A Poset View of the Major Index

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Abstract

We introduce the Major MacMahon map from $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{Z}[q]$, and show how this map interacts with the pyramid and bipyramid operators. When the Major MacMahon map is applied to the **ab**-index of a simplicial poset, it yields the q-analogue of $n!$ times the h-polynomial of the poset. Applying the map to the Boolean algebra gives the distribution of the major index on the symmetric group, a seminal result due to MacMahon. Similarly, when applied to the cross-polytope we obtain the distribution of one of the major indexes on signed permutations due to Reiner.

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1 Introduction

One hundred and one years ago in 1913 Major Percy Alexander MacMahon [9] (see also his collected works [11]) introduced the major index of a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of the multiset $M = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\}$ of size n to be the sum of the elements of its descent set, that is,

$$\mathrm{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i.$$

He showed that the distribution of this permutation statistic is given by the q-analogue of the multinomial Gaussian coefficient, that is, the following identity holds:

$$\sum_{\pi} q^{\mathrm{maj}(\pi)} = \frac{[n]!}{[\alpha_1]! \cdot [\alpha_2]! \cdots [\alpha_k]!} = \begin{bmatrix} n \\ \alpha \end{bmatrix}, \tag{1.1}$$

where π ranges over all permutations of the multiset M and α is the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Here $[n]! = [n] \cdot [n-1] \cdots [1]$ denotes the q-analogue of $n!$, where $[n] = 1 + q + \cdots + q^{n-1}$.

Many properties of the *descent set* of a permutation π, that is, $\mathrm{Des}(\pi) = \{i \,:\, \pi_i > \pi_{i+1}\}$, have been studied by encoding the set by its **ab**-word; see for instance [6, 12]. For a multiset

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permutation $\pi \in \mathfrak{S}_M$ the **ab**-*word* is given by $u(\pi) = u_1 u_2 \cdots u_{n-1}$, where $u_i = \mathbf{b}$ if $\pi_i > \pi_{i+1}$ and $u_i = \mathbf{a}$ otherwise.

Inspired by this definition, we introduce the *Major MacMahon map* Θ on the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ of non-commutative polynomials in the variables $\mathbf{a}$ and $\mathbf{b}$ to the ring $\mathbb{Z}[q]$ of polynomials in the variable q, by

$$\Theta(w) = \prod_{i \,:\, u_i = \mathbf{b}} q^i,$$

for a monomial $w = u_1 u_2 \cdots u_n$ and extend Θ to all of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by linearity. In short, the map Θ sends each variable $\mathbf{a}$ to 1 and the variables $\mathbf{b}$ to q to the power of its position, read from left to right. A Swedish example is $\Theta(\mathbf{abba}) = q^5$.

2 Chain enumeration and products of posets

Let P be a graded poset of rank $n+1$ with minimal element $\widehat{0}$, maximal element $\widehat{1}$ and rank function ρ. Let the rank difference be defined by $\rho(x, y) = \rho(y) - \rho(x)$. The *flag f-vector* entry f_S, for $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset $\{1, 2, \ldots, n\}$, is the number of chains $c = \{\widehat{0} = x_0 < x_1 < x_2 < \cdots < x_{k+1} = \widehat{1}\}$ such that the rank of the element x_i is s_i, that is, $\rho(x_i) = s_i$ for $1 \leq i \leq k$. The *flag h-vector* is defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

For a subset S of $\{1, 2, \ldots, n\}$ define two **ab**-polynomials of degree n by $u_S = u_1 u_2 \cdots u_n$ and $v_S = v_1 v_2 \cdots v_n$ by

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S, \end{cases} \quad \text{and} \quad v_i = \begin{cases} \mathbf{a} - \mathbf{b} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

The **ab**-*index* of the poset P is defined by the two equivalent expressions:

$$\Psi(P) = \sum_S f_S \cdot v_S = \sum_S h_S \cdot u_S,$$

where the two sums range over all subsets S of $\{1, 2, \ldots, n\}$. For more details on the **ab**-index, see [7] or the book [16, Section 3.17].

Recall that a graded poset P is *Eulerian* if every non-trivial interval has the same number of elements of even rank as odd rank. Equivalently, a poset is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in P. When the graded poset P is Eulerian then the **ab**-index $\Psi(P)$ can be written in terms of the non-commuting variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ and it is called the **cd**-index; see [2]. For an n-dimensional convex polytope V its face lattice $\mathscr{L}(V)$ is an Eulerian poset of rank $n+1$. In this case we write $\Psi(V)$ for the **ab**-index (**cd**-index) instead of the cumbersome $\Psi(\mathscr{L}(V))$.

There are also two products on graded posets that we will study. The first is the *Cartesian product*, defined by $P \times Q = \{(x, y) \,:\, x \in P, y \in Q\}$ with the order relation $(x, y) \leq_{P \times Q} (z, w)$ if

$x \leq_P z$ and $y \leq_Q w$. Note that the rank of the Cartesian product of two graded posets of ranks m and n is $m+n$. As a special case we define $\text{Pyr}(P) = P \times B_1$, where B_1 is the Boolean algebra of rank 1. The geometric reason for the notation Pyr is that this operation corresponds to the geometric operation of taking the pyramid of a polytope, that is, $\mathscr{L}(\text{Pyr}(V)) = \text{Pyr}(\mathscr{L}(V))$ for a polytope V.

The second product is the *dual diamond product*, defined by

$$P \diamond^* Q = (P - \{\widehat{1}_P\}) \times (Q - \{\widehat{1}_Q\}) \cup \{\widehat{1}\}.$$

The rank of the product $P \diamond^* Q$ is the sum of the ranks of P and Q minus one. This is the dual to the diamond product $\diamond$ defined by removing the minimal elements of the posets, taking the Cartesian product and then adjoining a new minimal element. The product $\diamond$ behaves well with the quasi-symmetric functions of type B. (See Sections 5 and 6.) However, we will dualize our presentation and keep working with the product $\diamond^*$.

Yet again, we have an important special case. We define $\text{Bipyr}(P) = P \diamond^* B_2$. The geometric motivation is the connection to the bipyramid of a polytope, that is, $\mathscr{L}(\text{Bipyr}(V)) = \text{Bipyr}(\mathscr{L}(V))$ for a polytope V.

3 Pyramids and bipyramids

Define on the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ of non-commutative polynomials in the variables $\mathbf{a}$ and $\mathbf{b}$ the two derivations G and D by

$$\begin{aligned} G(1) = 0, \quad G(\mathbf{a}) = \mathbf{ba}, \quad G(\mathbf{b}) = \mathbf{ab}, \\ D(1) = 0, \quad D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba}. \end{aligned}$$

Extend these two derivations to all of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by linearity. The *pyramid* and the *bipyramid operators* are given by

$$\text{Pyr}(w) = G(w) + w \cdot \mathbf{c} \quad \text{and} \quad \text{Bipyr}(w) = D(w) + \mathbf{c} \cdot w.$$

These two operators are suitably named, since for a graded poset P we have

$$\Psi(\text{Pyr}(P)) = \text{Pyr}(\Psi(P)) \quad \text{and} \quad \Psi(\text{Bipyr}(P)) = \text{Bipyr}(\Psi(P)).$$

For further details, see [7].

Theorem 3.1. *The Major MacMahon map Θ interacts with right multiplication by $\mathbf{c}$, the derivation G, the pyramid and the bipyramid operators as follows:*

$$\Theta(w \cdot \mathbf{c}) = (1 + q^{n+1}) \cdot \Theta(w), \tag{3.1}$$
$$\Theta(G(w)) = q \cdot [n] \cdot \Theta(w), \tag{3.2}$$
$$\Theta(\text{Pyr}(w)) = [n+2] \cdot \Theta(w), \tag{3.3}$$
$$\Theta(\text{Bipyr}(w)) = [2] \cdot [n+1] \cdot \Theta(w), \tag{3.4}$$

where w is a homogeneous $\mathbf{ab}$-polynomial of degree n.

Proof. It is enough to prove the four identities for an **ab**-monomial w of degree n. Directly we have that $\Theta(w \cdot \mathbf{a}) = \Theta(w)$ and $\Theta(w \cdot \mathbf{b}) = q^{n+1} \cdot \Theta(w)$. Adding these two identities yields equation (3.1).

Assume that w consists of k **b**'s. We label the n letters of w as follows: The k **b**'s are labeled 1 through k reading from right to left, whereas the $n-k$ **a**'s are labeled $k+1$ through n reading left to right. As an example, the word $w = \mathbf{aababba}$ is written as $w_4 w_5 w_3 w_6 w_2 w_1 w_7$.

Identity (3.2) is a consequence of the following claim. Applying the derivation G only to the letter w_i and then applying the Major MacMahon map yields $q^i \cdot \Theta(w)$, that is,

$$\Theta(u \cdot G(w_i) \cdot v) = q^i \cdot \Theta(u \cdot w_i \cdot v), \tag{3.5}$$

where w is factored as $u \cdot w_i \cdot v$. To see this, first consider when $1 \leq i \leq k$. There are i **b**'s to the right of w_i including w_i itself. They each are shifted one step to the right when replacing $w_i = \mathbf{b}$ with $G(\mathbf{b}) = \mathbf{ab}$ and hence we gain a factor of q^i. The second case is when $k+1 \leq i \leq n$. Then w_i is an **a** and is replaced by **ba** under the derivation G. Assume that there are j **b**'s to the right of w_i. When these j **b**'s are shifted one step to the right they contribute a factor of q^j. We also create a new **b**. It has $i-k-1$ **a**'s to the left and $k-j$ **b**'s to the left. Hence the position of the new **b** is $(i-k-1)+(k-j)+1 = i-j$ and thus its contribution is q^{i-j}. Again the factor is given by $q^j \cdot q^{i-j} = q^i$, proving the claim. Now by summing over these n cases, identity (3.2) follows. Identity (3.3) is the sum of identities (3.1) and (3.2).

To prove identity (3.4), we use a different labeling of the monomial w. This time label the k **b**'s with the subscripts 0 through $k-1$, rather than 1 through k. That is, in our example $w = \mathbf{aababba}$ is now labeled as $w_4 w_5 w_2 w_6 w_1 w_0 w_7$. We claim that for $w = u \cdot w_i \cdot v$ we have that

$$\Theta(u \cdot D(w_i) \cdot v) = q^i \cdot [2] \cdot \Theta(w).$$

The first case is $0 \leq i \leq k-1$. Then $w_i = \mathbf{b}$ has i **b**'s to its right. Thus when replacing **b** with **ba** there are i **b**'s that are shifted one step, giving the factor q^i. Similarly, when replacing w_i with **ab**, there are $i+1$ **b**'s that are shifted one step, giving the factor q^{i+1}. The sum of the two factors is $q^i \cdot [2]$. The second case is $k+1 \leq i \leq n$. It is as the second case above when replacing w_i with **ba**, yielding the factor q^i. When replacing w_i with **ab** there is one more shift, giving q^{i+1}. Adding these two subcases completes the proof of the claim.

It is straightforward to observe that

$$\Theta(\mathbf{c} \cdot w) = q^k \cdot [2] \cdot \Theta(w).$$

Calling this the case $i = k$, the identity (3.4) follows by summing the $n+1$ cases $0 \leq i \leq n$. ◻

Iterating equations (3.3) and (3.4) we obtain that the Major MacMahon map of the **ab**-index of the n-dimensional simplex Δ_n and the n-dimensional cross-polytope C_n^*.

Corollary 3.2. *The n-dimensional simplex Δ_n and the n-dimensional cross-polytope C_n^* satisfy*

$$\Theta(\Psi(\Delta_n)) = [n+1]!,$$
$$\Theta(\Psi(C_n^*)) = [2]^n \cdot [n]!.$$

4 Simplicial posets

A graded poset P is *simplicial* if all of its lower order intervals are Boolean, that is, for all elements $x < \widehat{1}$ the interval $[\widehat{0}, x]$ is isomorphic to the Boolean algebra $B_{\rho(x)}$. It is well-known that all the flag information of a simplicial poset of rank $n+1$ is contained in the f-vector $(f_0, f_1, \ldots, f_n)$, where $f_0 = 1$ and $f_i = f_{\{i\}}$ for $1 \leq i \leq n$. The h-vector, equivalently, the h-polynomial $h(P) = h_0 + h_1 \cdot q + \cdots + h_n \cdot q^n$ of a simplicial poset P, is defined by the polynomial relation

$$h(q) = \sum_{i=0}^{n} f_i \cdot q^i \cdot (1-q)^{n-i}.$$

See for instance [19, Section 8.3]. The h-polynomial and the bipyramid operation interact as follows:

$$h(\text{Bipyr}(P)) = (1+q) \cdot h(P).$$

We can now evaluate the Major MacMahon map on the **ab**-index of a simplicial poset.

Theorem 4.1. *For a simplicial poset P of rank $n+1$ the following identity holds:*

$$\Theta(\Psi(P)) = [n]! \cdot h(P). \tag{4.1}$$

Proof. Let $B_n \cup \{\widehat{1}\}$ denote the Boolean algebra B_n with a new maximal element added. Note that $B_n \cup \{\widehat{1}\}$ is indeed a simplicial poset and its h-polynomial is 1. Furthermore, equation (4.1) holds for $B_n \cup \{\widehat{1}\}$ since

$$\Theta(\Psi(B_n \cup \{\widehat{1}\})) = \Theta(\Psi(B_n) \cdot \mathbf{a}) = \Theta(\Psi(B_n)) = [n]! = [n]! \cdot h(B_n \cup \{\widehat{1}\}).$$

Also, if (4.1) holds for a poset P then it also holds for $\text{Bipyr}(P)$, since we have

$$\Theta(\Psi(\text{Bipyr}(P))) = [2] \cdot [n+1] \cdot \Theta(\Psi(P)) = [2] \cdot [n+1] \cdot [n]! \cdot h(P) = [n+1]! \cdot h(\text{Bipyr}(P)).$$

Observe that both sides of (4.1) are linear in the h-polynomial. Hence to prove it for any simplicial poset P it is enough to prove it for a basis of the span of all simplicial posets of rank $n+1$. Such a basis is given by the posets

$$\mathcal{B}_n = \left\{\text{Bipyr}^i(B_{n-i} \cup \{\widehat{1}\})\right\}_{0 \leq i \leq n}.$$

This is a basis since the polynomials $h(\text{Bipyr}^i(B_{n-i} \cup \{\widehat{1}\})) = (1+q)^i$, for $0 \leq i \leq n$, are a basis for polynomials in the variable q of degree at most n.

Finally, since every element in the basis is built up by iterating bipyramids of the posets $B_n \cup \{\widehat{1}\}$, the theorem holds for all simplicial posets. $\square$

Observe that the poset $\text{Bipyr}^i(B_{n-i} \cup \{\widehat{1}\})$ is the face lattice of the simplicial complex consisting of the 2^i facets of the n-dimensional cross-polytope in the cone $x_1, \ldots, x_{n-i} \geq 0$.

For an Eulerian simplicial poset P, the h-vector is symmetric, that is, $h_i = h_{n-i}$. In other words, the h-polynomial is palindromic. Stanley [15] introduced the *simplicial shelling components*, that is, the **cd**-polynomials $\check{\Phi}_{n,i}$ such that the **cd**-index of an Eulerian simplicial poset P of rank $n+1$ is given by

$$\Psi(P) = \sum_{i=0}^{n} h_i \cdot \check{\Phi}_{n,i}. \tag{4.2}$$

These **cd**-polynomials satisfy the recursion $\check{\Phi}_{n,0} = \Psi(B_n) \cdot \mathbf{c}$ and $\check{\Phi}_{n,i} = G(\check{\Phi}_{n-1,i-1})$; see [7, Section 8]. The Major MacMahon map of these polynomials is described by the next result.

Corollary 4.2. *The Major MacMahon map of the simplicial shelling components is given by*

$$\Theta(\check{\Phi}_{n,i}) = q^i \cdot [2(n-i)] \cdot [n-1]!.$$

Proof. When $i = 0$ we have $\Theta(\check{\Phi}_{n,0}) = \Theta(\Psi(B_n) \cdot \mathbf{c}) = (1 + q^n) \cdot [n]! = [2n] \cdot [n-1]!$. Also when $i \geq 1$ we obtain $\Theta(\check{\Phi}_{n,i}) = \Theta(G(\check{\Phi}_{n-1,i-1})) = q \cdot [n-1] \cdot \Theta(\check{\Phi}_{n-1,i-1}) = q^i \cdot [2(n-i)] \cdot [n-1]!$. □

We end with the following observation.

Theorem 4.3. *For an Eulerian poset P of rank $n+1$, the polynomial $[2]^{\lceil n/2 \rceil}$ divides $\Theta(\Psi(P))$.*

Proof. It is enough to show this result for a **cd**-monomial w of degree n. A **c** in an odd position i of w yields a factor of $1 + q^i$. A **d** that covers an odd position i of w yields either $q^{i-1} + q^i$ or $q^i + q^{i+1}$. Each of these polynomials contributes a factor of $1 + q$. The result follows since there are $\lceil n/2 \rceil$ odd positions. □

5 The Cartesian product of posets

We now study how the Major MacMahon map behaves under the Cartesian product. Recall that for a graded poset P the **ab**-index $\Psi(P)$ encodes the flag f-vector information of the poset P. There is another encoding of this information as a quasi-symmetric function. For further information about quasi-symmetric functions, see [17, Section 7.19].

A composition α of n is a list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. Let $\mathrm{Comp}(n)$ denote the set of compositions of n. There are three natural bijections between **ab**-monomials u of degree n, subsets S of the set $\{1, 2, \ldots, n\}$ and compositions of $n+1$. Given a composition $\alpha \in \mathrm{Comp}_{n+1}$ we have the subset S_α, the **ab**-monomial u_α and the **ab**-polynomial v_α defined by

$$\begin{aligned} S_\alpha &= \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}, \\ u_\alpha &= \mathbf{a}^{\alpha_1 - 1} \cdot \mathbf{b} \cdot \mathbf{a}^{\alpha_2 - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot \mathbf{a}^{\alpha_k - 1}, \\ v_\alpha &= (\mathbf{a} - \mathbf{b})^{\alpha_1 - 1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\alpha_2 - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\alpha_k - 1}. \end{aligned}$$

For S a subset of $\{1, 2, \ldots, n\}$ let $\mathrm{co}(S)$ denote associated composition.

The *monomial quasi-symmetric function* M_α is defined as the sum

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} t_{i_1}^{\alpha_1} \cdot t_{i_2}^{\alpha_2} \cdots t_{i_k}^{\alpha_k}.$$

A second basis is given by the *fundamental quasi-symmetric function* L_α defined as

$$L_\alpha = \sum_{S_\alpha \subseteq T \subseteq \{1,2,\ldots,n\}} M_{\mathrm{co}(T)}.$$

Following [8] define an injective linear map $\gamma : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathrm{QSym}$ by

$$\gamma(v_\alpha) = M_\alpha,$$

for a composition α of $n \geq 1$. The image of γ is all quasi-symmetric functions without constant term. Moreover, the image of the **ab**-monomial u_α under γ is the fundamental quasi-symmetric function L_α, that is,

$$\gamma(u_\alpha) = L_\alpha.$$

Another way to encode the flag vectors of a poset P is by the *quasi-symmetric function* of the poset. It is quickly defined as $F(P) = \gamma(\Psi(P))$. A more poset-oriented definition is the following limit of sums over multichains:

$$F(P) = \lim_{k \longrightarrow \infty} \sum_{\widehat{0} = x_0 \leq x_1 \leq \cdots \leq x_k = \widehat{1}} t_1^{\rho(x_0,x_1)} \cdot t_2^{\rho(x_1,x_2)} \cdots t_k^{\rho(x_{k-1},x_k)}.$$

For more on the quasi-symmetric function of a poset, see [5].

The *stable principal specialization* of a quasi-symmetric function is the substitution $\mathrm{ps}(f) = f(1, q, q^2, \ldots)$. Note that this is a homeomorphism, that is, $\mathrm{ps}(f \cdot g) = \mathrm{ps}(f) \cdot \mathrm{ps}(g)$.

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ let α^* denote the reverse composition, that is, $\alpha^* = (\alpha_k, \ldots, \alpha_2, \alpha_1)$. This involution extends to an anti-automorphism on QSym by $M_\alpha^* \longmapsto M_{\alpha^*}$. Define ps^* by the relation $\mathrm{ps}^*(f) = \mathrm{ps}(f^*)$. Informally speaking, this corresponds to the substitution $\mathrm{ps}^*(f) = f(\ldots, q^2, q, 1)$.

Theorem 5.1. *For a homogeneous* **ab***-polynomial w of degree $n-1$ the Major MacMahon map is given by*

$$\Theta(w) = (1-q)^n \cdot [n]! \cdot \mathrm{ps}^*(\gamma(w)). \tag{5.1}$$

For a poset P of rank n this identity is

$$\Theta(\Psi(P)) = (1-q)^n \cdot [n]! \cdot \mathrm{ps}^*(F(P)). \tag{5.2}$$

Proof. It is enough to prove identity (5.1) for an **ab**-monomial w of degree $n-1$. Let α be the composition of n corresponding to the reverse monomial w^*. Furthermore, let $e(\alpha)$ be the sum $\sum_{i \in S_\alpha} (n-i)$. Note that $e(\alpha)$ is in fact the sum $\sum_{i \in S} i$, where S is the subset associated with the **ab**-monomial w. That is, we have $q^{e(\alpha)} = \Theta(w)$. Equation (5.1) follows from Lemma 7.19.10 in [17]. By applying the first identity to $\Psi(P)$, we obtain identity (5.2). ◻

Since the quasi-symmetric function is multiplicative under the Cartesian product, we have the next result.

Theorem 5.2. *For two posets P and Q of ranks m, respectively n, the following identity holds:*

$$\Theta(\Psi(P \times Q)) = \begin{bmatrix} m+n \\ n \end{bmatrix} \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \tag{5.3}$$

Proof. The proof is a direct verification as follows:

$$\begin{aligned} \Theta(\Psi(P \times Q)) &= (1-q)^{m+n} \cdot [m+n]! \cdot \mathrm{ps}(F(P^* \times Q^*)) \\ &= \begin{bmatrix} m+n \\ m \end{bmatrix} \cdot (1-q)^{m+n} \cdot [m]! \cdot [n]! \cdot \mathrm{ps}(F(P^*)) \cdot \mathrm{ps}(F(Q^*)) \\ &= \begin{bmatrix} m+n \\ m \end{bmatrix} \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \end{aligned}$$

□

6 The dual diamond product

Define the *quasi-symmetric function of type* B^* of a graded poset P to be the expression

$$F_{B^*}(P) = \sum_{\widehat{0} \leq x < \widehat{1}} F([\widehat{0}, x]) \cdot s^{\rho(x,\widehat{1})-1}.$$

This is an element of the algebra $\mathrm{QSym} \otimes \mathbb{Z}[s]$ which we view as the quasi-symmetric functions of type B^*. We view QSym_{B^*} as a subalgebra of $\mathbb{Z}[t_1, t_2, \ldots; s]$, which is quasi-symmetric in the variables $t_1, t_2, \ldots$. For instance, a basis for QSym_{B^*} is given by $M_\alpha \cdot s^i$ where α ranges over all compositions and i over all non-negative integers. Similar to the map $\gamma : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathrm{QSym}$, we define $\gamma_{B^*} : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathrm{QSym}_{B^*}$ by

$$\gamma_{B^*}\left((\mathbf{a}-\mathbf{b})^{\alpha_1 - 1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\alpha_2 - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\alpha_k - 1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{p}\right) = M_\alpha \cdot s^p,$$

where α is the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Similar to the relation $\gamma(\Psi(P)) = F(P)$, we have

$$\gamma_{B^*}(\Psi(P)) = F_{B^*}(P).$$

Furthermore, the type B^* quasi-symmetric function F_{B^*} is multiplicative respect to the product $\diamond^*$, that is, $F_{B^*}(P \diamond^* Q) = F_{B^*}(P) \cdot F_{B^*}(Q)$; see [8, Theorem 13.3].

Let f be a homogeneous quasi-symmetric function such that $f \cdot s^j$ is a quasi-symmetric function of type B^*. Define the *stable principal specialization* of the quasi-symmetric function $f \cdot s^j$ of type B^* to be $\mathrm{ps}_{B^*}(f \cdot s^j) = q^{\deg(f)} \cdot \mathrm{ps}^*(f)$, where $\mathrm{ps}^*(f) = \mathrm{ps}(f^*)$. This is the substitution $s = 1$, $t_k = q$, $t_{k-1} = q^2$, … as k tends to infinity, since $f(\ldots, q^3, q^2, q) = q^{\deg(f)} \cdot f(\ldots, q^2, q, 1)$. Especially, for a graded poset P we have

$$\mathrm{ps}_{B^*}(F_{B^*}(P)) = \sum_{\widehat{0} \leq x < \widehat{1}} q^{\rho(x)} \cdot \mathrm{ps}^*(F([\widehat{0}, x])). \tag{6.1}$$

Theorem 6.1. *For a graded poset P of rank $n+1$ the relationship between the Major MacMahon map and the stable principal specialization of type B^* is given by*

$$\Theta(\Psi(P)) = (1-q)^n \cdot [n]! \cdot \mathrm{ps}_{B^*}(F_{B^*}(P^*)). \tag{6.2}$$

Especially, for a homogeneous **ab***-polynomial w of degree n the Major MacMahon map is given by*

$$\Theta(w) = (1-q)^n \cdot [n]! \cdot \mathrm{ps}_{B^*}(\gamma_{B^*}(w^*)). \tag{6.3}$$

Proof. For the poset P we have

$$\begin{aligned}
\mathrm{ps}^*(F(P)) &= \lim_{k\to\infty} \sum_{\widehat{0}=x_0\leq x_1\leq\cdots\leq x_k=\widehat{1}} \left(q^{k-1}\right)^{\rho(x_0,x_1)} \cdots \left(q^2\right)^{\rho(x_{k-3},x_{k-2})} \cdot q^{\rho(x_{k-2},x_{k-1})} \cdot 1^{\rho(x_{k-1},x_k)} \\
&= \lim_{k\to\infty} \sum_{\widehat{0}=x_0\leq x_1\leq\cdots\leq x_k=\widehat{1}} q^{\rho(x_{k-1})} \cdot \left(q^{k-2}\right)^{\rho(x_0,x_1)} \cdots q^{\rho(x_{k-3},x_{k-2})} \cdot 1^{\rho(x_{k-2},x_{k-1})} \\
&= \sum_{\widehat{0}\leq x\leq\widehat{1}} q^{\rho(x)} \cdot \mathrm{ps}^*(F([\widehat{0},x])) \\
&= \sum_{\widehat{0}\leq x<\widehat{1}} q^{\rho(x)} \cdot \mathrm{ps}^*(F([\widehat{0},x])) + q^{n+1} \cdot \mathrm{ps}^*(F(P)).
\end{aligned}$$

Rearranging terms yields

$$\begin{aligned}
\sum_{\widehat{0}\leq x<\widehat{1}} q^{\rho(x)} \cdot \mathrm{ps}^*(F([\widehat{0},x])) &= (1-q^{n+1}) \cdot \mathrm{ps}^*(F(P)) \\
&= (1-q^{n+1}) \cdot \mathrm{ps}(F(P^*)) \\
&= (1-q^{n+1}) \cdot \frac{\Theta(\Psi(P))}{(1-q)^{n+1} \cdot [n+1]!} \\
&= \frac{\Theta(\Psi(P))}{(1-q)^n \cdot [n]!}.
\end{aligned}$$

Combining the last identity with (6.1) yields the desired result. □

Theorem 6.2. *For two graded posets P and Q of ranks $m+1$, respectively $n+1$, the identity holds:*

$$\Theta(\Psi(P \diamond^* Q)) = \begin{bmatrix} m+n \\ n \end{bmatrix} \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \tag{6.4}$$

Proof. The proof is a direct verification as follows:

$$\begin{aligned}
\Theta(\Psi(P \diamond^* Q)) &= (1-q)^{m+n} \cdot [m+n]! \cdot \mathrm{ps}_{B^*}(F_{B^*}(P^* \diamond^* Q^*)) \\
&= \begin{bmatrix} m+n \\ m \end{bmatrix} \cdot (1-q)^{m+n} \cdot [m]! \cdot [n]! \cdot \mathrm{ps}_{B^*}(F_{B^*}(P^*)) \cdot \mathrm{ps}_{B^*}(F_{B^*}(Q^*)) \\
&= \begin{bmatrix} m+n \\ m \end{bmatrix} \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).
\end{aligned}$$

□

7 Permutations

One connection between permutations and posets is via the concept of R-labelings. For more details, see [16, Section 3.14]. Let $\mathcal{E}(P)$ be the set of all cover relations of P, that is, $\mathcal{E}(P) = \{(x,y) \in P^2 : x \prec y\}$. A graded poset P has an *R-labeling* if there is a map $\lambda : \mathcal{E}(P) \longrightarrow \Lambda$, where Λ is a linearly ordered set, such that in every interval $[x,y]$ in P there is a unique maximal chain $c = \{x = x_0 \prec x_1 \prec \cdots \prec x_k = y\}$ such that $\lambda(x_0,x_1) \leq \lambda(x_1,x_2) \leq \cdots\cdots \leq \lambda(x_{k-1},x_k)$.

For a maximal chain c in the poset P of rank n, let $\lambda(c)$ denote the list $(\lambda(x_0,x_1), \lambda(x_1,x_2), \ldots, \lambda(x_{k-1},x_k))$. The *Jordan–Hölder set* of P, denoted by $JH(P)$, is the set of all the lists $\lambda(c)$ where c ranges over all maximal chains of P. The descent set of a list of labels $\lambda(c)$ is the set of positions where there are descents in the list. Similarly, we define the descent word of $\lambda(c)$ to be $u_{\lambda(c)} = u_1 u_2 \cdots u_{n-1}$ where $u_i = \mathbf{b}$ if $\lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})$ and $u_i = \mathbf{a}$ otherwise.

The bridge between posets and permutations is given by the next result.

Theorem 7.1. *For an R-labeling λ of a graded poset P we have that*

$$\Psi(P) = \sum_{c} u_{\lambda(c)},$$

where the sum is over the Jordan–Hölder set $JH(P)$.

This is a reformulation of a result of Björner and Stanley [3, Theorem 2.7]. The reformulation can be found in [6, Lemma 3.1].

As a corollary we obtain MacMahon's classical result on the major index on a multiset; see [9]. For a composition α of n let $\mathfrak{S}_\alpha$ denote all the permutations of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\}$.

Corollary 7.2 (MacMahon)**.** *For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n the following identity holds:*

$$\sum_{\pi \in \mathfrak{S}_\alpha} q^{\mathrm{maj}(\pi)} = \frac{[n]!}{[\alpha_1]! \cdot [\alpha_2]! \cdots [\alpha_k]!}.$$

Proof. Let P_i denote the chain of rank α_i for $i = 1, \ldots, k$. Furthermore, label all the cover relations in P_i with i. Let L denote the distributive lattice $P_1 \times P_2 \times \cdots \times P_k$. Furthermore, let L inherit an R-labeling from its factors, that is, if $x = (x_1, x_2, \ldots, x_k) \prec (y_1, y_2, \ldots, y_k) = y$ let the label $\lambda(x,y)$ be the unique coordinate i such that $x_i \prec y_i$. Observe that the Jordan–Hölder set of L is $\mathfrak{S}_\alpha$. Direct computation yields $\Psi(P_i) = \mathbf{a}^{\alpha_i - 1}$, so the Major MacMahon map is $\Theta(\Psi(P_i)) = 1$. Iterating Theorem 5.2 evaluates the Major MacMahon map on L:

$$\sum_{\pi \in \mathfrak{S}_\alpha} q^{\mathrm{maj}(\pi)} = \Theta\left(\sum_{\pi \in \mathfrak{S}_\alpha} u(\pi)\right) = \Theta\left(\Psi(L)\right) = \begin{bmatrix} n \\ \alpha \end{bmatrix}. \qquad \square$$

For a vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ of positive integers let an $\mathbf{r}$-signed permutation be a list $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1}) = ((j_1, \pi_1), (j_2, \pi_2), \ldots, (j_n, \pi_n), 0)$ such that $\pi_1 \pi_2 \cdots \pi_n$ is a permutation in the symmetric group $\mathfrak{S}_n$ and the sign j_i is from the set $S_{\pi_i} = \{-1\} \cup \{2, \ldots, r_{\pi_i}\}$. On the set of labels

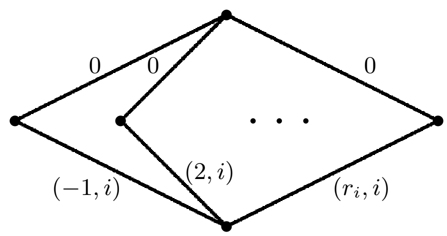

Figure 1: The poset P_i with its R-labeling used in the proof of Corollary 7.3.

$\Lambda = \{(j,i) : 1 \leq i \leq n, j \in S_i\} \cup \{0\}$ we use the lexicographic order with the extra condition that $0 < (j,i)$ if and only if $0 < j$. Denote the set of $\mathbf{r}$-signed permutations by $\mathfrak{S}_n^{\mathbf{r}}$. The descent set of an $\mathbf{r}$-signed permutation σ is the set $\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and the major index is defined as $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$. Similar to Corollary 7.2, we have the following result.

Corollary 7.3. *The distribution of the major index for* $\mathbf{r}$*-signed permutations is given by*

$$\sum_{\sigma \in \mathfrak{S}_n^{\mathbf{r}}} q^{\mathrm{maj}(\sigma)} = [n]! \cdot \prod_{i=1}^{n} (1 + (r_i - 1) \cdot q).$$

Proof. The proof is the same as Corollary 7.2 except we replace the chains with the posets P_i in Figure 1. Note that $\Psi(P_i) = \mathbf{a} + (r_i - 1) \cdot \mathbf{b}$. Let L be the lattice $L = P_1 \diamond^* P_2 \diamond^* \cdots \diamond^* P_n$. Let L inherit the labels of the cover relations from its factors with the extra condition that the cover relations attached to the maximal element receive the label 0. This is an R-labeling and the labels of the maximal chains are exactly the $\mathbf{r}$-signed permutations. □

For signed permutations, that is, $\mathbf{r} = (2, 2, \ldots, 2)$, the above result follows from an identity due to Reiner [13, Equation (5)].

8 Concluding remarks

We suggest the following q,t-extension of the Major MacMahon map Θ. Define $\Theta^{q,t} : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}[q,t]$ by

$$\Theta^{q,t}(w) = \Theta(w) \cdot w_{|\mathbf{a}=1, \mathbf{b}=t} = \prod_{i \,:\, u_i = \mathbf{b}} q^i \cdot t, \tag{8.1}$$

for an $\mathbf{ab}$-monomial $w = u_1 u_2 \cdots u_n$. Applying this map to the $\mathbf{ab}$-index of the Boolean algebra yields one of the four types of q-Eulerian polynomials:

$$\Theta^{q,t}(\Psi(B_n)) = A_n^{\mathrm{maj},\mathrm{des}}(q,t) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)}.$$

The following identity has been attributed to Carlitz [4], but goes back to MacMahon [10, Volume 2, Chapter IV, §462],

$$\sum_{k \geq 0} [k+1]^n \cdot t^k = \frac{A_n^{\mathrm{maj},\mathrm{des}}(q,t)}{\prod_{j=0}^{n} (1 - t \cdot q^j)}. \tag{8.2}$$

For recent work on the q-Eulerian polynomials, see Shareshian and Wachs [14]. It is natural to ask if there is a poset approach to identity (8.2).

In the second half of Section 7, before Corollary 7.3, we offer one way to define a major index for signed permutations. However, there are several different ways to extend the major index to signed permutations. Two of our favorites are [1, 18].

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