

Optimal Output Feedback Architecture for Triangular LQG Problems

Takashi Tanaka and Pablo A. Parrilo

Abstract—Distributed control problems under some specific information constraints can be formulated as (possibly infinite dimensional) convex optimization problems. The underlying motivation of this work is to develop an understanding of the optimal decision making architecture for such problems. In this paper, we particularly focus on the N -player triangular LQG problems and show that the optimal output feedback controllers have attractive state space realizations. The optimal controller can be synthesized using a set of stabilizing solutions to $2N$ linearly coupled algebraic Riccati equations, which turn out to be easily solvable under reasonable assumptions.

NOMENCLATURE

- $\mathbf{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := C(sI - A)^{-1}B + D$ represents a proper rational transfer function matrix.
- The space of matrix valued functions \mathbf{G} such that $\|\mathbf{G}\| \triangleq \sqrt{\langle \mathbf{G}, \mathbf{G} \rangle} < \infty$ is denoted by L_2 , where $\langle \mathbf{G}_1, \mathbf{G}_2 \rangle \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \mathbf{G}_1^*(j\omega) \mathbf{G}_2(j\omega) d\omega$. L_2 can be written as $L_2 = H_2 \oplus H_2^\perp$, where H_2 is the subspace of functions \mathbf{G} analytic in $\text{Re}(s) > 0$. The H_2 norm of a function $\mathbf{G} \in H_2$ can be computed using the same formula $\|\mathbf{G}\| \triangleq \sqrt{\langle \mathbf{G}, \mathbf{G} \rangle}$. The Hardy space H_∞ will be also used in this paper.
- Let I_n be the $n = n_1 + \dots + n_N$ dimensional identity matrix. Denote by E^i the i -th block column of I_n , and by E_i the i -th block row of I_n . We also write $E^{\uparrow i}$ to denote the first $n_1 + \dots + n_i$ columns of I_n , while $E^{\downarrow i}$ is the last $n_i + \dots + n_N$ columns of I_n . Similarly, $E_{\uparrow i}$ denotes the first $n_1 + \dots + n_i$ rows of I_n , while $E_{\downarrow i}$ is the last $n_i + \dots + n_N$ rows of I_n . For a general matrix M , shorthand notations such as $M_{\uparrow i} = E_{\uparrow i} M$, $M^{\uparrow i} = M E^{\uparrow i}$, $M_{\uparrow i}^{\downarrow j} = E_{\uparrow i} M E^{\downarrow j}$ will be also used.
- \mathcal{I} specifies a sparsity structure of matrices, or matrix-valued functions. If M has $N \times N$ subblocks, $M \in \mathcal{I}_{LBT}$ means that M is lower block triangular. If $M \in \mathcal{I}_{\downarrow i}^{\uparrow i}$, then M has nonzero components only in the $(\cdot)^{\uparrow i}_{\downarrow i}$ subblock. If $M \in \mathcal{I}_i^j$, only (i, j) -th subblock can be nonzero.
- $(X, K) = ARE_p(A, B, F, H)$ represents a solution of algebraic Riccati equation $A^T X + X A - (X B + F^T H) \Psi^{-1} (X B + F^T H)^T + F^T F = 0$ with $K = -\Psi^{-1} (X B + F^T H)^T$ where $\Psi \triangleq H^T H$. Similarly, $(Y, L) = ARE_d(A, C, W, V)$ represents a solution of $A Y + Y A^T - (C Y + V W^T)^T \Phi^{-1} (C Y + V W^T) + W W^T = 0$ with $L = -(C Y + V W^T)^T \Phi^{-1}$ where $\Phi = V V^T$.
- $\text{row}\{M_1, \dots, M_n\} \triangleq [M_1 \ \dots \ M_n]$, $\text{col}\{M_1, \dots, M_n\} \triangleq [M_1^T \ \dots \ M_n^T]^T$.

I. INTRODUCTION

It is widely recognized that tractability of distributed control problems greatly depends on the information structure underlying the problem. If the information structure is arbitrary, the problem can be hopelessly hard as demonstrated by an iconic example by Witsenhausen in 1968. In contrast, many tractability results initiated by Ho and Chu [1] suggest that distributed control problems seem much more accessible when decision makers form a hierarchy in terms of their ability to observe and control the physical system. Currently, a unification via the *quadratic invariance (QI)* introduced by [7] is known to capture a wide class of distributed control problems that can be formulated as (infinite dimensional) convex optimization problems. Unfortunately, the QI framework does not immediately lead us to an explicit form of the optimal solution, and as a result, state space realizations of the optimal controllers remain unknown for many QI optimal control problems. This paper derives a state space realization of the solution to the triangular LQG problem, which is a special case but an important instance of the QI optimal control problems.

The triangular LQG problem is formulated as follows. Suppose that the transfer function of the system to be controlled is given by

$$\mathbf{G} = \left[\begin{array}{cc|cc} \mathbf{G}_{11} & \mathbf{G}_{12} & W & B \\ \mathbf{G}_{21} & \mathbf{G}_{22} & 0 & H \\ \hline C & V & 0 & 0 \end{array} \right] \triangleq \left[\begin{array}{c|cc} A & W & B \\ F & 0 & H \\ \hline C & V & 0 \end{array} \right].$$

Matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ are partitioned according to $n = n_1 + \dots + n_N$, $m = m_1 + \dots + m_N$, $p = p_1 + \dots + p_N$, and $A, B, C \in \mathcal{I}_{LBT}$ with respect to this partitioning. The injected noise w , performance output z , control input u , and observation output y are related by $\text{col}\{z, y\} = \mathbf{G} \text{col}\{w, u\}$. A controller transfer function $\mathbf{K} \in \mathcal{I}_{LBT}$ needs to be designed so that $u = \mathbf{K}y$ minimizes the H_2 norm of the closed loop transfer function from w to z .

Problem 1: Find a state space realization of the optimal solution \mathbf{K}_{opt} to the problem:

$$\min \|\mathbf{G}_{11} + \mathbf{G}_{12} \mathbf{K} (I - \mathbf{K} \mathbf{G}_{22})^{-1} \mathbf{G}_{21}\|^2 \quad (1a)$$

$$\text{s.t. } \mathbf{K} \in \mathcal{I}_{LBT} \text{ and stabilizing.} \quad (1b)$$

Problem 1 can be interpreted as a distributed control problem under a particular information constraint shown in Fig. 1. We make some natural assumptions on system matrices A, B, C, F, H, W and V so that Problem 1 is well-posed.

Assumption 1: 1. For every $i \in \{1, 2, \dots, N\}$, (A_{ii}, B_{ii}) is stabilizable and H has full column rank.

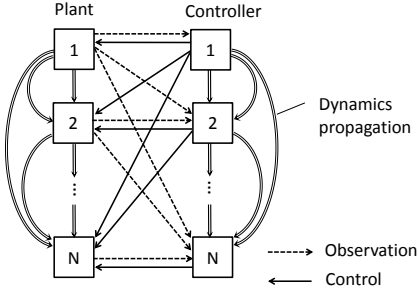


Fig. 1. $A, B, C \in \mathcal{I}_{LBT}$ means that system dynamics propagates only downward on the chain of local subsystems (defined by A_{ii}, B_{ii}, C_{ii}). Each row of $\mathbf{K} \in \mathcal{I}_{LBT}$ can be considered as an independent controller, who controls local subsystem based on the observations of the outputs of upstream subsystems. Alternatively, each column of \mathbf{K} can be seen as an independent controller, who observes local output and controls downstream subsystems. Another valid interpretation is to view $\mathbf{K} \in \mathcal{I}_{LBT}$ as a sum of local controllers $\mathbf{K}_i \in \mathcal{I}_{\downarrow i}^{\uparrow i}$, who observes upstream subsystems and controls downstream systems (the above figure).

2. $\begin{bmatrix} A - j\omega I & B \\ F & H \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.
3. For every $i \in \{1, 2, \dots, N\}$, (C_{ii}, A_{ii}) is detectable and V has full row rank.
4. $\begin{bmatrix} A - j\omega I & W \\ C & V \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

Due to the *quadratic invariance* property [7], Problem 1 can be written as a convex optimization problem by introducing a particular re-parametrization [9]. It is also straightforward to see by *vectorization* [8] that Problem 1 admits a unique and rational solution in this parameter domain, and hence that \mathbf{K}_{opt} is also rational. In this paper, we further show that the optimal controller has a fascinating state space structure, which can be easily synthesized by solving a set of linearly coupled Riccati equations. This work extends recent progress in the understanding of state space solutions to distributed control problems [2]–[4], [10], [12].

II. SUMMARY OF THE RESULT

Under Assumption 1 and 2 (the second assumption will be discussed later), a state space model of \mathbf{K}_{opt} can be constructed. This requires to determine N controller gains K_1, \dots, K_N and N observer gains L_1, \dots, L_N by finding a set of stabilizing solutions to algebraic Riccati equations:

$$(X_1, K_1) = ARE_p(A, B, F, H) \quad (2a)$$

$$(X_i, K_i) = ARE_p(A + L_{i-1}C_{\uparrow i-1}, B^{\downarrow i}, -H^{\downarrow i-1}K_{i-1}, H^{\downarrow i}), \quad i \in \{2, 3, \dots, N\} \quad (2b)$$

$$(Y_N, L_N) = ARE_d(A, C, W, V) \quad (2c)$$

$$(Y_i, L_i) = ARE_d(A + B^{\downarrow i+1}K_{i+1}, C_{\uparrow i}, -L_{i+1}V_{\uparrow i+1}, V_{\uparrow i}), \quad i \in \{1, 2, \dots, N-1\}. \quad (2d)$$

Notice that (2a) and (2c) can be solved independently since they require problem data A, B, C, F, H, W and V only, while the remaining $2N - 2$ Riccati equations (2b) and (2d) have dependencies on each other. By carefully looking at their substructures, we show that unknown variables can be sequentially determined as shown in Fig. 2. Apparently,

concepts of control and estimation are highly symmetric in this synthesis.

It is convenient to introduce nN dimensional square incidence matrices ζ and μ to describe the architecture of \mathbf{K}_{opt} . Define ζ by dividing it into $N \times N$ sub-blocks and setting its (i, j) -th sub-block to I_n if $i \geq j$ and to zero otherwise. Matrix μ is defined as the inverse of ζ .

Theorem 1: Under Assumption 1 and 2, (2) admits a unique set of solutions $(X_i, K_i), (Y_i, L_i), i = 1, \dots, N$ such that each of them is a stabilizing solution to the corresponding Riccati equation in (2). Using these solutions, the optimal controller to Problem 1 can be written as $\mathbf{K}_{opt} = \begin{bmatrix} A_K & B_K \\ C_K & 0 \end{bmatrix}$ where A_K, B_K, C_K are defined by

$$A_K = I \otimes A + \text{diag}\{L_1 C_{\uparrow 1}, \dots, L_N C_{\uparrow N}\} + \zeta \text{diag}\{B^{\downarrow 1} K_1, \dots, B^{\downarrow N} K_N\} \mu \quad (3a)$$

$$B_K = -\text{col}\{L_1 E_{\uparrow 1}, \dots, L_N E_{\uparrow N}\} \quad (3b)$$

$$C_K = \text{row}\{E^{\downarrow 1} K_1, \dots, E^{\downarrow N} K_N\} \mu. \quad (3c)$$

Moreover, the optimal value of Problem 1 can be written as $J_{opt}^2 = J_{cnt}^2 + J_{dcnt}^2$ where

$$J_{cnt}^2 = \text{tr} W^T X_1 W + \text{tr} \Psi K_1 Y_N K_1^T$$

$$J_{dcnt}^2 = \sum_{j=1}^{N-1} \text{tr} (H K_1 - H^{\downarrow j+1} K_{j+1}) Y_j (H K_1 - H^{\downarrow j+1} K_{j+1})^T.$$

Note that J_{cnt} can be interpreted as the optimal cost when the controller is designed without information constraints (compare with the result of standard H_2 control). The price to pay to impose information constraints as in Fig. 1 is precisely given by J_{dcnt} . The optimal controller given in Theorem 1 turns out to be a certainty equivalent controller. That is, if $\text{col}\{x^{K_1}(t), \dots, x^{K_N}(t)\}$ is the state of the optimal controller, then $x^{K_i}(t)$ can be interpreted as the least mean square estimate of $x(t)$ based on the observations of outputs of upstream subsystems (see Appendix E). Nevertheless, as Fig. 2 shows, controller and observer gain must be jointly designed when $N \geq 2$. The well-known *separation principle* holds only in an exceptional circumstance of $N = 1$, where controller and observer gains can be designed separately.

III. COUPLED RICCATI EQUATIONS

Since the proof of optimality of the proposed controller is closely related to the solvability of the set of Riccati equations (2), we study its solution procedure in this section. We introduce the following partitioning of unknown matrices X_i, Y_i, K_i and L_i for every $i \in \{1, 2, \dots, N\}$:

$$X_i = \begin{bmatrix} \tilde{X}_i & \bar{X}_i \\ \bar{X}_i^T & \hat{X}_i \end{bmatrix}, \quad K_i = \begin{bmatrix} \bar{K}_i & \hat{K}_i \end{bmatrix},$$

$$Y_i = \begin{bmatrix} \hat{Y}_i & \bar{Y}_i \\ \bar{Y}_i^T & \tilde{Y}_i \end{bmatrix}, \quad L_i = \begin{bmatrix} \hat{L}_i \\ \bar{L}_i \end{bmatrix} \quad (4)$$

where $\hat{X}_i = (X_i)_{\downarrow i}^{\downarrow i}$, $\hat{Y}_i = (Y_i)_{\uparrow i}^{\uparrow i}$, $\hat{K}_i = K_i E^{\downarrow i}$, $\hat{L}_i = E_{\uparrow i} L_i$. In particular, $X_1 = \hat{X}_1, K_1 = \hat{K}_1$ and $Y_N = \hat{Y}_N, L_N = \hat{L}_N$. Matrices \bar{K}_i and \bar{L}_i are further partitioned as

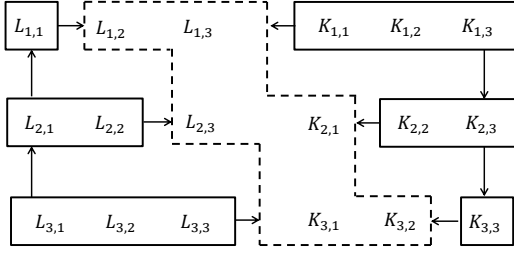


Fig. 2. This diagram illustrates the order in which controller/observer gains are determined when $N = 3$. Suppose $K_{i,j}$ is the i -th player's controller gain acting on his state estimate of x_j (i.e., $K_i = \text{row}\{K_{i,1}, K_{i,2}, K_{i,3}\}$). Similarly, suppose $L_{i,j}$ is the i -th player's estimator gain to update his state estimate of x_j (i.e., $L_i = \text{col}\{L_{i,1}, L_{i,2}, L_{i,3}\}$). The solution process starts by solving (2a) to generate $K_{1,1}, K_{1,2}, K_{1,3}$. This recursively allows one to solve for $K_{2,2}, K_{2,3}$ and then for $K_{3,3}$. Estimator gains can be also found by first solving (2c) and then proceed backward on the chain. Finally, the remaining gains (in the dotted line) is computed at once by solving a system of linear equations. This procedure is a natural extension of the case of $N = 2$ reported in [4].

$\hat{K}_i = \text{row}\{\hat{K}_i^a, \hat{K}_i^b\}$, $\hat{L}_i = \text{col}\{\hat{L}_i^b, \hat{L}_i^a\}$ where $\hat{K}_i^a = K_i E^i$ and $\hat{L}_i^a = E_i L_i$. Also introduce

$$\hat{A}_i^K \triangleq A_{\downarrow i}^{\downarrow i} + B_{\downarrow i}^{\downarrow i} \hat{K}_i, \quad \bar{A}_i^K \triangleq A_{\downarrow i}^{\uparrow i-1} + B_{\downarrow i}^{\downarrow i} \bar{K}_i$$

$$\hat{A}_i^L \triangleq A_{\uparrow i}^{\uparrow i} + \hat{L}_i C_{\uparrow i}^{\uparrow i}, \quad \bar{A}_i^L \triangleq A_{\downarrow i+1}^{\uparrow i} + \bar{L}_i C_{\uparrow i}^{\uparrow i}$$

$$A_{1,0}^{KL} \triangleq A + BK_1, \quad A_{N+1,N}^{KL} \triangleq A + L_N C$$

$$A_{i,i-1}^{KL} \triangleq A + B^{\downarrow i} K_i + L_{i-1} C_{\uparrow i-1} \text{ for } 2 \leq i \leq N.$$

Equation (2b) can be written as

$$A_{i,i-1}^{KL} X_i + X_i A_{i,i-1}^{KL} + \Sigma_i = 0 \quad (5a)$$

$$K_i^T \Psi_{\downarrow i}^{\downarrow i} + X_i B_{\downarrow i}^{\downarrow i} - K_{i-1}^T \Psi_{\downarrow i-1}^{\downarrow i} = 0 \quad (5b)$$

where $\Sigma_i = (H^{\downarrow i} K_i - H^{\downarrow i-1} K_{i-1})^T (H^{\downarrow i} K_i - H^{\downarrow i-1} K_{i-1})$ and (2d) is rearranged as

$$A_{i+1,i}^{KL} Y_i + Y_i A_{i+1,i}^{KL} + \Pi_i = 0 \quad (6a)$$

$$\Phi_{\uparrow i}^{\uparrow i} L_i^T + C_{\uparrow i} Y_i - \Phi_{\downarrow i}^{\downarrow i+1} L_{i+1}^T = 0 \quad (6b)$$

where $\Pi_i = (L_i V_{\uparrow i} - L_{i+1} V_{\uparrow i+1})(L_i V_{\uparrow i} - L_{i+1} V_{\uparrow i+1})^T$. Now, all unknown variables can be determined by the following three-step procedure, which also visualized in Fig. 3.

A. Step 1: Sequential solving of Riccati subequations

In this step, sub-matrices indicated by “ $\hat{\cdot}$ ” in (4) are determined. Notice that $\hat{X}_1, \hat{K}_1, \hat{Y}_N, \hat{L}_N$ are directly obtained by solving (2a) and (2c). To compute \hat{X}_i, \hat{K}_i for $i \in \{2, 3, \dots, N\}$, focus on the lower-right $(\cdot)_{\downarrow i}^{\downarrow i}$ sub-block of (5a) and $(\cdot)_{\downarrow i}$ subblock of (5b). They are by themselves Riccati equations with respect to (\hat{X}_i, \hat{K}_i) :

$$(\hat{X}_i, \hat{K}_i) = \text{ARE}_p(A_{\downarrow i}^{\downarrow i}, B_{\downarrow i}^{\downarrow i}, -H^{\downarrow i-1} \hat{K}_{i-1}^b, H^{\downarrow i}). \quad (7)$$

Since the right hand side contains \hat{K}_{i-1}^b , these Riccati equations need to be solved in the forward (ascending) order on the chain. Similarly, \hat{Y}_i, \hat{L}_i for $i \in \{1, 2, \dots, N-1\}$

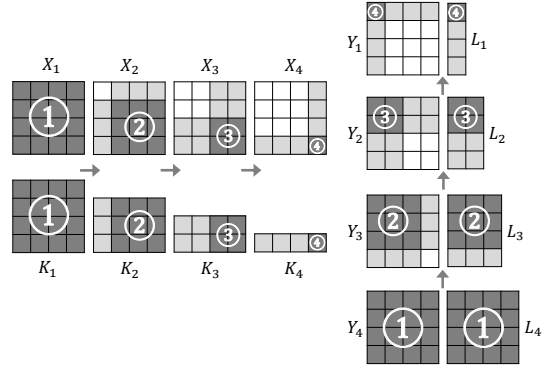


Fig. 3. Sequential process to determine unknown variables. Step 1: Entries with dark gray are computed by sequentially solving Riccati subequations. Step 2: Entries with light gray are computed at once by solving a linear system. Step 3: White entries are obtained by solving Lyapunov equations.

can be computed by focusing on the upper-left $(\cdot)_{\uparrow i}^{\uparrow i}$ sub-block of (6a) and $(\cdot)_{\uparrow i}^{\uparrow i}$ sub-block of (6b). They are Riccati equations with respect to (\hat{Y}_i, \hat{L}_i) :

$$(\hat{Y}_i, \hat{L}_i) = \text{ARE}_d(A_{\uparrow i}^{\uparrow i}, C_{\uparrow i}^{\uparrow i}, -\hat{L}_{i+1}^b V_{\uparrow i+1}, V_{\uparrow i}). \quad (8)$$

Since the right hand side contains \hat{L}_{i+1}^b , they need to be solved in the backward (descending) order in the chain.

Proposition 1: Under Assumption 1, algebraic Riccati equations (7) and (8) admit a unique positive semidefinite solution, which is also stabilizing.

B. Step 2: Solving a linear system

In this step, we compute components with “ $\bar{\cdot}$ ”. By looking at the upper-right $(\cdot)_{\uparrow i-1}^{\downarrow i}$ subblock of (5a) and the upper $(\cdot)_{\uparrow i-1}$ subblock of (5b), as well as the upper-right $(\cdot)_{\uparrow i}^{\downarrow i+1}$ subblock of (6a) and the right $(\cdot)^{\downarrow i+1}$ subblock of (6b), we obtain

$$\bar{K}_i^T \Psi_{\downarrow i}^{\downarrow i} + \bar{X}_i B_{\downarrow i}^{\downarrow i} - \text{row}\{\bar{K}_{i-1}, \hat{K}_{i-1}^a\}^T \Psi_{\downarrow i-1}^{\downarrow i} = 0 \quad (9a)$$

$$\hat{A}_{i-1}^L \bar{X}_i + \bar{X}_i \hat{A}_i^K + \bar{A}_{i-1}^L \bar{X}_i + \text{row}\{\bar{K}_{i-1}, \hat{K}_{i-1}^a\}^T (\Psi_{\downarrow i-1}^{\downarrow i-1} \hat{K}_{i-1}^b - \Psi_{\downarrow i-1}^{\downarrow i} \hat{K}_i) = 0 \quad (9b)$$

for every $i \in \{2, 3, \dots, N\}$ and

$$\Phi_{\uparrow i}^{\uparrow i} \bar{L}_i^T + C_{\uparrow i}^{\uparrow i} \bar{Y}_i^T - \Phi_{\uparrow i}^{\uparrow i+1} \text{col}\{\hat{L}_{i+1}^a, \bar{L}_{i+1}\}^T = 0 \quad (9c)$$

$$\hat{A}_i^L \bar{Y}_i + \bar{Y}_i \hat{A}_{i+1}^K + \hat{Y}_i \bar{A}_{i+1}^K + (\hat{L}_{i+1}^b \Phi_{\uparrow i+1}^{\uparrow i+1} - \hat{L}_i \Phi_{\uparrow i}^{\uparrow i+1}) \text{col}\{\hat{L}_{i+1}^a, \bar{L}_{i+1}\}^T = 0 \quad (9d)$$

for every $i \in \{1, 2, \dots, N-1\}$. Since $\hat{X}_i, \hat{K}_i, \hat{Y}_i, \hat{L}_i$ are computed in the previous step, these are linear equations with respect to $\bar{X}_i, \bar{K}_i, i \in \{2, 3, \dots, N\}$ and $\bar{Y}_i, \bar{L}_i, i \in \{1, 2, \dots, N-1\}$. There are precisely the same number of linear constraints as the number of real unknowns. Unfortunately, we are currently not aware of a theoretical guarantee for the non-singularity of (9). Hence at this point, we have to make an additional assumption:

Assumption 2: The linear system (9) with respect to $\bar{X}_i, \bar{K}_i, i \in \{2, 3, \dots, N\}$ and $\bar{Y}_i, \bar{L}_i, i \in \{1, 2, \dots, N-1\}$ admit a unique solution.

When $N = 2$, it is shown in [4] that the linear system (9) admits a unique solution under Assumption 1 and thus Assumption 2 is unnecessary. It must be addressed in the future whether this generalizes to $N > 2$. Our numerical studies indicate that, when problem data is randomly generated to satisfy Assumption 1, (9) is usually a well-conditioned linear system.

C. Step 3: Solving Lyapunov equations

Finally, \tilde{X}_i for $i \in \{2, 3, \dots, N\}$ and \tilde{Y}_i for $i \in \{1, 2, \dots, N-1\}$ are computed by looking at $(\cdot)_{\uparrow i-1}^{\uparrow i-1}$ sub-block of (5a) and $(\cdot)_{\downarrow i+1}^{\downarrow i+1}$ sub-block of (6a).

$$\begin{aligned} & \hat{A}_{i-1}^{L^T} \tilde{X}_i + \tilde{X}_i \hat{A}_{i-1}^L + \bar{A}_{i-1}^{L^T} \bar{X}_i^T + \bar{X}_i \bar{A}_{i-1}^L - \bar{K}_i^T \Psi_{\downarrow i}^{\downarrow i} \bar{K}_i \\ & + \text{row}\{\bar{K}_{i-1}, \hat{K}_{i-1}^a\}^T \Psi_{\downarrow i-1}^{\downarrow i-1} \text{row}\{\bar{K}_{i-1}, \hat{K}_{i-1}^a\} = 0 \\ & i \in \{2, 3, \dots, N\} \end{aligned} \quad (10a)$$

$$\begin{aligned} & \hat{A}_{i+1}^K \tilde{Y}_i + \tilde{Y}_i \hat{A}_{i+1}^{K^T} + \bar{A}_{i+1}^K \bar{Y}_i + \bar{Y}_i \bar{A}_{i+1}^{K^T} - \bar{L}_i \Phi_{\uparrow i}^{\uparrow i} \bar{L}_i^T \\ & + \text{col}\{\hat{L}_{i+1}^a, \bar{L}_{i+1}\} \Phi_{\uparrow i+1}^{\uparrow i+1} \text{col}\{\hat{L}_{i+1}^a, \bar{L}_{i+1}\}^T = 0 \\ & i \in \{1, 2, \dots, N-1\} \end{aligned} \quad (10b)$$

Since all other quantities are known by the previous step, these are Lyapunov equations with respect to \tilde{X}_i and \tilde{Y}_i , which can be easily solved. Since \hat{A}_i^K and \hat{A}_i^L are Hurwitz stable (guaranteed by Proposition 1), they admit a unique solution.

Proposition 2: Under Assumption 1 and 2, the set of algebraic Riccati equations (2) admit a unique tuple of positive semidefinite solutions $X_i, Y_i, i \in \{1, 2, \dots, N\}$. Moreover, they are stabilizing solutions.

Proof: It is clear from Theorem 3 that (2a) and (2c) admit unique positive semidefinite solutions which are stabilizing. To see why solutions constructed in Step 1, 2 and 3 above are positive semidefinite, notice that under Assumption 1 and 2, the algorithm produces a unique set of variables satisfying (5a) and (6a). Furthermore, notice that

$$A_{i,i-1}^{KL} = \begin{bmatrix} & \hat{A}_{i-1}^L & 0 \\ A_{\downarrow i}^{\uparrow i-1} + B_{\downarrow i}^{\downarrow i} \bar{K}_i + \bar{L}_{i-1} C_{\uparrow i-1}^{\uparrow i-1} & \hat{A}_i^K & \end{bmatrix}$$

is a stable matrix since its diagonal blocks are stable. Hence, due to the inertia property of a Lyapunov equation, X_i and Y_i must be positive semidefinite. They are indeed stabilizing solutions since $A_{i,i-1}^{KL}$ is a stable matrix. ■

IV. DERIVATION OF MAIN RESULT

A. Stability

Notice that the closed-loop transfer function is given by

$$\begin{aligned} \mathbf{G}^{cl} &= \begin{bmatrix} \mathbf{G}_{11}^{cl} & \mathbf{G}_{12}^{cl} \\ \mathbf{G}_{21}^{cl} & \mathbf{G}_{22}^{cl} \end{bmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{W} & \mathcal{B} \\ \mathcal{F} & 0 & H \\ \mathcal{C} & V & 0 \end{array} \right] \\ &\triangleq \left[\begin{array}{cc|cc} A_K & B_K C & B_K V & 0 \\ BC_K & A & W & B \\ \hline HC_K & F & 0 & H \\ 0 & C & V & 0 \end{array} \right]. \end{aligned} \quad (11)$$

$$\begin{array}{cccccc} S_{0,1} & \subset & S_{0,2} & \subset & \cdots & \subset & S_{0,N} & \subset & S_{0,N+1} \\ \cup & & \cup & & & & \cup & & \cup \\ S_{1,1} & \subset & S_{1,2} & \subset & \cdots & \subset & S_{1,N} & \subset & S_{1,N+1} \\ & & \cup & & & & \cup & & \cup \\ & & S_{2,2} & \subset & \cdots & \subset & S_{2,N} & \subset & S_{2,N+1} \\ & & & & & & \cup & & \cup \\ & & & & \ddots & & \vdots & & \vdots \\ & & & & & & \cup & & \cup \\ & & & & & & S_{N,N} & \subset & S_{N,N+1} \end{array}$$

Fig. 4. Inclusion relation diagram among subspaces $S_{i,j}$ defined by (20). For convenience, we also define $S_{0,j} \triangleq \{\mathbf{Q} E_{\uparrow j} \mathbf{G}_{21}^{cl} : \mathbf{Q} \in H_2\}$, $S_{i,N+1} \triangleq \{\mathbf{G}_{12}^{cl} E^{\downarrow i} \mathbf{Q} : \mathbf{Q} \in H_2\}$, $S_{0,N+1} \triangleq H_2$.

To see that \mathbf{K}_{opt} is a stabilizing controller, we need to verify that \mathcal{A} is a stable matrix. Let $\bar{\zeta}$ be an $n(N+1)$ -dimensional square matrix whose (i,j) -th sub-block is I_n if $i \geq j$ and is zero otherwise. Also define $\bar{\mu} = \bar{\zeta}^{-1}$. One can easily check that a similarity transformation gives

$$\bar{\mu} \mathcal{A} \bar{\zeta} = \begin{bmatrix} A_{1,0}^{KL} & * & * & * & * \\ & A_{2,1}^{KL} & * & * & * \\ & & \ddots & * & * \\ & & & A_{N,N-1}^{KL} & * \\ 0 & & & & A_{N+1,N}^{KL} \end{bmatrix}. \quad (12)$$

This is a block upper-triangular matrix. Moreover, all diagonal blocks are stable matrices as we saw in the proof of Proposition 2. This shows the stability of \mathcal{A} .

B. Optimal Controller Characterization

If \mathbf{K}_{opt} is the optimal solution to Problem 1, then any perturbation $\mathbf{K} = \mathbf{K}_{opt} + \mathbf{K}'$ such that $\mathbf{K}' \in \mathcal{I}_{LBT}$ only degrades control performance. Due to the uniqueness of the rational solution to Problem 1, it is sufficient for us to show that $\mathbf{K}' = 0$ is the optimal solution to the modified H_2 optimal control problem

$$\begin{aligned} \min & \|\mathbf{G}_{11}^{cl} + \mathbf{G}_{12}^{cl} \mathbf{K}' (I - \mathbf{G}_{22}^{cl} \mathbf{K}')^{-1} \mathbf{G}_{21}^{cl}\| \\ \text{s.t.} & \mathbf{K}' \text{ is stabilizing and } \mathbf{K}' \in \mathcal{I}_{LBT}. \end{aligned}$$

Since $\mathbf{G}_{22}^{cl} \in H_2 \cap \mathcal{I}_{LBT}$, the subspace \mathcal{I}_{LBT} is quadratically invariant under \mathbf{G}_{22}^{cl} . This means that all stabilizing controllers are parametrized by the structured Youla parameter $\mathbf{Q} \triangleq -\mathbf{K}' (I - \mathbf{G}_{22}^{cl} \mathbf{K}')^{-1} \in H_\infty \cap \mathcal{I}_{LBT}$. Hence, the above statement is equivalent to that $\mathbf{Q} = 0$ is the optimal solution to the model matching problem¹

$$\begin{aligned} \min & \|\mathbf{G}_{11}^{cl} - \mathbf{G}_{12}^{cl} \mathbf{Q} \mathbf{G}_{21}^{cl}\| \\ \text{s.t.} & \mathbf{Q} \in H_2 \cap \mathcal{I}_{LBT}. \end{aligned} \quad (18)$$

Since the non-rectangular constraint \mathcal{I}_{LBT} is inconvenient to work with, we use an alternative characterization of the same statement using rectangular blocks $\mathcal{I}_{\downarrow i}^{\uparrow i}$, $i \in \{1, 2, \dots, N\}$.

¹The condition $\mathbf{Q} \in H_\infty \cap \mathcal{I}_{LBT}$ can be replaced by $\mathbf{Q} \in H_2 \cap \mathcal{I}_{LBT}$ without loss of generality. Recall that under Assumption 1, H has full column rank, V has full row rank, and $\mathbf{G}_{11}^{cl} \in H_2$. This means that \mathbf{Q} must be in H_2 so that the value of (18) is bounded.

$$\mathcal{K}_i \triangleq E_{\downarrow i} \text{row}\{0, \dots, 0, \overset{i\text{-th block}}{E^{\downarrow i+1} K_{i+1} - E^{\downarrow i} K_i}, \dots, \overset{(N-1)\text{-th block}}{E^{\downarrow N} K_N - E^{\downarrow N-1} K_{N-1}}, \overset{N\text{-th block}}{-E^{\downarrow N} K_N}, \overset{(N+1)\text{-th block}}{E^{\downarrow i} K_i}\} \quad (13)$$

$$\mathcal{L}_j \triangleq \text{col}\{\overset{1\text{st block}}{L_1 E_{\uparrow 1}}, \dots, \overset{(j-1)\text{-th block}}{L_{j-1} E_{\uparrow j-1}}, \overset{j\text{-th block}}{L_j E_{\uparrow j}}, \dots, \overset{N\text{-th block}}{L_j E_{\uparrow j}}, \overset{(N+1)\text{-th block}}{L_j E_{\uparrow j}}\} E_{\uparrow j} \quad (14)$$

$$\tilde{J}_i \triangleq \text{row}\{0, \dots, 0, \overset{(i-1)\text{-th block}}{I}, 0, \dots, 0, \overset{(N+1)\text{-th block}}{-I}\}, \hat{J}_j \triangleq \text{col}\{0, \dots, 0, \overset{(j+1)\text{-th block}}{I}, \dots, \overset{(N+1)\text{-th block}}{I}\} \quad (15)$$

$$\mathbf{U}_i \triangleq \left[\begin{array}{c|c} A_{i,i-1}^{KL} & B_{U_i} \\ \hline C_{U_i} & D_{U_i} \end{array} \right], \mathbf{M}_i^{-1} \triangleq \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}^{\downarrow i} \\ \hline -\Psi_{\downarrow i}^{\downarrow i-\frac{1}{2}} \mathcal{K}_i & \Psi_{\downarrow i}^{\downarrow i-\frac{1}{2}} \end{array} \right] \forall i \in \{1, 2, \dots, N\} \text{ where } B_{U_i} = B^{\downarrow i} \Psi_{\downarrow i}^{\downarrow i-\frac{1}{2}}, \quad (16)$$

$$C_{U_i} = \begin{cases} F + H K_1 & \text{if } i = 1 \\ \Psi_{\downarrow i-1}^{\downarrow i-1-\frac{1}{2}} E_{\downarrow i-1} (E^{\downarrow i} K_i - E^{\downarrow i-1} K_{i-1}) & \text{if } 2 \leq i \leq N \end{cases}, D_{U_i} = \begin{cases} H \Psi^{-\frac{1}{2}} & \text{if } i = 1 \\ \Psi_{\downarrow i-1}^{\downarrow i-1-\frac{1}{2}} E_{\downarrow i-1} E^{\downarrow i} \Psi_{\downarrow i}^{\downarrow i-\frac{1}{2}} & \text{if } 2 \leq i \leq N \end{cases}$$

$$\mathbf{V}_j \triangleq \left[\begin{array}{c|c} A_{j+1,j}^{KL} & B_{V_j} \\ \hline C_{V_j} & D_{V_j} \end{array} \right], \mathbf{N}_j^{-1} \triangleq \left[\begin{array}{c|c} \mathcal{A} & -\mathcal{L}_j \Phi_{\uparrow j}^{\uparrow j-\frac{1}{2}} \\ \hline C_{\uparrow j} & \Phi_{\uparrow j}^{\uparrow j-\frac{1}{2}} \end{array} \right] \forall j \in \{1, 2, \dots, N\} \text{ where } C_{V_j} = \Phi_{\uparrow j}^{\uparrow j-\frac{1}{2}} C_{\uparrow j}, \quad (17)$$

$$B_{V_j} = \begin{cases} W + L_N V & \text{if } j = N \\ (L_j E_{\uparrow j} - L_{j+1} E_{\uparrow j+1}) E^{\uparrow j+1} \Phi_{\uparrow j+1}^{\uparrow j+1-\frac{1}{2}} & \text{if } 1 \leq j \leq N-1 \end{cases}, D_{V_j} = \begin{cases} \Phi^{-\frac{1}{2}} V & \text{if } i = N \\ \Phi_{\uparrow j}^{\uparrow j-\frac{1}{2}} E_{\uparrow j} E^{\uparrow j+1} \Phi_{\uparrow j+1}^{\uparrow j+1-\frac{1}{2}} & \text{if } 1 \leq i \leq N-1 \end{cases}$$

Proposition 3: $\mathbf{Q} = 0$ is the optimal solution to (18) if and only if for every $i \in \{1, 2, \dots, N\}$, $\mathbf{Q}_i = 0$ is the optimal solution to the model matching problem

$$\begin{aligned} \min \quad & \|\mathbf{G}_{11}^{cl} - \mathbf{G}_{12}^{cl} \mathbf{Q}_i \mathbf{G}_{21}^{cl}\| \\ \text{s.t.} \quad & \mathbf{Q}_i \in H_2 \cap \mathcal{I}_{\downarrow i}^{\uparrow i}. \end{aligned} \quad (19)$$

One approach to find a solution to the model matching problem (19) is to apply the projection theorem [5]. Define subspaces $S_{i,j}$ of H_2 for $1 \leq i \leq j \leq N$ by

$$S_{i,j} \triangleq \{\mathbf{G}_{12}^{cl} E^{\downarrow i} \mathbf{Q} E_{\uparrow i} \mathbf{G}_{21}^{cl} : \mathbf{Q} \in H_2\}. \quad (20)$$

Proposition 3 implies that, in order to infer that \mathbf{K}_{opt} is the optimal controller, it suffices to prove that $\mathbf{Q} = 0$ is the minimizer of $\|\mathbf{G}_{11}^{cl} - \mathbf{G}_{12}^{cl} E^{\downarrow i} \mathbf{Q} E_{\uparrow i} \mathbf{G}_{21}^{cl}\|$ over $\mathbf{Q} \in H_2$ for every $i \in \{1, 2, \dots, N\}$. Equivalently, it needs to be shown that $\pi_{S_{i,i}}(\mathbf{G}_{11}^{cl}) = 0$ for every $i \in \{1, 2, \dots, N\}$, where $\pi_{S_{i,j}} : H_2 \rightarrow S_{i,j}$ is the projection operator.

C. Nested Projections

It is clear that the inclusion relations in Fig. 4 hold among subspaces defined by (20). We are going to exploit this diagram to find an explicit representation of $\pi_{S_{i,j}}(\mathbf{G}_{11}^{cl})$. Recall the following fact:

Theorem 2: (Nested Projections, see e.g., [6]) Let S_1, S_2, \dots, S_N be subspaces of a Hilbert space such that $S_N \subset S_{N-1} \subset \dots \subset S_2 \subset S_1$. Then $\pi_{S_N} = \pi_{S_N} \circ \pi_{S_{N-1}} \circ \dots \circ \pi_{S_2} \circ \pi_{S_1}$.

According to Fig. 4, Theorem 2 suggests that $\pi_{S_{i,i}}$ can be computed as, for instance,

$$\pi_{S_{i,i}} = \pi_{S_{i,i}} \circ \pi_{S_{i,i+1}} \circ \dots \circ \pi_{S_{i,N}} \circ \pi_{S_{i-1,N}} \circ \dots \circ \pi_{S_{1,N}}. \quad (21)$$

Understanding $\pi_{S_{i,i}}$ as a composition of stepwise projections is convenient in the following presentation, since each projection step can be associated with one of $2N$ Riccati equations in (2). To be precise, we consider writing $S_{i,j}$ using an ‘‘orthonormal’’ basis. Recall that a rational function

$\mathbf{U} \in H_{\infty}$ is said to be *inner* if $\mathbf{U}^* \mathbf{U} = I$ and *co-inner* if $\mathbf{U} \mathbf{U}^* = I$. It turns out that each subspace can be written as

$$S_{i,j} = \{\mathbf{U}_1 \cdots \mathbf{U}_i \mathbf{M}_i^{-1} \mathbf{Q} \mathbf{N}_j^{-1} \mathbf{V}_j \cdots \mathbf{V}_N : \mathbf{Q} \in H_2\} \quad (22)$$

where the explicit form of inner functions $\mathbf{U}_1, \dots, \mathbf{U}_N$, co-inner functions $\mathbf{V}_1, \dots, \mathbf{V}_N$ and other necessary quantities are given in (13)-(17). The above expression is obtained by repeated applications of a particular type of spectral factorizations (Lemma 2 in Appendix D). Each application of the factorization requires a solution to one of the Riccati equations in (2).

Writing $S_{i,j}$ in the form of (22) makes nested projections easier. Suppose that the projection of \mathbf{G}_{11}^{cl} onto $S_{i,j}$ can be written in the form of

$$\pi_{S_{i,j}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \cdots \mathbf{U}_i \tilde{\mathbf{P}}_i \mathbf{V}_j \cdots \mathbf{V}_N \in S_{i,j}$$

for some $\tilde{\mathbf{P}}_i$. Then it is easy to check that the subsequent projection is given by

$$\pi_{S_{i+1,j}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \cdots \mathbf{U}_{i+1} \tilde{\mathbf{P}}_{i+1} \mathbf{V}_j \cdots \mathbf{V}_N \in S_{i+1,j}$$

where $\tilde{\mathbf{P}}_{i+1}$ is chosen to satisfy the optimality condition

$$\left\langle \mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i - \tilde{\mathbf{P}}_{i+1}, \mathbf{M}_{i+1}^{-1} \mathbf{Q} \mathbf{N}_j^{-1} \right\rangle = 0 \quad \forall \mathbf{Q} \in H_2.$$

Details can be found in Lemma 3 in Appendix D. Also, notice that every projection generates a ‘‘residual term’’ as

$$\begin{aligned} \pi_{S_{i,j}}(\mathbf{G}_{11}^{cl}) &= \pi_{S_{i+1,j}}(\mathbf{G}_{11}^{cl}) + \mathbf{U}_1 \cdots \mathbf{U}_i \mathbf{R}_{(i,j) \rightarrow (i+1,j)} \mathbf{V}_j \cdots \mathbf{V}_N \\ \pi_{S_{i,j}}(\mathbf{G}_{11}^{cl}) &= \pi_{S_{i,j-1}}(\mathbf{G}_{11}^{cl}) + \mathbf{U}_1 \cdots \mathbf{U}_i \mathbf{R}_{(i,j) \rightarrow (i,j-1)} \mathbf{V}_j \cdots \mathbf{V}_N. \end{aligned}$$

The H_2 norm of residual terms will be used later to compute the optimal value of Problem 1. Finally, all the above operations can be performed at the state space level, as summarized in Lemma 1.

$$\|\mathbf{R}_{(i-1,j) \rightarrow (i,j)}\|^2 = \begin{cases} \text{tr} W^T X_1 W & \text{if } i = 1, j = 1 \\ \text{tr} \Phi_{\uparrow j}^{\downarrow i} L_j^T X_1 L_j & \text{if } i = 1, j \geq 2 \\ \text{tr} (L_{i-1} V_{\uparrow i-1} - L_j V_{\uparrow j})^T X_1 (L_{i-1} V_{\uparrow i-1} - L_j V_{\uparrow j}) & \text{if } i \geq 2 \end{cases} \quad (23)$$

$$\|\mathbf{R}_{(i,j+1) \rightarrow (i,j)}\|^2 = \begin{cases} \text{tr} F Y_N F^T & \text{if } i = N, j = N \\ \text{tr} \Psi_{\downarrow i}^{\uparrow j} K_i Y_N K_i^T & \text{if } i \leq N-1, j = N \\ \text{tr} (H^{\downarrow i} K_i - H^{\downarrow j+1} K_{j+1}) Y_j (H^{\downarrow i} K_i - H^{\downarrow j+1} K_{j+1})^T & \text{if } j \leq N-1. \end{cases} \quad (24)$$

Lemma 1: The projection of \mathbf{G}_{11}^{cl} onto any subspace $S_{i,j}$ in Fig. 4 is given by $\pi_{S_{i,j}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_i \mathbf{P}'_{i,j} \mathbf{V}_j \cdots \mathbf{V}_{N-1} \mathbf{V}_N$ where

$$\mathbf{P}'_{i,j} = \begin{bmatrix} \mathcal{A} & \Lambda_j \\ \Gamma_i & 0 \end{bmatrix} \quad (25)$$

$$\Gamma_i = \begin{cases} \mathcal{F} & \text{if } i = 0 \\ -\Psi_{\downarrow i}^{\uparrow \frac{1}{2}} \mathcal{K}_i & \text{if } 1 \leq i \leq N \end{cases}, \Lambda_j = \begin{cases} \mathcal{W} & \text{if } j = N+1 \\ -\mathcal{L}_j \Phi_{\uparrow j}^{\downarrow \frac{1}{2}} & \text{if } 1 \leq j \leq N \end{cases}.$$

Moreover,

$$\mathbf{R}_{(i-1,j) \rightarrow (i,j)} = \begin{bmatrix} \frac{A_{i,i-1}^{KL}}{C_{U_i}} & -\tilde{J}_i \Lambda_j \\ & 0 \end{bmatrix}$$

$$\mathbf{R}_{(i,j+1) \rightarrow (i,j)} = \begin{bmatrix} \frac{A_{j+1,j}^{KL}}{-\Gamma_i \tilde{J}_j} & B_{V_j} \\ & 0 \end{bmatrix}.$$

Proof: See Appendix D. ■

D. Proof of Optimality

We are now ready to prove that $\pi_{S_{i,i}}(\mathbf{G}_{11}^{cl}) = 0$ for every $i \in \{1, 2, \dots, N\}$. Combined with Proposition 3, this completes the proof of optimality of the proposed controller.

Proof: (of Theorem 1) We have verified the existence and uniqueness of the stabilizing solution to (2) in Section III. By Lemma 1, for every $i \in \{1, 2, \dots, N\}$, we have

$$\mathbf{P}'_{i,i} = \begin{bmatrix} \mathcal{A} & -\mathcal{L}_i \Phi_{\uparrow i}^{\downarrow \frac{1}{2}} \\ -\Psi_{\downarrow i}^{\uparrow \frac{1}{2}} \mathcal{K}_i & 0 \end{bmatrix}.$$

Apply a state space transformation defined by $\bar{\mu}$ and $\bar{\zeta}$. As we have observed in (12), $\bar{\mu} \mathcal{A} \bar{\zeta}$ is an upper block triangular matrix. Also, it is straightforward to check that all $(\cdot)_{\downarrow i+1}$ sub-blocks of $\bar{\mu} \mathcal{L}_i \Phi_{\uparrow i}^{\downarrow \frac{1}{2}}$ are zero. Furthermore, it is possible to show that all $(\cdot)_{\uparrow i}$ sub-blocks of $\Psi_{\downarrow i}^{\uparrow \frac{1}{2}} \mathcal{K}_i \bar{\zeta}$ are zero. Hence $\mathbf{P}'_{i,i} = 0$. Therefore, for every $i \in \{1, 2, \dots, N\}$, $\pi_{S_{i,i}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_i \mathbf{P}'_{i,i} \mathbf{V}_1 \cdots \mathbf{V}_{N-1} \mathbf{V}_N = 0$. By Proposition 3, this implies that the proposed controller is the optimal solution to Problem 1.

Since we have shown that \mathbf{G}_{11}^{cl} is the optimal closed loop transfer function, the optimal cost is given by computing its H_2 norm. To obtain more explicit expression, consider a nested projection $\pi_{S_{1,1}} = \pi_{S_{1,1}} \circ \cdots \circ \pi_{S_{1,N}} \circ \pi_{S_{1,N+1}}$. It is possible to write

$$\mathbf{G}_{11}^{cl} = \pi_{S_{1,1}}(\mathbf{G}_{11}^{cl}) + \mathbf{R}_{(0,N+1) \rightarrow (1,N+1)} + \mathbf{U}_1 \mathbf{R}_{(1,N+1) \rightarrow (1,N)} + \sum_{j=1}^{N-1} \mathbf{U}_1 \mathbf{R}_{(1,j+1) \rightarrow (1,j)} \mathbf{V}_{j+1} \cdots \mathbf{V}_N.$$

Since $\pi_{S_{1,1}}(\mathbf{G}_{11}^{cl}) = 0$ and all residual terms are orthogonal, the optimal cost $J_{opt} = \|\mathbf{G}_{11}^{cl}\|$ can be decomposed as

$$\|\mathbf{G}_{11}^{cl}\|^2 = \|\mathbf{R}_{(0,N+1) \rightarrow (1,N+1)}\|^2 + \|\mathbf{R}_{(1,N+1) \rightarrow (1,N)}\|^2 + \sum_{j=1}^{N-1} \|\mathbf{R}_{(1,j+1) \rightarrow (1,j)}\|^2.$$

Each term can be written more explicitly using the fact (23) (24). This proves $J_{opt}^2 = J_{cnt}^2 + J_{dcnt}^2$. ■

V. CONCLUSION AND FUTURE WORK

In this paper, we have presented a state-space realization of the optimal output feedback controller for the N -player triangular LQG problem. We have derived a set of algebraic Riccati equations to be solved to construct the optimal controller. Solvability of Riccati equations, namely non-singularity of the linear system (9), must be verified in the future work.

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APPENDIX

A. Quadratic Invariance and Convexity

This section gives a brief review of the notion of *Quadratic Invariance* and how an optimal control problem can be formulated as an infinite dimensional convex optimization problem. For brevity, our discussion here is rather informal; for a thorough introduction, readers are referred to [7]. The H_2 optimal control formulated in Problem 1 is a nonconvex optimization problem respect to \mathbf{K} . A natural approach is to introduce the *Youla parametrization*, which has been historically used to convexify centralized control problems. Youla parameterization is particularly simple if \mathbf{G} is stable, and is given by $\mathbf{Q} = -\mathbf{K}(I - \mathbf{G}_{22}\mathbf{K})^{-1}$. The inverse in this expression is guaranteed to exist on the domain of stabilizing \mathbf{K} by the generalized Nyquist criterion [11]. Conversely, the corresponding controller \mathbf{K} can be recovered from \mathbf{Q} by $\mathbf{K} = -(I - \mathbf{Q}\mathbf{G}_{22})^{-1}\mathbf{Q}$. The objective function is expressed as $\|\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{Q}\mathbf{G}_{21}\|^2$, which is clearly convex with respect to the new parameter \mathbf{Q} . Moreover, the requirement that \mathbf{K} is stabilizing is translated in the new domain as the requirement that \mathbf{Q} is stable (i.e., $\mathbf{Q} \in H_\infty$)², which is also a convex constraint. The information constraint $\mathbf{K} \in \mathcal{I}$, however, results in a nonconvex constraints on the \mathbf{Q} domain, unless \mathcal{I} is *quadratically invariant* under \mathbf{G}_{22} .

Definition 1: (Quadratic Invariance) Let \mathcal{U}, \mathcal{Y} be vector spaces. Suppose \mathbf{G} is a linear mapping from \mathcal{U} to \mathcal{Y} and \mathcal{I} is a set of linear maps from \mathcal{Y} to \mathcal{U} . Then \mathcal{I} is called quadratically invariant under \mathbf{G} if $\mathbf{K} \in \mathcal{I} \Rightarrow \mathbf{K}\mathbf{G}\mathbf{K} \in \mathcal{I}$. The significance of \mathcal{I} being quadratic invariant is that, the condition $\mathbf{K} \in \mathcal{I}$ can be translated to $\mathbf{Q} \in \mathcal{I}$ on the new domain, which is a convex constraint³.

Since we are considering triangular LQG problems in this paper, \mathbf{G}_{22} has a lower block triangular structure ($\mathbf{G}_{22} \in \mathcal{I}_{LBT}$), while the information constraint on the controller implies that the controller transfer function is also lower block triangular ($\mathbf{K} \in \mathcal{I}_{LBT}$). It is easy to verify that the space of transfer function matrices with sparsity pattern \mathcal{I}_{LBT} is indeed quadratically invariant under \mathbf{G}_{22} . Therefore, Problem 1 can be recast as the following infinite dimensional convex optimization problem.

$$\min \|\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{Q}\mathbf{G}_{21}\| \quad (26a)$$

$$\text{s.t } \mathbf{Q} \in H_\infty \cap \mathcal{I}_{LBT}. \quad (26b)$$

An optimization problem of this form is called the *model matching problem*.

²This can be intuitively understood via the *internal model principle* [11].

³This fact can be understood from the following informal observation: if $\mathbf{K} \in \mathcal{I}$ and \mathcal{I} is QI under \mathbf{G}_{22} , the Neumann series expansion gives

$$\mathbf{Q} = -\mathbf{K}(I - \mathbf{G}_{22}\mathbf{K})^{-1} = -\mathbf{K} - \mathbf{K}\mathbf{G}_{22}\mathbf{K} - \mathbf{K}(\mathbf{G}_{22}\mathbf{K})^2 - \dots$$

Since we can inductively show that every term in the last expression belongs to \mathcal{I} , $\mathbf{Q} \in \mathcal{I}$.

If \mathbf{G}_{22} is not stable, Youla parameter must be constructed using coprime factors of \mathbf{G}_{22} [13]. In this case, parametrization of stabilizing controllers subject to the information constraint is more involved. Nevertheless, if \mathcal{I} is quadratically invariant under \mathbf{G}_{22} , the information constraints can be recast as linear constraints on the Youla parameter [9]. Therefore, even in this case, the optimal H_2 control problem subject to information constraints can be reformulated as a convex optimization problem.

B. Centralized H_2 Optimal Control

Assume that A is a Hurwitz matrix and that $\mathbf{G}_{11} \in H_2$. In this case, the optimal solution \mathbf{Q} to the H_2 model matching problem (26) must be in H_2 , since otherwise $\|\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{Q}\mathbf{G}_{21}\|$ is unbounded under Assumption 1. Hence the centralized H_2 optimal control problem can be cast as a simple H_2 model matching problem

$$\min \|\mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{Q}\mathbf{G}_{21}\| \quad (27a)$$

$$\text{s.t } \mathbf{Q} \in H_2. \quad (27b)$$

One approach to find a solution to the model matching problem is to apply the projection theorem [5]. Let H be a Hilbert space and S be a closed nonempty subspace of H . If $a \in H$, then there exists an element $x_0 \in S$ such that $\|a - x_0\| \leq \|a - x\|$ for all $x \in S$. Such an element x_0 is called a projection of a onto S , and is denoted by $x_0 = \pi_S(a)$. It can be shown that $x_0 = \pi_S(a)$ if and only if $x_0 \in S$ and $\langle a - x_0, x \rangle = 0$ for all $x \in S$.

In the model matching problem (27), a set $S \triangleq \{\mathbf{G}_{12}\mathbf{Q}\mathbf{G}_{21} : \mathbf{Q} \in H_2\}$ defines a closed nonempty subspace of H_2 . Hence the optimal solution to (27) can be found by computing a projection of \mathbf{G}_{11} onto S . To this end, a coprime factorization technique of rational functions, particularly the *inner-outer factorization*, is useful. We first recall the following facts. Proofs can be found in [14].

Theorem 3: Suppose (A, B) is stabilizable, H has full column rank, and $\begin{bmatrix} A - j\omega I & B \\ F & H \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$. Then algebraic Riccati equation $(X, K) = ARE_p(A, B, F, H)$ has a unique positive semidefinite solution. Moreover, it is stabilizing (that is, $A_{cl} \triangleq A + BK$ is a Hurwitz stable matrix).

Theorem 4: (1). (Right Inner-Outer Factorization)

Assume $\mathbf{G}_{12} = \begin{bmatrix} A & B \\ F & H \end{bmatrix}$ is stabilizable and

$\begin{bmatrix} A - j\omega I & B \\ F & H \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

Then there exists a right coprime factorization $\mathbf{G}_{12} = \mathbf{U}\mathbf{M}^{-1}$ such that \mathbf{U} is inner and \mathbf{M} is stably invertible.

A particular realization of such factorization is

$$\mathbf{U} = \begin{bmatrix} A + BK & B\Psi^{-\frac{1}{2}} \\ F + HK & H\Psi^{-\frac{1}{2}} \end{bmatrix}, \quad \mathbf{M}^{-1} = \begin{bmatrix} A & B \\ -\Psi^{\frac{1}{2}}K & \Psi^{\frac{1}{2}} \end{bmatrix}$$

where $\Psi = H^T H$ and $(X, K) = ARE_p(A, B, F, H)$.

(2). (Left Inner-Outer Factorization)

Assume $\mathbf{G}_{21} = \left[\begin{array}{c|c} A & W \\ \hline C & V \end{array} \right]$ is detectable and $\left[\begin{array}{c|c} A - j\omega I & W \\ \hline C & V \end{array} \right]$ has full row rank for all $\omega \in \mathbb{R}$. Then there exists a left coprime factorization $\mathbf{G}_{21} = \mathbf{N}^{-1}\mathbf{V}$ such that \mathbf{V} is co-inner and \mathbf{N} is stably invertible. A particular realization of such factorization is

$$\mathbf{V} = \left[\begin{array}{c|c} A + LC & W + LV \\ \hline \Phi^{-\frac{1}{2}}C & \Phi^{-\frac{1}{2}}V \end{array} \right], \quad \mathbf{N}^{-1} = \left[\begin{array}{c|c} A & -L\Phi^{\frac{1}{2}} \\ \hline C & \Phi^{\frac{1}{2}} \end{array} \right]$$

where $\Phi = VV^T$ and $(Y, L) = ARE_d(A, C, W, V)$.

Using \mathbf{U} and \mathbf{V} , the subspace S can be expressed as $S = \{\mathbf{UPV} : \mathbf{P} \in H_2\}$, where $\mathbf{P} = \mathbf{M}^{-1}\mathbf{QN}^{-1}$ is a new parameter. Since \mathbf{M} and \mathbf{N} are stably invertible, requiring that $\mathbf{Q} \in H_2$ is equivalent to requiring $\mathbf{P} \in H_2$.

Suppose the projection of \mathbf{G}_{11} onto S can be written as $\mathbf{UP}'\mathbf{V}$. By the optimality condition $\langle \mathbf{G}_{11} - \mathbf{UP}'\mathbf{V}, \mathbf{UPV} \rangle = 0 \quad \forall \mathbf{P} \in H_2$, one obtains $\mathbf{U}^*\mathbf{G}_{11}\mathbf{V}^* - \mathbf{P}' \in H_2^\perp$. Hence it must be that $\mathbf{P}' = \pi_{H_2}(\mathbf{U}^*\mathbf{G}_{11}\mathbf{V}^*)$. Moreover, straightforward state space manipulations show that

$$\mathbf{P}' = \pi_{H_2}(\mathbf{U}^*\mathbf{G}_{11}\mathbf{V}^*) = \left[\begin{array}{c|c} A & -L\Phi^{\frac{1}{2}} \\ \hline -\Psi^{\frac{1}{2}}K & 0 \end{array} \right]. \quad (28)$$

From this result, one can recover the optimal centralized H_2 controller.

$$\mathbf{K}_{opt} = \left[\begin{array}{c|c} A + BK + LC & -L \\ \hline K & 0 \end{array} \right].$$

It is also possible to show that the optimal control performance is

$$J_{cent} = \text{tr}(W^T X W) + \text{tr}(\Psi K Y K^T). \quad (29)$$

C. Proof of Proposition 1

By Theorem 3, $(\hat{X}_1, \hat{K}_1) = ARE_p(A, B, F, H)$ has a unique positive semidefinite solution that is also stabilizing; in particular $A_{\downarrow 1}^{\downarrow 1} + B_{\downarrow 1}^{\downarrow 1}\hat{K}_1$ is stable. Thus it suffices to show that if $A_{\downarrow i}^{\downarrow i} + B_{\downarrow i}^{\downarrow i}\hat{K}_i$ is stable for some $i \in \{1, 2, \dots, N-1\}$, then the algebraic Riccati equation

$$(\hat{X}_{i+1}, \hat{K}_{i+1}) = ARE_p(A_{\downarrow i+1}^{\downarrow i+1}, B_{\downarrow i+1}^{\downarrow i+1}, -H^{\downarrow i}\hat{K}_i^b, H^{\downarrow i+1}) \quad (30)$$

has a unique positive semidefinite solution that is also stabilizing. Since $(A_{\downarrow i+1}^{\downarrow i+1}, B_{\downarrow i+1}^{\downarrow i+1})$ is stabilizable by Assumption 1, by Theorem 3, the only way that (30) fails to have a unique positive semidefinite solution is that $\left[\begin{array}{c|c} A_{\downarrow i+1}^{\downarrow i+1} - j\omega I & B_{\downarrow i+1}^{\downarrow i+1} \\ \hline -H^{\downarrow i}\hat{K}_i^b & H^{\downarrow i+1} \end{array} \right] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$ for some $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \neq 0$.

Notice that z_1 cannot be zero as this implies z_2 is also zero due to the assumption that $H^{\downarrow i+1}$ has full column rank. Introducing $\tilde{z}_1 = \begin{bmatrix} 0 \\ z_1 \end{bmatrix}$, $\tilde{z}_2 = \begin{bmatrix} 0 \\ z_2 \end{bmatrix}$, we have

that $\left[\begin{array}{c|c} A_{\downarrow i}^{\downarrow i} - j\omega I & B_{\downarrow i}^{\downarrow i} \\ \hline -H^{\downarrow i}\hat{K}_i & H^{\downarrow i} \end{array} \right] \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = 0$. Left-multiplied by $\left[\begin{array}{c|c} I & -B_{\downarrow i}^{\downarrow i}(\Psi_{\downarrow i}^{\downarrow i})^{-1}H^{\downarrow i T} \end{array} \right]$, this reduces to $j\omega\tilde{z}_1 = (A_{\downarrow i}^{\downarrow i} +$

$B_{\downarrow i}^{\downarrow i}\hat{K}_i)\tilde{z}_1$, $\tilde{z}_1 \neq 0$. However, this contradicts that stability of $A_{\downarrow i}^{\downarrow i} + B_{\downarrow i}^{\downarrow i}\hat{K}_i$. This shows that (7) has a unique positive semidefinite and stabilizing solution for all $i \in \{2, \dots, N\}$. A similar argument can be applied to (8) as well.

D. Proof of Lemma 1

Lemma 2: (1). For every $i \in \{1, 2, \dots, N\}$, \mathbf{U}_i is an inner function and $\mathbf{G}_{12}^{cl}E^{\downarrow i} = \mathbf{U}_1\mathbf{U}_2 \cdots \mathbf{U}_i\mathbf{M}_i^{-1}$.

(2). For every $j \in \{1, 2, \dots, N\}$, \mathbf{V}_j is a co-inner function and $E_{\uparrow j}\mathbf{G}_{21}^{cl} = \mathbf{N}_j^{-1}\mathbf{V}_j\mathbf{V}_{j+1} \cdots \mathbf{V}_N$.

Proof: (1). When $i = 1$,

$$\begin{aligned} \mathbf{U}_1\mathbf{M}_1^{-1} &= \left[\begin{array}{c|c|c} A_{1,0}^{KL} & -BK_1 & B \\ \hline 0 & A & B \\ \hline F + HK_1 & -HK_1 & H \end{array} \right] \\ &= \left[\begin{array}{c|c|c} A_{1,0}^{KL} & 0 & 0 \\ \hline 0 & A & B \\ \hline F + HK_1 & \mathcal{F} & H \end{array} \right] = \mathbf{G}_{12}^{cl}. \end{aligned}$$

From the first to the second expression, a similarity transformation was applied by multiplying the state space ‘‘A’’ matrix by $\left[\begin{array}{c|c} I & \tilde{J}_1 \\ \hline 0 & I \end{array} \right]$ from the left and by its inverse from the right. The fact that K_1 is a solution to (2a) is exploited to obtain ‘‘0’’ at the upper-right corner of the state space ‘‘A’’ matrix. The last step was obtained by eliminating uncontrollable states. When $i \in \{2, 3, \dots, N\}$,

$$\begin{aligned} \mathbf{U}_i\mathbf{M}_i^{-1} &= \left[\begin{array}{c|c|c} A_{i,i-1}^{KL} & -B^{\downarrow i}\mathcal{K}_i & B^{\downarrow i} \\ \hline 0 & A & B^{\downarrow i} \\ \hline C_{U_i} & -\Psi_{\downarrow i-1}^{\downarrow i-1}E_{\downarrow i-1}E^{\downarrow i}\mathcal{K}_i & \Psi_{\downarrow i-1}^{\downarrow i-1}E_{\downarrow i-1}E^{\downarrow i} \end{array} \right] \\ &= \left[\begin{array}{c|c|c} A_{i,i-1}^{KL} & 0 & 0 \\ \hline 0 & A & B^{\downarrow i} \\ \hline C_{U_i} & -\Psi_{\downarrow i-1}^{\downarrow i-1}\mathcal{K}_{i-1} & \Psi_{\downarrow i-1}^{\downarrow i-1}E_{\downarrow i-1}E^{\downarrow i} \end{array} \right] \\ &= \mathbf{M}_{i-1}^{-1}E_{\downarrow i-1}E^{\downarrow i}. \end{aligned}$$

From the first to the second expression, a similarity transformation was applied by multiplying by $\left[\begin{array}{c|c} I & \tilde{J}_i \\ \hline 0 & I \end{array} \right]$ from the left and by its inverse from the right. Combining the above results, one obtains $\mathbf{G}_{12}^{cl}E^{\downarrow i} = \mathbf{U}_1\mathbf{U}_2 \cdots \mathbf{U}_i\mathbf{M}_i^{-1}$. To see that \mathbf{U}_i is inner, notice that

$$\begin{aligned} A_{i,i-1}^{KL T} X_i + X_i A_{i,i-1}^{KL} + C_{U_i}^T C_{U_i} &= 0 \\ D_{U_i}^T C_{U_i} + B_{U_i}^T X_i &= 0 \end{aligned}$$

follows from (5). Since $D_{U_i}^T D_{U_i} = I$, by Corollary 13.30 in [14], \mathbf{U}_i is an inner function.

(2). When $j = N$,

$$\begin{aligned} \mathbf{N}_N^{-1}\mathbf{V}_N &= \left[\begin{array}{c|c|c} A & -\mathcal{L}_N C & -\mathcal{L}_N V \\ \hline 0 & A_{N+1,N}^{KL} & W + L_N V \\ \hline C & C & V \end{array} \right] \\ &= \left[\begin{array}{c|c|c} A & 0 & W \\ \hline 0 & A_{N+1,N}^{KL} & W + L_N V \\ \hline C & 0 & V \end{array} \right] = \mathbf{G}_{21}^{cl}. \end{aligned}$$

From the first to the second expression, a similarity transformation was applied by multiplying the state space ‘‘A’’ matrix

by $\begin{bmatrix} I & \hat{J}_N \\ 0 & I \end{bmatrix}$ from the left and by its inverse from the right. The last step was obtained by eliminating unobservable states. When $j \in \{1, 2, \dots, N-1\}$,

$$\begin{aligned} \mathbf{N}_j^{-1} \mathbf{V}_j &= \left[\begin{array}{cc|c} \mathcal{A} & -\mathcal{L}_j C_{\uparrow j} & -\mathcal{L}_j E_{\uparrow j} E^{\uparrow j+1} \Phi_{\uparrow j+1}^{\uparrow j+1 \frac{1}{2}} \\ 0 & A_{j+1,j}^{KL} & B_{V_j} \\ \hline C_{\uparrow j} & C_{\uparrow j} & E_{\uparrow j} E^{\uparrow j+1} \Phi_{\uparrow j+1}^{\uparrow j+1 \frac{1}{2}} \end{array} \right] \\ &= \left[\begin{array}{cc|c} \mathcal{A} & 0 & -\mathcal{L}_{j+1} \Phi_{\uparrow j+1}^{\uparrow j+1 \frac{1}{2}} \\ 0 & A_{j+1,j}^{KL} & B_{V_j} \\ \hline C_{\uparrow j} & 0 & E_{\uparrow j} E^{\uparrow j+1} \Phi_{\uparrow j+1}^{\uparrow j+1 \frac{1}{2}} \end{array} \right] \\ &= E_{\uparrow j} E^{\uparrow j+1} \mathbf{N}_{i+1}^{-1}. \end{aligned}$$

From the first to the second expression, a similarity transformation was applied by multiplying by $\begin{bmatrix} I & \hat{J}_j \\ 0 & I \end{bmatrix}$ from the left and by its inverse from the right. Combining the above results, one obtains $E_{\uparrow j} \mathbf{G}_{21}^{cl} = \mathbf{N}_j^{-1} \mathbf{V}_j \mathbf{V}_{j+1} \cdots \mathbf{V}_N$. To see that \mathbf{V}_j is co-inner, or equivalently that \mathbf{V}_j^T is inner, notice that

$$\begin{aligned} A_{j+1,j}^{KL} Y_j + Y_j A_{j+1,j}^{KL T} + B_{V_j} B_{V_j}^T &= 0 \\ D_{V_j} B_{V_j}^T + C_{V_j} Y_j &= 0 \end{aligned}$$

follows from (6). Since $D_{V_j} D_{V_j}^T = I$, \mathbf{V}_j is a co-inner function. ■

Lemma 3: Let Γ_i and Λ_j be given as in Lemma 1, while Γ and Λ be any matrices. Let $0 \leq i < j \leq N+1$.

(1). If $\mathbf{F} = \mathbf{U}_1 \cdots \mathbf{U}_i \tilde{\mathbf{P}}_i \mathbf{V}_j \cdots \mathbf{V}_N \in S_{i,j}$ where

$$\tilde{\mathbf{P}}_i \triangleq \left[\begin{array}{c|c} \mathcal{A} & \Lambda \\ \hline \Gamma_i & 0 \end{array} \right],$$

then $\pi_{S_{i+1,j}}(\mathbf{F}) = \mathbf{U}_1 \cdots \mathbf{U}_{i+1} \tilde{\mathbf{P}}_{i+1} \mathbf{V}_j \cdots \mathbf{V}_N$.

(2). If $\mathbf{F} = \mathbf{U}_1 \cdots \mathbf{U}_i \hat{\mathbf{P}}_j \mathbf{V}_j \cdots \mathbf{V}_N \in S_{i,j}$ where

$$\hat{\mathbf{P}}_j \triangleq \left[\begin{array}{c|c} \mathcal{A} & \Lambda_j \\ \hline \Gamma & 0 \end{array} \right],$$

then $\pi_{S_{i,j-1}}(\mathbf{F}) = \mathbf{U}_1 \cdots \mathbf{U}_i \hat{\mathbf{P}}_{i-1} \mathbf{V}_{j-1} \cdots \mathbf{V}_N$.

Proof: (1). We show that if $\tilde{\mathbf{P}}_i$ is in the assumed form, then so is $\hat{\mathbf{P}}_{i+1}$. By the optimality condition, $\pi_{S_{i+1,j}}(\mathbf{F})$ satisfies

$$\langle \mathbf{F} - \pi_{S_{i+1,j}}(\mathbf{F}), \mathbf{U}_1 \cdots \mathbf{U}_{i+1} \mathbf{M}_{i+1}^{-1} \mathbf{Q}_{i+1,j} \mathbf{N}_j^{-1} \mathbf{V}_j \cdots \mathbf{V}_N \rangle = 0$$

for all $\mathbf{Q}_{i+1,j} \in H_2$. Hence $\tilde{\mathbf{P}}_{i+1}$ must satisfy

$$\langle \mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i - \tilde{\mathbf{P}}_{i+1}, \mathbf{M}_{i+1}^{-1} \mathbf{Q}_{i+1,j} \mathbf{N}_j^{-1} \rangle = 0 \quad \forall \mathbf{Q}_{i+1,j} \in H_2.$$

Such $\tilde{\mathbf{P}}_{i+1}$ is given by $\tilde{\mathbf{P}}_{i+1} = \pi_{H_2}(\mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i)$, since in this case, the inner product of $\mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i - \tilde{\mathbf{P}}_{i+1} \in H_2^\perp$ and $\mathbf{M}_{i+1}^{-1} \mathbf{Q}_{i+1,j} \mathbf{N}_j^{-1} \in H_2$ is zero. The projection $\pi_{H_2}(\mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i)$ is computed as follows. For every $i \in$

$\{0, 1, \dots, N-1\}$,

$$\begin{aligned} \mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i &= \left[\begin{array}{cc|c} -A_{i+1,i}^{KL T} & C_{U_{i+1}}^T \Gamma_i & 0 \\ 0 & \mathcal{A} & \Lambda \\ \hline -B_{U_{i+1}}^T & D_{U_{i+1}}^T \Gamma_i & 0 \end{array} \right] \\ &= \left[\begin{array}{cc|c} -A_{i+1,i}^{KL T} & 0 & * \\ 0 & \mathcal{A} & \Lambda \\ \hline * & \Gamma_{i+1} & 0 \end{array} \right] \end{aligned}$$

To obtain the last expression, a similarity transformation is applied to the state space matrices by left-multiplying by $\begin{bmatrix} I & -X_{i+1} \tilde{J}_{i+1} \\ 0 & I \end{bmatrix}$ and right-multiplying by its inverse. Notice that the Riccati equation (2a) or (2b) appears on the upper right block of the state space ‘‘A’’ matrix, and hence this component is zero. Moreover, since \mathcal{A} is stable and $-A_{1,0}^{KL T}$ is anti-stable, its projection onto H_2 is given by

$$\tilde{\mathbf{P}}_{i+1} = \pi_{H_2}(\mathbf{U}_{i+1}^* \tilde{\mathbf{P}}_i) = \left[\begin{array}{c|c} \mathcal{A} & \Lambda \\ \hline \Gamma_{i+1} & 0 \end{array} \right].$$

(2). Similarly, we show that if $\hat{\mathbf{P}}_j$ is in the assumed form, then so is $\hat{\mathbf{P}}_{j-1}$. From the optimality condition, it is possible to infer that $\hat{\mathbf{P}}_{j-1} = \pi_{H_2}(\hat{\mathbf{P}}_j \mathbf{V}_{j-1}^*)$. For every $j \in \{1, 2, \dots, N+1\}$,

$$\begin{aligned} \hat{\mathbf{P}}_j \mathbf{V}_{j-1}^* &= \left[\begin{array}{cc|c} \mathcal{A} & \Lambda_j B_{V_{j-1}}^T & \Lambda_j D_{V_{j-1}}^T \\ 0 & -A_{j,j-1}^{KL T} & -C_{V_{j-1}}^T \\ \hline \Gamma & 0 & 0 \end{array} \right] \\ &= \left[\begin{array}{cc|c} \mathcal{A} & 0 & \Lambda_{j-1} \\ 0 & -A_{j,j-1}^{KL T} & * \\ \hline \Gamma & * & 0 \end{array} \right]. \end{aligned}$$

A similarity transformation is applied to the state space matrices by left-multiplying by $\begin{bmatrix} I & -\hat{J}_{j-1} Y_{j-1} \\ 0 & I \end{bmatrix}$ and right-multiplying by its inverse. Since $-A_{j,j-1}^{KL T}$ is anti-stable, its projection onto H_2 is $\hat{\mathbf{P}}_{j-1} = \pi_{H_2}(\hat{\mathbf{P}}_j \mathbf{V}_{j-1}^*) = \left[\begin{array}{c|c} \mathcal{A} & \Lambda_{j-1} \\ \hline \Gamma & 0 \end{array} \right]$. ■

Proof of Lemma 1 is by induction. When $i = 0$ and $j = N+1$, the identity (25) clearly holds since $\mathbf{P}'_{0,N+1} = \mathbf{G}_{11}^{cl}$. So suppose (25) holds for some (i, j) such that $0 \leq i < j \leq N+1$. By the nested projection, $\pi_{S_{i+1,j}}(\mathbf{G}_{11}^{cl}) = \pi_{S_{i+1,j}}(\mathbf{F})$, where $\mathbf{F} = \pi_{S_{i,j}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \cdots \mathbf{U}_i \mathbf{P}_{i,j} \mathbf{V}_j \cdots \mathbf{V}_N$. Applying Lemma 3 (1), we have that $\pi_{S_{i+1,j}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \cdots \mathbf{U}_{i+1} \hat{\mathbf{P}}_{i+1,j} \mathbf{V}_j \cdots \mathbf{V}_N$. Hence, we have verified that (25) holds for $(i+1, j)$. Similarly, by the nested projection, $\pi_{S_{i,j-1}}(\mathbf{G}_{11}^{cl}) = \pi_{S_{i,j-1}}(\mathbf{F})$. Applying Lemma 3 (2), we have that $\pi_{S_{i,j-1}}(\mathbf{G}_{11}^{cl}) = \mathbf{U}_1 \cdots \mathbf{U}_i \hat{\mathbf{P}}_{i,j-1} \mathbf{V}_{j-1} \cdots \mathbf{V}_N$. Hence, we have verified that (25) holds for $(i, j-1)$. This proves that the identity (25) holds for every subspace in Fig. 4. Finally, state space expressions for $\mathbf{R}_{(i-1,j) \rightarrow (i,j)} = \mathbf{P}'_{i-1,j} - \mathbf{U}_i \mathbf{P}'_{i,j}$ and $\mathbf{R}_{(i,j+1) \rightarrow (i,j)} = \mathbf{P}'_{i,j+1} - \mathbf{P}'_{i,j} \mathbf{V}_j$ are obtained by straightforward state space manipulations.

E. Certainty Equivalence

We verify that $x^{K_i}(t)$ can be interpreted as the least mean square estimate of $x(t)$ conditioned on the observations of outputs of upstream subsystems. In other words, if $\mathbf{H}_{y_j} : w \mapsto y_{\uparrow j}$ and $\mathbf{F} : w \mapsto x$ are given, a transfer function $\mathbf{Q}_{x_j} : y_{\uparrow j} \mapsto x^{K_j}$ defined by the proposed controller minimizes $\|\mathbf{F} - \mathbf{Q}_{x_j} \mathbf{H}_{y_j}\|$ over H_2 . Notice that $\mathbf{H}_{y_j} = E_{\uparrow j} \mathbf{G}_{21}^{cl}$,

$$\mathbf{F} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{W} \\ \hline \text{row}\{0, \dots, 0, I\} & 0 \end{array} \right], \mathbf{Q}_{x_j} = \left[\begin{array}{c|c} A_K & B_K E_{\uparrow j} \\ \hline E_j & 0 \end{array} \right].$$

By the optimality condition, it suffices to check that $\pi_{S_{0,j}}(\mathbf{F}) = \mathbf{Q}_{x_j} \mathbf{H}_{y_j}$. Applying Lemma 3 (2) repeatedly, the LHS becomes $\pi_{S_{0,j}}(\mathbf{F}) = \hat{\mathbf{P}}_j \mathbf{V}_j \cdots \mathbf{V}_N$ where $\hat{\mathbf{P}}_j = \left[\begin{array}{c|c} \mathcal{A} & -\mathcal{L}_j \Phi_{\uparrow j}^{\uparrow j \frac{1}{2}} \\ \hline \text{row}\{0, \dots, 0, I\} & 0 \end{array} \right]$, while the RHS is

$$\begin{aligned} \mathbf{Q}_{x_j} \mathbf{H}_{y_j} &= \mathbf{Q}_{x_j} \mathbf{N}_j^{-1} \mathbf{V}_j \cdots \mathbf{V}_N \\ &= \left[\begin{array}{c|c} \mathcal{A} & -\mathcal{L}_j \Phi_{\uparrow j}^{\uparrow j \frac{1}{2}} \\ \hline \text{row}\{E_j, 0\} & 0 \end{array} \right] \mathbf{V}_j \cdots \mathbf{V}_N. \end{aligned}$$

To see $\hat{\mathbf{P}}_j = \mathbf{Q}_{x_j} \mathbf{H}_{y_j}$, notice that

$$\begin{aligned} \hat{\mathbf{P}}_j - \mathbf{Q}_{x_j} \mathbf{H}_{y_j} &= \left[\begin{array}{c|c} \mathcal{A} & -\mathcal{L}_j \Phi_{\uparrow j}^{\uparrow j \frac{1}{2}} \\ \hline \text{row}\{-E_j, I\} & 0 \end{array} \right] \\ &= \left[\begin{array}{c|c} \bar{\mu} \mathcal{A} \bar{\zeta} & -\bar{\mu} \mathcal{L}_j \Phi_{\uparrow j}^{\uparrow j \frac{1}{2}} \\ \hline \text{row}\{-E_j, I\} \zeta & 0 \end{array} \right]. \quad (31) \end{aligned}$$

The upper right block of (31) has nonzero matrices only on the first j subblocks, while the first j subblocks of the lower left block of (31) are all zero matrices. Since $\bar{\mu} \mathcal{A} \bar{\zeta}$ is upper-triangular as observed in (12), all controllable states are not observable and (31) is identically zero. This proves $\pi_{S_{0,j}}(\mathbf{F}) = \mathbf{Q}_{x_j} \mathbf{H}_{y_j}$.