our invariant for $x = 2a$ distinguishes them (4992 on 10_{41}, 4896 on 10_{94}), while we get 25 on both for $x = a$. We did not manage to find a pair of knots with the same Homfly invariant wich are distinguished by at least one of our two invariants. On the 11-crossings knots 11_{280} and 11_{439} with the same Kauffman polynomials, which are the braid closures of $112\bar{1}2\bar{3}\bar{2}1\bar{2}\bar{2}3\bar{4}3\bar{4}$ and $\bar{1}\bar{1}22\bar{1}2\bar{1}2\bar{3}2\bar{3}$, our invariants for $x = a$ are 85 and 82, and for $x = 2a$ they are 22176 and 22048. On the other hand, we computed our invariants on Kanenobu's example of 4 distinct knots, presented as the closure of 3-braids, with the same Homfly and Kauffman polynomial, see [11], example at the end of §3; our invariants cannot distinguish them, getting as value 84 for $x = a$ and 7341576506136599985 for $x = 2a$. One possibility is therefore that these invariants both depend on the HOMFLY polynomial, in a way still to be discovered.

$Rolfsen$	$Braid$	$x = a$	$x = 2a$	$Name$
2_1^2	$\bar{1}\bar{1}$	$3a$	0	Hopf link
4_1^2	$3\bar{2}1\bar{2}\bar{3}\bar{2}\bar{1}\bar{2}$	$6a$	$16a$	Solomon's knot
5_1^2	$\bar{2}1\bar{2}1\bar{2}$	$15a$	$48a$	Whitehead link
6_3^2	$3\bar{2}1\bar{2}3\bar{2}\bar{1}\bar{2}$	$21a$	$128a$	
7_1^3	$3\bar{2}\bar{1}23\bar{2}1\bar{2}3$	9	9	
6_2^3	$2\bar{1}2\bar{1}2\bar{1}$	33	225	Borromean link
6_3^3	$2\bar{1}21\bar{2}1$	-3	21	
8_1^4	$5\bar{4}\bar{3}21\bar{4}32\bar{4}3\bar{4}5\bar{4}3\bar{2}\bar{1}\bar{2}3\bar{4}$	6	61	
8_2^4	$54\bar{3}214324 3\bar{4}5\bar{4}3\bar{2}\bar{1}\bar{2}34$	0	-3	Whitehead link

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