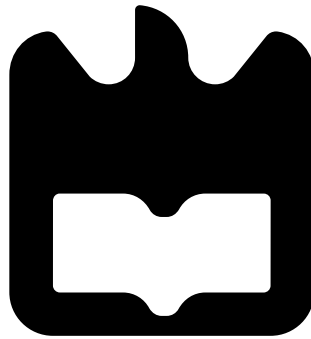




**Tatiana
Odzijewicz**

Generalized Fractional Calculus of Variations

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Cálculo das Variações Fraccional Generalizado

Tese de doutoramento apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações (PDMA 2009-2013) da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Associado com Agregação do Departamento de Matemática da Universidade de Aveiro, e co-orientação da Doutora Agnieszka Barbara Malinowska, Professora Auxiliar do Departamento de Matemática da Universidade Técnica de Białystok, Polónia.

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Resumo

Nesta tese de doutoramento apresentamos um cálculo das variações fraccional generalizado. Consideramos problemas variacionais com derivadas e integrais fraccionais generalizados e estudamo-los usando métodos directos e indirectos. Em particular, obtemos condições necessárias de optimalidade de Euler–Lagrange para o problema fundamental e isoperimétrico, condições de transversalidade e teoremas de Noether. Demonstramos a existência de soluções, num espaço de funções apropriado, sob condições do tipo de Tonelli. Terminamos mostrando a existência de valores próprios, e correspondentes funções próprias ortogonais, para problemas de Sturm–Liouville.

Palavras chave

cálculo das variações, condições necessárias de optimalidade do tipo de Euler–Lagrange, métodos directos, problemas isoperimétricos, teorema de Noether, cálculo fraccional, problema de Sturm–Liouville.

**2010 Mathematics Subject
Classification:**

26A33; 49K05; 49K21.

Abstract

In this thesis we introduce a generalized fractional calculus of variations. We consider variational problems containing generalized fractional integrals and derivatives and study them using standard (indirect) and direct methods. In particular, we obtain necessary optimality conditions of Euler–Lagrange type for the fundamental and isoperimetric problems, natural boundary conditions, and Noether theorems. Existence of solutions is shown under Tonelli type conditions. Moreover, we apply our results to prove existence of eigenvalues, and corresponding orthogonal eigenfunctions, to fractional Sturm–Liouville problems.

Keywords

calculus of variations, necessary optimality conditions of Euler–Lagrange type, direct methods, isoperimetric problem, Noether’s theorem, fractional calculus, Sturm–Liouville problem.

2010 Mathematics Subject Classification:

26A33; 49K05; 49K21.

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Introduction

This thesis is dedicated to the generalized fractional calculus of variations and its main task is to unify and extend results concerning the standard fractional variational calculus, that are available in the literature. My adventure with the subject started on the first year of my PhD Doctoral Programme, when I studied the course, given by my present supervisor Delfim F. M. Torres, called *Calculus of Variations and Optimal Control*. He described me an idea of the fractional calculus and showed that one can consider variational problems with non-integer operators. Fractional integrals and derivatives can be defined in different ways, and consequently in each case one must consider different variational problems. Therefore, my supervisor suggested me to study more general operators, that by choosing special kernels, reduce to the standard fractional integrals and derivatives. Finally, this interest resulted in my PhD thesis entitled *Generalized Fractional Calculus of Variations*.

The calculus of variations is a mathematical research field that was born in 1696 with the solution to the brachistochrone problem (see, e.g., [104]) and is focused on finding extremal values of functionals [33, 45, 48, 104]. Usually, considered functionals are given in the form of an integral that involves an unknown function and its derivatives. Variational problems are particularly attractive because of their many-fold applications, e.g., in physics, engineering, and economics; the variational integral may represent an action, energy, or cost functional [39, 105]. The calculus of variations possesses also important connections with other fields of mathematics, e.g., with the particularly important in this work— fractional calculus.

Fractional calculus, i.e., the calculus of non-integer order derivatives, has also its origin in the 1600s. It is a generalization of (integer) differential calculus, allowing to define derivatives (and integrals) of real or complex order [51, 91, 98]. During three centuries the theory of fractional derivatives developed as a pure theoretical field of mathematics, useful only for mathematicians. However, in the last few decades, fractional problems have received an increasing attention of many researchers. As mentioned in [16], *Science Watch of Thomson Reuters* identified the subject as an *Emerging Research Front* area. Fractional derivatives are non-local operators and are historically applied in the study of non-local or time dependent processes [91]. The first and well established application of fractional calculus in Physics was in the framework of anomalous diffusion, which is related to features observed in many physical systems. Here we can mention the report [71] demonstrating that fractional equations work as a complementary tool in the description of anomalous transport processes. Within the fractional approach it is possible to include external fields in a straightforward manner. As a consequence, in a short period of time the list of applications expanded. Applications include chaotic dynamics [107], material sciences [66], mechanics of fractal and complex media [26, 63], quantum mechanics [47, 61], physical kinetics [108], long-range dissipation [102], long-range interaction [101, 103], just to mention a few. This diversity of applications makes the fractional calculus an important subject, which requires serious attention and strong interest.

The calculus of variations and the fractional calculus are connected since the XIX century. Indeed, in 1823 Niels Henrik Abel applied the fractional calculus to the solution of an integral equation that arises in the formulation of the tautochrone problem. This problem, sometimes also called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. It turns out that the cycloid is the isochrone as well as the brachistochrone curve, solving simultaneously the brachistochrone problem of the calculus of variations and Abel's fractional problem [1]. It was however only in the XX century that both areas joined in a unique research field: the fractional calculus of variations.

The fractional calculus of variations consists in extremizing (minimizing or maximizing) functionals whose Lagrangians contain fractional integrals and derivatives. It was born in 1996-97, when Riewe derived Euler–Lagrange fractional differential equations and showed how non-conservative systems in mechanics can be described using fractional derivatives [96, 97]. It is a remarkable result since frictional and non-conservative forces are beyond the usual macroscopic variational treatment and, consequently, beyond the most advanced methods of classical mechanics [60]. Recently, several different approaches have been developed to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives. Results include problems depending on Caputo fractional derivatives, Riemann–Liouville fractional derivatives, Riesz fractional derivatives and others [7–10, 12, 18, 19, 21, 22, 29, 37, 42, 43, 52, 62, 67, 69, 72, 73, 77, 80, 100]. For the state of the art of the fractional calculus of variations we refer the reader to the recent book [70].

A more general unifying perspective to the subject is, however, possible, by considering fractional operators depending on general kernels [4, 57, 78, 79]. In this work we follow such an approach, developing a generalized fractional calculus of variations. We consider problems, where the Lagrangians depend not only on classical derivatives but also on generalized fractional operators. Moreover, we discuss even more general problems, where also classical integrals are substituted by generalized fractional integrals and obtain general theorems, for several types of variational problems, which are valid for rather arbitrary operators and kernels. As special cases, one obtains the recent results available in the literature of fractional variational calculus [35, 36, 46, 53, 70].

This thesis consists of two parts. The first one, named Synthesis, gives preliminary definitions and properties of fractional operators under consideration (Chapter 1). Moreover, it briefly describes recent results on the fractional calculus of variations (Chapter 2). The second one, called Original Work, contains new results published during my PhD project in peer reviewed international journals, as chapters in books, or in the conference proceedings [23, 24, 41, 77–83, 86–89]. It is divided in three chapters. We begin with Chapter 3, where we apply standard methods to solve several problems of the generalized fractional calculus of variations. We consider problems with Lagrangians depending on classical derivatives, generalized fractional integrals and generalized fractional derivatives. We obtain necessary optimality conditions for the basic and isoperimetric problems, as well as natural boundary conditions for free boundary value problems. In addition, we prove a generalized fractional counterpart of Noether's theorem. We consider the case of one and several independent variables. Moreover, each section contains illustrative optimization problems. Chapter 4 is dedicated to direct methods in the fractional calculus of variations. We prove a generalized fractional Tonelli's theorem, showing existence of minimizers for fractional variational functionals. Then we obtain necessary optimality conditions for minimizers. Several illus-

trative examples are presented. In the last Chapter 5 we show a certain application of the fractional variational calculus. More precisely, we prove existence of eigenvalues and corresponding eigenfunctions for the fractional Sturm–Liouville problem using variational methods. Moreover, we show two theorems concerning the lowest eigenvalue and illustrate our results through an example. We finish the thesis with a conclusion, pointing out important directions of future research.

Part I
Synthesis

Chapter 1

Fractional Calculus

Fractional calculus is a generalization of (integer) differential calculus, in the sense that it deals with derivatives of real or complex order. It was introduced on 30th September 1695. On that day, Leibniz wrote a letter to L'Hôpital, raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives. L'Hôpital wanted to know the result for the derivative of order $n = 1/2$. Leibniz replied that “*one day, useful consequences will be drawn*” and, in fact, his vision became a reality. However, the study of non-integer order derivatives did not appear in the literature until 1819, when Lacroix presented a definition of fractional derivative based on the usual expression for the n th derivative of the power function [59]. Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., non-integer) differential operators were introduced: the Grunwald–Letnikov, Riemann–Liouville, Hadamard, Caputo, Riesz [47, 51, 91, 98] and the more recent notions of Cresson [29], Katugampola [49], Klimek [52], Kilbas [50] or variable order fractional operators introduced by Samko and Ross in 1993 [99].

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in [4] and later studied in [24, 57, 78, 79, 82, 87, 89]. Precisely, authors considered general operators, which by choosing special kernels, reduce to the standard fractional operators. However, other nonstandard kernels can also be considered as particular cases.

This chapter presents preliminary definitions and facts of classical, variable order and generalized fractional operators.

1.1 One-dimensional Fractional Calculus

We begin with basic facts on the one-dimensional classical, variable order, and generalized fractional operators.

1.1.1 Classical Fractional Operators

In this section, we present definitions and properties of the one-dimensional fractional integrals and derivatives under consideration. The reader interested in the subject is referred to the books [51, 53, 91, 98].

Definition 1 (Left and right Riemann–Liouville fractional integrals). *We define the left and the right Riemann–Liouville fractional integrals ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) by*

$${}_aI_t^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \quad t \in (a, b], \quad (1.1)$$

and

$${}_tI_b^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)d\tau}{(\tau-t)^{1-\alpha}}, \quad t \in [a, b), \quad (1.2)$$

respectively. Here $\Gamma(\alpha)$ denotes Euler’s Gamma function. Note that, ${}_aI_t^\alpha[f]$ and ${}_tI_b^\alpha[f]$ are defined a.e. on (a, b) for $f \in L^1(a, b; \mathbb{R})$.

One can also define fractional integral operators in the frame of Hadamard setting. In the following, we present definitions of Hadamard fractional integrals.

Definition 2 (Left and right Hadamard fractional integrals). *We define the left-sided and right-sided Hadamard integrals of fractional order $\alpha \in \mathbb{R}$ ($\alpha > 0$) by*

$${}_aJ_t^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)d\tau}{\tau}, \quad t > a$$

and

$${}_tJ_b^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{\tau}{t}\right)^{\alpha-1} \frac{f(\tau)d\tau}{\tau}, \quad t < b,$$

respectively.

Definition 3 (Left and right Riemann–Liouville fractional derivatives). *The left Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{R}$ ($0 < \alpha < 1$) of a function f , denoted by ${}_aD_t^\alpha[f]$, is defined by*

$$\forall t \in (a, b), \quad {}_aD_t^\alpha[f](t) := \frac{d}{dt} {}_aI_t^{1-\alpha}[f](t).$$

Similarly, the right Riemann–Liouville fractional derivative of order α of a function f , denoted by ${}_tD_b^\alpha[f]$, is defined by

$$\forall t \in [a, b), \quad {}_tD_b^\alpha[f](t) := -\frac{d}{dt} {}_tI_b^{1-\alpha}[f](t).$$

As we can see below, Riemann–Liouville fractional integral and differential operators of power functions return power functions.

Property 1 (cf. Property 2.1 [51]). *Now, let $1 > \alpha, \beta > 0$. Then the following identities hold:*

$$\begin{aligned} {}_aI_t^\alpha[(\tau-a)^{\beta-1}](t) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, \\ {}_aD_t^\alpha[(\tau-a)^{\beta-1}](t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \end{aligned}$$

$${}_t I_b^\alpha [(b - \tau)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (b - t)^{\beta + \alpha - 1},$$

and

$${}_t D_b^\alpha [(b - \tau)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (b - t)^{\beta - \alpha - 1}.$$

Definition 4 (Left and right Caputo fractional derivatives). *The left and the right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ ($0 < \alpha < 1$) are given by*

$$\forall t \in (a, b), \quad {}_a^C D_t^\alpha [f](t) := {}_a I_t^{1-\alpha} \left[\frac{d}{dt} f \right] (t)$$

and

$$\forall t \in [a, b), \quad {}_t^C D_b^\alpha [f](t) := -{}_t I_b^{1-\alpha} \left[\frac{d}{dt} f \right] (t),$$

respectively.

Let $0 < \alpha < 1$ and $f \in AC([a, b]; \mathbb{R})$. Then the Riemann–Liouville and Caputo fractional derivatives satisfy relations

$${}_a^C D_t^\alpha [f](t) = {}_a D_t^\alpha [f](t) - \frac{f(a)}{(t - a)^\alpha \Gamma(1 - \alpha)}, \quad (1.3)$$

$${}_t^C D_b^\alpha [f](t) = -{}_t D_b^\alpha [f](t) + \frac{f(b)}{(b - t)^\alpha \Gamma(1 - \alpha)}, \quad (1.4)$$

that can be found in [51]. Moreover, for Riemann–Liouville fractional integrals and derivatives, the following composition rules hold

$$({}_a I_t^\alpha \circ {}_a D_t^\alpha) [f](t) = f(t), \quad (1.5)$$

$$({}_t I_b^\alpha \circ {}_t D_b^\alpha) [f](t) = f(t). \quad (1.6)$$

Note that, if $f(a) = 0$, then (1.3) and (1.5) give

$$({}_a I_t^\alpha \circ {}_a^C D_t^\alpha) [f](t) = ({}_a I_t^\alpha \circ {}_a D_t^\alpha) [f](t) = f(t), \quad (1.7)$$

and if $f(b) = 0$, then (1.4) and (1.6) imply that

$$({}_t I_b^\alpha \circ {}_t^C D_b^\alpha) [f](t) = ({}_t I_b^\alpha \circ {}_t D_b^\alpha) [f](t) = f(t). \quad (1.8)$$

The following assertion shows that Riemann–Liouville fractional integrals satisfy semi-group property.

Property 2 (cf. Lemma 2.3 [51]). *Let $1 > \alpha, \beta > 0$ and $f \in L^r(a, b; \mathbb{R})$, ($1 \leq r \leq \infty$). Then, equations*

$$\left({}_a I_t^\alpha \circ {}_a I_t^\beta \right) [f](t) = {}_a I_t^{\alpha+\beta} [f](t),$$

and

$$\left({}_t I_b^\alpha \circ {}_t I_b^\beta \right) [f](t) = {}_t I_b^{\alpha+\beta} [f](t)$$

are satisfied.

Next results show that, for certain classes of functions, Riemann–Liouville fractional derivatives and Caputo fractional derivatives are left inverse operators of Riemann–Liouville fractional integrals.

Property 3 (cf. Lemma 2.4 [51]). *If $1 > \alpha > 0$ and $f \in L^r(a, b; \mathbb{R})$, ($1 \leq r \leq \infty$), then the following is true:*

$$({}_a D_t^\alpha \circ {}_a I_t^\alpha)[f](t) = f(t),$$

$$({}_t D_b^\alpha \circ {}_t I_b^\alpha)[f](t) = f(t).$$

Property 4 (cf. Lemma 2.21 [51]). *Let $1 > \alpha > 0$. If f is continuous on the interval $[a, b]$, then*

$$({}_a^C D_t^\alpha \circ {}_a I_t^\alpha)[f](t) = f(t),$$

$$({}_t^C D_b^\alpha \circ {}_t I_b^\alpha)[f](t) = f(t).$$

For r -Lebesgue integrable functions, Riemann–Liouville fractional integrals and derivatives satisfy the following composition properties.

Property 5 (cf. Property 2.2 [51]). *Let $1 > \alpha > \beta > 0$ and $f \in L^r(a, b; \mathbb{R})$, ($1 \leq r \leq \infty$). Then, relations*

$$\left({}_a D_t^\beta \circ {}_a I_t^\alpha \right) [f](t) = {}_a I_t^{\alpha-\beta} [f](t),$$

and

$$\left({}_t D_b^\beta \circ {}_t I_b^\alpha \right) [f](t) = {}_t I_b^{\alpha-\beta} [f](t)$$

are satisfied.

In classical calculus, integration by parts formula relates the integral of a product of functions to the integral of their derivative and antiderivative. As we can see below, this formula works also for fractional derivatives, however it changes the type of differentiation: left Riemann–Liouville fractional derivatives are transformed to right Caputo fractional derivatives.

Property 6 (cf. Lemma 2.19 [53]). *Assume that $0 < \alpha < 1$, $f \in AC([a, b]; \mathbb{R})$ and $g \in L^r(a, b; \mathbb{R})$ ($1 \leq r \leq \infty$). Then, the following integration by parts formula holds:*

$$\int_a^b f(t) {}_a D_t^\alpha [g](t) dt = \int_a^b g(t) {}_t^C D_b^\alpha [f](t) dt + f(t) {}_a I_t^{1-\alpha} [g](t) \Big|_{t=a}^{t=b}. \quad (1.9)$$

Let us recall the following property yielding boundedness of Riemann–Liouville fractional integral in the space $L^r(a, b; \mathbb{R})$ (cf. Lemma 2.1, formula 2.1.23, from the monograph by Kilbas et al. [51]).

Property 7. *The fractional integral ${}_a I_t^\alpha$ is bounded in space $L^r(a, b; \mathbb{R})$ for $\alpha \in (0, 1)$ and $r \geq 1$*

$$\|{}_a I_t^\alpha [f]\|_{L^r} \leq K_\alpha \|f\|_{L^r}, \quad K_\alpha = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \quad (1.10)$$

1.1.2 Variable Order Fractional Operators

In 1993, Samko and Ross [99] proposed an interesting generalization of fractional operators. They introduced the study of fractional integration and differentiation when the order is not a constant but a function. Afterwards, several works were dedicated to variable order fractional operators, their applications and interpretations [11, 28, 65]. In particular, Samko's variable order fractional calculus turns out to be very useful in mechanics and in the theory of viscous flows [28, 34, 65, 90, 94, 95]. Indeed, many physical processes exhibit fractional-order behavior that may vary with time or space [65]. The paper [28] is devoted to the study of a variable-order fractional differential equation that characterizes some problems in the theory of viscoelasticity. In [34] the authors analyze the dynamics and control of a nonlinear variable viscoelasticity oscillator, and two controllers are proposed for the variable order differential equations that track an arbitrary reference function. The work [90] investigates the drag force acting on a particle due to the oscillatory flow of a viscous fluid. The drag force is determined using the variable order fractional calculus, where the order of derivative vary according to the dynamics of the flow. In [95] a variable order differential equation for a particle in a quiescent viscous liquid is developed. For more on the application of variable order fractional operators to the modeling of dynamic systems, we refer the reader to the recent review article [94].

Let us introduce the following triangle:

$$\Delta := \{(t, \tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\},$$

and let $\alpha(t, \tau) : \Delta \rightarrow [0, 1]$ be such that $\alpha \in C^1(\bar{\Delta}; \mathbb{R})$.

Definition 5 (Left and right Riemann–Liouville integrals of variable order). *Operator*

$${}_a I_t^{\alpha(\cdot, \cdot)}[f](t) := \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1} f(\tau) d\tau \quad (t > a)$$

is the left Riemann–Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$, while

$${}_t I_b^{\alpha(\cdot, \cdot)}[f](t) := \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))} (\tau - t)^{\alpha(\tau, t) - 1} f(\tau) d\tau \quad (t < b)$$

is the right Riemann–Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$.

The following example gives a variable order fractional integral for the power function $(t - a)^\gamma$.

Example 1 (cf. Equation 4 of [99]). *Let $\alpha(t, \tau) = \alpha(t)$ be a function depending only on variable t , $0 < \alpha(t) < 1$ for almost all $t \in (a, b)$ and $\gamma > -1$. Then,*

$${}_a I_t^{\alpha(\cdot)}(t - a)^\gamma = \frac{\Gamma(\gamma + 1)(t - a)^{\gamma + \alpha(t)}}{\Gamma(\gamma + \alpha(t) + 1)}. \quad (1.11)$$

Next we define two types of variable order fractional derivatives.

Definition 6 (Left and right Riemann–Liouville derivatives of variable order). *The left Riemann–Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ of a function f is defined by*

$$\forall t \in (a, b), \quad {}_a D_t^{\alpha(\cdot, \cdot)}[f](t) := \frac{d}{dt} {}_a I_t^{1-\alpha(\cdot, \cdot)}[f](t),$$

while the right Riemann–Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ is defined by

$$\forall t \in [a, b), \quad {}_t D_b^{\alpha(\cdot, \cdot)}[f](t) := -\frac{d}{dt} {}_t I_b^{1-\alpha(\cdot, \cdot)}[f](t).$$

Definition 7 (Left and right Caputo derivatives of variable fractional order). *The left Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$ is defined by*

$$\forall t \in (a, b), \quad {}_a^C D_t^{\alpha(\cdot, \cdot)}[f](t) := {}_a I_t^{1-\alpha(\cdot, \cdot)} \left[\frac{d}{dt} f \right] (t),$$

while the right Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$ is given by

$$\forall t \in [a, b), \quad {}_t^C D_b^{\alpha(\cdot, \cdot)}[f](t) := -{}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\frac{d}{dt} f \right] (t).$$

1.1.3 Generalized Fractional Operators

This section presents definitions of one-dimensional generalized fractional operators. In special cases, these operators simplify to the classical Riemann–Liouville fractional integrals, and Riemann–Liouville and Caputo fractional derivatives. As before,

$$\Delta := \{(t, \tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}.$$

Definition 8 (Generalized fractional integrals of Riemann–Liouville type). *Let us consider a function k defined almost everywhere on Δ with values in \mathbb{R} . For any function f defined almost everywhere on (a, b) with value in \mathbb{R} , the generalized fractional integral operator K_P is defined for almost all $t \in (a, b)$ by:*

$$K_P[f](t) = \lambda \int_a^t k(t, \tau) f(\tau) d\tau + \mu \int_t^b k(\tau, t) f(\tau) d\tau, \quad (1.12)$$

with $P = \langle a, t, b, \lambda, \mu \rangle$, $\lambda, \mu \in \mathbb{R}$.

In particular, for suitably chosen kernels $k(t, \tau)$ and sets P , kernel operators K_P , reduce to the classical or variable order fractional integrals of Riemann–Liouville type, and classical fractional integrals of Hadamard type.

Example 2. (a) *Let $k^\alpha(t - \tau) = \frac{1}{\Gamma(\alpha)}(t - \tau)^{\alpha-1}$ and $0 < \alpha < 1$. If $P = \langle a, t, b, 1, 0 \rangle$, then*

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau =: {}_a I_t^\alpha[f](t)$$

is the left Riemann–Liouville fractional integral of order α ; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau =: {}_t I_b^\alpha[f](t)$$

is the right Riemann–Liouville fractional integral of order α .

(b) For $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1}$ and $P = \langle a, t, b, 1, 0 \rangle$,

$$K_P[f](t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1} f(\tau) d\tau =: {}_a I_t^{\alpha(\cdot, \cdot)}[f](t)$$

is the left Riemann–Liouville fractional integral of order $\alpha(\cdot, \cdot)$ and for $P = \langle a, t, b, 0, 1 \rangle$

$$K_P[f](t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))} (\tau - t)^{\alpha(\tau, t) - 1} f(\tau) d\tau =: {}_t I_b^{\alpha(\cdot, \cdot)}[f](t)$$

is the right Riemann–Liouville fractional integral of order $\alpha(t, \tau)$.

(c) For any $0 < \alpha < 1$, kernel $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha)} \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{1}{\tau}$ and $P = \langle a, t, b, 1, 0 \rangle$, the general operator K_P reduces to the left Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau) d\tau}{\tau} =: {}_a J_t^\alpha[f](t);$$

and for $P = \langle a, t, b, 0, 1 \rangle$ operator K_P reduces to the right Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{\tau}{t}\right)^{\alpha-1} \frac{f(\tau) d\tau}{\tau} =: {}_t J_b^\alpha[f](t).$$

(d) Generalized fractional integrals can be also reduced to, e.g., Riesz, Katugampola or Kilbas fractional operators. Their definitions can be found in [49–51].

The generalized differential operators A_P and B_P are defined with the help of the operator K_P .

Definition 9 (Generalized fractional derivative of Riemann–Liouville type). *The generalized fractional derivative of Riemann–Liouville type, denoted by A_P , is defined by*

$$A_P = \frac{d}{dt} \circ K_P.$$

The next differential operator is obtained by interchanging the order of the operators in the composition that defines A_P .

Definition 10 (Generalized fractional derivative of Caputo type). *The general kernel differential operator of Caputo type, denoted by B_P , is given by*

$$B_P = K_P \circ \frac{d}{dt}.$$

Example 3. *The standard Riemann–Liouville and Caputo fractional derivatives (see, e.g., [51, 53, 91, 98]) are easily obtained from the general kernel operators A_P and B_P , respectively. Let $k^\alpha(t - \tau) = \frac{1}{\Gamma(1-\alpha)} (t - \tau)^{-\alpha}$, $\alpha \in (0, 1)$. If $P = \langle a, t, b, 1, 0 \rangle$, then*

$$A_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha} f(\tau) d\tau =: {}_a D_t^\alpha[f](t)$$

is the standard left Riemann–Liouville fractional derivative of order α , while

$$B_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau =: {}^C D_t^\alpha[f](t)$$

is the standard left Caputo fractional derivative of order α ; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$-A_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} f(\tau) d\tau =: {}_t D_b^\alpha[f](t)$$

is the standard right Riemann–Liouville fractional derivative of order α , while

$$-B_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} f'(\tau) d\tau =: {}^C D_b^\alpha[f](t)$$

is the standard right Caputo fractional derivative of order α .

1.2 Multidimensional Fractional Calculus

In this section, we introduce notions of classical, variable order and generalized partial fractional integrals and derivatives, in a multidimensional finite domain. They are natural generalizations of the corresponding fractional operators of Section 1.1.1. Furthermore, similarly as in the integer order case, computation of partial fractional derivatives and integrals is reduced to the computation of one-variable fractional operators. Along the work, for $i = 1, \dots, n$, let a_i, b_i and α_i be numbers in \mathbb{R} and $t = (t_1, \dots, t_n)$ be such that $t \in \Omega_n$, where $\Omega_n = (a_1, b_1) \times \dots \times (a_n, b_n)$ is a subset of \mathbb{R}^n . Moreover, let us define the following sets:

$$\Delta_i := \{(t_i, \tau) \in \mathbb{R}^2 : a_i \leq \tau < t_i \leq b_i\}, \quad i = 1, \dots, n.$$

1.2.1 Classical Partial Fractional Integrals and Derivatives

In this section we present definitions of classical partial fractional integrals and derivatives. Interested reader can find more details in Section 24.1 of the book [98].

Definition 11 (Left and right Riemann–Liouville partial fractional integrals). *Let $t \in \Omega_n$. The left and the right partial Riemann–Liouville fractional integrals of order $\alpha_i \in \mathbb{R}$ ($\alpha_i > 0$) with respect to the i th variable t_i are defined by*

$${}_{a_i} I_{t_i}^{\alpha_i}[f](t) := \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} \frac{f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau}{(t_i - \tau)^{1-\alpha_i}}, \quad t_i > a_i, \quad (1.13)$$

and

$${}_{t_i} I_{b_i}^{\alpha_i}[f](t) := \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} \frac{f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau}{(\tau - t_i)^{1-\alpha_i}}, \quad t_i < b_i, \quad (1.14)$$

respectively.

Definition 12 (Left and right Riemann–Liouville partial fractional derivatives). *Let $t \in \Omega_n$. The left partial Riemann–Liouville fractional derivative of order $\alpha_i \in \mathbb{R}$ ($0 < \alpha_i < 1$) of a function f , with respect to the i th variable t_i is defined by*

$$\forall t_i \in (a_i, b_i], \quad {}_{a_i}D_{t_i}^{\alpha_i}[f](t) := \frac{\partial}{\partial t_i} {}_{a_i}I_{t_i}^{1-\alpha_i}[f](t).$$

Similarly, the right partial Riemann–Liouville fractional derivative of order α_i of a function f , with respect to the i th variable t_i is defined by

$$\forall t_i \in [a_i, b_i), \quad {}_{t_i}D_{b_i}^{\alpha_i}[f](t) := -\frac{\partial}{\partial t_i} {}_{t_i}I_{b_i}^{1-\alpha_i}[f](t).$$

Definition 13 (Left and right Caputo partial fractional derivatives). *Let $t \in \Omega_n$. The left and the right partial Caputo fractional derivatives of order $\alpha_i \in \mathbb{R}$ ($0 < \alpha_i < 1$) of a function f , with respect to the i th variable t_i are given by*

$$\forall t_i \in (a_i, b_i], \quad {}^C D_{a_i}^{\alpha_i}[f](t) := {}_{a_i}I_{t_i}^{1-\alpha_i} \left[\frac{\partial}{\partial t_i} f \right] (t),$$

and

$$\forall t_i \in [a_i, b_i), \quad {}^C D_{b_i}^{\alpha_i}[f](t) := -{}_{t_i}I_{b_i}^{1-\alpha_i} \left[\frac{\partial}{\partial t_i} f \right] (t),$$

respectively.

1.2.2 Variable Order Partial Fractional Integrals and Derivatives

In this section, we introduce the notions of partial fractional operators of variable order. In the following let us assume that $\alpha_i : \Delta_i \rightarrow [0, 1]$, $\alpha_i \in C^1(\bar{\Delta}; \mathbb{R})$, $i = 1, \dots, n$, $t \in \Omega_n$ and $f : \Omega_n \rightarrow \mathbb{R}$.

Definition 14. *The left Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot, \cdot)$ with respect to the i th variable t_i , is given by*

$${}_{a_i}I_{t_i}^{\alpha_i(\cdot, \cdot)}[f](t) := \int_{a_i}^{t_i} \frac{1}{\Gamma(\alpha_i(t_i, \tau))} (t_i - \tau)^{\alpha_i(t_i, \tau)-1} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau,$$

$t_i > a_i$, while

$${}_{t_i}I_{b_i}^{\alpha_i(\cdot, \cdot)}[f](t) := \int_{t_i}^{b_i} \frac{1}{\Gamma(\alpha_i(\tau, t_i))} (\tau - t_i)^{\alpha_i(\tau, t_i)-1} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau,$$

$t_i < b_i$, is the right Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot, \cdot)$ with respect to variable t_i .

Definition 15. *The left Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot, \cdot)$, with respect to the i th variable t_i , is given by*

$$\forall t_i \in (a_i, b_i], \quad {}_{a_i}D_{t_i}^{\alpha_i(\cdot, \cdot)}[f](t) = \frac{\partial}{\partial t_i} {}_{a_i}I_{t_i}^{1-\alpha_i(\cdot, \cdot)}[f](t)$$

while the right Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot, \cdot)$, with respect to the i th variable t_i , is defined by

$$\forall t_i \in [a_i, b_i], \quad {}_{t_i}D_{b_i}^{\alpha_i(\cdot, \cdot)}[f](t) = -\frac{\partial}{\partial t_i} {}_{t_i}I_{b_i}^{1-\alpha_i(\cdot, \cdot)}[f](t).$$

Definition 16. The left Caputo partial derivative of variable fractional order $\alpha_i(\cdot, \cdot)$, with respect to the i th variable t_i , is defined by

$$\forall t_i \in (a_i, b_i], \quad {}_a^C D_{t_i}^{\alpha_i(\cdot, \cdot)}[f](t) = {}_a I_{t_i}^{1-\alpha_i(\cdot, \cdot)} \left[\frac{\partial}{\partial t_i} f \right] (t),$$

while the right Caputo partial derivative of variable fractional order $\alpha_i(\cdot, \cdot)$, with respect to the i th variable t_i , is given by

$$\forall t_i \in [a_i, b_i), \quad {}_t^C D_{b_i}^{\alpha_i(\cdot, \cdot)}[f](t) = -{}_t I_{b_i}^{1-\alpha_i(\cdot, \cdot)} \left[\frac{\partial}{\partial t_i} f \right] (t).$$

Note that, if $\alpha_i(\cdot, \cdot)$ is a constant function, then the partial operators of variable fractional order are reduced to corresponding partial integrals and derivatives of constant order introduced in Section 1.2.1.

1.2.3 Generalized Partial Fractional Operators

Let us assume that $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ are in \mathbb{R}^n . We shall present definitions of generalized partial fractional integrals and derivatives. Let $k_i : \Delta_i \rightarrow \mathbb{R}$, $i = 1, \dots, n$ and $t \in \Omega_n$.

Definition 17 (Generalized partial fractional integral). For any function f defined almost everywhere on Ω_n with value in \mathbb{R} , the generalized partial integral K_{P_i} is defined for almost all $t_i \in (a_i, b_i)$ by:

$$\begin{aligned} K_{P_i}[f](t) := & \lambda_i \int_{a_i}^{t_i} k_i(t_i, \tau) f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau \\ & + \mu_i \int_{t_i}^{b_i} k_i(\tau, t_i) f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau, \end{aligned}$$

where $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$.

Definition 18 (Generalized partial fractional derivative of Riemann–Liouville type). The generalized partial fractional derivative of Riemann–Liouville type with respect to the i th variable t_i is given by

$$A_{P_i} := \frac{\partial}{\partial t_i} \circ K_{P_i}.$$

Definition 19 (Generalized partial fractional derivative of Caputo type). The generalized partial fractional derivative of Caputo type with respect to the i th variable t_i is given by

$$B_{P_i} := K_{P_i} \circ \frac{\partial}{\partial t_i}.$$

Example 4. Similarly, as in the one-dimensional case, partial operators K , A and B reduce to the standard partial fractional integrals and derivatives. The left- or right-sided Riemann–Liouville partial fractional integral with respect to the i th variable t_i is obtained by choosing the kernel $k_i^\alpha(t_i, \tau) = \frac{1}{\Gamma(\alpha_i)}(t_i - \tau)^{\alpha_i-1}$. That is,

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{\alpha_i-1} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{a_i}I_{t_i}^{\alpha_i}[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, and

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{\alpha_i-1} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{t_i}I_{b_i}^{\alpha_i}[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 0, 1 \rangle$. The standard left- and right-sided Riemann–Liouville and Caputo partial fractional derivatives with respect to i th variable t_i are received by choosing the kernel $k_i^\alpha(t_i, \tau) = \frac{1}{\Gamma(1-\alpha_i)}(t_i - \tau)^{-\alpha_i}$. If $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, then

$$A_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \frac{\partial}{\partial t_i} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{a_i}D_{t_i}^{\alpha_i}[f](t),$$

$$B_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{a_i}^C D_{t_i}^{\alpha_i}[f](t).$$

If $P_i = \langle a_i, t_i, b_i, 0, 1 \rangle$, then

$$-A_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \frac{\partial}{\partial t_i} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{t_i}D_{b_i}^{\alpha_i}[f](t),$$

$$-B_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau =: {}_{t_i}^C D_{b_i}^{\alpha_i}[f](t).$$

Moreover, one can easily check, that also variable order partial fractional integrals and derivatives are particular cases of operators K_{P_i} , A_{P_i} and B_{P_i} .

Chapter 2

Fractional Calculus of Variations

The calculus of variations is a beautiful and useful field of mathematics that deals with problems of determining extrema (maxima or minima) of functionals. For the first time, serious attention of scientists was directed to the variational calculus in 1696, when Johann Bernoulli asked about the curve with specified endpoints, lying in a vertical plane, for which the time taken by a material point sliding without friction and under gravity from one end to the other is minimal. This problem gained interest of such scientists as Leibniz, Newton or L'Hospital and was called brachistochrone problem. Afterwards, a student of Bernoulli, the brilliant Swiss mathematician Leonhard Euler, considered the problem of finding a function extremizing (minimizing or maximizing) an integral

$$\mathcal{J}(y) = \int_a^b F(y(t), \dot{y}(t), t) dt \quad (2.1)$$

subject to the boundary conditions

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b \quad (2.2)$$

with $y \in C^2([a, b]; \mathbb{R})$, $a, b, y_a, y_b \in \mathbb{R}$ and $F(u, v, t)$ satisfying some smoothness properties. He proved that curve $y(t)$ must satisfy the following necessary condition, so-called Euler–Lagrange equation:

$$\frac{\partial F(y(t), \dot{y}(t), t)}{\partial u} - \frac{d}{dt} \left(\frac{\partial F(y(t), \dot{y}(t), t)}{\partial v} \right) = 0. \quad (2.3)$$

Solutions of equation (2.3) are usually called extremals. It is important to remark that the calculus of variations is a very interesting topic because of its numerous applications in geometry and differential equations, in mechanics and physics, and in areas as diverse as engineering, medicine, economics, and renewable resources [27].

In the next example we give a simple application of the calculus of variations. Precisely, we present the soap bubble problem, stated by Euler in 1744.

Example 5 (cf. Example 14.1 [27]). *In the soap bubble problem we want to find a surface of rotation, spanned by two concentric rings of radius A and B , which has the minimum area. This wish is confirmed by experiment and is based on d'Alembert principle. In the sense of*

the calculus of variations, we can formulate the soap bubble problem in the following way: we want to minimize the variational functional

$$\mathcal{J}(y) = \int_a^b y(t) \sqrt{1 + \dot{y}(t)^2} dt \quad \text{subject to } y(a) = A, \quad y(b) = B.$$

This is a special case of problem (2.1)-(2.2) with $F(u, v, t) = u\sqrt{1 + v^2}$. Let $y(t) > 0 \forall t$. It is not difficult to verify that the Euler–Lagrange equation is given by

$$\ddot{y}(t) = \frac{1 + \dot{y}(t)^2}{y(t)}$$

and its solution is the catenary curve given by

$$y(t) = k \cosh\left(\frac{t + c}{k}\right),$$

where c, k are certain constants.

This thesis is devoted to the fractional calculus of variations and its generalizations. Therefore, in the next sections we present basic results of the non-integer variational calculus. Let us precise, that along the work we will understand $\partial_i F$ as the partial derivative of function F with respect to its i th argument.

2.1 Fractional Euler–Lagrange Equations

Within the years, several methods were proposed to solve mechanical problems with non-conservative forces, e.g., Rayleigh dissipation function method, technique introducing an auxiliary coordinate or approach including the microscopic details of the dissipation directly in the Lagrangian. Although, all mentioned methods are correct, they are not as direct and simple as it is in the case of conservative systems. In the notes from 1996-1997, Riewe presented a new approach to nonconservative forces [96, 97]. He claimed that friction forces follow from Lagrangians containing terms proportional to fractional derivatives. Precisely, for $y : [a, b] \rightarrow \mathbb{R}^r$ and $\alpha_i, \beta_j \in [0, 1]$, $i = 1, \dots, N$, $j = 1, \dots, N'$, he considered the following energy functional:

$$\mathcal{J}(y) = \int_a^b F\left({}_a D_t^{\alpha_1}[y](t), \dots, {}_a D_t^{\alpha_N}[y](t), {}_t D_b^{\beta_1}[y](t), \dots, {}_t D_b^{\beta_{N'}}[y](t), y(t), t\right) dt,$$

with r , N and N' being natural numbers. Using the fractional variational principle he obtained the following Euler–Lagrange equation:

$$\sum_{i=1}^N {}_t D_b^{\alpha_i} [\partial_i F] + \sum_{i=1}^{N'} {}_a D_t^{\beta_i} [\partial_{i+N} F] + \partial_{N'+N+1} F = 0. \quad (2.4)$$

Riewe illustrated his results through the classical problem of linear friction.

Example 6 ([97]). *Let us consider the following Lagrangian:*

$$F = \frac{1}{2}m\dot{y}^2 - V(y) + \frac{1}{2}\gamma i \left({}_a D_t^{\frac{1}{2}}[y] \right)^2, \quad (2.5)$$

where the first term in the sum represents kinetic energy, the second one potential energy, the last one is linear friction energy and $i^2 = -1$. Using (2.4) we can obtain the Euler–Lagrange equation for a Lagrangian containing derivatives of order one and order $\frac{1}{2}$:

$$\frac{\partial F}{\partial y} + {}_t D_b^{\frac{1}{2}} \left[\frac{\partial F}{\partial {}_a D_t^{\frac{1}{2}}[y]} \right] - \frac{d}{dt} \frac{\partial F}{\partial \dot{y}} = 0,$$

which, in the case of Lagrangian (2.5), becomes

$$m\ddot{y} = -\gamma i \left({}_t D_b^{\frac{1}{2}} \circ {}_a D_t^{\frac{1}{2}} \right) [y] - \frac{\partial V(y)}{\partial y}.$$

In order to obtain the equation with linear friction, $m\ddot{y} + \gamma\dot{y} + \frac{\partial V}{\partial y} = 0$, Riewe suggested considering an infinitesimal time interval, that is, the limiting case $a \rightarrow b$, while keeping $a < b$.

After the works of Riewe several authors contributed to the theory of the fractional variational calculus. First, let us point out the approach discussed by Klimek in [52]. It was suggested to study symmetric fractional derivatives of order α ($0 < \alpha < 1$) defined as follows:

$$\mathcal{D}^\alpha := \frac{1}{2} {}_a D_t^\alpha + \frac{1}{2} {}_t D_b^\alpha.$$

In contrast to the left and right fractional derivatives, operator \mathcal{D}^α is symmetric for the scalar product given by

$$\langle f|g \rangle := \int_a^b \overline{f(t)}g(t) dt,$$

that is,

$$\langle \mathcal{D}^\alpha[f]|g \rangle = \langle f|\mathcal{D}^\alpha[g] \rangle.$$

With this notion for the fractional derivative, for $\alpha_i \in (0, 1)$ and $y : [a, b] \rightarrow \mathbb{R}^r$, $i = 1, \dots, N$, Klimek considered the following action functional:

$$\mathcal{J}(y) = \int_a^b F(\mathcal{D}^{\alpha_1}[y](t), \dots, \mathcal{D}^{\alpha_N}[y](t), y(t), t) dt. \quad (2.6)$$

Using the fractional variational principle, she derived the Euler–Lagrange equation given by

$$\partial_{N+1}F + \sum_{i=1}^N \mathcal{D}^{\alpha_i} [\partial_i F] = 0. \quad (2.7)$$

As an example Klimek considered the following variational functional

$$\mathcal{J}(y) = \int_a^b 2m\dot{y}^2(t) - \gamma i \left(\mathcal{D}^{\frac{1}{2}}[y](t) \right)^2 - V(y(t)) dt$$

and under appropriate assumptions arrived to the equation with linear friction

$$m\ddot{y} = -\frac{\partial V}{\partial y} - \gamma\dot{y}. \quad (2.8)$$

Another type of problems, containing Riemann–Liouville fractional derivatives, was discussed by Klimek in [53]:

$$\mathcal{J}(y) = \int_a^b F({}_a D_t^{\alpha_1}[y](t), \dots, {}_a D_t^{\alpha_N}[y](t), y(t), t) dt$$

and the Euler–Lagrange equation

$$\partial_{N+1} F + \sum_{i=1}^N {}^C D_b^{\alpha_i} [\partial_i F] = 0 \quad (2.9)$$

including fractional derivatives of the Caputo type was obtained.

The next examples are borrowed from [53].

Example 7 (cf. Example 4.1.1 of [53]). *Let $0 < \alpha < 1$ and y be a minimizer of the functional*

$$\mathcal{J}(y) = \int_a^b \frac{1}{2} y(t) {}_a D_t^\alpha [y](t) dt.$$

Then y is a solution to the following Euler–Lagrange equation:

$$\frac{1}{2} ({}_a D_t^\alpha [y] + {}^C D_b^\alpha [y]) = 0.$$

Example 8 (cf. Example 4.1.2 of [53]). *Let $0 < \alpha < 1$. The model of harmonic oscillator, in the framework of classical mechanics, is connected to an action*

$$\mathcal{J}(y) = \int_a^b \left[-\frac{1}{2} y'^2(t) + \frac{\omega^2}{2} y^2(t) \right] dt, \quad (2.10)$$

and is determined by the following equation

$$y'' + \omega^2 y = 0. \quad (2.11)$$

If in functional (2.10) instead of derivative of order one we put a derivative of fractional order α , then

$$\mathcal{J}(y) = \int_a^b \left[-\frac{1}{2} ({}_a D_t^\alpha [y](t))^2 + \frac{\omega^2}{2} y^2(t) \right] dt$$

and by (2.9) the Euler–Lagrange equation has the following form:

$$-{}^C D_b^\alpha [{}_a D_t^\alpha [y]] + \omega^2 y = 0. \quad (2.12)$$

If $\alpha \rightarrow 1^+$, then equation (2.12) reduces to (2.11). The proof of this fact, as well as solutions to fractional harmonic oscillator equation (2.12), can be found in [53].

2.2 Fractional Embedding of Euler–Lagrange Equations

The notion of embedding introduced in [30] is an algebraic procedure providing an extension of classical differential equations over an arbitrary vector space. This formalism is developed in the framework of stochastic processes [30], non-differentiable functions [31], and fractional equations [29]. The general scheme of embedding theories is the following: (i) fix a vector space V and a mapping $\iota : C^0([a, b], \mathbb{R}^n) \rightarrow V$; (ii) extend differential operators over V ; (iii) extend the notion of integral over V . Let (ι, D, J) be a given embedding formalism, where a linear operator $D : V \rightarrow V$ takes place for a generalized derivative on V , and a linear operator $J : V \rightarrow \mathbb{R}$ takes place for a generalized integral on V . The embedding procedure gives two different ways, a priori, to generalize Euler–Lagrange equations. The first (pure algebraic) way is to make a direct embedding of the Euler–Lagrange equation. The second (analytic) is to embed the Lagrangian functional associated to the equation and to derive, by the associated calculus of variations, the Euler–Lagrange equation for the embedded functional. A natural question is then the problem of coherence between these two extensions:

COHERENCE PROBLEM. *Let (ι, D, J) be a given embedding formalism. Do we have equivalence between the Euler–Lagrange equation which gives the direct embedding and the one received from the embedded Lagrangian system?*

As shown in the work [29] for standard fractional differential calculus, the answer to the question above is known to be negative. To be more precise, let us define the following operator first introduced in [29].

Definition 20 (Fractional operator of order (α, β)). *Let $a, b \in \mathbb{R}$, $a < b$ and $\mu \in \mathbb{C}$. We define the fractional operator of order (α, β) , with $\alpha > 0$ and $\beta > 0$, by*

$$\mathcal{D}_\mu^{\alpha, \beta} = \frac{1}{2} \left[{}_a D_t^\alpha - {}_t D_b^\beta \right] + \frac{i\mu}{2} \left[{}_a D_t^\alpha + {}_t D_b^\beta \right]. \quad (2.13)$$

In particular, for $\alpha = \beta = 1$ one has $\mathcal{D}_\mu^{1,1} = \frac{d}{dt}$. Moreover, for $\mu = -i$ we recover the left Riemann–Liouville fractional derivative of order α ,

$$\mathcal{D}_{-i}^{\alpha, \beta} = {}_a D_t^\alpha,$$

and for $\mu = i$ the right Riemann–Liouville fractional derivative of order β :

$$\mathcal{D}_i^{\alpha, \beta} = -{}_t D_b^\beta.$$

Now, let us consider the following variational functional:

$$\mathcal{J}(y) = \int_a^b F(\mathcal{D}_\mu^{\alpha, \beta}[y](t), y(t), t) dt$$

defined on the space of continuous functions such that ${}_a D_t^\alpha[y]$ together with ${}_t D_b^\beta[y]$ exist and $y(a) = y_a$, $y(b) = y_b$. Using the direct embedding procedure, the Euler–Lagrange equation derived by Cresson is

$$\mathcal{D}_{-\mu}^{\beta, \alpha} [\partial_1 F] = \partial_2 F. \quad (2.14)$$

Using the variational principle in derivation of the Euler–Lagrange equation, one has

$$\mathcal{D}_\mu^{\alpha, \beta} [\partial_1 F] = \partial_2 F. \quad (2.15)$$

Reader can easily notice that, in general, there is a difference between equations (2.14) and (2.15) i.e., they are not coherent. Cresson claimed [29] that this lack of coherence has the following sources:

- the set of variations in the method of variational principle is too large and therefore it does not give correct answer; one should find the corresponding constraints for the variations;
- there is a relation between lack of coherence and properties of the operator used to generalize the classical derivative.

Let us observe that coherence between (2.14) and (2.15) is restored in the case when $\alpha = \beta$ and $\mu = 0$. This type of coherence is called time reversible coherence. For a deeper discussion of the subject we refer the reader to [29].

In this chapter we presented few results of the fractional calculus of variations. A comprehensive study of the subject can be found in the books [53, 70].

Part II

Original Work

Chapter 3

Standard Methods in Fractional Variational Calculus

The model problem of this chapter is to find an admissible function giving a minimum value to the integral functional, which depends on an unknown function (or functions) of one or several variables and its generalized fractional derivatives and/or generalized fractional integrals. In order to answer this question, we will make use of the standard methods in the fractional calculus of variations (see e.g., [70]). Namely, by analogy to the classical variational calculus (see e.g., [33]), the approach that we call standard, is first to prove Euler–Lagrange equations, find their solutions and then to check if they are minimizers. It is important to remark that standard methods suffer an important disadvantage. Precisely, solvability of Euler–Lagrange equations is assumed, which is not the case in direct methods that are going to be presented later (see Chapter 4).

Now, before we describe briefly an arrangement of this chapter, we define the concept of minimizer. Let $(X, \|\cdot\|)$ be normed linear space and \mathcal{I} be a functional defined on a nonempty subset \mathcal{A} of X . Moreover, let us introduce the following set: if $\bar{y} \in \mathcal{A}$ and $\delta > 0$, then

$$\mathcal{N}_\delta(\bar{y}) := \{y \in \mathcal{A} : \|y - \bar{y}\| < \delta\}$$

is called neighborhood of \bar{y} in \mathcal{A} .

Definition 21. *Function $\bar{y} \in \mathcal{A}$ is called minimizer of \mathcal{I} if there exists a neighborhood $\mathcal{N}_\delta(\bar{y})$ of \bar{y} such that*

$$\mathcal{I}(\bar{y}) \leq \mathcal{I}(y), \quad \text{for all } y \in \mathcal{N}_\delta(\bar{y}).$$

Note that any function $y \in \mathcal{N}_\delta(\bar{y})$ can be represented in a convenient way as a perturbation of \bar{y} . Precisely,

$$\forall y \in \mathcal{N}_\delta(\bar{y}), \quad \exists \eta \in \mathcal{A}_0, \quad y = \bar{y} + h\eta, \quad |h| \leq \varepsilon,$$

where $0 < \varepsilon < \frac{\delta}{\|\eta\|}$ and \mathcal{A}_0 is a suitable set of functions η such that

$$\mathcal{A}_0 = \{\eta \in X : \bar{y} + h\eta \in \mathcal{A}, \quad |h| \leq \varepsilon\}.$$

We begin the chapter with Section 3.1, where we prove generalized integration by parts formula and boundedness of generalized fractional integral from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$.

In Section 3.2 we consider the one-dimensional fundamental problem with generalized fractional operators and obtain an appropriate Euler–Lagrange equation. Then, we prove

that under some convexity assumptions on Lagrangian, every solution to the Euler–Lagrange equation is automatically a solution to our problem. Moreover, as corollaries, we obtain results for problems of the constant and variable order fractional variational calculus and discuss some illustrative examples.

In Section 3.3 we study variational problems with free end points and besides Euler–Lagrange equations we prove natural boundary conditions. As particular cases we obtain natural boundary conditions for problems with constant and variable order fractional operators.

Section 3.4 is devoted to generalized fractional isoperimetric problems. We want to find functions that minimize an integral functional subject to given boundary conditions and isoperimetric constraints. We prove necessary optimality conditions and, as corollaries, we obtain Euler–Lagrange equations for isoperimetric problems with constant and variable order fractional operators. Furthermore, we illustrate our results through several examples.

In Section 3.5 we prove a generalized fractional counterpart of Noether’s theorem. Assuming invariance of the functional, we prove that any extremal must satisfy a certain generalized fractional equation. Corresponding results are obtained for functionals with constant and variable order fractional operators.

Section 3.6 is dedicated to variational problems defined by the use of the generalized fractional integral instead of the classical integral. We obtain Euler–Lagrange equations and discuss several examples.

Finally, in Section 3.7 we study multidimensional fractional variational problems with generalized partial operators. We begin with the proofs of integration by parts formulas for generalized partial fractional integrals and derivatives. Next, we use these results to show Euler–Lagrange equations for the fundamental problem. Moreover, we prove a generalized fractional Dirichlet’s principle, necessary optimality condition for the isoperimetric problem and Noether’s theorem. We finish the chapter with some conclusions.

3.1 Properties of Generalized Fractional Integrals

This section is devoted to properties of generalized fractional operators. We begin by proving in Section 3.1.1 that the generalized fractional operator K_P is bounded and linear. Later, in Section 3.1.2, we give integration by parts formulas for generalized fractional operators.

3.1.1 Boundedness of Generalized Fractional Operators

Along the work, we assume that $1 < p < \infty$ and that q is an adjoint of p , that is $\frac{1}{p} + \frac{1}{q} = 1$. Let us prove the following theorem yielding boundedness of the generalized fractional integral K_P from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$.

Theorem 1. *Let us assume that $k \in L^q(\Delta; \mathbb{R})$. Then, K_P is a linear bounded operator from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$.*

Proof. The linearity is obvious. We will show that K_P is bounded from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$. Considering only the first term of K_P , let us prove that the following inequality holds for any $f \in L^p(a, b; \mathbb{R})$:

$$\left(\int_a^b \left| \int_a^t k(t, \tau) f(\tau) d\tau \right|^q dt \right)^{1/q} \leq \|k\|_{L^q(\Delta, \mathbb{R})} \|f\|_{L^p}. \quad (3.1)$$

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Using Fubini's theorem, we have $k(t, \cdot) \in L^q(a, t; \mathbb{R})$ for almost all $t \in (a, b)$. Then, applying Hölder's inequality, we have

$$\left| \int_a^t k(t, \tau) f(\tau) d\tau \right|^q \leq \left[\left(\int_a^t |k(t, \tau)|^q d\tau \right)^{\frac{1}{q}} \left(\int_a^t |f(\tau)|^p d\tau \right)^{\frac{1}{p}} \right]^q \leq \int_a^t |k(t, \tau)|^q d\tau \|f\|_{L^p}^q \quad (3.2)$$

for almost all $t \in (a, b)$. Hence, integrating equation (3.2) on the interval (a, b) , we obtain inequality (3.1). The proof is completed using the same strategy on the second term in the definition of K_P . \square

Corollary 1. *If $\frac{1}{p} < \alpha < 1$, then ${}_a I_t^\alpha$ is a linear bounded operator from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$.*

Proof. Let us denote $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha)}(t - \tau)^{\alpha-1}$. For $\frac{1}{p} < \alpha < 1$ there exist a constant $C \in \mathbb{R}$ such that for almost all $t \in (a, b)$

$$\int_a^t |k^\alpha(t, \tau)|^q d\tau \leq C. \quad (3.3)$$

Integrating (3.3) on the (a, b) we have $k^\alpha(t, \tau) \in L^q(\Delta; \mathbb{R})$. Therefore, applying Theorem 1 with $P = \langle a, t, b, 1, 0 \rangle$ operator ${}_a I_t^\alpha$ is linear bounded from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$. \square

Next result shows that with the use of Theorem 1, one can prove that variable order fractional integral is a linear bounded operator.

Corollary 2. *Let $\alpha : \Delta \rightarrow [\delta, 1]$ with $\delta > \frac{1}{p}$. Then ${}_a I_t^{\alpha(\cdot, \cdot)}$ is linear bounded operator from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$.*

Proof. Let us denote $k^\alpha(t, \tau) = (t - \tau)^{\alpha(t, \tau)-1} / \Gamma(\alpha(t, \tau))$. We have just to prove that $k^\alpha \in L^q(\Delta, \mathbb{R})$ in order to use Theorem 1. Let us note that since α is with values in $[\delta, 1]$ with $\delta > 0$, then $1/(\Gamma \circ \alpha)$ is bounded. Hence, we have just to prove that $(\Gamma \circ \alpha)k^\alpha \in L^q(\Delta, \mathbb{R})$. We have two different cases: $b - a \leq 1$ and $b - a > 1$.

In the first case, for any $(t, \tau) \in \Delta$, we have $0 < t - \tau \leq 1$ and $q(\delta - 1) > -1$. Then:

$$\int_a^t (t - \tau)^{q(\alpha(t, \tau)-1)} d\tau \leq \int_a^t (t - \tau)^{q(\delta-1)} d\tau = \frac{(t - a)^{q(\delta-1)+1}}{q(\delta-1)+1} \leq \frac{1}{q(\delta-1)+1}.$$

In the second case, for almost all $(t, \tau) \in \Delta \cap (a, a + 1) \times (a, b)$, we have $0 < t - \tau \leq 1$. Consequently, we conclude in the same way that:

$$\int_a^t (t - \tau)^{q(\alpha(t, \tau)-1)} d\tau \leq \frac{1}{q(\delta-1)+1}.$$

Still in the second case, for almost all $(t, \tau) \in \Delta \cap (a + 1, b) \times (a, b)$, we have $\tau < t - 1$ or $t - 1 \leq \tau \leq t$. Then:

$$\int_a^t (t - \tau)^{q(\alpha(t, \tau)-1)} d\tau = \int_a^{t-1} (t - \tau)^{q(\alpha(t, \tau)-1)} d\tau + \int_{t-1}^t (t - \tau)^{q(\alpha(t, \tau)-1)} d\tau \leq b - a - 1 + \frac{1}{q(\delta-1)+1}.$$

Consequently, in any case, there exist a constant $C \in \mathbb{R}$ such that for almost all $t \in (a, b)$:

$$\int_a^t |k^\alpha(t, \tau)|^q d\tau \leq C. \quad (3.4)$$

Finally, $k^\alpha \in L^q(\Delta, \mathbb{R})$. □

3.1.2 Generalized Fractional Integration by Parts

In this section we obtain formula of integration by parts for the generalized fractional calculus. Our results are particularly useful with respect to applications in dynamic optimization, where the derivation of the Euler–Lagrange equations uses, as a key step in the proof, integration by parts (see e.g., the proof of Theorem 3 in Section 3.2).

In our setting, integration by parts changes a given parameter set P into its dual P^* . The term *duality* comes from the fact that $P^{**} = P$.

Definition 22 (Dual parameter set). *Given a parameter set $P = \langle a, t, b, \lambda, \mu \rangle$ we denote by P^* the parameter set $P^* = \langle a, t, b, \mu, \lambda \rangle$. We say that P^* is the dual of P .*

Our first formula of fractional integration by parts involves the operator K_P .

Theorem 2. *Let us assume that $k \in L^q(\Delta; \mathbb{R})$. Then, the operator K_{P^*} defined by*

$$K_{P^*}[f](t) = \mu \int_a^t k(t, \tau) f(\tau) d\tau + \lambda \int_t^b k(\tau, t) f(\tau) d\tau \quad (3.5)$$

is a linear bounded operator from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$. Moreover, the following integration by parts formula holds:

$$\int_a^b f(t) \cdot K_P[g](t) dt = \int_a^b g(t) \cdot K_{P^*}[f](t) dt, \quad (3.6)$$

for any $f, g \in L^p(a, b; \mathbb{R})$.

Proof. Using Theorem 1, we obtain that K_{P^*} is a linear bounded operator from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$. The second part is easily proved using Fubini’s theorem. Indeed, considering only the first term of K_P , the following equality holds for any $f, g \in L^p(a, b; \mathbb{R})$:

$$\lambda \int_a^b f(t) \cdot \int_a^t k(t, \tau) g(\tau) d\tau dt = \lambda \int_a^b g(\tau) \cdot \int_\tau^b k(t, \tau) f(t) dt d\tau.$$

The proof is completed by using the same strategy on the second part of the definition of K_P . □

The next example shows that one cannot relax the hypotheses of Theorem 2.

3.1. PROPERTIES OF GENERALIZED FRACTIONAL INTEGRALS

Example 9. Let $P = \langle 0, t, 1, 1, -1 \rangle$, $f = g = 1$, and $k(t, \tau) = \frac{t^2 - \tau^2}{(t^2 + \tau^2)^2}$. Direct calculations show that

$$\begin{aligned} \int_0^1 K_P[1] dt &= \int_0^1 \left(\int_0^t \frac{t^2 - \tau^2}{(t^2 + \tau^2)^2} d\tau - \int_t^1 \frac{\tau^2 - t^2}{(t^2 + \tau^2)^2} d\tau \right) dt \\ &= \int_0^1 \left(\int_0^1 \frac{t^2 - \tau^2}{(t^2 + \tau^2)^2} d\tau \right) dt = \int_0^1 \frac{1}{t^2 + 1} dt = \frac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 K_{P^*}[1] d\tau &= \int_0^1 \left(- \int_0^\tau \frac{\tau^2 - t^2}{(t^2 + \tau^2)^2} dt + \int_\tau^1 \frac{t^2 - \tau^2}{(t^2 + \tau^2)^2} dt \right) d\tau \\ &= - \int_0^1 \left(\int_0^1 \frac{\tau^2 - t^2}{(t^2 + \tau^2)^2} dt \right) d\tau = - \int_0^1 \frac{1}{\tau^2 + 1} d\tau = -\frac{\pi}{4}. \end{aligned}$$

Therefore, the integration by parts formula (3.6) does not hold. Observe that in this case $\int_0^1 \int_0^1 |k(t, \tau)|^2 d\tau dt = \infty$.

For the classical Riemann–Liouville fractional integrals the following result holds.

Corollary 3. Let $\frac{1}{p} < \alpha < 1$. If $f, g \in L_p(a, b; \mathbb{R})$, then

$$\int_a^b g(t) \cdot {}_a I_t^\alpha [f](t) dt = \int_a^b f(t) \cdot {}_t I_b^\alpha [g](t) dt. \quad (3.7)$$

Proof. Let $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha)}(t - \tau)^{\alpha-1}$. Using the same reasoning as in the proof of Corollary 1, one has $k^\alpha \in L^q(\Delta; \mathbb{R})$. Therefore, (3.7) follows from (3.6). \square

Furthermore, Riemann–Liouville integrals of variable order satisfy the following integration by parts formula.

Corollary 4. Let $\alpha : \Delta \rightarrow [\delta, 1]$ with $\delta > \frac{1}{p}$ and let $f, g \in L_p(a, b; \mathbb{R})$. Then

$$\int_a^b g(t) \cdot {}_a I_t^{\alpha(\cdot, \cdot)} [f](t) dt = \int_a^b f(t) \cdot {}_t I_b^{\alpha(\cdot, \cdot)} [g](t) dt. \quad (3.8)$$

Proof. Let $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha(t, \tau))}(t - \tau)^{\alpha(t, \tau)-1}$. Similarly as in the proof of Corollary 2, for $\alpha : \Delta \rightarrow [\delta, 1]$ with $\delta > \frac{1}{p}$ one has $k^\alpha \in L^q(\Delta; \mathbb{R})$. Therefore, (3.8) follows from (3.6). \square

3.2 Fundamental Problem

For $P = \langle a, t, b, \lambda, \mu \rangle$, let us consider the following functional:

$$\begin{aligned} \mathcal{I} : \mathcal{A}(y_a, y_b) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) dt, \end{aligned} \quad (3.9)$$

where

$$\mathcal{A}(y_a, y_b) := \{y \in C^1([a, b]; \mathbb{R}) : y(a) = y_a, y(b) = y_b, \text{ and } K_P[y], B_P[y] \in C([a, b]; \mathbb{R})\},$$

\dot{y} denotes the classical derivative of y , K_P is the generalized fractional integral operator with kernel belonging to $L^q(\Delta; \mathbb{R})$, $B_P = K_P \circ \frac{d}{dt}$ and F is the Lagrangian function, of class C^1 :

$$\begin{aligned} F : \mathbb{R}^4 \times [a, b] &\longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) &\longmapsto F(x_1, x_2, x_3, x_4, t). \end{aligned} \quad (3.10)$$

Moreover, we assume that

- $K_{P^*} [\tau \mapsto \partial_2 F(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C([a, b]; \mathbb{R})$,
- $t \mapsto \partial_3 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in C^1([a, b]; \mathbb{R})$,
- $K_{P^*} [\tau \mapsto \partial_4 F(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C^1([a, b]; \mathbb{R})$.

The next result gives a necessary optimality condition of the Euler–Lagrange type for the problem of finding a function minimizing functional (3.9).

Theorem 3. *Let $\bar{y} \in \mathcal{A}(y_a, y_b)$ be a minimizer of functional (3.9). Then, \bar{y} satisfies the following Euler–Lagrange equation:*

$$\frac{d}{dt} [\partial_3 F(\star_{\bar{y}})(t)] + A_{P^*} [\tau \mapsto \partial_4 F(\star_{\bar{y}})(\tau)](t) = \partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\tau \mapsto \partial_2 F(\star_{\bar{y}})(\tau)](t), \quad (3.11)$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$, for $t \in (a, b)$.

Proof. Because $\bar{y} \in \mathcal{A}(y_a, y_b)$ is a minimizer of (3.9), we have

$$\mathcal{I}(\bar{y}) \leq \mathcal{I}(\bar{y} + h\eta),$$

for any $|h| \leq \varepsilon$ and every $\eta \in \mathcal{A}(0, 0)$. Let us define the following function

$$\begin{aligned} \phi_{\bar{y}, \eta} : [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{I}(\bar{y} + h\eta) = \int_a^b F(\bar{y}(t) + h\eta(t), K_P[\bar{y} + h\eta](t), \dot{\bar{y}}(t) + h\dot{\eta}(t), B_P[\bar{y} + h\eta](t), t) dt. \end{aligned}$$

Since $\phi_{\bar{y}, \eta}$ is of class C^1 on $[-\varepsilon, \varepsilon]$ and

$$\phi_{\bar{y}, \eta}(0) \leq \phi_{\bar{y}, \eta}(h), \quad |h| \leq \varepsilon,$$

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we deduce that

$$\phi'_{\bar{y},\eta}(0) = \left. \frac{d}{dh} \mathcal{I}(\bar{y} + h\eta) \right|_{h=0} = 0.$$

Hence, by the theorem of differentiation under an integral sign and by the chain rule we get

$$\int_a^b (\partial_1 F(\star_{\bar{y}})(t) \cdot \eta(t) + \partial_2 F(\star_{\bar{y}})(t) \cdot K_P[\eta](t) + \partial_3 F(\star_{\bar{y}})(t) \cdot \dot{\eta}(t) + \partial_4 F(\star_{\bar{y}})(t) \cdot B_P[\eta](t)) dt = 0.$$

Finally, Theorem 2 yields

$$\begin{aligned} \int_a^b (\partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\tau \mapsto \partial_2 F(\star_{\bar{y}})(\tau)](t)) \cdot \eta(t) \\ + (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*} [\tau \mapsto \partial_4 F(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}(t) dt = 0, \end{aligned}$$

and applying the classical integration by parts formula and the fundamental lemma of the calculus of variations (see e.g., [44]) we obtain (3.11). \square

Remark 1. From now, in order to simplify the notation, for T, S being fractional operators we will write shortly

$$T [\partial_i F(y(\tau), T[y](\tau), \dot{y}(\tau), S[y](\tau), \tau)]$$

instead of

$$T [\tau \mapsto \partial_i F(y(\tau), T[y](\tau), \dot{y}(\tau), S[y](\tau), \tau)], \quad i = 1, \dots, 5.$$

Example 10. Let $P = \langle 0, t, 1, 1, 0 \rangle$. Consider the problem of minimizing the following functional:

$$\mathcal{I}(y) = \int_0^1 (K_P[y](t) + t)^2 dt$$

subject to the given boundary conditions

$$y(0) = -1 \quad \text{and} \quad y(1) = -1 - \int_0^1 u(1 - \tau) d\tau,$$

where the kernel k of K_P is such that $k(t, \tau) = h(t - \tau)$ with $h \in C^1([0, 1]; \mathbb{R})$ and $h(0) = 1$. Here the resolvent u is related to the kernel h by $u(t) = \mathcal{L}^{-1} \left[\frac{1}{s\tilde{h}(s)} - 1 \right] (t)$, $\tilde{h}(s) = \mathcal{L}[h](s)$, where \mathcal{L} and \mathcal{L}^{-1} are the direct and the inverse Laplace operators, respectively. We apply Theorem 3 with Lagrangian F given by $F(x_1, x_2, x_3, x_4, t) = (x_2 + t)^2$. Because

$$y(t) = -1 - \int_0^t u(t - \tau) d\tau$$

is the solution to the Volterra integral equation of the first kind (see, e.g., Equation 16, p.114 of [92])

$$K_P[y](t) + t = 0,$$

it satisfies our generalized Euler–Lagrange equation (3.11), that is,

$$K_{P^*} [K_P[y](\tau) + \tau] (t) = 0, \quad t \in (a, b).$$

In particular, for the kernel $h(t - \tau) = e^{-(t-\tau)}$ and the boundary conditions are $y(0) = -1$, $y(1) = -2$, the solution is $y(t) = -1 - t$.

Remark 2 (cf. Theorem 2.2.3 of [104]). *If the functional (3.9) does not depend on K_P and B_P , then Theorem 3 reduces to the classical result: if $\bar{y} \in C^2([a, b]; \mathbb{R})$ is a solution to the problem of minimizing the functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), \dot{y}(t), t) dt, \quad \text{subject to } y(a) = y_a, \quad y(b) = y_b,$$

then \bar{y} satisfies the Euler–Lagrange equation

$$\partial_1 F(\bar{y}(t), \dot{\bar{y}}(t), t) - \frac{d}{dt} \partial_2 F(\bar{y}(t), \dot{\bar{y}}(t), t) = 0, \quad \text{for all } t \in (a, b).$$

Remark 3. *In the particular case when functional (3.9) does not depend on the integer derivative of function y , we obtain from Theorem 3 the following result: if $\bar{y} \in \mathcal{A}(y_a, y_b)$ is a solution to the problem of minimizing the functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), K_P[y](t), B_P[y](t), t) dt,$$

subject to $y(a) = y_a$ and $y(b) = y_b$, then \bar{y} satisfies the Euler–Lagrange equation

$$\begin{aligned} & A_{P^*} [\partial_4 F(\bar{y}(\tau), K_P[\bar{y}](\tau), B_P[\bar{y}](\tau), \tau)] (t) \\ &= \partial_1 F(\bar{y}(t), K_P[\bar{y}](t), B_P[\bar{y}](t), t) + K_{P^*} [\partial_2 F(\bar{y}(\tau), K_P[\bar{y}](\tau), B_P[\bar{y}](\tau), \tau)] (t), \quad t \in (a, b). \end{aligned}$$

This extends some of the recent results of [4].

Corollary 5. *Let $0 < \alpha < \frac{1}{q}$ and let $\bar{y} \in C^1([a, b]; \mathbb{R})$ be a solution to the problem of minimizing the functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t) dt, \quad (3.12)$$

subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, where

- $F \in C^1(\mathbb{R}^4 \times [a, b]; \mathbb{R})$,
- functions $t \mapsto \partial_1 F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t)$,
 ${}_t I_b^{1-\alpha} [\partial_2 F(y(\tau), {}_a I_\tau^{1-\alpha}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^\alpha[y](\tau), \tau)]$ are continuous on $[a, b]$,
- functions $t \mapsto \partial_3 F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t)$,
 ${}_t I_b^{1-\alpha} [\partial_4 F(y(\tau), {}_a I_\tau^{1-\alpha}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^\alpha[y](\tau), \tau)]$ are continuously differentiable on $[a, b]$.

3.2. FUNDAMENTAL PROBLEM

Then, the following Euler–Lagrange equation holds

$$\begin{aligned} \frac{d}{dt} & \left(\partial_3 F(\bar{y}(t), {}_a I_t^{1-\alpha}[\bar{y}](t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t) \right) \\ & - {}_t D_b^\alpha \left[\partial_4 F(\bar{y}(\tau), {}_a I_\tau^{1-\alpha}[\bar{y}](\tau), \dot{\bar{y}}(\tau), {}^C D_\tau^\alpha[\bar{y}](\tau), \tau) \right] (t) \\ & = \partial_1 F(\bar{y}(t), {}_a I_t^{1-\alpha}[\bar{y}](t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t) \\ & + {}_t I_b^\alpha \left[\partial_2 F(\bar{y}(\tau), {}_a I_\tau^{1-\alpha}[\bar{y}](\tau), \dot{\bar{y}}(\tau), {}^C D_\tau^\alpha[\bar{y}](\tau), \tau) \right] (t), \quad t \in (a, b). \end{aligned} \quad (3.13)$$

Proof. The intended Euler–Lagrange equation follows from (3.11) by choosing $P = \langle a, t, b, 1, 0 \rangle$ and the kernel $k^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha)}(t - \tau)^{-\alpha}$. Note that for $0 < \alpha < \frac{1}{q}$, we have $k^\alpha \in L^q(\Delta; \mathbb{R})$. \square

In Example 11 we make use of the Mittag–Leffler function of one parameter. Let $\alpha > 0$. We recall that the Mittag–Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

This function appears naturally in the solutions of fractional differential equations, as a generalization of the exponential function [25]. Indeed, while a linear second order ordinary differential equation with constant coefficients presents an exponential function as solution, in the fractional case the Mittag–Leffler functions emerge [51].

Example 11. Let $0 < \alpha < \frac{1}{q}$. Consider problem of minimizing the functional

$$\mathcal{I}(y) = \int_0^1 \sqrt{1 + (\dot{y}(t) + {}^C D_t^\alpha[y](t) - 1)^2} dt \quad (3.14)$$

subject to the following boundary conditions:

$$y(0) = 0 \text{ and } y(1) = \int_0^1 E_{1-\alpha} [-(1 - \tau)^{1-\alpha}] d\tau. \quad (3.15)$$

Function F of Corollary 5 is given by $F(x_1, x_2, x_3, x_4, t) = \sqrt{1 + (x_3 + x_4 - 1)^2}$. One can easily check that (see [51] p.324)

$$y(t) = \int_0^t E_{1-\alpha} [-(t - \tau)^{1-\alpha}] d\tau \quad (3.16)$$

satisfies $\dot{y}(t) + {}^C D_t^\alpha[y](t) \equiv 1$. Moreover, it satisfies

$$\frac{d}{dt} \left(\frac{\dot{y}(t) + {}^C D_t^\alpha[y](t) - 1}{\sqrt{1 + (\dot{y}(t) + {}^C D_t^\alpha[y](t) - 1)^2}} \right) - {}_t D_b^\alpha \left[\frac{\dot{y}(\tau) + {}^C D_\tau^\alpha[y](\tau) - 1}{\sqrt{1 + (\dot{y}(\tau) + {}^C D_\tau^\alpha[y](\tau) - 1)^2}} \right] (t) = 0,$$

for all $t \in (a, b)$. We conclude that (3.16) is a candidate for function giving a minimum to problem (3.14)–(3.15).

Corollary 6 (cf. [3]). *Let $0 < \alpha < \frac{1}{q}$ and let \mathcal{I} be the functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), \dot{y}(t), \lambda_a^C D_t^\alpha[y](t) + \mu_t^C D_b^\alpha[y](t), t) dt, \quad (3.17)$$

where λ and μ are real numbers, and let $\bar{y} \in C^1([a, b]; \mathbb{R})$ be a minimizer of \mathcal{I} among all functions satisfying boundary conditions $y(a) = y_a$, $y(b) = y_b$. Moreover, we assume that

- $F \in C^1(\mathbb{R}^3 \times [a, b]; \mathbb{R})$,
- functions $t \mapsto \partial_2 F(y(t), \dot{y}(t), \lambda_a^C D_t^\alpha[y](t) + \mu_t^C D_b^\alpha[y](t), t)$,
 ${}_a I_t^{1-\alpha} [\partial_3 F(y(\tau), \dot{y}(\tau), \lambda_a^C D_\tau^\alpha[y](\tau) + \mu_\tau^C D_b^\alpha[y](\tau), \tau)]$,
 and ${}_t I_b^{1-\alpha} [\partial_3 F(y(\tau), \dot{y}(\tau), \lambda_a^C D_\tau^\alpha[y](\tau) + \mu_\tau^C D_b^\alpha[y](\tau), \tau)]$ are continuously differentiable on $[a, b]$.

Then, \bar{y} satisfies the Euler–Lagrange equation

$$\begin{aligned} & \lambda_t D_b^\alpha [\partial_3 F(\bar{y}(\tau), \dot{\bar{y}}(\tau), \lambda_a^C D_\tau^\alpha[\bar{y}](\tau) + \mu_\tau^C D_b^\alpha[\bar{y}](\tau), \tau)](t) \\ & + \mu_a D_t^\alpha [\partial_3 F(\bar{y}(\tau), \dot{\bar{y}}(\tau), \lambda_a^C D_\tau^\alpha[\bar{y}](\tau) + \mu_\tau^C D_b^\alpha[\bar{y}](\tau), \tau)](t) \\ & + \partial_1 F(\bar{y}(t), \dot{\bar{y}}(t), \lambda_a^C D_t^\alpha[\bar{y}](t) + \mu_t^C D_b^\alpha[\bar{y}](t), t) \\ & - \frac{d}{dt} (\partial_2 F(\bar{y}(t), \dot{\bar{y}}(t), \lambda_a^C D_t^\alpha[\bar{y}](t) + \mu_t^C D_b^\alpha[\bar{y}](t), t)) = 0 \end{aligned} \quad (3.18)$$

for all $t \in (a, b)$.

Proof. Choose $P = \langle a, t, b, \lambda, -\mu \rangle$ and $k^\alpha(t - \tau) = \frac{1}{\Gamma(1-\alpha)}(t - \tau)^{-\alpha}$. Then, for $0 < \alpha < \frac{1}{q}$ kernel k^α is in $L^q(\Delta; \mathbb{R})$, the operator B_P reduces to the sum of the left and right Caputo fractional derivatives and (3.18) follows from (3.11). \square

Corollary 7. *Let us consider the problem of minimizing a functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t) dt \quad (3.19)$$

subject to boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad (3.20)$$

where \dot{y} , ${}_a I_t^{1-\alpha(\cdot, \cdot)}[y]$, ${}_a^C D_t^{\alpha(\cdot, \cdot)}[y] \in C([a, b]; \mathbb{R})$ and $\alpha : \Delta \rightarrow [0, 1 - \delta]$ with $\delta > \frac{1}{p}$. Moreover, we assume that

- $F \in C^1(\mathbb{R}^4 \times [a, b], \mathbb{R})$
- function $t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_2 F(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau) \right]$ is continuous on $[a, b]$,
- functions $t \mapsto \partial_3 F(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t)$
 and ${}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_4 F(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau) \right]$ are continuously differentiable on $[a, b]$.

3.2. FUNDAMENTAL PROBLEM

Then, if $\bar{y} \in C^1([a, b]; \mathbb{R})$ is a solution to problem (3.19)–(3.20), it necessarily satisfies

$$\begin{aligned} & \partial_1 F \left(y(t), {}_a I_t^{1-\alpha(\cdot)}[y](t), \dot{y}, {}_a^C D_t^{\alpha(\cdot)}[y](t), t \right) \\ & \quad - \frac{d}{dt} \partial_3 F \left(y(t), {}_a I_t^{1-\alpha(\cdot)}[y](t), \dot{y}, {}_a^C D_t^{\alpha(\cdot)}[y](t), t \right) \\ & \quad + {}_t I_b^{1-\alpha(\cdot)} \left[\partial_2 F \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot)}[y](\tau), \tau \right) \right] (t) \\ & \quad + {}_t D_b^{\alpha(\cdot)} \left[\partial_4 F \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot)}[y](\tau), \tau \right) \right] (t) = 0 \end{aligned} \quad (3.21)$$

for all $t \in (a, b)$.

Proof. For $\alpha : \Delta \rightarrow [0, 1 - \delta]$ with $\delta > \frac{1}{p}$ we have that $k^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)}$ is in $L^q(\Delta; \mathbb{R})$. Therefore, from (3.11) follows (3.21). \square

In the next example $\alpha(t)$ is a function defined on the interval $[a, b]$ and taking values in the set $[0, 1 - \delta]$, where $\delta > \frac{1}{p}$. As before, we assume that $\alpha \in C^1([a, b]; \mathbb{R})$

Example 12. Consider the following problem:

$$\begin{aligned} \mathcal{J}(y) &= \int_a^b \left({}_a^C D_t^{\alpha(\cdot)}[y](t) \right)^2 + \left({}_a I_t^{1-\alpha(\cdot)}[y](t) - \frac{\xi(t - \tau)^{1-\alpha(t)}}{\Gamma(2 - \alpha(t))} \right)^2 dt \longrightarrow \min, \\ & y(a) = \xi, \quad y(b) = \xi, \end{aligned}$$

for a given real ξ . Because $\mathcal{J}(y) \geq 0$ for any function y and $\mathcal{J}(\bar{y}) = 0$ for the admissible function $\bar{y} = \xi$ (use relation (1.11) for $\gamma = 0$, the linearity of operator ${}_a I_t^{1-\alpha(\cdot)}$, and the definition of left Caputo derivative of a variable fractional order), we conclude that \bar{y} is the global minimizer to the problem. It is straightforward to check that \bar{y} satisfies our variable order fractional Euler–Lagrange equation (3.21).

Next result gives a sufficient condition assuring that solution of (3.11) is indeed minimizer of (3.9).

Theorem 4. Let $\bar{y} \in \mathcal{A}(y_a, y_b)$ satisfies (3.11) and $(x_1, x_2, x_3, x_4) \mapsto F(x_1, x_2, x_3, x_4, t)$ be convex for every $t \in [a, b]$. Then \bar{y} is a minimizer of (3.9).

Proof. Let us assume that $\bar{y} \in \mathcal{A}(y_a, y_b)$ satisfies equation (3.11) and that $(x_1, x_2, x_3, x_4) \mapsto F(x_1, x_2, x_3, x_4, t)$ is convex for every $t \in [a, b]$. Then, for every $y \in \mathcal{A}(y_a, y_b)$ we have

$$\begin{aligned} \mathcal{I}(y) &\geq \mathcal{I}(\bar{y}) + \int_a^b \left(\partial_1 F \cdot (y - \bar{y}) + \partial_2 F \cdot (K_P[y] - K_P[\bar{y}]) \right. \\ &\quad \left. + \partial_3 F \cdot (\dot{y} - \dot{\bar{y}}) + \partial_4 F \cdot (B_P[y] - B_P[\bar{y}]) \right) dt, \end{aligned}$$

where $\partial_i F$ are taken in $(\bar{y}, K_P[\bar{y}], \dot{\bar{y}}, B_P[\bar{y}], t)$, $i = 1, 2, 3, 4$. Having in mind that $y(a) - \bar{y}(a) = y(b) - \bar{y}(b) = 0$, and using the classical integration by parts formula as well as Theorem 2 one has

$$\mathcal{I}(y) \geq \mathcal{I}(\bar{y}) + \int_a^b \left(\partial_1 F + K_{P^*}[\partial_2 F] - \frac{d}{dt}(\partial_3 F) - A_{P^*}[\partial_4 F] \right) (y - \bar{y}) dt,$$

where, as before, $\partial_i F$ are taken in $(\bar{y}, K_P[\bar{y}], \dot{\bar{y}}, B_P[\bar{y}], t)$, $i = 1, 2, 3, 4$. Finally, by Euler–Lagrange equation (3.11), we have $\mathcal{I}(y) \geq \mathcal{I}(\bar{y})$. \square

3.3 Free Initial Boundary

Let us define the following set

$$\mathcal{A}(y_b) := \{y \in C^1([a, b]; \mathbb{R}) : y(a) \text{ is free, } y(b) = y_b, \text{ and } K_P[y], B_P[y] \in C([a, b]; \mathbb{R})\},$$

and let \bar{y} be a minimizer of functional (3.9) on $\mathcal{A}(y_b)$, i.e., now

$$\begin{aligned} \mathcal{I} : \mathcal{A}(y_b) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) dt. \end{aligned} \quad (3.22)$$

Because

$$\mathcal{I}(\bar{y}) \leq \mathcal{I}(\bar{y} + h\eta),$$

for any $|h| \leq \varepsilon$ and every $\eta \in \mathcal{A}(0)$, we obtain as in the proof of Theorem 3 that

$$\begin{aligned} \int_a^b (\partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 F(\star_{\bar{y}})(\tau)](t)) \cdot \eta(t) \\ + (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}(t) dt = 0, \forall \eta \in \mathcal{A}(0), \end{aligned}$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$. Moreover, having in mind that $\eta(b) = 0$ and using the classical integration by parts formula, we find that

$$\begin{aligned} \int_a^b \left(\partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 F(\star_{\bar{y}})(\tau)](t) - \frac{d}{dt} (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)) \right) \cdot \eta(t) dt \\ + \partial_3 F(\star_{\bar{y}})(t) \cdot \eta(t)|_a + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t) \cdot \eta(t)|_a = 0, \forall \eta \in \mathcal{A}(0). \end{aligned}$$

Now, using the fundamental lemma of calculus of variations (see e.g., [44]) and the fact that $\eta(a)$ is arbitrary, we obtain

$$\begin{cases} \frac{d}{dt} [\partial_3 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)] = \partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 F(\star_{\bar{y}})(\tau)](t), \\ \partial_3 F(\star_{\bar{y}})(t)|_a + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)|_a = 0. \end{cases}$$

We have just proved the following result.

Theorem 5. *If $\bar{y} \in \mathcal{A}(y_b)$ is a solution to the problem of minimizing functional (3.22) on the set $\mathcal{A}(y_b)$, then \bar{y} satisfies the Euler–Lagrange equation (3.11). Moreover, the extra boundary condition*

$$\partial_3 F(\star_{\bar{y}})(t)|_a + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)|_a = 0 \quad (3.23)$$

holds with $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$.

3.3. FREE INITIAL BOUNDARY

Similarly as it is in the theory of the classical calculus of variations we will call (3.23) the generalized fractional natural boundary condition.

Corollary 8 (cf. [2]). *Let $0 < \alpha < \frac{1}{q}$ and \mathcal{I} be the functional given by*

$$\mathcal{I}(y) = \int_a^b F(y(t), {}_a^C D_t^\alpha[y](t), t) dt,$$

where $F \in C^1(\mathbb{R}^2 \times [a, b]; \mathbb{R})$, and ${}_a I_t^{1-\alpha} [\partial_2 F(y(\tau), {}_a^C D_\tau^\alpha[y](\tau), \tau)]$ is continuously differentiable on $[a, b]$. If $\bar{y} \in C^1([a, b]; \mathbb{R})$ is a minimizer of \mathcal{I} among all functions satisfying the boundary condition $y(b) = y_b$, then \bar{y} satisfies the Euler–Lagrange equation

$$\partial_1 F(\bar{y}(t), {}_a^C D_t^\alpha[\bar{y}](t), t) + {}_t D_b^\alpha [\partial_2 F(\bar{y}(\tau), {}_a^C D_\tau^\alpha[\bar{y}](\tau), \tau)](t) = 0$$

and the fractional natural boundary condition

$${}_t I_b^{1-\alpha} [\partial_2 F(\bar{y}(\tau), {}_a^C D_\tau^\alpha[\bar{y}](\tau), \tau)](t)|_a = 0$$

holds for all $t \in (a, b)$.

Proof. Corollary 8 follows from Theorem 5 with $P = \langle a, t, b, 1, 0 \rangle$ and $k^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha)}(t - \tau)^{-\alpha}$. \square

Corollary 9. *Suppose that $\alpha : \Delta \rightarrow [0, 1 - \delta]$, $\delta > \frac{1}{p}$ and \mathcal{I} is the functional given by (3.19). If $\bar{y} \in C^1([a, b]; \mathbb{R})$ is a minimizer to \mathcal{I} satisfying boundary condition $y(b) = y_b$ and being such that ${}_a I_t^{1-\alpha(\cdot, \cdot)}[y], {}_a^C D_t^{\alpha(\cdot, \cdot)}[y] \in C([a, b]; \mathbb{R})$, then \bar{y} is a solution to the Euler–Lagrange equation*

$$\begin{aligned} \partial_1 F(\bar{y}(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[\bar{y}](t), \dot{\bar{y}}, {}_a^C D_t^{\alpha(\cdot, \cdot)}[\bar{y}](t), t) - \frac{d}{dt} \partial_3 F(\bar{y}(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[\bar{y}](t), \dot{\bar{y}}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[\bar{y}](t), t) \\ + {}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_2 F(\bar{y}(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[\bar{y}](\tau), \dot{\bar{y}}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[\bar{y}](\tau), \tau) \right](t) \\ + {}_t D_b^{\alpha(\cdot, \cdot)} \left[\partial_4 F(\bar{y}(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[\bar{y}](\tau), \dot{\bar{y}}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[\bar{y}](\tau), \tau) \right](t) = 0 \end{aligned}$$

and the natural boundary condition

$$\begin{aligned} \partial_3 F(\bar{y}(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[\bar{y}](t), \dot{\bar{y}}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[\bar{y}](t), t) \Big|_a \\ + {}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_4 F(\bar{y}(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[\bar{y}](\tau), \dot{\bar{y}}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[\bar{y}](\tau), \tau) \right](t) \Big|_a = 0 \end{aligned}$$

holds for all $t \in (a, b)$.

Proof. Corollary 9 is an easy consequence of Theorem 5. \square

Remark 4. *Observe that if the functional (3.22) is independent of the operator K_P , that is we have the following problem:*

$$\int_a^b F(y(t), \dot{y}(t), B_P[y](t), t) dt \longrightarrow \min, \quad y(b) = y_b$$

($y(a)$ free), then the optimality conditions (3.11) and (3.23) reduce, respectively, to

$$\begin{aligned} \partial_1 F(\bar{y}(t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) - \frac{d}{dt} \partial_2 F(\bar{y}(t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) \\ - A_{P^*} [\partial_3 F(\bar{y}(\tau), \dot{\bar{y}}(\tau), B_P[\bar{y}](\tau), \tau)](t) = 0 \end{aligned}$$

and $\partial_2 F(\bar{y}(t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)|_a + K_{P^*} [\partial_3 F(\bar{y}(\tau), \dot{\bar{y}}(\tau), B_P[\bar{y}](\tau), \tau)](t)|_a = 0$ for all $t \in (a, b)$.

3.4 Isoperimetric Problem

One of the earliest problems in geometry is the isoperimetric problem, already considered by the ancient Greeks. It consists to find, among all closed curves of a given length, the one which encloses the maximum area. The general problem for which one integral is to be given a fixed value, while another is to be made a maximum or a minimum, is nowadays part of the calculus of variations [45, 104]. Such *isoperimetric problems* have found a broad class of important applications throughout the centuries, with numerous useful implications in astronomy, geometry, algebra, analysis, and engineering [20, 32]. For recent advancements on the study of isoperimetric problems see [13, 14, 40] and references therein. Here we consider isoperimetric problems with generalized fractional operators. Similarly to Section 3.2 and 3.3, we deal with integrands involving both the generalized Caputo fractional derivative and the generalized fractional integral, as well as the classical derivative.

Let $P = \langle a, t, b, \lambda, \mu \rangle$. We define the following functional:

$$\begin{aligned} \mathcal{J} : \mathcal{A}(y_a, y_b) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b G(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) dt, \end{aligned} \quad (3.24)$$

where by \dot{y} we understand the classical derivative of y , K_P is the generalized fractional integral operator with kernel belonging to $L^q(\Delta; \mathbb{R})$, $B_P = K_P \circ \frac{d}{dt}$ and G is a Lagrangian of class C^1 :

$$\begin{aligned} G : \mathbb{R}^4 \times [a, b] &\longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) &\longmapsto G(x_1, x_2, x_3, x_4, t). \end{aligned}$$

Moreover, we assume that

- $K_{P^*} [\partial_2 G(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C([a, b]; \mathbb{R})$,
- $t \mapsto \partial_3 G(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in C^1([a, b]; \mathbb{R})$,
- $K_{P^*} [\partial_4 G(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C^1([a, b]; \mathbb{R})$.

The first problem in this section, is to find a minimizer of functional (3.9) subject to the isoperimetric constraint $\mathcal{J}(y) = \xi$. In the next theorem we provide necessary optimality condition for this type of problems.

Theorem 6. *Suppose that \bar{y} is a minimizer of functional \mathcal{I} in the class*

$$\mathcal{A}_\xi(y_a, y_b) := \{y \in \mathcal{A}(y_a, y_b) : \mathcal{J}(y) = \xi\}.$$

3.4. ISOPERIMETRIC PROBLEM

Then there exists a real constant λ_0 , such that, for $H = F - \lambda_0 G$, equation

$$\frac{d}{dt} [\partial_3 H(\star_{\bar{y}})(t)] + A_{P^*} [\partial_4 H(\star_{\bar{y}})(\tau)](t) = \partial_1 H(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 H(\star_{\bar{y}})(\tau)](t), \quad t \in (a, b) \quad (3.25)$$

holds, provided that

$$\frac{d}{dt} [\partial_3 G(\star_{\bar{y}})(t)] + A_{P^*} [\partial_4 G(\star_{\bar{y}})(\tau)](t) \neq \partial_1 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 G(\star_{\bar{y}})(\tau)](t), \quad t \in (a, b) \quad (3.26)$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$.

Proof. By hypothesis (3.26) and the fundamental lemma of the calculus of variations (see e.g., [44]) we can choose $\eta_2 \in \mathcal{A}(0, 0)$ so that

$$\begin{aligned} \int_a^b (\partial_1 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 G(\star_{\bar{y}})(\tau)](t)) \cdot \eta_2(t) \\ + (\partial_3 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 G(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}_2(t) dt = 1. \end{aligned}$$

With this function η_2 and an arbitrary $\eta_1 \in \mathcal{A}(0, 0)$, let us define functions

$$\begin{aligned} \phi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] &\longrightarrow \mathbb{R} \\ (h_1, h_2) &\longmapsto \mathcal{I}(\bar{y} + h_1 \eta_1 + h_2 \eta_2) \end{aligned}$$

and

$$\begin{aligned} \psi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] &\longrightarrow \mathbb{R} \\ (h_1, h_2) &\longmapsto \mathcal{J}(\bar{y} + h_1 \eta_1 + h_2 \eta_2) - \xi. \end{aligned}$$

Observe that $\psi(0, 0) = 0$ and

$$\begin{aligned} \frac{\partial \psi}{\partial h_2} \Big|_{(0,0)} = \int_a^b (\partial_1 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 G(\star_{\bar{y}})(\tau)](t)) \cdot \eta_2(t) \\ + (\partial_3 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 G(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}_2(t) dt = 1. \end{aligned}$$

According to the implicit function theorem we can find $\varepsilon_0 > 0$ and a function $s \in C^1([-\varepsilon_0, \varepsilon_0]; \mathbb{R})$ with $s(0) = 0$ such that

$$\psi(h_1, s(h_1)) = 0, \quad \forall h_1 \in [-\varepsilon_0, \varepsilon_0]$$

which implies that $\bar{y} + h_1 \eta_1 + s(h_1) \eta_2 \in \mathcal{A}_\xi(y_a, y_b)$. We also have

$$\frac{\partial \psi}{\partial h_1} + \frac{\partial \psi}{\partial h_2} \cdot s'(h_1) = 0, \quad \forall h_1 \in [-\varepsilon_0, \varepsilon_0]$$

and hence

$$s'(0) = - \frac{\partial \psi}{\partial h_1} \Big|_{(0,0)}.$$

Because $\bar{y} \in \mathcal{A}(y_a, y_b)$ is a minimizer of \mathcal{I} we have

$$\phi(0, 0) \leq \phi(h_1, s(h_1)), \quad \forall h_1 \in [-\varepsilon_0, \varepsilon_0]$$

and then

$$\frac{\partial \phi}{\partial h_1} \Big|_{(0,0)} + \frac{\partial \phi}{\partial h_2} \Big|_{(0,0)} \cdot s'(0) = 0.$$

Letting $\lambda_0 = \frac{\partial \phi}{\partial h_2} \Big|_{(0,0)}$ be the Lagrange multiplier we find

$$\frac{\partial \phi}{\partial h_1} \Big|_{(0,0)} - \lambda_0 \frac{\partial \psi}{\partial h_1} \Big|_{(0,0)} = 0$$

or, in other words,

$$\begin{aligned} & \int_a^b \{[(\partial_1 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 F(\star_{\bar{y}})(\tau)](t)) \cdot \eta_1(t) + (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 F(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}_1(t)] \\ & - \lambda_0 [(\partial_1 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 G(\star_{\bar{y}})(\tau)](t)) \cdot \eta_1(t) + (\partial_3 G(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 G(\star_{\bar{y}})(\tau)](t)) \cdot \dot{\eta}_1(t)]\} dt \\ & = 0. \end{aligned}$$

Finally, applying one more time the fundamental lemma of the calculus of variations we obtain (3.25). \square

Example 13. Let $P = \langle 0, t, 1, 1, 0 \rangle$. Consider the problem

$$\begin{aligned} \mathcal{I}(y) &= \int_0^1 (K_P[y](t) + t)^2 dt \longrightarrow \min, \\ \mathcal{J}(y) &= \int_0^1 t K_P[y](t) dt = \xi, \\ y(0) &= \xi - 1, \quad y(1) = (\xi - 1) \left(1 + \int_0^1 u(1 - \tau) d\tau \right), \end{aligned}$$

where the kernel k is such that $k(t, \tau) = h(t - \tau)$ with $h \in C^1([0, 1]; \mathbb{R})$, $h(0) = 1$ and $K_{P^*}[id](t) \neq 0$ (notation id means identity transformation i.e., $id(t) = t$). Here the resolvent u is related to the kernel h in the same way as in Example 10. Since $K_{P^*}[id](t) \neq 0$, there is no solution to the Euler–Lagrange equation for functional \mathcal{J} . The augmented Lagrangian H of Theorem 6 is given by $H(x_1, x_2, t) = (x_2 + t)^2 - \lambda_0 t x_2$. Function

$$y(t) = (\xi - 1) \left(1 + \int_0^t u(t - \tau) d\tau \right)$$

is the solution to the Volterra integral equation of the first kind $K_P[y](t) = (\xi - 1)t$ (see, e.g., Equation 16, p. 114 of [92]) and for $\lambda_0 = 2\xi$ satisfies our optimality condition (3.25):

$$K_{P^*} [2(K_P[y](\tau) + \tau) - 2\xi\tau](t) = 0. \quad (3.27)$$

The solution of (3.27) subject to the given boundary conditions depends on the particular choice for the kernel. For example, let $h^\alpha(t - \tau) = e^{\alpha(t - \tau)}$. Then the solution of (3.27) subject

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to the boundary conditions $y(0) = \xi - 1$ and $y(1) = (\xi - 1)(1 - \alpha)$ is $y(t) = (\xi - 1)(1 - \alpha t)$ (cf. p. 15 of [92]). If $h^\alpha(t - \tau) = \cos(\alpha(t - \tau))$, then the boundary conditions are $y(0) = \xi - 1$ and $y(1) = (\xi - 1)(1 + \alpha^2/2)$, and the extremal is $y(t) = (\xi - 1)(1 + \alpha^2 t^2/2)$ (cf. p. 46 of [92]).

Borrowing different kernels from the book [92], many other examples of dynamic optimization problems can be explicitly solved by application of the results of this section.

As particular cases of our problem (3.9), (3.24), one obtains previously studied fractional isoperimetric problems with Caputo derivatives.

Corollary 10 (cf. [80]). *Let $\bar{y} \in C^1([a, b]; \mathbb{R})$ be a minimizer to the functional*

$$\mathcal{I}(y) = \int_a^b F(y(t), \dot{y}(t), {}^C D_t^\alpha[y](t), t) dt$$

subject to an isoperimetric constraint of the form

$$\mathcal{J}(y) = \int_a^b G(y(t), \dot{y}(t), {}^C D_t^\alpha[y](t), t) dt = \xi,$$

and boundary conditions

$$y(a) = y_a, \quad y(b) = y_b,$$

where $0 < \alpha < \frac{1}{q}$, and functions F, G are such that

- $F, G \in C^1(\mathbb{R}^3 \times [a, b]; \mathbb{R})$,
- $t \mapsto \partial_2 F(y(t), \dot{y}(t), {}^C D_t^\alpha[y](t), t)$, $t \mapsto \partial_2 G(y(t), \dot{y}(t), {}^C D_t^\alpha[y](t), t)$, ${}_t I_b^{1-\alpha} [\partial_3 F(y(\tau), \dot{y}(\tau), {}^C D_\tau^\alpha[y](\tau), \tau)]$ and ${}_t I_b^{1-\alpha} [\partial_3 G(y(\tau), \dot{y}(\tau), {}^C D_\tau^\alpha[y](\tau), \tau)]$ are continuously differentiable on $[a, b]$.

If \bar{y} is such that

$$\begin{aligned} \partial_1 G(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t) - \frac{d}{dt} (\partial_2 G(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t)) \\ + {}_t D_b^\alpha [\partial_3 G(\bar{y}(\tau), \dot{\bar{y}}(\tau), {}^C D_\tau^\alpha[\bar{y}](\tau), \tau)](t) \neq 0, \end{aligned} \quad (3.28)$$

then there exists a constant λ_0 such that \bar{y} satisfies

$$\begin{aligned} \partial_1 H(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t) - \frac{d}{dt} (\partial_2 H(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\alpha[\bar{y}](t), t)) \\ + {}_t D_b^\alpha [\partial_3 H(\bar{y}(\tau), \dot{\bar{y}}(\tau), {}^C D_\tau^\alpha[\bar{y}](\tau), \tau)](t) = 0. \end{aligned} \quad (3.29)$$

with $H = F - \lambda_0 G$.

Proof. The result follows from Theorem 6. □

Example 14. Let $\alpha \in \left(0, \frac{1}{q}\right)$ and $\xi \in \mathbb{R}$. Consider the following fractional isoperimetric problem:

$$\begin{aligned} \mathcal{I}(y) &= \int_0^1 (\dot{y}(t) + {}^C_0 D_t^\alpha [y](t))^2 dt \longrightarrow \min \\ \mathcal{J}(y) &= \int_0^1 (\dot{y}(t) + {}^C_0 D_t^\alpha [y](t)) dt = \xi \\ y(0) &= 0, \quad y(1) = \int_0^1 E_{1-\alpha} \left(-(1-\tau)^{1-\alpha} \right) \xi d\tau. \end{aligned} \tag{3.30}$$

In our example (3.30) the function H of Corollary 10 is given by

$$H(y(t), \dot{y}(t), {}^C_0 D_t^\alpha [y](t), t) = (\dot{y}(t) + {}^C_0 D_t^\alpha [y](t))^2 - \lambda_0 (\dot{y}(t) + {}^C_0 D_t^\alpha [y](t)).$$

One can easily check that function

$$y(t) = \int_0^t E_{1-\alpha} \left(-(t-\tau)^{1-\alpha} \right) \xi d\tau \tag{3.31}$$

- is such that (3.28) holds;
- satisfies $\dot{y}(t) + {}^C_0 D_t^\alpha [y](t) = \xi$ (see, e.g., p. 328, Example 5.24 [51]).

Moreover, (3.31) satisfies the Euler–Lagrange equation (3.29) for $\lambda_0 = 2\xi$, i.e.,

$$-\frac{d}{dt} (2(\dot{y}(t) + {}^C_0 D_t^\alpha [y](t)) - 2\xi) + {}_t D_1^\alpha [(2(\dot{y}(\tau) + {}^C_0 D_\tau^\alpha [y](\tau)) - 2\xi)] = 0$$

We conclude that (3.31) is an extremal for the isoperimetric problem (3.30).

Let us consider two cases.

1. Choose $\xi = 1$. When $\alpha \rightarrow 0$ one gets from (3.30) the classical isoperimetric problem

$$\begin{aligned} \mathcal{I}(y) &= \int_0^1 (\dot{y}(t) + y(t))^2 dt \longrightarrow \min \\ \mathcal{J}(y) &= \int_0^1 y(t) dt = \frac{1}{e} \\ y(0) &= 0 \quad y(1) = 1 - \frac{1}{e}. \end{aligned} \tag{3.32}$$

Our extremal (3.31) is then reduced to the classical extremal $y(t) = 1 - e^{-t}$ of the isoperimetric problem (3.32).

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2. Let $\alpha = \frac{1}{2}$. Then (3.30) gives the following fractional isoperimetric problem:

$$\begin{aligned} \mathcal{I}(y) &= \int_0^1 \left(\dot{y}(t) + {}_0^C D_t^{\frac{1}{2}}[y](t) \right)^2 dt \longrightarrow \min \\ \mathcal{J}(y) &= \int_0^1 \left(\dot{y}(t) + {}_0^C D_t^{\frac{1}{2}}[y](t) \right) dt = \xi \\ y(0) &= 0, \quad y(1) = \xi \left(\operatorname{erfc}(1)e + \frac{2}{\sqrt{\pi}} - 1 \right), \end{aligned} \quad (3.33)$$

where erfc is the complementary error function defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-t^2) dt.$$

The extremal (3.31) for the particular fractional isoperimetric problem (3.33) is

$$y(t) = \xi \left(e^t \operatorname{erfc}(\sqrt{t}) - \frac{2\sqrt{t}}{\sqrt{\pi}} - 1 \right).$$

Corollary 11. Let us consider the problem of minimizing functional (3.19) subject to an isoperimetric constraint

$$\mathcal{J}(y) = \int_a^b G \left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t \right) dt = \xi \quad (3.34)$$

and the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad (3.35)$$

where $\dot{y}, {}_a I_t^{1-\alpha(\cdot, \cdot)}[y], {}_a^C D_t^{\alpha(\cdot, \cdot)}[y] \in C([a, b]; \mathbb{R})$ and $\alpha : \Delta \rightarrow [0, 1 - \delta]$ with $\delta > \frac{1}{p}$. Moreover, we assume that

- $G \in C^1(\mathbb{R}^4 \times [a, b]; \mathbb{R})$,
- ${}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_2 G \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau \right) \right]$ is continuous on $[a, b]$,
- $t \mapsto \partial_3 G \left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t \right)$ and ${}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_4 G \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau \right) \right]$ are continuously differentiable on $[a, b]$.

If $\bar{y} \in C^1([a, b]; \mathbb{R})$ is a solution to problem (3.19), (3.34), (3.35), then there exists real number λ_0 such that, for $H = F - \lambda_0 G$, we have

$$\partial_1 H - \frac{d}{dt} \partial_3 H + {}_t I_b^{1-\alpha(t, \cdot)}[\partial_2 H] + {}_t D_b^{\alpha(t, \cdot)}[\partial_4 H] = 0,$$

provided that

$$\partial_1 G - \frac{d}{dt} \partial_3 G + {}_t I_b^{1-\alpha(t, \cdot)}[\partial_2 G] + {}_t D_b^{\alpha(t, \cdot)}[\partial_4 G] \neq 0.$$

Here, functions $\partial_i H$ and $\partial_i G$, $i = 1, 2, 3, 4$, are evaluated at $\left(\bar{y}(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[\bar{y}](t), \dot{\bar{y}}, {}_a^C D_t^{\alpha(\cdot, \cdot)}[\bar{y}](t), t \right)$.

Proof. The result follows from Theorem 6. \square

Theorem 6 can be easily extended to r subsidiary conditions of integral type. Let G_k , $k = 1, \dots, r$, be Lagrangians of class C^1 :

$$\begin{aligned} G_k : \quad \mathbb{R}^4 \times [a, b] &\longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) &\longmapsto G_k(x_1, x_2, x_3, x_4, t). \end{aligned}$$

and let

$$\begin{aligned} \mathcal{J}_k : \mathcal{A}(y_a, y_b) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b G_k(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) dt, \end{aligned} \quad (3.36)$$

where \dot{y} denotes the classical derivative of y , K_P is generalized fractional integral operator with the kernel belonging to $L^q(\Delta; \mathbb{R})$ and $B_P = K_P \circ \frac{d}{dt}$. Moreover, we assume that

- $K_{P^*} [\partial_2 G_k(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C([a, b]; \mathbb{R})$,
- $t \mapsto \partial_3 G_k(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in C^1([a, b]; \mathbb{R})$,
- $K_{P^*} [\partial_4 G_k(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C^1([a, b]; \mathbb{R})$.

Suppose that $\xi = (\xi_1, \dots, \xi_r)$ and define

$$\mathcal{A}_\xi(y_a, y_b) := \{y \in \mathcal{A}(y_a, y_b) : \mathcal{J}_k[y] = \xi_k, k = 1 \dots, r\}.$$

Next theorem gives necessary optimality condition for a minimizer of functional \mathcal{I} subject to r isoperimetric constraints.

Theorem 7. *Let \bar{y} be a minimizer of \mathcal{I} in the class $\mathcal{A}_\xi(y_a, y_b)$. If one can find functions $\eta_1, \dots, \eta_r \in \mathcal{A}(0, 0)$ such that the matrix $A = (a_{kl})$, with*

$$\begin{aligned} a_{kl} := \int_a^b (\partial_1 G_k(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 G_k(\star_{\bar{y}})(\tau)](t)) \cdot \eta_l(\tau) \\ + (\partial_3 G_k(\star_{\bar{y}})(t) + K_{P^*} [\partial_4 G_k(\star_{\bar{y}})(\tau)](t)) \cdot \eta_l(t) dt, \end{aligned}$$

has rank equal to r , then there exist $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that, for $H = F - \sum_{k=1}^r \lambda_k G_k$, minimizer \bar{y} satisfies

$$\frac{d}{dt} [\partial_3 H(\star_{\bar{y}})(t) + A_{P^*} [\partial_4 H(\star_{\bar{y}})(\tau)](t)] = \partial_1 H(\star_{\bar{y}})(t) + K_{P^*} [\partial_2 H(\star_{\bar{y}})(\tau)](t), \quad t \in (a, b), \quad (3.37)$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$.

Proof. Let us define

$$\begin{aligned} \phi : [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon_1, \varepsilon_1] \times \dots \times [-\varepsilon_r, \varepsilon_r] &\longrightarrow \mathbb{R} \\ (h_0, h_1, \dots, h_r) &\longmapsto \mathcal{I}(\bar{y} + h_0 \eta_0 + h_1 \eta_1 + \dots + h_r \eta_r) \end{aligned}$$

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and

$$\begin{aligned} \psi_k : [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon_1, \varepsilon_1] \times \cdots \times [-\varepsilon_r, \varepsilon_r] &\longrightarrow \mathbb{R} \\ (h_0, h_1, \dots, h_r) &\longmapsto \mathcal{J}_k(\bar{y} + h_0\eta_0 + h_1\eta_1 + \cdots + h_r\eta_r) - \xi_k \end{aligned}$$

Observe that ϕ, ψ_k are functions of class $C^1([-\varepsilon_0, \varepsilon_0] \times \cdots \times [-\varepsilon_r, \varepsilon_r]; \mathbb{R})$, $A = \left(\frac{\partial \psi_k}{\partial h_i} \Big|_0 \right)$ and that $\psi_k(0, 0, \dots, 0) = 0$. Moreover, because \bar{y} is a minimizer of functional \mathcal{I} we have

$$\phi(0, \dots, 0) \leq \phi(h_0, h_1, \dots, h_r).$$

From the classical multiplier theorem, there are $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that

$$\nabla \phi(0, \dots, 0) + \sum_{k=1}^r \lambda_k \nabla \psi_k(0, \dots, 0) = 0, \quad (3.38)$$

From (3.38), we can compute $\lambda_1, \dots, \lambda_r$, turning out to be independent of the choice of $\eta_0 \in \mathcal{A}(0, 0)$. Finally, by the fundamental lemma of the calculus of variations, we arrive to (3.37). \square

3.5 Noether's Theorem

Emmy Noether's classical work [74] from 1918 states that a conservation law in variational mechanics follow whenever the Lagrangian function is invariant under a one-parameter continuous group of transformations, that transform dependent and/or independent variables. This result not only unifies conservation laws but also suggests a way to discover new ones. In this section we consider variational problems that depend on generalized fractional integrals and derivatives. Following the methods used in [29, 42, 43, 48] we apply Euler–Lagrange equations to formulate a generalized fractional version of Noether's theorem without transformation of time. We start by introducing the notion generalized fractional extremal and one-parameter family of infinitesimal transformations.

Definition 23. *The function $y \in C^1([a, b]; \mathbb{R})$, with $K_P[y], B_P[y] \in C([a, b]; \mathbb{R})$, that is a solution to equation (3.11) is said to be a generalized fractional extremal.*

We consider a one-parameter family of transformations of the form $\hat{y}(t) = \phi(\theta, t, y(t))$, where ϕ is a map of class C^2 :

$$\begin{aligned} \phi : [-\varepsilon, \varepsilon] \times [a, b] \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\theta, t, x) &\longmapsto \phi(\theta, t, x), \end{aligned}$$

such that $\phi(0, t, x) = x$. Note that, using Taylor's expansion of $\phi(\theta, t, y(t))$ in a neighborhood of 0 one has

$$\hat{y}(t) = \phi(0, t, y(t)) + \theta \frac{\partial}{\partial \theta} \phi(0, t, y(t)) + o(\theta),$$

provided that $\theta \in [-\varepsilon, \varepsilon]$. Moreover, having in mind that $\phi(0, t, y(t)) = y(t)$ and denoting $\xi(t, y(t)) = \frac{\partial}{\partial \theta} \phi(0, t, y(t))$, one has

$$\hat{y}(t) = y(t) + \theta \xi(t, y(t)) + o(\theta), \quad (3.39)$$

so that, for $\theta \in [-\varepsilon, \varepsilon]$ the linear approximation to the transformation is

$$\hat{y}(t) \approx y(t) + \theta \xi(t, y(t)).$$

Now, let us introduce the notion of invariance.

Definition 24. We say that the Lagrangian F is invariant under the one-parameter family of infinitesimal transformations (3.39), where ξ is such that $t \mapsto \xi(t, y(t)) \in C^1([a, b]; \mathbb{R})$ with $K_P[\tau \mapsto \xi(\tau, y(\tau))], B_P[\tau \mapsto \xi(\tau, y(\tau))] \in C([a, b]; \mathbb{R})$ if

$$F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) = F(\hat{y}(t), K_P[\hat{y}](t), \dot{\hat{y}}(t), B_P[\hat{y}](t), t), \quad (3.40)$$

for all $\theta \in [-\varepsilon, \varepsilon]$, and all $y \in C^1([a, b]; \mathbb{R})$ with $K_P[y], B_P[y] \in C([a, b]; \mathbb{R})$.

In order to state Noether's theorem in a compact form, we introduce the following two bilinear operators:

$$\mathbf{D}[f, g] := f \cdot A_{P^*}[g] + g \cdot B_P[f], \quad (3.41)$$

$$\mathbf{I}[f, g] := -f \cdot K_{P^*}[g] + g \cdot K_P[f]. \quad (3.42)$$

Theorem 8 (Generalized Fractional Noether's Theorem). *Let F be invariant under the one parameter family of infinitesimal transformations (3.39). Then for every generalized fractional extremal the following equality holds*

$$\frac{d}{dt} (\xi(t, y(t)) \cdot \partial_3 F(\star_y)(t)) + \mathbf{D}[\xi(t, y(t)), \partial_4 F(\star_y)(t)] + \mathbf{I}[\xi(t, y(t)), \partial_2 F(\star_y)(t)] = 0, \quad t \in (a, b), \quad (3.43)$$

where $(\star_y)(t) = (y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t)$.

Proof. By equation (3.40) one has

$$\left. \frac{d}{d\theta} \left[F(\hat{y}(t), K_P[\hat{y}](t), \dot{\hat{y}}(t), B_P[\hat{y}](t), t) \right] \right|_{\theta=0} = 0$$

The usual chain rule implies

$$\begin{aligned} \partial_1 F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} \hat{y}(t) + \partial_2 F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} K_P[\hat{y}](t) \\ + \partial_3 F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} \dot{\hat{y}}(t) + \partial_4 F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} B_P[\hat{y}](t) \Big|_{\theta=0} = 0. \end{aligned}$$

By linearity of K_P, B_P differentiating with respect to θ , and putting $\theta = 0$ we have

$$\begin{aligned} \partial_1 F(\star_y)(t) \cdot \xi(t, y(t)) + \partial_2 F(\star_y)(t) \cdot K_P[\tau \mapsto \xi(\tau, y(\tau))](t) \\ + \partial_3 F(\star_y)(t) \cdot \frac{d}{dt} \xi(t, y(t)) + \partial_4 F(\star_y)(t) \cdot B_P[\tau \mapsto \xi(\tau, y(\tau))](t) = 0. \end{aligned}$$

Now, using generalized Euler-Lagrange equation (3.11) and formulas (3.41) and (3.42) one arrives to (3.43). \square

Example 15. Let $P = \langle a, t, b, \lambda, \mu \rangle$ and $y \in C^1([a, b]; \mathbb{R})$ with $B_P[y] \in C([a, b]; \mathbb{R})$. Consider Lagrangian $F(B_P[y](t), t)$ and transformations

$$\hat{y}(t) = y(t) + \varepsilon c + o(\varepsilon), \quad (3.44)$$

3.5. NOETHER'S THEOREM

where c is a constant. Then, we have

$$F(B_P[y](t), t) = F(B_P[\hat{y}](t), t)$$

Therefore, F is invariant under (3.44) and the generalized fractional Noether's theorem indicates that

$$A_{P^*}[\partial_1 F(B_P[y](\tau), \tau)](t) = 0, \quad t \in (a, b), \quad (3.45)$$

along any generalized fractional extremal y . Notice that equation (3.45) can be written in the form

$$\frac{d}{dt}(K_{P^*}[\partial_1 F(B_P[y](\tau), \tau)](t)) = 0, \quad (3.46)$$

that is, quantity $K_{P^*}[\partial_1 F(B_P[y](\tau), \tau)]$ is conserved along all generalized fractional extremals and this quantity, following the classical approach, can be called a generalized fractional constant of motion.

Similarly to previous sections, one can obtain from Theorem 8 results regarding to constant and variable order fractional integrals and derivatives.

Corollary 12. *If for any $y \in C^1([a, b]; \mathbb{R})$ the following equality is satisfied*

$$F(y(t), \dot{y}(t), \lambda_a^C D_t^\alpha[y](t) + \mu_t^C D_b^\alpha[y](t), t) = F(\hat{y}(t), \dot{\hat{y}}(t), \lambda_a^C D_t^\alpha[\hat{y}](t) + \mu_t^C D_b^\alpha[\hat{y}](t), t),$$

where \hat{y} is the family (3.39), then we have

$$\begin{aligned} & \frac{d}{dt} (\xi(t, y(t)) \cdot \partial_2 F(y(t), \dot{y}(t), \lambda_a^C D_t^\alpha[y](t) + \mu_t^C D_b^\alpha[y](t), t)) \\ & - \xi(t, y(t)) \cdot (\lambda_t D_b^\alpha [\partial_3 F(y(\tau), \dot{y}(\tau), \lambda_a^C D_\tau^\alpha[y](\tau) + \mu_\tau^C D_b^\alpha[y](\tau), \tau)](t) \\ & \quad \mu_a D_t^\alpha [\partial_3 F(y(\tau), \dot{y}(\tau), \lambda_a^C D_\tau^\alpha[y](\tau) + \mu_\tau^C D_b^\alpha[y](\tau), \tau)](t)) \\ & + \partial_3 F(y(t), \dot{y}(t), \lambda_a^C D_t^\alpha[y](t) + \mu_t^C D_b^\alpha[y](t), t) \cdot (\lambda_a^C D_t^\alpha[\xi(\tau, y(\tau))](t) + \mu_t^C D_b^\alpha[\xi(\tau, y(\tau))](t)) = 0 \end{aligned}$$

along all solutions of the Euler–Lagrange equation (3.18).

Proof. Corollary 12 is a simple consequence of Theorem 8. □

Corollary 13. *Let $y \in C^1([a, b]; \mathbb{R})$ with $I_t^{1-\alpha(\cdot, \cdot)}[y], {}^C D_t^{\alpha(\cdot, \cdot)}[y] \in C([a, b]; \mathbb{R})$ and suppose that*

$$F(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t) = F(\hat{y}(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[\hat{y}](t), \dot{\hat{y}}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[\hat{y}](t), t)$$

where \hat{y} is the family (3.39) such that $t \mapsto \xi(t, y(t)) \in C^1([a, b]; \mathbb{R})$ and $I_t^{1-\alpha(\cdot, \cdot)}[\tau \mapsto \xi(\tau, y(\tau))], {}_a^C D_t^{\alpha(\cdot, \cdot)}[\tau \mapsto \xi(\tau, y(\tau))] \in C([a, b]; \mathbb{R})$. Then all solutions of the Euler–Lagrange equation (3.21) must satisfy

$$\begin{aligned} & \frac{d}{dt} \left(\xi(t, y(t)) \cdot \partial_3 F \left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t \right) \right) \\ & - \xi(t, y(t)) \cdot {}_t D_b^{\alpha(\cdot, \cdot)} \left[\partial_4 F \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau \right) \right] (t) \\ & + \partial_4 F \left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t \right) \cdot {}_a^C D_t^{\alpha(\cdot, \cdot)}[\xi(\tau, y(\tau))](t) \\ & - \xi(t, y(t)) \cdot {}_t I_b^{1-\alpha(\cdot, \cdot)} \left[\partial_2 F \left(y(\tau), {}_a I_\tau^{1-\alpha(\cdot, \cdot)}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^{\alpha(\cdot, \cdot)}[y](\tau), \tau \right) \right] (t) \\ & + \partial_2 F \left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t \right) \cdot {}_a I_t^{1-\alpha(\cdot, \cdot)}[\xi(\tau, y(\tau))](t) = 0, \quad t \in (a, b). \end{aligned}$$

Proof. Corollary 13 can be easily obtained from Theorem 8. \square

Corollary 14. *Suppose that $y \in C^1([a, b]; \mathbb{R})$ and for family (3.39) one has*

$$F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t) dt = F(\hat{y}(t), {}_a I_t^{1-\alpha}[\hat{y}](t), \dot{\hat{y}}(t), {}_a^C D_t^\alpha[\hat{y}](t), t).$$

Then

$$\begin{aligned} \frac{d}{dt} & (\xi(t, y(t)) \cdot \partial_2 F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t)) \\ & - \xi(t, y(t)) \cdot {}_t D_b^\alpha [\partial_4 F(y(\tau), {}_a I_\tau^{1-\alpha}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^\alpha[y](\tau), \tau)] \\ & + \partial_4 F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t) \cdot {}_a^C D_t^\alpha [\xi(\tau, y(\tau))] \\ & - \xi(t, y(t)) \cdot {}_t I_b^{1-\alpha} [\partial_2 F(y(\tau), {}_a I_\tau^{1-\alpha}[y](\tau), \dot{y}(\tau), {}_a^C D_\tau^\alpha[y](\tau), \tau)] \\ & + \partial_2 F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t) \cdot {}_a I_t^{1-\alpha} [\xi(\tau, y(\tau))] = 0 \end{aligned}$$

along any solution of the Euler–Lagrange equation (3.13).

Proof. Corollary 14 can be easily obtained from Theorem 8. \square

3.6 Variational Calculus in Terms of a Generalized Integral

In this section, we develop a generalized fractional calculus of variations, by considering very general problems, where the classical integrals are substituted by generalized fractional integrals, and the Lagrangians depend not only on classical derivatives but also on generalized fractional operators. By choosing particular operators and kernels, one obtains the recent results available in the literature of mathematical physics [35, 36, 46].

Let $R = \langle a, b, b, 1, 0 \rangle$, $P = \langle a, t, b, \lambda, \mu \rangle$ and consider the problem of finding a function \bar{y} that gives minimum value to the functional

$$\begin{aligned} \mathcal{I} : \mathcal{A}(y_a, y_b) & \longrightarrow \mathbb{R} \\ y & \longmapsto K_R [F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t)](b), \end{aligned} \quad (3.47)$$

where K_R and K_P are generalized fractional integrals with kernels $k(x, t)$ and $h(t, \tau)$, respectively, being elements of $L^q(\Delta; \mathbb{R})$, $B_P = K_P \circ \frac{d}{dt}$ and F is a Lagrangian of class C^1 :

$$\begin{aligned} F : \mathbb{R}^4 \times [a, b] & \longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) & \longmapsto F(x_1, x_2, x_3, x_4, t). \end{aligned}$$

Moreover, we assume that

- $t \mapsto k(b, t) \cdot \partial_1 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in C([a, b]; \mathbb{R})$,
- $K_{P^*} [k(b, \tau) \partial_2 F(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C([a, b]; \mathbb{R})$,
- $t \mapsto k(b, t) \cdot \partial_3 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in C^1([a, b]; \mathbb{R})$,
- $K_{P^*} [k(b, \tau) \partial_4 F(y(\tau), K_P[y](\tau), \dot{y}(\tau), B_P[y](\tau), \tau)] \in C^1([a, b]; \mathbb{R})$.

Theorem 9. *If $\bar{y} \in \mathcal{A}(y_a, y_b)$ is a minimizer of functional (3.47), then \bar{y} satisfies the generalized Euler–Lagrange equation*

$$\begin{aligned} & k(b, t) \cdot \partial_1 F(\star_{\bar{y}})(t) - \frac{d}{dt} (\partial_3 F(\star_{\bar{y}})(t) \cdot k(b, t)) \\ & - A_{P^*} [k(b, \tau) \cdot \partial_4 F(\star_{\bar{y}})(\tau)](t) + K_{P^*} [k(b, \tau) \cdot \partial_2 F(\star_{\bar{y}})(\tau)](t) = 0, \quad t \in (a, b), \end{aligned} \quad (3.48)$$

where $(\star_{\bar{y}}) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$.

Proof. One can prove Theorem 9 in the similar way as Theorem 3. \square

Example 16. *Let $R = \langle 0, 1, 1, 1, 0 \rangle$, and $P = \langle 0, t, 1, 1, 0 \rangle$. Consider the following problem:*

$$\begin{aligned} \mathcal{J}(y) &= K_R \left[tK_P[y](t) + \sqrt{1 - (K_P[y](t))^2} \right] (1) \longrightarrow \min, \\ y(0) &= 1, \quad y(1) = \frac{\sqrt{2}}{4} + \int_0^1 u(1 - \tau) \frac{1}{(1 + \tau^2)^{\frac{3}{2}}} d\tau, \end{aligned}$$

with kernel h such that $h(t, \tau) = l(t - \tau)$, $l \in C^1([0, 1]; \mathbb{R})$ and $l(0) = 1$. Here, the resolvent $u(t)$ is related to the kernel $l(t)$ by $u(t) = \mathcal{L}^{-1} \left[\frac{1}{sl(s)} - 1 \right] (t)$, $\tilde{l}(s) = \mathcal{L}[l(t)](s)$, where \mathcal{L} and \mathcal{L}^{-1} are the direct and the inverse Laplace operators, respectively. We apply Theorem 9 with Lagrangian F given by $F(x_1, x_2, x_3, x_4, t) = tx_2 + \sqrt{1 - x_2^2}$. Because

$$y(t) = \frac{1}{(1 + t^2)^{\frac{3}{2}}} + \int_0^t u(t - \tau) \frac{1}{(1 + \tau^2)^{\frac{3}{2}}} d\tau$$

is the solution to the Volterra integral equation of first kind (see, e.g., Equation 16, p. 114 of [92])

$$K_P[y](t) = \frac{t\sqrt{1 + t^2}}{1 + t^2},$$

it satisfies our generalized Euler–Lagrange equation (3.48), that is,

$$K_{P^*} \left[k(1, \tau) \left(\frac{-K_{P^*}[y](\tau)}{\sqrt{1 - (K_{P^*}[y](\tau))^2}} + \tau \right) \right] (t) = 0.$$

In particular, for the kernel $l^\beta(t - \tau) = \cosh(\beta(t - \tau))$, the boundary conditions are $y(0) = 1$ and $y(1) = 1 + \beta^2(1 - \sqrt{2})$, and the solution is $y(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}} + \beta^2 \left(1 - \sqrt{1 + t^2} \right)$ (cf. p. 22 in [92]).

The following corollary gives an extension of the main result of [37].

Corollary 15. *If $\bar{y} \in C^1([a, b]; \mathbb{R})$ is a solution to the problem of minimizing*

$$\mathcal{J}(y) = {}_a I_b^\alpha \left[F \left(y(t), \dot{y}(t), {}^C D_t^\beta [y](t), t \right) \right] (b),$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b,$$

where $\alpha, \beta \in (0, \frac{1}{q})$, $F \in C^2(\mathbb{R}^3 \times [a, b]; \mathbb{R})$, then

$$\begin{aligned} & \partial_1 F(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\beta [\bar{y}](t), t) \cdot (b-t)^{\alpha-1} \\ & - \frac{d}{dt} \left(\partial_2 F(\bar{y}(t), \dot{\bar{y}}(t), {}^C D_t^\beta [\bar{y}](t), t) \cdot (b-t)^{\alpha-1} \right) \\ & + {}_t D_b^\beta \left[(b-\tau)^{\alpha-1} \cdot \partial_3 F(\bar{y}(\tau), \dot{\bar{y}}(\tau), {}^C D_\tau^\beta [\bar{y}](\tau), \tau) \right] = 0, \quad t \in (a, b). \end{aligned}$$

If the Lagrangian of functional (3.47) does not depend on generalized fractional operators B and K , then Theorem 9 gives the following result: if $y \in C^1([a, b]; \mathbb{R})$ is a solution to the problem of extremizing

$$\mathcal{I}(y) = \int_a^b F(y(t), \dot{y}(t), t) k(b, t) dt \quad (3.49)$$

subject to $y(a) = y_a$ and $y(b) = y_b$, then

$$\partial_1 F(y(t), \dot{y}(t), t) - \frac{d}{dt} \partial_2 F(y(t), \dot{y}(t), t) = \frac{1}{k(b, t)} \cdot \frac{d}{dt} k(b, t) \partial_2 F(y(t), \dot{y}(t), t). \quad (3.50)$$

We recognize on the right hand side of (3.50) the generalized weak dissipative parameter

$$\delta(t) = \frac{1}{k(b, t)} \cdot \frac{d}{dt} k(b, t).$$

Now, let us consider two examples of Lagrangians associated to functional (3.49).

Example 17. As a first example, let us consider kernel $k^\alpha(b, t) = e^{\alpha(b-t)}$ and the Lagrangian

$$L(y(t), \dot{y}(t), t) = \frac{1}{2} m \dot{y}^2(t) - V(y(t)),$$

where $V(y)$ is the potential energy and m stands for mass. The Euler–Lagrange equation (3.50) gives the following second-order ordinary differential equation:

$$\ddot{y}(t) - \alpha \dot{y}(t) = -\frac{1}{m} V'(y(t)). \quad (3.51)$$

Equation (3.51) coincides with (14) of [46], obtained by modification of Hamilton’s principle.

Next example extends some of the recent results of [35, 36], where the fractional action-like variational approach (FALVA) was proposed to model dynamical systems. FALVA functionals are particular cases of (3.49), where the fractional time integral introduces only one parameter α .

Example 18. Let us consider the Caldirola–Kanai Lagrangian [35, 36]

$$L(y(t), \dot{y}(t), t) = m(t) \left(\frac{\dot{y}^2(t)}{2} - \omega^2 \frac{y^2(t)}{2} \right), \quad (3.52)$$

which describes a dynamical oscillatory system with exponentially increasing time-dependent mass, where ω is the frequency and $m(t) = m_0 e^{-\gamma b} e^{\gamma t} = \bar{m}_0 e^{\gamma t}$, $\bar{m}_0 = m_0 e^{-\gamma b}$. Using our generalized FALVA Euler–Lagrange equation (3.50) with kernel $k(b, t)$ to Lagrangian (3.52), we obtain

$$\ddot{y}(t) + (\delta(t) + \gamma) \dot{y}(t) + \omega^2 y(t) = 0. \quad (3.53)$$

We note that there is a small inconsistency in [35], regarding the coefficient of $\dot{y}(t)$ in (3.53).

3.7 Generalized Variational Calculus of Several Variables

Variational problems with functionals depending on several variables arise, for example, in mechanics problems, which involve systems with infinite number of degrees of freedom, like a vibrating elastic solid. Fractional variational problems involving multiple integrals have been already studied in different contexts. We can mention here [8, 17, 29, 77] where the multidimensional fractional Euler–Lagrange equations for a field were obtained, or [68] where first and second fractional Noether–type theorems are proved. In this section we present a more general approach to the subject by considering functionals depending on generalized fractional operators.

3.7.1 Multidimensional Generalized Fractional Integration by Parts

In this section, it is of our interest to obtain integration by parts formula for generalized fractional operators. We shall denote by $t = (t_1, \dots, t_n)$ a point in Ω_n , where $\Omega_n = (a_1, b_1) \times \dots \times (a_n, b_n)$, and by $dt = dt_1 \dots dt_n$. Throughout this subsection $i \in \{1, \dots, n\}$ is arbitrary but fixed.

Theorem 10. *Let $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$ be the parameter set and let K_{P_i} be the generalized partial fractional integral with k_i being a difference kernel such that $k_i \in L^1(0, b_i - a_i; \mathbb{R})$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$, $f, \eta \in C(\bar{\Omega}_n; \mathbb{R})$, then the generalized partial fractional integrals satisfy the following identity:*

$$\int_{\Omega_n} f(t) \cdot K_{P_i}[\eta](t) dt = \int_{\Omega_n} \eta(t) \cdot K_{P_i^*}[f](t) dt,$$

where P_i^* is the dual of P_i .

Proof. Let $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$ and $f, \eta \in C(\bar{\Omega}_n; \mathbb{R})$. Since f and η , are continuous functions on $\bar{\Omega}_n$, they are bounded on $\bar{\Omega}_n$ i.e., there exist real numbers $C, D > 0$ such that $|f(t)| \leq C$, $|\eta(t)| \leq D$, for all $t \in \bar{\Omega}_n$. Therefore,

$$\int_{\Omega_n} \left(\int_{a_i}^{t_i} |\lambda_i k_i(t_i - \tau)| \cdot |f(t)| \cdot |\eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n)| d\tau \right. \\ \left. + \int_{t_i}^{b_i} |\mu_i k_i(\tau - t_i)| \cdot |f(t)| \cdot |\eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n)| d\tau \right) dt$$

$$\begin{aligned}
 &\leq C \cdot D \int_{\Omega_n} \left(\int_{a_i}^{t_i} |\lambda_i k_i(t_i - \tau)| d\tau + \int_{t_i}^{b_i} |\mu_i k_i(\tau - t_i)| d\tau \right) dt \\
 &\leq C \cdot D (|\mu_i| + |\lambda_i|) \|k_i\|_{L^1(0, b_i - a_i; \mathbb{R})} \int_{\Omega_n} dt \\
 &= C \cdot D (|\mu_i| + |\lambda_i|) \|k_i\|_{L^1(0, b_i - a_i; \mathbb{R})} \cdot \prod_{i=1}^n (b_i - a_i) < \infty
 \end{aligned}$$

Hence, we can use Fubini's theorem to change the order of integration in the iterated integrals:

$$\begin{aligned}
 &\int_{\Omega_n} f(t) \cdot K_{P_i}[\eta](t) dt_n \dots dt_1 = \int_{\Omega_n} \left(\lambda_i \int_{a_i}^{t_i} f(t) k_i(t_i - \tau) \eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau \right. \\
 &\quad \left. + \mu_i \int_{t_i}^{b_i} f(t) k_i(\tau - t_i) \eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) d\tau \right) dt_n \dots dt_1 \\
 &= \int_{\Omega_n} \left(\lambda_i \int_{\tau}^{b_i} f(t) k_i(t_i - \tau) \eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) dt_i \right. \\
 &\quad \left. + \mu_i \int_{a_i}^{\tau} f(t) k_i(\tau - t_i) \eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) dt_i \right) dt_n \dots dt_{i-1} d\tau dt_{i+1} \dots dt_1 \\
 &= \int_{\Omega_n} \eta(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_n) \left(\lambda_i \int_{\tau}^{b_i} f(t) k_i(t_i - \tau) dt_i \right. \\
 &\quad \left. + \mu_i \int_{a_i}^{\tau} f(t) k_i(\tau - t_i) dt_i \right) dt_n \dots dt_{i-1} d\tau dt_{i+1} \dots dt_1 \\
 &= \int_{\Omega_n} \eta(t) \cdot K_{P_i^*}[f](t) dt_n \dots dt_1.
 \end{aligned}$$

□

As corollaries, we obtain the following integration by parts formulas for constant and variable order fractional integrals.

Corollary 16. *Let $0 < \alpha_i < 1$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $f, \eta \in C(\bar{\Omega}_n; \mathbb{R})$, then the following formula holds*

$$\int_{\Omega_n} f(t) \cdot {}_{a_i}I_{t_i}^{\alpha_i}[\eta](t) dt = \int_{\Omega_n} \eta(t) \cdot {}_{t_i}I_{b_i}^{\alpha_i}[f](t) dt. \quad (3.54)$$

Corollary 17. *Suppose that $\alpha : [0, b_i - a_i] \rightarrow [0, 1]$, and that $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ are such that $f, \eta \in C(\bar{\Omega}_n; \mathbb{R})$. Then,*

$$\int_{\Omega_n} f(t) \cdot {}_{a_i}I_{t_i}^{\alpha(\cdot)}[\eta](t) dt = \int_{\Omega_n} \eta(t) \cdot {}_{t_i}I_{b_i}^{\alpha(\cdot)}[f](t) dt. \quad (3.55)$$

3.7. GENERALIZED VARIATIONAL CALCULUS OF SEVERAL VARIABLES

Theorem 11 (Generalized Fractional Integration by Parts for Several Variables). *Let $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$ be the parameter set and $f, \eta \in C^1(\bar{\Omega}_n; \mathbb{R})$. Moreover, let $B_{P_i} = \frac{d}{dt} \circ K_{P_i}$, where K_{P_i} is the generalized partial fractional integral with difference kernel, i.e., $k_i = k_i(t_i - \tau)$ such that $k_i \in L_1(0, b_i - a_i; \mathbb{R})$, and $K_{P_i^*}[f] \in C^1(\bar{\Omega}_n; \mathbb{R})$. Then*

$$\int_{\Omega_n} f(t) \cdot B_{P_i}[\eta](t) dt = \int_{\partial\Omega_n} \eta(t) \cdot K_{P_i^*}[f](t) \cdot \nu^i d(\partial\Omega_n) - \int_{\Omega_n} \eta(t) \cdot A_{P_i^*}[f](t) dt,$$

where ν^i is the outward pointing unit normal to $\partial\Omega_n$.

Proof. By the definition of generalized partial Caputo fractional derivative, Theorem 10 and the standard integration by parts formula (see, e.g., [38]) one has

$$\begin{aligned} \int_{\Omega_n} f(t) \cdot B_{P_i}[\eta](t) dt &= \int_{\Omega_n} f(t) \cdot K_{P_i} \left[\frac{\partial \eta}{\partial t_i} \right] (t) dt = \int_{\Omega_n} \frac{\partial \eta(t)}{\partial t_i} \cdot K_{P_i^*}[f](t) dt \\ &= \int_{\partial\Omega_n} \eta(t) \cdot K_{P_i^*}[f](t) \cdot \nu^i d(\partial\Omega_n) - \int_{\Omega_n} \eta(t) \cdot \frac{\partial}{\partial t_i} K_{P_i^*}[f](t) dt \\ &= \int_{\partial\Omega_n} \eta(t) \cdot K_{P_i^*}[f](t) \cdot \nu^i d(\partial\Omega_n) - \int_{\Omega_n} \eta(t) \cdot A_{P_i^*}[f](t) dt. \end{aligned}$$

□

Next corollaries present multidimensional integration by parts formulas for constant and variable order fractional derivatives.

Corollary 18. *If $0 < \alpha_i < 1$, functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ are such that $f, \eta \in C^1(\bar{\Omega}_n; \mathbb{R})$ and ${}_t I_{b_i}^{1-\alpha_i}[f] \in C^1(\bar{\Omega}_n; \mathbb{R})$, then*

$$\int_{\Omega_n} f(t) \cdot {}^C D_{t_i}^{\alpha_i}[\eta](t) dt = \int_{\partial\Omega_n} \eta(t) \cdot {}_t I_{b_i}^{1-\alpha_i}[f](t) \cdot \nu^i d(\partial\Omega_n) - \int_{\Omega_n} \eta(t) \cdot {}_t D_{b_i}^{\alpha_i}[f](t) dt.$$

Corollary 19. *If $\alpha_i : [0, b_i - a_i] \rightarrow [0, 1]$, functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ are such that $f, \eta \in C^1(\bar{\Omega}_n; \mathbb{R})$ and ${}_t I_{b_i}^{\alpha_i(\cdot)}[f] \in C^1(\bar{\Omega}_n; \mathbb{R})$, then*

$$\int_{\Omega_n} f(t) \cdot {}^C D_{t_i}^{\alpha_i(\cdot)}[\eta](t) dt = \int_{\partial\Omega_n} \eta(t) \cdot {}_t I_{b_i}^{\alpha_i(\cdot)}[f](t) \cdot \nu^i d(\partial\Omega_n) - \int_{\Omega_n} \eta \cdot {}_t D_{b_i}^{\alpha_i(\cdot)}[f](t) dt.$$

3.7.2 Fundamental Problem

In this subsection, we use the notion of generalized fractional gradient.

Definition 25. *Let $n \in \mathbb{N}$ and $P = (P_1, \dots, P_n)$, $P_i = \langle a_i, t, b_i, \lambda_i, \mu_i \rangle$. We define generalized fractional gradient of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to the generalized fractional operator T by*

$$\nabla_{T_P}[f] := \sum_{i=1}^n e_i \cdot T_{P_i}[f],$$

where $\{e_i : i = 1, \dots, n\}$ denotes the standard basis in \mathbb{R}^n .

For $n \in \mathbb{N}$ let us assume that $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$ and $P = (P_1, \dots, P_n)$, $y : \mathbb{R}^n \rightarrow \mathbb{R}$, and $\zeta : \partial\Omega_n \rightarrow \mathbb{R}$ is a given function. Consider the following functional:

$$\begin{aligned} \mathcal{I} : \mathcal{A}(\zeta) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_{\Omega_n} F(y(t), \nabla_{K_P}[y](t), \nabla[y](t), \nabla_{B_P}[y](t), t) dt \end{aligned} \quad (3.56)$$

where

$$\mathcal{A}(\zeta) := \{y \in C^1(\bar{\Omega}_n; \mathbb{R}) : y|_{\partial\Omega_n} = \zeta, K_{P_i}[y], B_{P_i}[y] \in C(\bar{\Omega}_n; \mathbb{R}), i = 1, \dots, n\},$$

∇ denotes the classical gradient operator, ∇_{K_P} and ∇_{B_P} are generalized fractional gradient operators such that K_{P_i} is the generalized partial fractional integral with the kernel $k_i = k_i(t_i - \tau)$, $k_i \in L^1(0, b_i - a_i; \mathbb{R})$ and B_{P_i} is the generalized partial fractional derivative of Caputo type satisfying $B_{P_i} = K_{P_i} \circ \frac{\partial}{\partial t_i}$, for $i = 1, \dots, n$. Moreover, we assume that F is a Lagrangian of class C^1 :

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R}^{3n} \times \bar{\Omega}_n &\longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) &\longmapsto F(x_1, x_2, x_3, x_4, t), \end{aligned}$$

and

- $K_{P_i}^* [\partial_{1+i} F(y(\tau), \nabla_{K_P}[y](\tau), \nabla[y](\tau), \nabla_{B_P}[y](\tau), \tau)] \in C(\bar{\Omega}_n; \mathbb{R})$,
- $t \mapsto \partial_{1+n+i} F(y(t), \nabla_{K_P}[y](t), \nabla[y](t), \nabla_{B_P}[y](t), t) \in C^1(\bar{\Omega}_n; \mathbb{R})$,
- $K_{P_i}^* [\partial_{1+2n+i} F(y(\tau), \nabla_{K_P}[y](\tau), \nabla[y](\tau), \nabla_{B_P}[y](\tau), \tau)] \in C^1(\bar{\Omega}_n; \mathbb{R})$,

where $i = 1, \dots, n$.

The following theorem states that if a function minimizes functional (3.56), then it necessarily must satisfy (3.57). This means that equation (3.57) determines candidates to solve problem of minimizing functional (3.56).

Theorem 12. *Suppose that $\bar{y} \in \mathcal{A}(\zeta)$ is a minimizer of (3.56). Then, \bar{y} satisfies the following generalized Euler–Lagrange equation:*

$$\begin{aligned} \partial_1 F(\star_{\bar{y}})(t) + \sum_{i=1}^n \left(K_{P_i}^* [\partial_{1+i} F(\star_{\bar{y}})(\tau)](t) \right. \\ \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} F(\star_{\bar{y}})(t)) - A_{P_i}^* [\partial_{1+2n+i} F(\star_{\bar{y}})(\tau)](t) \right) = 0, \quad t \in \Omega_n, \end{aligned} \quad (3.57)$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), \nabla_{K_P}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{B_P}[\bar{y}](t), t)$.

Proof. Let $\bar{y} \in \mathcal{A}(\zeta)$ be a minimizer of (3.56). Then, for any $|h| \leq \varepsilon$ and every $\eta \in \mathcal{A}(0)$, it satisfies

$$\mathcal{I}(\bar{y}) \leq \mathcal{I}(\bar{y} + h\eta).$$

Now, let us define the following function

$$\begin{aligned} \phi_{\bar{y}, \eta} : [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{I}(\bar{y} + h\eta) = \int_{\Omega_n} F(\bar{y}(t) + h\eta(t), \nabla_{K_P}[\bar{y} + h\eta](t), \nabla[\bar{y} + h\eta](t), \nabla_{B_P}[\bar{y} + h\eta](t), t) dt. \end{aligned}$$

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Because $\phi_{\bar{y},\eta} \in C^1([-\varepsilon, \varepsilon]; \mathbb{R})$ and

$$\phi_{\bar{y},\eta}(0) \leq \phi_{\bar{y},\eta}(h), \quad |h| \leq \varepsilon,$$

one has

$$\phi'_{\bar{y},\eta}(0) = \left. \frac{d}{dh} \mathcal{I}(\bar{y} + h\eta) \right|_{h=0} = 0.$$

Moreover, using the chain rule, we obtain

$$\begin{aligned} \int_{\Omega_n} \partial_1 F(\star_{\bar{y}})(t) \cdot \eta(t) + \sum_{i=1}^n \left(\partial_{1+i} F(\star_{\bar{y}})(t) \cdot K_{P_i}[\eta](t) \right. \\ \left. + \partial_{1+n+i} F(\star_{\bar{y}})(t) \cdot \frac{\partial \eta(t)}{\partial t_i} + \partial_{1+2n+i} F(\star_{\bar{y}})(t) \cdot B_{P_i}[\eta](t) \right) dt = 0. \end{aligned}$$

Finally, Theorem 10 implies that

$$\begin{aligned} \int_{\Omega_n} \left(\partial_1 F(\star_{\bar{y}})(t) + \sum_{i=1}^n K_{P_i^*} [\partial_{1+i} F(\star_{\bar{y}})(\tau)](t) \right) \cdot \eta(t) \\ + \sum_{i=1}^n \left(\partial_{1+n+i} F(\star_{\bar{y}})(t) + K_{P_i^*} [\partial_{1+2n+i} F(\star_{\bar{y}})(\tau)](t) \right) \cdot \frac{\partial \eta(t)}{\partial t_i} dt = 0 \end{aligned}$$

and by the classical integration by parts formula (see e.g., [38]) and the fundamental lemma of the calculus of variations (see e.g., Theorem 1.24, [33]), we arrive to (3.57). \square

Example 19. Consider a motion of medium whose displacement may be described as a scalar function $y(t, x)$, where $x = (x^1, x^2)$. For example, this function may represent the transverse displacement of a membrane. Suppose that the kinetic energy T and the potential energy V of the medium are given by

$$T \left(\frac{\partial y(t, x)}{\partial t} \right) = \frac{1}{2} \int \rho(x) \left(\frac{\partial y(t, x)}{\partial t} \right)^2 dx,$$

$$V(y) = \frac{1}{2} \int k(x) |\nabla[y](t, x)|^2 dx,$$

where $\rho(x)$ is the mass density and $k(x)$ is the stiffness, both assumed positive. Then, the classical action functional is

$$\mathcal{I}(y) = \frac{1}{2} \int_{\Omega} \left(\rho(x) \left(\frac{\partial y(t, x)}{\partial t} \right)^2 - k(x) |\nabla[y](t, x)|^2 \right) dx dt.$$

We shall illustrate what are the Euler–Lagrange equations when the Lagrangian density depends on generalized fractional derivatives. When we have the Lagrangian with the kinetic term depending on the operator B_{P_1} , with $P_1 = \langle a_1, t, b_1, \lambda, \mu \rangle$, then the fractional action functional has the form

$$\mathcal{I}(y) = \frac{1}{2} \int_{\Omega_3} \left[\rho(x) (B_{P_1}[y](t, x))^2 - k(x) |\nabla[y](t, x)|^2 \right] dx dt. \quad (3.58)$$

The generalized fractional Euler–Lagrange equation satisfied by an extremal of (3.58) is

$$-\rho(x)A_{P_1^*}[B_{P_1}[y](\tau, s)](t, x) - \nabla[k(s)\nabla[y](\tau, s)](t, x) = 0.$$

If ρ and k are constants, then equation

$$\rho A_{P_1^*}[B_{P_1}[y](\tau, s)](t, x) + c^2 \Delta[y](t, x) = 0,$$

where $c^2 = k/\rho$, can be called the generalized time-fractional wave equation. Now, assume that the kinetic energy and the potential energy depend on B_{P_1} and ∇_{B_P} operators, respectively, where $P = (P_2, P_3)$. Then, the action functional for the system has the form

$$\mathcal{I}(y) = \frac{1}{2} \int_{\Omega_3} [\rho (B_{P_1}[y](t, x))^2 - k |\nabla_{B_P}[y](t, x)|^2] dx dt. \quad (3.59)$$

The generalized fractional Euler–Lagrange equation satisfied by an extremal of (3.59) is

$$-\rho A_{P_1^*}[B_{P_1}[y](\tau, s)](t, x) + \sum_{i=2}^3 A_{P_i^*}[B_{P_i}[y](\tau, s)](t, x) = 0.$$

If ρ and k are constants, then

$$A_{P_1^*}[B_{P_1}[y](\tau, s)](t, x) - c^2 \left(\sum_{i=2}^3 A_{P_i^*}[k B_{P_i}[y](\tau, s)](t, x) \right) = 0$$

can be called the generalized space- and time-fractional wave equation.

Corollary 20. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 1)^n$ and let $\bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R})$ be a minimizer of the functional

$$\mathcal{I}(y) = \int_{\Omega_n} F(y(t), \nabla_{I^{1-\alpha}}[y](t), \nabla[y](t), \nabla_{D^\alpha}[y](t), t) dt \quad (3.60)$$

satisfying

$$y(t)|_{\partial\Omega_n} = \zeta(t), \quad (3.61)$$

where $\zeta : \partial\Omega_n \rightarrow \mathbb{R}$ is a given function,

$$\nabla_{I^{1-\alpha}} = \sum_{i=1}^n e_i \cdot a_i I_{t_i}^{1-\alpha_i}, \quad \nabla_{D^\alpha} = \sum_{i=1}^n e_i \cdot {}^C D_{t_i}^{\alpha_i},$$

F is of class C^1 and

- $t_{b_i}^{1-\alpha_i} [\partial_{1+i} F(y(\tau), \nabla_{I^{1-\alpha}}[y](\tau), \nabla[y](\tau), \nabla_{D^\alpha}[y](\tau), \tau)]$ is continuous on $\bar{\Omega}_n$,
- $t \mapsto \partial_{1+n+i} F(y(t), \nabla_{I^{1-\alpha}}[y](t), \nabla[y](t), \nabla_{D^\alpha}[y](t), t)$ is continuously differentiable on $\bar{\Omega}_n$,
- $t_{b_i}^{1-\alpha_i} [\partial_{1+2n+i} F(y(\tau), \nabla_{I^{1-\alpha}}[y](\tau), \nabla[y](\tau), \nabla_{D^\alpha}[y](\tau), \tau)]$ is continuously differentiable on $\bar{\Omega}_n$.

Then, \bar{y} satisfies the following fractional Euler–Lagrange equation:

$$\begin{aligned} & \partial_1 F(\bar{y}(t), \nabla_{I^{1-\alpha}}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{D^\alpha}[\bar{y}](t), t) \\ & + \sum_{i=1}^n \left({}_t I_{b_i}^{1-\alpha_i} [\partial_{1+i} F(\bar{y}(\tau), \nabla_{I^{1-\alpha}}[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_{D^\alpha}[\bar{y}](\tau), \tau)](t) \right. \\ & \quad \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} F(\bar{y}(t), \nabla_{I^{1-\alpha}}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{D^\alpha}[\bar{y}](t), t)) \right. \\ & \quad \left. + {}_t D_{b_i}^{\alpha_i} [\partial_{1+2n+i} F(\bar{y}(\tau), \nabla_{I^{1-\alpha}}[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_{D^\alpha}[\bar{y}](\tau), \tau)](t) \right) = 0, \quad t \in \Omega_n. \end{aligned} \quad (3.62)$$

Corollary 21. For $i = 1, \dots, n$, suppose that $\alpha_i : [0, b_i - a_i] \rightarrow [0, 1]$ and let

$$\nabla_I = \sum_{i=1}^n e_i \cdot {}_{a_i} I_{t_i}^{1-\alpha_i(\cdot)}, \quad \nabla_D = \sum_{i=1}^n e_i \cdot {}_{a_i}^C D_{t_i}^{\alpha_i(\cdot)}.$$

If $\bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R})$ minimizes the functional

$$\mathcal{I}(y) = \int_{\Omega_n} F(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t) dt, \quad (3.63)$$

subject to the boundary condition

$$y(t)|_{\partial\Omega_n} = \zeta(t),$$

where $\zeta : \partial\Omega_n \rightarrow \mathbb{R}$ is a given function, ${}_{a_i} I_{t_i}^{1-\alpha_i(\cdot)}[y], {}_{a_i}^C D_{t_i}^{\alpha_i(\cdot)} \in C(\bar{\Omega}_n; \mathbb{R})$, F is of class C^1 and

- ${}_t I_{b_i}^{1-\alpha_i(\cdot)} [\partial_{1+i} F(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuous on $\bar{\Omega}_n$,
- $t \mapsto \partial_{1+n+i} F(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t)$ is continuously differentiable on $\bar{\Omega}_n$,
- ${}_t I_{b_i}^{1-\alpha_i(\cdot)} [\partial_{1+2n+i} F(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuously differentiable on $\bar{\Omega}_n$,

then \bar{y} satisfies the following equation

$$\begin{aligned} & \partial_1 F(\bar{y}(t), \nabla_I[\bar{y}](t), \nabla[\bar{y}](t), \nabla_D[\bar{y}](t), t) \\ & + \sum_{i=1}^n \left({}_t I_{b_i}^{1-\alpha_i(\cdot)} [\partial_{1+i} F(\bar{y}(\tau), \nabla_I[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_D[\bar{y}](\tau), \tau)](t) \right. \\ & \quad \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} F(\bar{y}(t), \nabla_I[\bar{y}](t), \nabla[\bar{y}](t), \nabla_D[\bar{y}](t), t)) \right. \\ & \quad \left. + {}_t D_{b_i}^{\alpha_i(\cdot)} [\partial_{1+2n+i} F(\bar{y}(\tau), \nabla_I[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_D[\bar{y}](\tau), \tau)](t) \right) = 0, \quad t \in \Omega_n. \end{aligned} \quad (3.64)$$

Theorem 13. Suppose that $\bar{y} \in \mathcal{A}(\zeta)$ satisfies (3.57) and function $(x_1, x_2, x_3, x_4) \rightarrow F(x_1, x_2, x_3, x_4, t)$ is convex for every $t \in \Omega_n$. Then, \bar{y} is a minimizer of functional (3.56).

Proof. Let $\bar{y} \in \mathcal{A}(\zeta)$ be a function satisfying equation (3.57) and such that $(x_1, x_2, x_3, x_4) \rightarrow F(x_1, x_2, x_3, x_4, t)$ is convex for every $t \in \Omega_n$. Then, the following inequality holds:

$$\mathcal{I}(y) \geq \mathcal{I}(\bar{y}) + \int_{\Omega_n} \left(\partial_1 F(y - \bar{y}) + \sum_{i=1}^n \left[\partial_{1+i} F \cdot K_{P_i} [y - \bar{y}] + \partial_{1+n+i} F \frac{\partial}{\partial t_i} [y - \bar{y}] + \partial_{1+2n+i} F \cdot B_{P_i} [y - \bar{y}] \right] \right) dt,$$

where functions $\partial_i F$ are evaluated at $(\bar{y}, \nabla_{K_P} [\bar{y}], \nabla [\bar{y}], \nabla_{B_P} [\bar{y}], t)$, for $i = 1, \dots, 3n+1$. Moreover, using the classical integration by parts formula, as well as Theorem 10 and the fact that $y - \bar{y}|_{\partial\Omega_n} = 0$, we obtain

$$\mathcal{I}(y) \geq \mathcal{I}(\bar{y}) + \int_{\Omega_n} \left(\partial_1 F + \sum_{i=1}^n \left[K_{P_i}^* [\partial_{1+i} F] + \frac{\partial}{\partial t_i} (\partial_{1+n+i} F) + A_{P_i}^* [\partial_{1+2n+i} F] \right] \right) (y - \bar{y}) dt.$$

Finally, applying equation (3.57), we have $\mathcal{I}(y) \geq \mathcal{I}(\bar{y})$ for any $y \in \mathcal{A}(\zeta)$ and the proof is complete. \square

3.7.3 Dirichlet's Principle

One of the most important variational principles for PDEs is Dirichlet's principle for the Laplace equation. We shall present its generalized fractional counterpart.

We show that the solution of the generalized fractional boundary value problem

$$\begin{cases} \sum_{i=1}^n A_{P_i}^* [B_{P_i} [y]] = 0 & \text{in } \Omega_n, \\ y = \zeta & \text{on } \partial\Omega_n, \end{cases} \quad (3.65)$$

$$(3.66)$$

can be characterized as a minimizer of the following variational functional

$$\mathcal{I}(y) = \int_{\Omega_n} |\nabla_{B_P} [y]|^2 dt \quad (3.67)$$

on the set $\mathcal{A}(\zeta)$, where

$$\nabla_{B_P} = \sum_{i=1}^n e_i \cdot B_{P_i}$$

is the generalized fractional gradient operator such that the partial derivatives B_{P_i} have kernels $k_i = k_i(t_i - \tau)$, $k_i \in L^1(0, b_i - a_i; \mathbb{R})$, and parameter sets given by $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$, $i = 1, \dots, n$.

Remark 5. *In the following we assume that both problems, (3.65)–(3.66) and minimization of (3.67) on the set $\mathcal{A}(\zeta)$, have solutions.*

Theorem 14 (Generalized Fractional Dirichlet's Principle). *Suppose that $\bar{y} \in \mathcal{A}(\zeta)$. Then \bar{y} solves the boundary value problem (3.65)–(3.66) if and only if \bar{y} satisfies*

$$\mathcal{I}(\bar{y}) = \min_{y \in \mathcal{A}(\zeta)} \mathcal{I}(y). \quad (3.68)$$

Proof. Theorem 14 is a simple consequence of Theorem 12 and Theorem 13. \square

Theorem 15. *There exists at most one solution $\bar{y} \in \mathcal{A}(\zeta)$ to problem (3.65)–(3.66).*

Proof. Let $\bar{y} \in \mathcal{A}(\zeta)$ be a solution to problem (3.65)–(3.66). Assume that \hat{y} is another solution to problem (3.65)–(3.66). Then, $w = \bar{y} - \hat{y} \neq 0$ and

$$0 = - \int_{\Omega_n} w \cdot \sum_{i=1}^n A_{P_i^*} [B_{P_i}[w]] dt.$$

By classical integration by parts formula and Theorem 10 and since $w|_{\partial\Omega_n} = 0$ we have

$$0 = \int_{\Omega_n} w \cdot \sum_{i=1}^n (B_{P_i}[w])^2 dt = \int_{\Omega_n} |\nabla_{B_P}[w]|^2 dt.$$

Note that $|\nabla_{B_P}[w]|^2$ is a positive definite quantity. The volume integral of a positive definite quantity is equal to zero only in the case when this quantity is zero itself throughout the volume. Thus $\nabla_{B_P}[w] = 0$. Since w is twice continuously differentiable and $k_i \in L^1(0, b_i - a_i; \mathbb{R})$ we have

$$\frac{\partial}{\partial t_i} w(t) = 0, \quad i = 1, \dots, n$$

i.e., $\nabla[w] = 0$. Because $w = 0$ on $\partial\Omega_n$, we deduce that $w = 0$. In other words $\bar{y} = \hat{y}$. \square

3.7.4 Isoperimetric Problem

Suppose that $y : \mathbb{R}^n \rightarrow \mathbb{R}$, $P = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$, $P = (P_1, \dots, P_n)$ and $\zeta : \partial\Omega_n \rightarrow \mathbb{R}$ is a given curve. Let us define the following functional

$$\begin{aligned} \mathcal{J} : \mathcal{A}(\zeta) &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_{\Omega_n} G(y(t), \nabla_{K_P}[y](t), \nabla[y](t), \nabla_{B_P}[y](t), t) dt, \end{aligned} \quad (3.69)$$

where operators $\nabla_{K_P}, \nabla, \nabla_{B_P}$ and function G are the same as in the case of functional (3.56). The next theorem gives a necessary optimality condition for a function to be a minimizer of functional (3.56) subject to isoperimetric constraint $\mathcal{J}(y) = \xi$.

Theorem 16. *Let us assume that \bar{y} minimizes functional (3.56) on the following set:*

$$\mathcal{A}_\xi(\zeta) := \{y \in \mathcal{A}(\zeta) : \mathcal{J}(y) = \xi\}.$$

Then, one can find a real constant λ_0 such that, for $H = F - \lambda_0 G$, equation

$$\begin{aligned} \partial_1 H(\star \bar{y})(t) + \sum_{i=1}^n \left(K_{P_i^*} [\partial_{1+i} H(\star \bar{y})(\tau)](t) \right. \\ \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(\star \bar{y})(t)) - A_{P_i^*} [\partial_{1+2n+i} H(\star \bar{y})(\tau)](t) \right) = 0 \end{aligned} \quad (3.70)$$

holds, provided that

$$\begin{aligned} \partial_1 G(\star_{\bar{y}})(t) + \sum_{i=1}^n \left(K_{P_i^*} [\partial_{1+i} G(\star_{\bar{y}})(\tau)](t) \right. \\ \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} G(\star_{\bar{y}})(t)) - A_{P_i^*} [\partial_{1+2n+i} G(\star_{\bar{y}})(\tau)](t) \right) \neq 0, \quad (3.71) \end{aligned}$$

where $(\star_{\bar{y}})(t) = (\bar{y}(t), \nabla_{K_P} [\bar{y}](t), \nabla [\bar{y}](t), \nabla_{B_P} [\bar{y}](t), t)$.

Proof. The fundamental lemma of the calculus of variations, and hypothesis (3.71), imply that there exists $\eta_2 \in \mathcal{A}(0)$ so that

$$\begin{aligned} \int_{\Omega_n} \left(\partial_1 G(\star_{\bar{y}})(t) + \sum_{i=1}^n K_{P_i^*} [\partial_{1+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \eta_2(t) \\ + \sum_{i=1}^n \left(\partial_{1+n+i} G(\star_{\bar{y}})(t) + K_{P_i^*} [\partial_{1+2n+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \frac{\partial \eta_2(t)}{\partial t_i} dt = 1. \end{aligned}$$

Now, with function η_2 and an arbitrary $\eta_1 \in \mathcal{A}(0)$, let us define

$$\begin{aligned} \phi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] &\longrightarrow \mathbb{R} \\ (h_1, h_2) &\longmapsto \mathcal{I}(\bar{y} + h_1 \eta_1 + h_2 \eta_2) \end{aligned}$$

and

$$\begin{aligned} \psi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] &\longrightarrow \mathbb{R} \\ (h_1, h_2) &\longmapsto \mathcal{J}(\bar{y} + h_1 \eta_1 + h_2 \eta_2) - \xi \end{aligned}$$

Notice that, $\psi(0, 0) = 0$ and that

$$\begin{aligned} \frac{\partial \psi}{\partial h_2} \Big|_{(0,0)} = \int_{\Omega_n} \left(\partial_1 G(\star_{\bar{y}})(t) + \sum_{i=1}^n K_{P_i^*} [\partial_{1+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \eta_2(t) \\ + \sum_{i=1}^n \left(\partial_{1+n+i} G(\star_{\bar{y}})(t) + K_{P_i^*} [\partial_{1+2n+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \frac{\partial \eta_2(t)}{\partial t_i} dt = 1. \end{aligned}$$

The implicit function theorem implies, that there is $\varepsilon_0 > 0$ and a function $s \in C^1([-\varepsilon_0, \varepsilon_0]; \mathbb{R})$ with $s(0) = 0$ such that

$$\psi(h_1, s(h_1)) = 0, \quad |h_1| \leq \varepsilon_0,$$

and then $\bar{y} + h_1 \eta_1 + s(h_1) \eta_2 \in \mathcal{A}_\xi(\zeta)$. Moreover,

$$\frac{\partial \psi}{\partial h_1} + \frac{\partial \psi}{\partial h_2} \cdot s'(h_1) = 0, \quad |h_1| \leq \varepsilon_0,$$

and then

$$s'(0) = - \frac{\partial \psi}{\partial h_1} \Big|_{(0,0)}.$$

3.7. GENERALIZED VARIATIONAL CALCULUS OF SEVERAL VARIABLES

Because $\bar{y} \in \mathcal{A}(\zeta)$ is a minimizer of \mathcal{I} we have

$$\phi(0, 0) \leq \phi(h_1, s(h_1)), \quad |h_1| \leq \varepsilon_0,$$

and hence

$$\left. \frac{\partial \phi}{\partial h_1} \right|_{(0,0)} + \left. \frac{\partial \phi}{\partial h_2} \right|_{(0,0)} \cdot s'(0) = 0.$$

Letting $\lambda_0 = \left. \frac{\partial \phi}{\partial h_2} \right|_{(0,0)}$ be the Lagrange multiplier we find

$$\left. \frac{\partial \phi}{\partial h_1} \right|_{(0,0)} - \lambda_0 \left. \frac{\partial \psi}{\partial h_1} \right|_{(0,0)} = 0$$

or, in other words,

$$\begin{aligned} \int_{\Omega_n} & \left[\left(\partial_1 F(\star_{\bar{y}})(t) + \sum_{i=1}^n K_{P_i^*} [\partial_{1+i} F(\star_{\bar{y}})(\tau)](t) \right) \cdot \eta_1(t) \right. \\ & \left. + \sum_{i=1}^n \left(\partial_{1+n+i} F(\star_{\bar{y}})(t) + K_{P_i^*} [\partial_{1+2n+i} F(\star_{\bar{y}})(\tau)](t) \right) \cdot \frac{\partial \eta_1(t)}{\partial t_i} \right] \\ & - \lambda_0 \left[\left(\partial_1 G(\star_{\bar{y}})(t) + \sum_{i=1}^n K_{P_i^*} [\partial_{1+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \eta_1(t) \right. \\ & \left. + \sum_{i=1}^n \left(\partial_{1+n+i} G(\star_{\bar{y}})(t) + K_{P_i^*} [\partial_{1+2n+i} G(\star_{\bar{y}})(\tau)](t) \right) \cdot \frac{\partial \eta_1(t)}{\partial t_i} \right] dt = 0. \end{aligned}$$

Finally, applying the integration by parts formula for classical and fractional derivatives, and by the fundamental lemma of the calculus of variations we obtain (3.70). \square

Corollary 22. *Let us assume that $\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 1)^n$ and $\bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R})$ is minimizer of functional (3.60) subject to the isoperimetric constraint $\mathcal{J}(y) = \xi$, where*

$$\mathcal{J}(y) = \int_{\Omega_n} G(y(t), \nabla_{I^{1-\alpha}}[y](t), \nabla[y](t), \nabla_{D^\alpha}[y](t), t) dt \quad (3.72)$$

and boundary condition (3.61). Moreover

- G is of class C^1 ,
- $t_i I_{b_i}^{1-\alpha_i} [\partial_{1+i} G(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuous on $\bar{\Omega}_n$,
- $t \mapsto \partial_{1+n+i} G(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t)$ is continuously differentiable on $\bar{\Omega}_n$,
- $t_i I_{b_i}^{1-\alpha_i} [\partial_{1+2n+i} G(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuously differentiable on $\bar{\Omega}_n$.

Then, if \bar{y} is not an extremal for functional (3.72), we can find $\lambda_0 \in \mathbb{R}$ such that the following equation:

$$\begin{aligned} & \partial_1 H(\bar{y}(t), \nabla_{I^{1-\alpha}}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{D^\alpha}[\bar{y}](t), t) \\ & + \sum_{i=1}^n \left({}_t I_{b_i}^{1-\alpha_i} [\partial_{1+i} H(\bar{y}(\tau), \nabla_{I^{1-\alpha}}[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_{D^\alpha}[\bar{y}](\tau), \tau)](t) \right. \\ & \quad \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(\bar{y}(t), \nabla_{I^{1-\alpha}}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{D^\alpha}[\bar{y}](t), t)) \right. \\ & \quad \left. + {}_{t_i} D_{b_i}^{\alpha_i} [\partial_{1+2n+i} H(\bar{y}(\tau), \nabla_{I^{1-\alpha}}[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_{D^\alpha}[\bar{y}](\tau), \tau)](t) \right) = 0, \quad t \in \Omega_n, \end{aligned}$$

is satisfied, where $H = F - \lambda_0 G$.

Corollary 23. Suppose that $\alpha_i : [0, b_i - a_i] \rightarrow [0, 1]$. If $\bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R})$ minimizes (3.63) subject to (3.61) and

$$\mathcal{J}(y) = \int_{\Omega_n} G(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t) dt = \xi,$$

where ${}_a I_{t_i}^{1-\alpha_i(\cdot)}[y], {}_a^C D_{t_i}^{\alpha_i(\cdot)} \in C(\bar{\Omega}_n; \mathbb{R})$, and

- G is of class C^1 ,
- ${}_t I_{b_i}^{1-\alpha_i(\cdot)} [\partial_{1+i} G(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuous on $\bar{\Omega}_n$,
- $t \mapsto \partial_{1+n+i} G(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t)$ is continuously differentiable on $\bar{\Omega}_n$,
- ${}_{t_i} I_{b_i}^{1-\alpha_i(t_i \cdot)} [\partial_{1+2n+i} G(y(\tau), \nabla_I[y](\tau), \nabla[y](\tau), \nabla_D[y](\tau), \tau)]$ is continuously differentiable on $\bar{\Omega}_n$,

for $i = 1, \dots, n$. Then, there is $\lambda_0 \in \mathbb{R}$ such that, for $H = F - \lambda_0 G$, \bar{y} is a solution to the following equation:

$$\begin{aligned} & \partial_1 H(\bar{y}(t), \nabla_I[\bar{y}](t), \nabla[\bar{y}](t), \nabla_D[\bar{y}](t), t) \\ & + \sum_{i=1}^n \left({}_t I_{b_i}^{1-\alpha_i(\cdot)} [\partial_{1+i} H(\bar{y}(\tau), \nabla_I[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_D[\bar{y}](\tau), \tau)](t) \right. \\ & \quad \left. - \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(\bar{y}(t), \nabla_I[\bar{y}](t), \nabla[\bar{y}](t), \nabla_D[\bar{y}](t), t)) \right. \\ & \quad \left. + {}_{t_i} D_{b_i}^{\alpha_i(\cdot)} [\partial_{1+2n+i} H(\bar{y}(\tau), \nabla_I[\bar{y}](\tau), \nabla[\bar{y}](\tau), \nabla_D[\bar{y}](\tau), \tau)](t) \right) = 0, \quad t \in \Omega_n, \end{aligned}$$

provided that \bar{y} is not a solution to Euler–Lagrange equation associated to \mathcal{J} .

3.7.5 Noether's Theorem

In Section 3.5 of this thesis, we have proved a generalized fractional version of Noether's theorem. That is, assuming invariance of Lagrangian under changes in the coordinate system, we showed that its extremal must satisfy equation (3.43). In this section, we prove a generalized multidimensional fractional Noether's theorem. As before, we start with definitions of extremal and invariance.

Definition 26. A function $y \in C^1(\bar{\Omega}_n; \mathbb{R})$ such that $K_{P_i}[y], B_{P_i}[y] \in C(\bar{\Omega}_n; \mathbb{R})$, $i = 1, \dots, n$ satisfying equation (3.57) is said to be a generalized fractional extremal.

We consider a one-parameter family of transformations of the form $\hat{y}(t) = \phi(\theta, t, y(t))$, where ϕ is a map of class C^2 :

$$\begin{aligned} \phi : [-\varepsilon, \varepsilon] \times \bar{\Omega}_n \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\theta, t, x) &\longmapsto \phi(\theta, t, x), \end{aligned}$$

such that $\phi(0, t, x) = x$. Note that, using Taylor's expansion of $\phi(\theta, t, y(t))$ in a neighborhood of 0 one has

$$\hat{y}(t) = \phi(0, t, y(t)) + \theta \frac{\partial}{\partial \theta} \phi(0, t, y(t)) + o(\theta),$$

provided that $\theta \in [-\varepsilon, \varepsilon]$. Moreover, having in mind that $\phi(0, t, y(t)) = y(t)$ and denoting $\xi(t, y(t)) = \frac{\partial}{\partial \theta} \phi(0, t, y(t))$, one has

$$\hat{y}(t) = y(t) + \theta \xi(t, y(t)) + o(\theta), \quad (3.73)$$

so that, for $\theta \in [-\varepsilon, \varepsilon]$ the linear approximation to the transformation is

$$\hat{y}(t) \approx y(t) + \theta \xi(t, y(t)).$$

Now, let us introduce the notion of invariance.

Definition 27. We say that the Lagrangian F is invariant under the one-parameter family of infinitesimal transformations (3.73), where ξ is such that $t \mapsto \xi(t, y(t)) \in C^1(\bar{\Omega}_n; \mathbb{R})$ with $K_{P_i}[\tau \mapsto \xi(\tau, y(\tau))], B_{P_i}[\tau \mapsto \xi(\tau, y(\tau))] \in C(\bar{\Omega}_n; \mathbb{R})$, $i = 1, \dots, n$, if

$$F(y(t), \nabla_{K_P}[y](t), \nabla[y](t), \nabla_{B_P}[y](t), t) = F(\hat{y}(t), \nabla_{K_P}[\hat{y}](t), \nabla[\hat{y}](t), \nabla_{B_P}[\hat{y}](t), t), \quad (3.74)$$

for all $\theta \in [-\varepsilon, \varepsilon]$, and all $y \in C^1(\bar{\Omega}_n; \mathbb{R})$ with $K_{P_i}[y], B_{P_i}[y] \in C(\bar{\Omega}_n; \mathbb{R})$, $i = 1, \dots, n$.

Similarly to Section 3.5, we want to state Noether's theorem in a compact form. For that we introduce the following two bilinear operators:

$$\mathbf{D}_i[f, g] := f \cdot A_{P_i^*}[g] + g \cdot B_{P_i}[f], \quad (3.75)$$

$$\mathbf{I}_i[f, g] := -f \cdot K_{P_i^*}[g] + g \cdot K_{P_i}[f], \quad (3.76)$$

where $i = 1, \dots, n$.

Now we are ready to state the generalized fractional Noether's theorem.

Theorem 17 (Generalized Multidimensional Fractional Noether's Theorem). *Let F be invariant under the one-parameter family of infinitesimal transformations (3.73). Then for every generalized fractional extremal the following equality holds*

$$\sum_{i=1}^n \mathbf{I}_i [\xi(t, y(t)), \partial_{1+i} F(\star_y)(t)] + \frac{\partial}{\partial t_i} (\xi(t, y(t)) \cdot \partial_{1+n+i} F(\star_y)(t)) + \mathbf{D}_i [\xi(t, y(t)), \partial_{1+2n+i} F(\star_y)(t)] = 0, \quad t \in \Omega_n, \quad (3.77)$$

where $(\star_y)(t) = (y(t), \nabla_{K_P}[y](t), \nabla[y](t), \nabla_{B_P}[y](t), t)$.

Proof. By equation (3.74) one has

$$\left. \frac{d}{d\theta} [F(\hat{y}(t), \nabla_{K_P}[\hat{y}](t), \nabla[\hat{y}](t), \nabla_{B_P}[\hat{y}](t), t)] \right|_{\theta=0} = 0$$

The usual chain rule implies

$$\begin{aligned} & \partial_1 F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} \hat{y}(t) + \sum_{i=1}^n \partial_{1+i} F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} K_{P_i}[\hat{y}](t) \\ & + \partial_{1+n+i} F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} \frac{\partial}{\partial t_i} \hat{y}(t) + \partial_{1+2n+i} F(\star_{\hat{y}})(t) \cdot \frac{d}{d\theta} B_{P_i}[\hat{y}](t) \Big|_{\theta=0} = 0. \end{aligned}$$

By linearity of K_{P_i} , B_{P_i} , $i = 1, \dots, n$ differentiating with respect to θ , and putting $\theta = 0$ we have

$$\begin{aligned} & \partial_1 F(\star_y)(t) \cdot \xi(t, y(t)) + \sum_{i=1}^n \partial_{1+i} F(\star_y)(t) \cdot K_{P_i}[\tau \mapsto \xi(\tau, y(\tau))](t) \\ & + \partial_{1+n+i} F(\star_y)(t) \cdot \frac{\partial}{\partial t_i} \xi(t, y(t)) + \partial_{1+2n+i} F(\star_y)(t) \cdot B_{P_i}[\tau \mapsto \xi(\tau, y(\tau))](t) = 0. \end{aligned}$$

Now, using generalized Euler-Lagrange equation (3.57) and formulas (3.75) and (3.76) we arrive to (3.77). \square

Example 20. *Let $c \in \mathbb{R}$, $P = (P_1, \dots, P_n)$ with $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$ and $y \in C^1(\bar{\Omega}_n; \mathbb{R})$ with $B_{P_i}[y] \in C(\bar{\Omega}_n; \mathbb{R})$, $i = 1, \dots, n$. We consider one-parameter family of infinitesimal transformations*

$$\hat{y}(t) = y(t) + \varepsilon c + o(\varepsilon), \quad (3.78)$$

and the Lagrangian $F(\nabla_{B_P}[y](t), t)$ Then we have

$$F(\nabla_{B_P}[y](t), t) = F(\nabla_{B_P}[\hat{y}](t), t).$$

Hence, F is invariant under (3.78) and Theorem 17 asserts that

$$\sum_{i=1}^n \mathbf{D}_i [c, \partial_{n+i} F(\nabla_{B_P}[y](t), t)] = 0. \quad (3.79)$$

Similarly to the previous sections, one can obtain from Theorem 17 results regarding to constant and variable order fractional integrals and derivatives.

3.8. CONCLUSION

Corollary 24. *Suppose that $\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 1)^n$ and that*

$$F(y(t), \nabla_{I^{1-\alpha}}[y](t), \nabla[y](t), \nabla_{D^\alpha}[y](t), t) = F(\hat{y}(t), \nabla_{I^{1-\alpha}}[\hat{y}](t), \nabla[\hat{y}](t), \nabla_{D^\alpha}[\hat{y}](t), t),$$

where $y \in C^1(\bar{\Omega}_n; \mathbb{R})$ and \hat{y} is the family (3.73). Then all solutions of the Euler–Lagrange equation (3.62) satisfy

$$\sum_{i=1}^n \mathbf{I}_i^{1-\alpha_i} [\xi, \partial_{1+i}F] + \frac{\partial}{\partial t_i} (\xi \cdot \partial_{1+n+i}F) + \mathbf{D}_i^{\alpha_i} [\xi, \partial_{1+2n+i}F] = 0,$$

where

$$\mathbf{D}_i^{\alpha_i}[f, g] := -f \cdot {}_t D_{b_i}^{\alpha_i}[g] + g \cdot {}_a_i^C D_{t_i}^{\alpha_i}[f],$$

$$\mathbf{I}_i^{1-\alpha_i}[f, g] := -f \cdot {}_t I_{b_i}^{1-\alpha_i}[g] + g \cdot {}_a_i I_{t_i}^{1-\alpha_i}[f], \quad i = 1, \dots, n,$$

function ξ is taken in $(t, y(t))$ and functions $\partial_j F$ are evaluated at $(y(t), \nabla_{I^{1-\alpha}}[y](t), \nabla[y](t), \nabla_{D^\alpha}[y](t), t)$ for $j = 1, \dots, 3n$.

Corollary 25. *Let us assume that $y \in C^1(\bar{\Omega}_n; \mathbb{R})$ with ${}_a_i I_{t_i}^{1-\alpha_i(\cdot)}[y], {}_a_i^C D_{t_i}^{\alpha_i(\cdot)}[y] \in C(\bar{\Omega}_n; \mathbb{R})$, $\alpha_i : [0, b_i - a_i] \rightarrow [0, 1]$, and that*

$$F(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t) = F(\hat{y}(t), \nabla_I[\hat{y}](t), \nabla[\hat{y}](t), \nabla_D[\hat{y}](t), t),$$

where \hat{y} is the family (3.73) such that $t \mapsto \xi(t, y(t)) \in C^1(\bar{\Omega}_n; \mathbb{R})$ with ${}_a_i I_{t_i}^{1-\alpha_i(\cdot)}[\tau \mapsto \xi(\tau, y(\tau))], {}_a_i^C D_{t_i}^{\alpha_i(\cdot)}[\tau \mapsto \xi(\tau, y(\tau))] \in C(\bar{\Omega}_n; \mathbb{R})$. Then

$$\sum_{i=1}^n \mathbf{I}_i^{1-\alpha_i(\cdot)} [\xi, \partial_{1+i}F] + \frac{\partial}{\partial t_i} (\xi \cdot \partial_{1+n+i}F) + \mathbf{D}_i^{\alpha_i(\cdot)} [\xi, \partial_{1+2n+i}F] = 0,$$

along any solution of Euler–Lagrange equation (3.64), where

$$\mathbf{D}_i^{\alpha_i(\cdot)}[f, g] := -f \cdot {}_t D_{b_i}^{\alpha_i(\cdot)}[g] + g \cdot {}_a_i^C D_{t_i}^{\alpha_i(\cdot)}[f],$$

$$\mathbf{I}_i^{1-\alpha_i(\cdot)}[f, g] := -f \cdot {}_t I_{b_i}^{1-\alpha_i(\cdot)}[g] + g \cdot {}_a_i I_{t_i}^{1-\alpha_i(\cdot)}[f], \quad i = 1, \dots, n,$$

function ξ is taken in $(t, y(t))$ and functions $\partial_j F$ are evaluated at $(y(t), \nabla_I[y](t), \nabla[y](t), \nabla_D[y](t), t)$ for $j = 1, \dots, 3n$.

3.8 Conclusion

In this chapter we unified, subsumed and significantly extended the necessary optimality conditions available in the literature of the fractional calculus of variations. It should be mentioned, however, that since fractional operators are nonlocal, it can be extremely challenging to find analytical solutions to fractional problems of the calculus of variations and, in many cases, solutions may not exist. Here, we gave several examples with analytic solutions, and many more can be found borrowing different kernels from the book [92]. On the other hand, one can easily choose examples for which the fractional Euler–Lagrange differential equations are hard to solve, and in that case one needs to use numerical methods [10]. However, in the absence of existence, the necessary conditions for extremality are vacuous: one can not

characterize an entity that does not exist in the first place. For solving a problem of the fractional calculus of variations one should proceed along the following three steps: (i) first, prove that a solution to the problem exists; (ii) second, verify the applicability of necessary optimality conditions; (iii) finally, apply the necessary conditions which identify the extremals (the candidates). Further elimination, if necessary, identifies the minimizer(s) of the problem. All three steps in the above procedure are crucial. As mentioned by Young in [106], the calculus of variations has born from the study of necessary optimality conditions, but any such theory is “naive” until the existence of minimizers is verified. Therefore, in the next chapter, we shall follow the direct approach, first proving that a solution exists and next finding candidates with the help of the necessary optimality conditions.

3.9 State of the Art

The results of this chapter were published in [75, 77–83, 86, 87] and were presented by the author at several international conferences:

- International Mathematical Symposium on Mathematical Theory of Networks and Systems, 9th July– 13th July 2012, Melbourne, Australia. Presentation: *A Generalized Fractional Calculus with Applications* [84]. Talk at invited session.
- 5th Symposium on Fractional Differentiation and Its Applications, 14th May – 17th May 2012, Nanjing, China. Presentation: *Generalized Fractional Green’s Theorem*. Talk at invited session.
- Transform methods and special functions, 20th October– 23rd October 2011, Sofia, Bulgaria. Presentation: *Fractional Variational Calculus of Variable Order*.
- SIAM Conference on Control and Its Applications, 8th July– 12th July 2011, Baltimore, United States of America. Presentation: *Fractional Variational Calculus with Classical and Combined Caputo Derivatives*. Talk at invited session.
- International Conference on Isoperimetric Problem of Queen Dido and its Mathematical Ramifications, 24th May– 29th May 2010, Tunis, Tunisia. Presentation: *Fractional Isoperimetric Problems*.
- Workshop on Variational Analysis and Applications, October 2011, Évora, Portugal. Presentation: *Fractional Variational Calculus with Classical and Combined Caputo Derivatives*.
- The 4th IFAC Workshop on Fractional Differentiation and its Applications, 18th October– 20th October 2010, Badajoz, Spain. Presentation: *Calculus of Variations with Fractional and Classical Derivatives* [76].
- 8th EUROPT Workshop on Advances in Continuous Optimization, 9th July – 10th July 2010, Aveiro, Portugal. Presentation: *Necessary Optimality Conditions for Fractional Variational Problem*.

Paper [87] entitled *Green’s theorem for generalized fractional derivatives* [85] received the Grunvald-Letnikov Award, Best Student Paper (Theory) at the conference ‘Fractional Differentiation and its Applications’ in May 2012, Nanjing, China. More detailed information can be found at <http://em.hhu.edu.cn/fda12/Awards.html>.

Chapter 4

Direct Methods in Fractional Calculus of Variations

In this chapter we study the fundamental problem of calculus of variations with a Lagrangian depending on generalized fractional integrals and generalized fractional derivatives. In contrast with the standard approach presented in Chapter 3, we use direct methods to address the problem of finding minima to generalized fractional functionals. First, we prove existence of solutions in an appropriate space of functions and under suitable assumptions of regularity, coercivity and convexity. Next, we proceed with application of an optimality condition, and finish examining the candidates to arrive to solution.

Let us briefly describe the main contents of the chapter. In Section 4.1 we prove a Tonelli type theorem ensuring existence of minimizers for generalized fractional functionals. We also give sufficient conditions for a regular Lagrangian and for a coercive functional. Section 4.2 is devoted to a necessary optimality condition for minimizers. In the last Section 4.3 we improve our results assuming more regularity of the Lagrangian and generalized fractional operators.

4.1 Existence of a Minimizer for a Generalized Functional

Let us recall that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. In this section, our aim is to give sufficient conditions ensuring the existence of a minimizer for the following generalized Lagrangian functional:

$$\begin{aligned} \mathcal{I}: \mathcal{A} &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) dt, \end{aligned}$$

where \mathcal{A} is a weakly closed subset of $W^{1,p}(a, b; \mathbb{R})$, \dot{y} denotes the derivative of y , K_P is the generalized fractional integral with a kernel in $L^q(\Delta; \mathbb{R})$, $B_P = \frac{d}{dt} \circ K_P$ is the generalized fractional Caputo derivative, $P = \langle a, t, b, \lambda, \mu \rangle$ is a set of parameters and F is a Lagrangian of class C^1 :

$$\begin{aligned} F: \mathbb{R}^4 \times [a, b] &\longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) &\longmapsto F(x_1, x_2, x_3, x_4, t). \end{aligned}$$

4.1.1 A Tonelli-type Theorem

In this section we state a Tonelli-type theorem ensuring the existence of a minimizer for \mathcal{I} with the help of general assumptions of regularity, coercivity and convexity. These three hypothesis are usual in the classical case, see [33]. Precisely:

Definition 28. A Lagrangian F is said to be regular if it satisfies:

- $t \mapsto F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^1(a, b; \mathbb{R});$
- $t \mapsto \partial_1 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^1(a, b; \mathbb{R});$
- $t \mapsto \partial_2 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^p(a, b; \mathbb{R});$
- $t \mapsto \partial_3 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^q(a, b; \mathbb{R});$
- $t \mapsto \partial_4 F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^p(a, b; \mathbb{R}),$

for any $y \in W^{1,p}(a, b; \mathbb{R})$:

Definition 29. Functional \mathcal{I} is said to be coercive on \mathcal{A} if it satisfies:

$$\lim_{\substack{\|y\|_{W^{1,p}} \rightarrow \infty \\ y \in \mathcal{A}}} \mathcal{I}(y) = +\infty.$$

We are now in position to state the following result:

Theorem 18 (Tonelli-type Theorem). *Let us assume that:*

- F is regular;
- \mathcal{I} is coercive on \mathcal{A} ;
- $F(\cdot, t)$ is convex on $(\mathbb{R}^d)^4$ for any $t \in [a, b]$.

Then, there exists a minimizer for \mathcal{I} .

Proof. Since F is regular, for any $y \in \mathcal{A}$, $t \mapsto F(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^1(a, b; \mathbb{R})$ and then $\mathcal{I}(y)$ exists in \mathbb{R} . Let us introduce a minimizing sequence $(y_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ satisfying:

$$\mathcal{I}(y_n) \longrightarrow \inf_{y \in \mathcal{A}} \mathcal{I}(y) < +\infty.$$

Since \mathcal{I} is coercive, $(y_n)_{n \in \mathbb{N}}$ is bounded in $W^{1,p}(a, b; \mathbb{R})$. Since $W^{1,p}(a, b; \mathbb{R})$ is a reflexive Banach space, it exists a subsequence of $(y_n)_{n \in \mathbb{N}}$ weakly convergent in $W^{1,p}(a, b; \mathbb{R})$. In the following, we still denote this subsequence by $(y_n)_{n \in \mathbb{N}}$ and we denote by \bar{y} its weak limit. Since \mathcal{A} is a weakly closed subset of $W^{1,p}(a, b; \mathbb{R})$, $\bar{y} \in \mathcal{A}$. Finally, using the convexity of F , we have for any $n \in \mathbb{N}$:

$$\begin{aligned} \mathcal{I}(y_n) \geq \mathcal{I}(\bar{y}) + \int_a^b & \partial_1 F \cdot (y_n(t) - \bar{y}(t)) + \partial_2 F \cdot (K_P[y_n](t) - K_P[\bar{y}](t)) \\ & + \partial_3 F \cdot (\dot{y}_n(t) - \dot{\bar{y}}(t)) + \partial_4 F \cdot (B_P[y_n](t) - B_P[\bar{y}](t)) dt, \end{aligned} \quad (4.1)$$

where $\partial_i F$ are taken in $(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$ for any $i = 1, 2, 3, 4$.

Now, from these four following facts:

- F is regular;
- $y_n \xrightarrow{W^{1,p}} \bar{y}$;
- K_P is linear bounded from $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$;
- the compact embedding $W^{1,p}(a, b; \mathbb{R}) \hookrightarrow C([a, b]; \mathbb{R})$ holds;

one can easily conclude that:

- $t \mapsto \partial_3 F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) \in L^q(a, b; \mathbb{R})$ and $\dot{y}_n \xrightarrow{L^p} \dot{\bar{y}}$;
- $t \mapsto \partial_4 F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) \in L^p(a, b; \mathbb{R})$ and $B_P[y_n] \xrightarrow{L^q} B_P[\bar{y}]$;
- $t \mapsto \partial_1 F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) \in L^1(a, b; \mathbb{R})$ and $y_n \xrightarrow{L^\infty} \bar{y}$;
- $t \mapsto \partial_2 F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) \in L^p(a, b; \mathbb{R})$ and $K_P[y_n] \xrightarrow{L^q} K[\bar{y}]$.

Finally, when n tends to ∞ in the inequality (4.1), we obtain:

$$\inf_{y \in \mathcal{A}} \mathcal{I}(y) \geq \mathcal{I}(\bar{y}) \in \mathbb{R},$$

which completes the proof. \square

The first two hypothesis of Theorem 18 are very general. Consequently, in Sections 4.1.2 and 4.1.3, we give concrete assumptions on F ensuring its regularity and the coercivity of \mathcal{I} .

The last hypothesis of convexity is strong. Nevertheless, from more regularity assumptions on F , on K_P and on B_P , we prove in Section 4.3 that we can provide versions of Theorem 18 with weaker convexity assumptions.

4.1.2 Sufficient Condition for a Regular Lagrangians

In this section we give a sufficient condition on F implying its regularity. First, for any $M \geq 1$, let us define the set \mathcal{P}_M of maps P such that for any $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R})^4 \times [a, b]$:

$$P(x_1, x_2, x_3, x_4, t) = \sum_{k=0}^N c_k(x_1, t) |x_2|^{d_{2,k}} |x_3|^{d_{3,k}} |x_4|^{d_{4,k}},$$

with $N \in \mathbb{N}$ and where, for any $k = 0, \dots, N$, $c_k : \mathbb{R} \times [a, b] \rightarrow \mathbb{R}^+$ is continuous and $(d_{2,k}, d_{3,k}, d_{4,k}) \in [0, q] \times [0, p] \times [0, q]$ satisfies $d_{2,k} + (q/p)d_{3,k} + d_{4,k} \leq (q/M)$.

The following lemma shows the interest of sets \mathcal{P}_M .

Lemma 1. *Let $M \geq 1$ and $P \in \mathcal{P}_M$. Then:*

$$\forall y \in W^{1,p}(a, b; \mathbb{R}), t \mapsto P(y(t), K_P[y](t), \dot{y}(t), B_P[y](t), t) \in L^M(a, b; \mathbb{R}).$$

Proof. For any $k = 0, \dots, N$, $c_k(u, t)$ is continuous and then it is in $L^\infty(a, b; \mathbb{R})$. We also have $|K_P[y]|^{d_{2,k}} \in L^{q/d_{2,k}}(a, b; \mathbb{R})$, $|\dot{y}|^{d_{3,k}} \in L^{p/d_{3,k}}(a, b; \mathbb{R})$ and $|B_P[y]|^{d_{4,k}} \in L^{q/d_{4,k}}(a, b; \mathbb{R})$. Consequently:

$$c_k(u, t) |K_P[y]|^{d_{2,k}} |\dot{y}|^{d_{3,k}} |B_P[y]|^{d_{4,k}} \in L^r(a, b; \mathbb{R}), \quad (4.2)$$

with $r = q/(d_{2,k} + (q/p)d_{3,k} + d_{4,k}) \geq M$. The proof is complete. \square

Then, from this previous Lemma, one can easily obtain the following proposition.

Proposition 1. *Let us assume that there exist $P_0 \in \mathcal{P}_1$, $P_1 \in \mathcal{P}_1$, $P_2 \in \mathcal{P}_p$, $P_3 \in \mathcal{P}_q$ and $P_4 \in \mathcal{P}_p$ such that for any $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R})^4 \times [a, b]$:*

- $|F(x_1, x_2, x_3, x_4, t)| \leq P_0(x_1, x_2, x_3, x_4, t);$
- $|\partial_1 F(x_1, x_2, x_3, x_4, t)| \leq P_1(x_1, x_2, x_3, x_4, t);$
- $|\partial_2 F(x_1, x_2, x_3, x_4, t)| \leq P_2(x_1, x_2, x_3, x_4, t);$
- $|\partial_3 F(x_1, x_2, x_3, x_4, t)| \leq P_3(x_1, x_2, x_3, x_4, t);$
- $|\partial_4 F(x_1, x_2, x_3, x_4, t)| \leq P_4(x_1, x_2, x_3, x_4, t).$

Then, F is regular.

This last proposition states that if the norms of F and of its partial derivatives are controlled from above by elements of \mathcal{P}_M , then F is regular. We will see some examples in Section 4.1.4.

4.1.3 Sufficient Condition for a Coercive Functionals

The definition of coercivity for functional \mathcal{I} is strongly dependent on the considered set \mathcal{A} . Consequently, in this section, we will consider an example of set \mathcal{A} and we will give a sufficient condition on F ensuring the coercivity of \mathcal{I} in this case.

Precisely, let us consider $y_a \in \mathbb{R}$ and $\mathcal{A} = W_a^{1,p}(a, b; \mathbb{R})$ where $W_a^{1,p}(a, b; \mathbb{R}) := \{y \in W^{1,p}(a, b; \mathbb{R}), y(a) = y_a\}$. From the compact embedding $W^{1,p}(a, b; \mathbb{R}) \hookrightarrow C([a, b]; \mathbb{R})$, $W_a^{1,p}(a, b; \mathbb{R})$ is weakly closed in $W^{1,p}(a, b; \mathbb{R})$.

An important consequence of such a choice of set \mathcal{A} is given by the following lemma.

Lemma 2. *There exist $A_0, A_1 \geq 0$ such that for any $y \in W_a^{1,p}(a, b; \mathbb{R})$:*

- $\|y\|_{L^\infty} \leq A_0 \|\dot{y}\|_{L^p} + A_1;$
- $\|K_P[y]\|_{L^q} \leq A_0 \|\dot{y}\|_{L^p} + A_1;$
- $\|B_P[y]\|_{L^q} \leq A_0 \|\dot{y}\|_{L^p} + A_1.$

Proof. The last inequality comes from the boundedness of K_P . Let us consider the second one. For any $y \in W_a^{1,p}(a, b; \mathbb{R})$, we have $\|y\|_{L^p} \leq \|y - y_a\|_{L^p} + \|y_a\|_{L^p} \leq (b-a) \|\dot{y}\|_{L^p} + (b-a)^{1/p} |y_a|$. We conclude using again the boundedness of K_P . Now, let us consider the first inequality. For any $y \in W_a^{1,p}(a, b; \mathbb{R})$, we have $\|y\|_{L^\infty} \leq \|y - y_a\|_{L^\infty} + \|y_a\|_{L^\infty} \leq \|\dot{y}\|_{L^1} + \|y_a\|_{L^\infty} \leq (b-a)^{1/q} \|\dot{y}\|_{L^p} + \|y_a\|_{L^\infty}$. Finally, we have just to define A_0 and A_1 as the maxima of the appearing constants. The proof is complete. \square

Precisely, Lemma 2 states the *affine domination* of the term $\|\dot{y}\|_{L^p}$ on the terms $\|y\|_{L^\infty}$, $\|K_P[y]\|_{L^q}$ and $\|B_P[y]\|_{L^q}$ for any $y \in W_a^{1,p}(a, b; \mathbb{R})$. This characteristic of $W_a^{1,p}(a, b; \mathbb{R})$ leads us to give the following sufficient condition for a coercive functional \mathcal{I} .

Proposition 2. *Let us assume that for any $(x_1, x_2, x_3, x_4, t) \in \mathbb{R}^4 \times [a, b]$:*

$$F(x_1, x_2, x_3, x_4, t) \geq c_0 |x_3|^p + \sum_{k=1}^N c_k |x_1|^{d_{1,k}} |x_2|^{d_{2,k}} |x_3|^{d_{3,k}} |x_4|^{d_{4,k}},$$

with $c_0 > 0$ and $N \in \mathbb{N}$ and where, for any $k = 1, \dots, N$, $c_k \in \mathbb{R}$ and $(d_{1,k}, d_{2,k}, d_{3,k}, d_{4,k}) \in \mathbb{R}^+ \times [0, q] \times [0, p] \times [0, q]$ satisfies:

$$d_{2,k} + (q/p)d_{3,k} + d_{4,k} \leq q \quad \text{and} \quad 0 \leq d_{1,k} + d_{2,k} + d_{3,k} + d_{4,k} < p.$$

Then, \mathcal{I} is coercive on $W_a^{1,p}(a, b; \mathbb{R})$.

Proof. Let us define $r_k = q/(d_{2,k} + d_{4,k} + (q/p)d_{3,k}) \geq 1$ and let r'_k denote the adjoint of r_k i.e., $r'_k = \frac{r_k}{r_k - 1}$. Using Hölder's inequality, one can easily prove that, for any $y \in W_a^{1,p}(a, b; \mathbb{R})$, we have:

$$\mathcal{I}(y) \geq c_0 \|\dot{y}\|_{L^p} - (b-a)^{1/r'} \sum_{k=1}^N |c_k| \|y\|_{L^\infty}^{d_{1,k}} \|K_P[y]\|_{L^q}^{d_{2,k}} \|\dot{y}\|_{L^p}^{d_{3,k}} \|B_P[y]\|_{L^q}^{d_{4,k}}.$$

From the "affine domination" of the term $\|\dot{y}\|_{L^p}$ on the terms $\|y\|_{L^\infty}$, $\|K_P[y]\|_{L^q}$ and $\|B_P[y]\|_{L^q}$ for any $y \in W_a^{1,p}(a, b; \mathbb{R})$ (see Lemma 2) and from the assumption $0 \leq d_{1,k} + d_{2,k} + d_{3,k} + d_{4,k} < p$, we obtain that:

$$\lim_{\substack{\|\dot{y}\|_{L^p} \rightarrow \infty \\ y \in W_a^{1,p}(a, b; \mathbb{R})}} \mathcal{I}(y) = +\infty.$$

Finally, from Lemma 2, we also have in $W_a^{1,p}(a, b; \mathbb{R})$:

$$\|\dot{y}\|_{L^p} \rightarrow \infty \iff \|y\|_{W^{1,p}} \rightarrow \infty.$$

Consequently, \mathcal{I} is coercive on $W_a^{1,p}(a, b; \mathbb{R})$. The proof is complete. \square

In this section, we have studied the case where \mathcal{A} is the weakly closed subset of $W^{1,p}(a, b; \mathbb{R})$ satisfying the initial condition $y(a) = y_a$. For other examples of set \mathcal{A} , let us note that all the results of this section are still valid when:

- \mathcal{A} is weakly closed subset of $W^{1,p}(a, b; \mathbb{R})$ satisfying a final condition in $t = b$;
- \mathcal{A} is weakly closed subset of $W^{1,p}(a, b; \mathbb{R})$ satisfying two boundary conditions in $t = a$ and in $t = b$.

4.1.4 Examples of Lagrangians

In this section, we give several examples of a convex Lagrangian F satisfying assumptions of Propositions 1 and 2. In consequence, they are examples of application of Theorem 18 in the case $\mathcal{A} = W_a^{1,p}(a, b; \mathbb{R})$.

Example 21. *The most classical examples of a Lagrangian are the quadratic ones. Let us consider the following:*

$$F(x_1, x_2, x_3, x_4, t) = c(t) + \frac{1}{2} \sum_{i=1}^4 |x_i|^2,$$

where $c : [a, b] \rightarrow \mathbb{R}$ is of class C^1 . One can easily check that F satisfies the assumptions of Propositions 1 and 2 with $p = q = 2$. Moreover, F satisfies the convexity hypothesis of Theorem 18. Consequently, one can conclude that there exists a minimizer of \mathcal{I} defined on $W_a^{1,2}(a, b; \mathbb{R})$.

Example 22. Let us consider $p = q = 2$ and let us still denote by F the Lagrangian defined in Example 21. To obtain a more general example, one can define a Lagrangian F_1 from F as a time-dependent homothetic transformation and/or translation of its variables. Precisely:

$$F_1(x_1, x_2, x_3, x_4, t) = F(c_1(t)x_1 + c_1^0(t), c_2(t)x_2 + c_2^0(t), c_3(t)x_3 + c_3^0(t), c_4(t)x_4 + c_4^0(t), t), \quad (4.3)$$

where $c_i : [a, b] \rightarrow \mathbb{R}$ and $c_i^0 : [a, b] \rightarrow \mathbb{R}$ are of class C^1 for any $i = 1, 2, 3, 4$. In this case, F_1 also satisfies convexity hypothesis of Theorem 18 and the assumptions of Proposition 1. Moreover, if c_3 is with values in \mathbb{R}^+ , then F_1 also satisfies the assumptions of Proposition 2.

One should be careful: this last remark is not available in more general context. Precisely, if a general Lagrangian F satisfies the convexity hypothesis of Theorem 18 and assumptions of Propositions 1 and 2, then Lagrangian F_1 obtained by (4.3) also satisfies the convexity hypothesis of Theorem 18 and the assumptions of Proposition 1. Nevertheless, the assumption of Proposition 2 can be lost by this process.

Example 23. We can also study quasi-linear examples given by Lagrangians of the type

$$F(x_1, x_2, x_3, x_4, t) = c(t) + \frac{1}{p} |x_3|^p + \sum_{i=1}^4 f_i(t) \cdot x_i,$$

where $c : [a, b] \rightarrow \mathbb{R}$ and for any $i = 1, 2, 3, 4$, $f_i : [a, b] \rightarrow \mathbb{R}$ are of class C^1 . In this case, F satisfies the assumptions of Propositions 1 and 2. Consequently, since F satisfies the convexity hypothesis of Theorem 18, one can conclude that there exists a minimizer of \mathcal{I} defined on $W_a^{1,p}(a, b; \mathbb{R})$.

The most important constraint in order to apply Theorem 18 is the convexity hypothesis. This is the reason why the previous examples concern convex quasi-polynomial Lagrangians. Nevertheless, in Section 4.3, we are going to provide some improved versions of Theorem 18 with weaker convexity assumptions. This will be allowed by more regularity hypotheses on F and/or on K_P and B_P . We refer to Section 4.3 for more details.

4.2 Necessary Optimality Condition for a Minimizer

In this section, we assume additionally that:

- F satisfies the assumptions of Proposition 1 (in particular, F is regular);
- \mathcal{A} satisfies the following condition:

$$\forall y \in \mathcal{A}, \forall \eta \in \mathcal{C}_c^\infty, \exists 0 < \varepsilon \leq 1, \forall |h| \leq \varepsilon, y + h\eta \in \mathcal{A}. \quad (4.4)$$

4.2. NECESSARY OPTIMALITY CONDITION FOR A MINIMIZER

The assumption on \mathcal{A} is satisfied if $\mathcal{A} + \mathcal{C}_c^\infty \subset \mathcal{A}$ (for example $\mathcal{A} = W_a^{1,p}(a, b; \mathbb{R})$ in Section 4.1.3). By \mathcal{C}_c^∞ we denote the space of infinitely differentiable functions compactly supported in (a, b) .

In the next theorem we will make use of the following Lemma.

Lemma 3. *Let $M \geq 1$ and $P \in \mathcal{P}_M$. Then, for any $y \in \mathcal{A}$ and any $\eta \in \mathcal{C}_c^\infty$, it exists $g \in L^M(a, b; \mathbb{R}^+)$ such that for any $h \in [-\varepsilon, \varepsilon]$:*

$$P(y + h\eta, K_P[y] + hK_P[\eta], \dot{y} + h\dot{\eta}, B_P[y] + hB_P[\eta], t) \leq g. \quad (4.5)$$

Proof. Indeed, for any $k = 0, \dots, N$, for almost all $t \in (a, b)$ and for any $h \in [-\varepsilon, \varepsilon]$, we have:

$$\begin{aligned} & c_k(y(t) + h\eta(t), t) |K_P[y](t) + hK_P[\eta](t)|^{d_{2,k}} |\dot{y}(t) + h\dot{\eta}(t)|^{d_{3,k}} |B_P[y](t) + hB_P[\eta](t)|^{d_{4,k}} \\ & \leq \bar{c}_k \underbrace{(|K_P[y](t)|^{d_{2,k}} + |K_P[\eta](t)|^{d_{2,k}})}_{\in L^{q/d_{2,k}}(a, b; \mathbb{R})} \underbrace{(|\dot{y}(t)|^{d_{3,k}} + |\dot{\eta}(t)|^{d_{3,k}})}_{\in L^{p/d_{3,k}}(a, b; \mathbb{R})} \underbrace{(|B_P[y](t)|^{d_{4,k}} + |B_P[\eta](t)|^{d_{4,k}})}_{\in L^{q/d_{4,k}}(a, b; \mathbb{R})}, \end{aligned} \quad (4.6)$$

where $\bar{c}_k = 2^{d_{2,k} + d_{3,k} + d_{4,k}} \max_{[a, b] \times [-\varepsilon, \varepsilon]} c_k(y(t) + h\eta(t), t)$ exists in \mathbb{R} because c_k , y and η are continuous. Since $d_{2,k} + (q/p)d_{3,k} + d_{4,k} \leq (q/M)$, the right term in inequality (4.6) is in $L^M(a, b; \mathbb{R}^+)$ and is independent of h . The proof is complete. \square

Finally, from this previous Lemma, we can prove the following result:

Theorem 19. *Let us assume that F satisfies assumptions of Proposition 1, \mathcal{I} is coercive and $F(\cdot, t)$ is convex on \mathbb{R}^4 for any $t \in [a, b]$. Then, the minimizer $\bar{y} \in \mathcal{A}$ of \mathcal{I} (given by Theorem 18) satisfies the generalized Euler-Lagrange equation*

$$\frac{d}{dt} (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_4 F(\star_{\bar{y}})(\tau)](t)) = \partial_1 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_2 F(\star_{\bar{y}})(\tau)](t), \quad (\text{GEL})$$

for almost all $t \in (a, b)$, where $(\star_{\bar{y}})(t) = (\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)$.

Proof. Since F satisfies the assumptions of Proposition 1, F is regular. Consequently, from Theorem 18, we know that \mathcal{I} admits a minimizer $\bar{y} \in \mathcal{A}$ and then

$$\mathcal{I}(\bar{y}) \leq \mathcal{I}(\bar{y} + h\eta), \quad (4.7)$$

for any $|h| \leq \varepsilon$ and every $\eta \in \mathcal{C}_c^\infty$. Let us define the following map:

$$\begin{aligned} \phi_{\bar{y}, \eta} : [-\varepsilon, \varepsilon] & \longrightarrow \mathbb{R} \\ h & \longmapsto \mathcal{I}(\bar{y} + h\eta) = \int_a^b \psi_{\bar{y}, \eta}(t, h) dt, \end{aligned}$$

where

$$\psi_{\bar{y}, \eta}(t, h) := F(\bar{y}(t) + h\eta(t), K_P[\bar{y}](t) + hK_P[\eta](t), \dot{\bar{y}}(t) + h\dot{\eta}(t), B_P[\bar{y}](t) + hB_P[\eta](t), t).$$

First we want to prove that the following term:

$$\lim_{h \rightarrow 0} \frac{\mathcal{I}(\bar{y} + h\eta) - \mathcal{I}(\bar{y})}{h} = \lim_{h \rightarrow 0} \frac{\phi_{\bar{y}, \eta}(h) - \phi_{\bar{y}, \eta}(0)}{h} = \phi'_{\bar{y}, \eta}(0) \quad (4.8)$$

exists in \mathbb{R} . In order to differentiate $\phi_{\bar{y},\eta}$, we use the theorem of differentiation under the integral sign. Indeed, we have for almost all $t \in (a, b)$ that $\psi_{\bar{y},\eta}(t, \cdot)$ is differentiable on $[-\varepsilon, \varepsilon]$ with:

$$\begin{aligned} & \forall h \in [-\varepsilon, \varepsilon], \frac{\partial \psi_{\bar{y},\eta}}{\partial h}(t, h) \\ &= \partial_1 F(\star_{\bar{y}+h\eta})(t) \cdot \eta(t) + \partial_2 F(\star_{\bar{y}+h\eta})(t) \cdot K_P[\eta](t) + \partial_3 F(\star_{\bar{y}+h\eta})(t) \cdot \dot{\eta}(t) + \partial_4 F(\star_{\bar{y}+h\eta})(t) \cdot B_P[\eta](t). \end{aligned} \quad (4.9)$$

Since F satisfies the assumptions of Proposition 1, from Lemma 3 there exist $g_1 \in L^1(a, b; \mathbb{R}^+)$, $g_2 \in L^p(a, b; \mathbb{R}^+)$, $g_3 \in L^q(a, b; \mathbb{R}^+)$ and $g_4 \in L^p(a, b; \mathbb{R}^+)$ such that for any $h \in [-\varepsilon, \varepsilon]$ and for almost all $t \in (a, b)$:

$$\left| \frac{\partial \psi_{\bar{y},\eta}}{\partial h}(t, h) \right| \leq g_1(t) |\eta(t)| + g_2(t) |K_P[\eta](t)| + g_3(t) |\dot{\eta}(t)| + g_4(t) |B_P[\eta](t)|. \quad (4.10)$$

Since $\eta \in L^\infty(a, b; \mathbb{R})$, $K_P[\eta] \in L^q(a, b; \mathbb{R})$, $\dot{\eta} \in L^p(a, b; \mathbb{R})$ and $B_P[\eta] \in L^q(a, b; \mathbb{R})$, we can conclude that the right term in inequality (4.10) is in $L^1(a, b; \mathbb{R}^+)$ and is independent of h . Consequently, we can use the theorem of differentiation under the integral sign and we obtain that $\phi_{\bar{y},\eta}$ is differentiable with

$$\forall h \in [-\varepsilon, \varepsilon], \phi'_{\bar{y},\eta}(h) = \int_a^b \frac{\partial \psi_{\bar{y},\eta}}{\partial h}(t, h) dt. \quad (4.11)$$

Furthermore, inequality (4.7) implies

$$\left. \frac{d}{dh} \phi_{\bar{y},\eta}(h) \right|_{h=0} = 0,$$

what can be written as

$$\int_a^b \partial_1 F(\star_{\bar{y}})(t) \cdot \eta(t) + \partial_2 F(\star_{\bar{y}})(t) \cdot K_P[\eta](t) + \partial_3 F(\star_{\bar{y}})(t) \cdot \dot{\eta}(t) + \partial_4 F(\star_{\bar{y}})(t) \cdot B_P[\eta](t) dt = 0.$$

Moreover, applying integration by parts formula (3.5) one has

$$\int_a^b (\partial_1 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_2 F(\star_{\bar{y}})(\tau)]) \cdot \eta(t) + (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_4 F(\star_{\bar{y}})(\tau)]) \cdot \dot{\eta}(t) dt = 0.$$

Then, taking an absolutely continuous anti-derivative w of $t \mapsto \partial_1 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_2 F(\star_{\bar{y}})(\tau)](t) \in L^1(a, b; \mathbb{R})$, we obtain using integration by parts that:

$$\int_a^b (\partial_3 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_4 F(\star_{\bar{y}})(\tau)](t) - w(t)) \cdot \dot{\eta}(t) dt = 0.$$

From the du Bois Reymond Lemma there exists a constant $C \in \mathbb{R}$ such that for almost all $t \in (a, b)$, we have:

$$\partial_3 F(\star_{\bar{y}})(t) + K^*[\partial_4 F(\star_{\bar{y}})(\tau)](t) = C + w(t). \quad (4.12)$$

4.3. SOME IMPROVEMENTS

Since the right term is absolutely continuous, we can differentiate it almost everywhere on (a, b) . Finally, we obtain that \bar{y} is a minimizer of \mathcal{I} . Then the following equation holds almost everywhere on (a, b) :

$$\frac{d}{dt}(\partial_3 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_4 F(\star_{\bar{y}})(\tau)](t)) = \partial_1 F(\star_{\bar{y}})(t) + K_{P^*}[\partial_2 F(\star_{\bar{y}})(\tau)](t). \quad (4.13)$$

The proof is complete. \square

We would like to remark that in the particular case, when $k^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha)}(t - \tau)^{-\alpha}$, $k^\alpha \in L^q(\Delta; \mathbb{R})$ and $P = \langle a, t, b, 1, 0 \rangle$, then Sections 4.1.1, 4.1.2, 4.1.3, 4.1.4 and 4.2 recover the case of the following fractional variational functional

$$\begin{aligned} \mathcal{I}: \mathcal{A} &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b F(y(t), {}_a I_t^{1-\alpha}[y](t), \dot{y}(t), {}_a^C D_t^\alpha[y](t), t) dt, \end{aligned}$$

studied in [23]. Moreover, if we choose the kernel $k^\alpha(t, \tau) = \frac{1}{\Gamma(1-\alpha(t, \tau))}(t - \tau)^{-\alpha(t, \tau)}$, $k^\alpha \in L^q(\Delta; \mathbb{R})$ and the parameter set $P = \langle a, t, b, 1, 0 \rangle$, then Sections 4.1.1, 4.1.2, 4.1.3, 4.1.4 and 4.2 restore the case of the variable order fractional variational functional

$$\begin{aligned} \mathcal{I}: \mathcal{A} &\longrightarrow \mathbb{R} \\ y &\longmapsto \int_a^b F\left(y(t), {}_a I_t^{1-\alpha(\cdot, \cdot)}[y](t), \dot{y}(t), {}_a^C D_t^{\alpha(\cdot, \cdot)}[y](t), t\right) dt. \end{aligned}$$

4.3 Some Improvements

In this section, we assume more regularity of the Lagrangian F and of the operators K_P and B_P . It allows to weaken the convexity assumption in Theorem 18 and/or the assumptions of Propositions 1 and 2.

4.3.1 A First Weaker Convexity Assumption

Let us assume that F satisfies the following condition:

$$(F(\cdot, x_2, x_3, x_4, t))_{(x_2, x_3, x_4, t) \in \mathbb{R}^3 \times [a, b]} \text{ is uniformly equicontinuous on } \mathbb{R}. \quad (4.14)$$

This condition has to be understood as:

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0, \forall (u, v) \in \mathbb{R}^2, |u - v| \leq \delta \implies \forall (x_2, x_3, x_4, t) \in \mathbb{R}^3 \times [a, b], \\ |F(u, x_2, x_3, x_4, t) - F(v, x_2, x_3, x_4, t)| \leq \varepsilon. \end{aligned} \quad (4.15)$$

For example, this condition is satisfied for a Lagrangian F with bounded $\partial_1 F$. In this case, we can prove the following improved version of Theorem 18:

Theorem 20. *Let us assume that:*

- F satisfies the condition given by (4.14);
- F is regular;

- \mathcal{I} is coercive on \mathcal{A} ;
- $L(x_1, \cdot, t)$ is convex on \mathbb{R}^3 for any $x_1 \in \mathbb{R}$ and for any $t \in [a, b]$.

Then, there exists a minimizer for \mathcal{I} .

Proof. Indeed, with the same proof of Theorem 18, we can construct a weakly convergent sequence $(y_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ satisfying

$$y_n \xrightarrow{W^{1,p}} \bar{y} \in \mathcal{A} \quad \text{and} \quad \mathcal{I}(y_n) \longrightarrow \inf_{y \in \mathcal{A}} \mathcal{I}(y) < +\infty.$$

Since the compact embedding $W^{1,p}(a, b; \mathbb{R}) \hookrightarrow C([a, b]; \mathbb{R})$ holds, we have $y_n \xrightarrow{C} \bar{y}$. Let $\varepsilon > 0$ and let us consider $\delta > 0$ given by (4.15). There exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\|y_n - \bar{y}\|_\infty \leq \delta$. So, for any $n \geq N$ and for $t \in (a, b)$:

$$|F(y_n(t), K_P[y_n](t), \dot{y}_n(t), B_P[y_n](t), t) - F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)| \leq \varepsilon.$$

Consequently, for any $n \geq N$, we have

$$\mathcal{I}(y_n) \geq \int_a^b F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) dt - (b - a)\varepsilon.$$

From the convexity hypothesis and using the same strategy as in the proof of Theorem 18, we have by passing to the limit on n

$$\inf_{y \in \mathcal{A}} \mathcal{I}(y) \geq \mathcal{I}(\bar{y}) - (b - a)\varepsilon.$$

The proof is complete since the previous inequality is true for any $\varepsilon > 0$. □

Such an improvement allows to give examples of a Lagrangian F without convexity on its first variable. Taking inspiration from Example 21, we can provide the following example.

Example 24. Let us consider $p = 2$ and $\mathcal{A} = W_a^{1,2}(a, b; \mathbb{R})$. Let us consider

$$F(x_1, x_2, x_3, x_4, t) = f(x_1, t) + \frac{1}{2} \sum_{i=2}^4 |x_i|^2,$$

for any $f : \mathbb{R} \times [a, b] \rightarrow \mathbb{R}$ of class C^1 with $\partial_1 f$ bounded (like sine and cosine functions). In this case, F satisfies the hypothesis of Theorem 20 and we can conclude with the existence of a minimizer of \mathcal{I} defined on \mathcal{A} .

4.3.2 A Second Weaker Convexity Assumption

In this section, we assume that K_P is moreover a linear bounded operator from $C([a, b]; \mathbb{R})$ to $C([a, b]; \mathbb{R})$. For example, this condition is satisfied by Riemann–Liouville fractional integrals (see [51, 98] for detailed proofs). We also assume that F satisfies the following condition:

$$(F(\cdot, \cdot, x_3, x_4, t))_{(x_3, x_4, t) \in \mathbb{R}^2 \times [a, b]} \quad \text{is uniformly equicontinuous on } \mathbb{R}^2. \quad (4.16)$$

4.3. SOME IMPROVEMENTS

This condition has to be understood as:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall (u, v) \in \mathbb{R}^2, \forall (u_0, v_0) \in \mathbb{R}^2 \quad |u - v| \leq \delta, |u_0 - v_0| \leq \delta \implies \forall (x_3, x_4, t) \in \mathbb{R}^2 \times [a, b], \\ |F(u, u_0, x_3, x_4, t) - F(v, v_0, x_3, x_4, t)| \leq \varepsilon. \quad (4.17)$$

For example, this condition is satisfied for a Lagrangian F with bounded $\partial_1 F$ and bounded $\partial_2 F$. In this case, we can prove the following improved version of Theorem 18:

Theorem 21. *Let us assume that:*

- F satisfies the condition given by (4.16);
- F is regular;
- \mathcal{I} is coercive on \mathcal{A} ;
- $L(x_1, x_2, \cdot, t)$ is convex on \mathbb{R}^2 for any $(x_1, x_2) \in \mathbb{R}$ and for any $t \in [a, b]$.

Then, there exists a minimizer for \mathcal{I} .

Proof. Indeed, with the same proof of Theorem 18, we can construct a weakly convergent sequence $(y_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ satisfying

$$y_n \xrightarrow{W^{1,p}} \bar{y} \in \mathcal{A} \quad \text{and} \quad \mathcal{I}(y_n) \longrightarrow \inf_{y \in \mathcal{A}} \mathcal{I}(y) < +\infty.$$

Since the compact embedding $W^{1,p}(a, b; \mathbb{R}) \hookrightarrow C([a, b]; \mathbb{R})$ holds, we have $y_n \xrightarrow{C} \bar{y}$ and since K_P is continuous from $C([a, b]; \mathbb{R})$ to $C([a, b]; \mathbb{R})$, we have $K_P[\bar{y}_n] \xrightarrow{C} K_P[\bar{y}]$. Let $\varepsilon > 0$ and let us consider $\delta > 0$ given by (4.17). There exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\|y_n - \bar{y}\|_\infty \leq \delta$ and $\|K_P[y_n] - K_P[\bar{y}]\|_\infty \leq \delta$. So, for any $n \geq N$ and for $t \in (a, b)$:

$$|F(y_n(t), K_P[y_n](t), \dot{y}_n(t), B_P[y_n](t), t) - F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t)| \leq \varepsilon.$$

Consequently, for any $n \geq N$, we have

$$\mathcal{I}(y_n) \geq \int_a^b F(\bar{y}(t), K_P[\bar{y}](t), \dot{\bar{y}}(t), B_P[\bar{y}](t), t) dt - (b - a)\varepsilon.$$

From the convexity hypothesis and using the same strategy as in the proof of Theorem 18, we have by passing to the limit on n :

$$\inf_{y \in \mathcal{A}} \mathcal{I}(y) \geq \mathcal{I}(\bar{y}) - (b - a)\varepsilon.$$

The proof is complete since the previous inequality is true for any $\varepsilon > 0$. □

Such an improvement allows to give examples of a Lagrangian F without convexity on its two first variables. Taking inspiration from Example 23, we can provide the following example.

Example 25. *Let us consider*

$$F(x_1, x_2, x_3, x_4, t) = c(t) \cos(x_1) \cdot \sin(x_2) + \frac{1}{p} |x_3|^p + f(t) \cdot x_4,$$

where $c : [a, b] \rightarrow \mathbb{R}$, $f : [a, b] \rightarrow \mathbb{R}$ are of class $C^1([a, b]; \mathbb{R})$. In this case, one can prove that F satisfies all hypothesis of Theorem 21 and then, we can conclude with the existence of a minimizer of \mathcal{I} defined on $W_a^{1,p}(a, b; \mathbb{R})$ for any $1 < p < \infty$ and $1 < q < \infty$.

4.4 Conclusion

In [23] existence results for fractional variational problems containing Caputo derivatives were given. This chapter extends those results to any linear operator K_P bounded from the space $L^p(a, b; \mathbb{R})$ to $L^q(a, b; \mathbb{R})$, having in mind that $B_P := K_P \circ \frac{d}{dt}$.

4.5 State of the Art

The results presented in this chapter are submitted [24]. For an important particular case see [23].

Chapter 5

Application to the Sturm–Liouville Problem

In 1836–1837 the French mathematicians Sturm (1803-1853) and Liouville (1809-1855) published series of articles initiating a new subtopic of mathematical analysis – the Sturm–Liouville theory. It deals with the general linear, second order ordinary differential equation of the form

$$\frac{d}{dt} \left(p(t) \frac{dy}{dt} \right) + q(t)y = \lambda w(t)y, \quad (5.1)$$

where $t \in [a, b]$, and in any particular problem functions $p(t)$, $q(t)$ and $w(t)$ are known. In addition, certain boundary conditions are attached to equation (5.1). For specific choices of the boundary conditions, nontrivial solutions of (5.1) exist only for particular values of the parameter $\lambda = \lambda^{(m)}$, $m = 1, 2, \dots$. Constants $\lambda^{(m)}$ are called eigenvalues and corresponding solutions $y^{(m)}(t)$ are called eigenfunctions. For a deeper discussion of the classical Sturm–Liouville theory we refer the reader to [44, 104].

Recently, many researchers focused their attention on certain generalizations of Sturm–Liouville problem. Namely, they are interested in equations of the type (5.1), however with fractional differential operators (see e.g., [5, 6, 54–56, 64, 93]). In this chapter, we develop the Sturm–Liouville theory by studying the Sturm–Liouville eigenvalue problem with Caputo fractional derivatives. We show that fractional variational principles are useful for the approximation of eigenvalues and eigenfunctions. Traditional Sturm–Liouville theory does not depend upon the calculus of variations, but stems from the theory of ordinary linear differential equations. However, the Sturm–Liouville eigenvalue problem is readily formulated as a constrained variational principle, and this formulation can be used to approximate the solutions. We emphasize that it has a special importance for the fractional Sturm–Liouville equation since fractional operators are nonlocal and it can be extremely challenging to find analytical solutions. Besides allowing convenient approximations many general properties of the eigenvalues can be derived using the variational principle.

5.1 Useful Lemmas

In this section, we present three lemmas that are used to prove existence of solutions for the fractional Sturm–Liouville problem.

Lemma 4. Let $\alpha \in (0, 1)$ and function $\gamma \in C([a, b]; \mathbb{R})$. If

$$\int_a^b \gamma(t) \frac{d}{dt} ({}^C D_t^\alpha [h](t)) dt = 0$$

for each $h \in C^1([a, b]; \mathbb{R})$ such that $\frac{d}{dt} {}^C D_t^\alpha [h] \in C([a, b]; \mathbb{R})$ and boundary conditions

$$h(a) = {}_a I_t^{1-\alpha} h(b) = 0$$

and

$${}^C D_t^\alpha [h](t)|_{t=a} = {}^C D_b^\alpha [h](t)|_{t=b} = 0$$

are fulfilled, then $\gamma(t) = c_0 + c_1 t$, where c_0, c_1 are some real constants.

Proof. Let us define function h as follows

$$h(t) := {}_a I_t^{1+\alpha} [\gamma(\tau) - c_0 - c_1 \tau] (t) \quad (5.2)$$

with constants fixed by the conditions

$${}_a I_t^2 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=b} = 0 \quad (5.3)$$

$${}_a I_t^1 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=b} = 0. \quad (5.4)$$

Observe that function h is continuous and fulfills the boundary conditions

$$h(a) = 0 \quad {}_a I_t^{1-\alpha} [h](t)|_{t=b} = {}_a I_t^2 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=b} = 0$$

and

$$\begin{aligned} {}^C D_t^\alpha [h](t)|_{t=a} &= {}_a D_t^\alpha [h](t)|_{t=a} = \frac{d}{dt} {}_a I_t^2 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=a} \\ &= {}_a I_t^1 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=a} = 0, \end{aligned}$$

$$\begin{aligned} {}^C D_t^\alpha [h](t)|_{t=b} &= {}_a D_t^\alpha [h](t)|_{t=b} = \frac{d}{dt} {}_a I_t^2 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=b} \\ &= {}_a I_t^1 [\gamma(\tau) - c_0 - c_1 \tau] (t)|_{t=b} = 0. \end{aligned}$$

In addition,

$$t \mapsto h'(t) = {}_a I_t^\alpha [\gamma(\tau) - c_0 - c_1 \tau] (t) \in C([a, b]; \mathbb{R})$$

$$t \mapsto \frac{d}{dt} {}^C D_t^\alpha [h](t) = \gamma(t) - c_0 - c_1 t \in C([a, b]; \mathbb{R}).$$

We also have

$$\begin{aligned} &\int_a^b (\gamma(t) - c_0 - c_1 t) \frac{d}{dt} ({}^C D_t^\alpha [h](t)) dt \\ &= \int_a^b (-c_0 - c_1 t) \frac{d}{dt} ({}^C D_t^\alpha [h](t)) dt \\ &= -c_0 \cdot {}^C D_t^\alpha [h](t)|_{t=a}^{t=b} - c_1 t \cdot {}^C D_t^\alpha [h](t)|_{t=a}^{t=b} + c_1 \cdot {}_a I_t^{1-\alpha} [h](t)|_{t=a}^{t=b} = 0. \end{aligned}$$

On the other hand,

$$\frac{d}{dt} ({}^C D_t^\alpha [h](t)) = \frac{d}{dt} {}^C D_t^\alpha [{}_a I_t^{1+\alpha} [\gamma(\tau) - c_0 - c_1 \tau] (s)] (t) = \gamma(t) - c_0 - c_1 t$$

and

$$0 = \int_a^b (\gamma(t) - c_0 - c_1 t) \frac{d}{dt} ({}_a^C D_t^\alpha [h](t)) dt = \int_a^b (\gamma(t) - c_0 - c_1 t)^2 dt.$$

Thus function γ is

$$\gamma(t) = c_0 + c_1 t.$$

The proof is complete. \square

Lemma 5. *Let $\alpha \in (\frac{1}{2}, 1)$, $\gamma \in C([a, b]; \mathbb{R})$ and ${}_a D_t^{1-\alpha}[\gamma] \in L^2(a, b; \mathbb{R})$. If*

$$\int_a^b \gamma(t) \frac{d}{dt} ({}_a^C D_t^\alpha [h](t)) dt = 0$$

for each $h \in C^1([a, b]; \mathbb{R})$ such that $h'' \in L^2(a, b; \mathbb{R})$, $\frac{d}{dt} {}_a^C D_t^\alpha [h] \in C([a, b]; \mathbb{R})$ and boundary conditions

$$h(a) = {}_a I_t^{1-\alpha} [h](b) = 0, \quad (5.5)$$

$${}_a^C D_t^\alpha [h](t)|_{t=a} = {}_a^C D_t^\alpha [h](t)|_{t=b} = 0 \quad (5.6)$$

are fulfilled, then $\gamma(t) = c_0 + c_1 t$, where c_0, c_1 are some real constants.

Proof. We define function h as in the proof of Lemma 4:

$$h(t) := {}_a I_t^{1+\alpha} [\gamma(\tau) - c_0 - c_1 \tau](t) \quad (5.7)$$

with constants fixed by the conditions (5.3) and (5.4). The proof of the lemma is analogous to that of the Lemma 4. In addition, for the second order derivative we have

$$\begin{aligned} h''(t) &= \frac{d}{dt} {}_a I_t^\alpha [\gamma(\tau) - c_0 - c_1 \tau](t) \\ &= {}_a D_t^{1-\alpha} [\gamma(\tau) - c_0 - c_1 \tau](t) \\ &= {}_a D_t^{1-\alpha} [\gamma](t) - (c_0 + c_1 a) \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} - c_1 \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Let us observe that for $\alpha > 1/2$

$$t \mapsto \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \in L^2(a, b; \mathbb{R})$$

$$t \mapsto \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \in C([a, b]; \mathbb{R}) \subset L^2(a, b; \mathbb{R}).$$

Thus, we conclude that $h'' \in L^2(a, b; \mathbb{R})$ and function h constructed in this proof fulfills all the assumptions of Lemma 5. The remaining part of the proof is analogous to that for Lemma 4. \square

Lemma 6. (a) *Let $\alpha \in (\frac{1}{2}, 1)$, functions $\gamma_j \in C([a, b]; \mathbb{R})$, $j = 1, 2, 3$ and ${}_a D_t^{1-\alpha}[\gamma_3] \in L^2(a, b; \mathbb{R})$. If*

$$\int_a^b \left(\gamma_1(t) h(t) + \gamma_2(t) {}_a^C D_t^\alpha [h](t) + \gamma_3(t) \frac{d}{dt} ({}_a^C D_t^\alpha [h](t)) \right) dt = 0 \quad (5.8)$$

for each $h \in C^1([a, b]; \mathbb{R})$, such that $h'' \in L^2(a, b; \mathbb{R})$ and $\frac{d}{dt} {}^C D_t^\alpha [h] \in C([a, b]; \mathbb{R})$, fulfilling boundary conditions

$$h(a) = {}_a I_t^{1-\alpha} [h](b) = 0, \quad (5.9)$$

$${}_a^C D_t^\alpha [h](t)|_{t=a} = {}_a^C D_t^\alpha [h](t)|_{t=b} = 0 \quad (5.10)$$

then $\gamma_3 \in C^1([a, b]; \mathbb{R})$.

(b) Let $\alpha \in (\frac{1}{2}, 1)$ and functions $\gamma_1, \gamma_2 \in C([a, b]; \mathbb{R})$. If

$$\int_a^b (\gamma_1(t)h(t) + \gamma_2(t) {}_a^C D_t^\alpha [h](t)) dt = 0 \quad (5.11)$$

for each $h \in C^1([a, b]; \mathbb{R})$, such that $h'' \in L^2(a, b; \mathbb{R})$ and $\frac{d}{dt} {}^C D_t^\alpha [h] \in C([a, b]; \mathbb{R})$, fulfilling boundary conditions (5.9), (5.10), then

$$-\gamma_1(t) - {}_t^C D_b^\alpha [\gamma_2](t) = 0.$$

Proof. Observe that integral (5.8) can be rewritten as follows:

$$\begin{aligned} & \int_a^b \left(\gamma_1(t)h(t) + \gamma_2(t) {}_a^C D_t^\alpha [h](t) + \gamma_3(t) \frac{d}{dt} {}^C D_t^\alpha [h](t) \right) dt \\ &= \int_a^b \left(-({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) - {}_a I_t^1 [\gamma_2](t) + \gamma_3(t) \right) \frac{d}{dt} {}^C D_t^\alpha [h](t) dx = 0. \end{aligned}$$

Due to the fact that relations

$$\left({}_a I_t^\alpha \circ {}_t I_b^1 \circ \frac{d}{dt} {}^C D_t^\alpha \right) [h](t) = h(t)$$

and

$$\left({}_t I_b^1 \circ \frac{d}{d\tau} {}^C D_t^\alpha \right) [h](t) = -{}_a^C D_t^\alpha [h](t)$$

are valid because function h fulfills boundary conditions (5.9),(5.10). Denote

$$\gamma(t) := -({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) - {}_a I_t^1 [\gamma_2](t) + \gamma_3(t).$$

It is clear that $\gamma \in C([a, b]; \mathbb{R})$ and ${}_a D_t^{1-\alpha} [\gamma] \in L^2(a, b; \mathbb{R})$. Thus, according to Lemma 5, there exist constants c_0 and c_1 such that

$$-({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) - {}_a I_t^1 [\gamma_2](t) + \gamma_3(t) = c_0 + c_1 t.$$

Let us note that function γ_3 is

$$\gamma_3(t) = ({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) + {}_a I_t^1 [\gamma_2](t) + c_0 + c_1 t.$$

Hence its first order derivative is continuous in $[a, b]$ and $\gamma_3 \in C^1([a, b]; \mathbb{R})$.

The proof of part (b) is similar. We write integral (5.11) as follows:

$$\begin{aligned} \int_a^b (\gamma_1(t)h(t) + \gamma_2(t) {}_a^C D_t^\alpha [h](t)) dt \\ = \int_a^b (-({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) - {}_a I_t^1 [\gamma_2](t)) \frac{d}{dt} {}_a^C D_t^\alpha [h](t) dt = 0. \end{aligned}$$

The function in brackets is continuous in $[a, b]$,

$${}_a D_t^{1-\alpha} [-({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](\tau) - {}_a I_t^1 [\gamma_2](\tau)](t) = -({}_a I_t^\alpha \circ {}_t I_b^\alpha) [\gamma_1](t) - {}_a I_t^\alpha [\gamma_2](t),$$

and

$$-({}_a I_t^\alpha \circ {}_t I_b^\alpha) [\gamma_1] - {}_a I_t^\alpha [\gamma_2] \in C([a, b]; \mathbb{R}) \subset L^2(a, b; \mathbb{R}),$$

so we again can apply Lemma 5 and obtain that there exist constants c_0 and c_1 such that

$$({}_a I_t^1 \circ {}_t I_b^\alpha) [\gamma_1](t) + {}_a I_t^1 [\gamma_2](t) = c_0 + c_1 t.$$

Thus functions $\gamma_{1,2}$ fulfill equation:

$${}_t^C D_b^\alpha [\gamma_2](t) + \gamma_1(t) = 0.$$

□

5.2 The Fractional Sturm–Liouville Problem

The crucial idea in the proof of main result of this chapter (Theorem 23) is to apply direct variational methods to the fractional Sturm–Liouville equation. Starting from the fractional Sturm–Liouville equation the approach is to find an associated functional and to use this to find approximations to the minimizers, which are necessarily solutions to the original equation. In the case of the fractional Sturm–Liouville equation an associated variational problem is the fractional isoperimetric problem which is defined in the following way:

$$\min \mathcal{I}(y) = \int_a^b F(y(t), {}_a^C D_t^\alpha [y](t), t) dt, \quad (5.12)$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b \quad (5.13)$$

and the isoperimetric constraint

$$\mathcal{J}(y) = \int_a^b G(y(t), {}_a^C D_t^\alpha [y](t), t) dt = \xi, \quad (5.14)$$

where $\xi \in \mathbb{R}$ is given, and

$$\begin{aligned} F : [a, b] \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (y, u, t) &\longmapsto F(y, u, t), \end{aligned}$$

$$\begin{aligned} G : [a, b] \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (y, u, t) &\longmapsto G(y, u, t) \end{aligned}$$

are functions of class C^1 , such that $\frac{\partial F}{\partial u}$, $\frac{\partial G}{\partial u}$ have continuous ${}_t D_b^\alpha$ derivatives.

Theorem 22 (cf. Theorem 3.3 [15]). *If $y \in C[a, b]$ with ${}_a^C D_t^\alpha[y] \in C([a, b]; \mathbb{R})$ is a minimizer for problem (5.12)–(5.14), then there exists a real constant λ such that, for $H = F + \lambda G$, the equation*

$$\frac{\partial H}{\partial y}(y(t), {}_a^C D_t^\alpha[y](t), t) + {}_t D_b^\alpha \left[\frac{\partial H}{\partial u}(y(t), {}_a^C D_t^\alpha[y](t), t) \right] = 0 \quad (5.15)$$

holds, provided that

$$\frac{\partial G}{\partial y}(y(t), {}_a^C D_t^\alpha[y](t), t) + {}_t D_b^\alpha \left[\frac{\partial G}{\partial u}(y(t), {}_a^C D_t^\alpha[y](t), t) \right] \neq 0.$$

5.2.1 Existence of Discrete Spectrum

We show that, similarly to the classical case, for the fractional Sturm–Liouville problem there exist an infinite monotonic increasing sequence of eigenvalues. Moreover, apart from multiplicative factors, to each eigenvalue there corresponds precisely one eigenfunction and eigenfunctions form an orthogonal set of solutions.

We shall use the following assumptions.

(H1) Let $\frac{1}{2} < \alpha < 1$ and p, q, w_α be given functions such that: p is of C^1 class and $p(t) > 0$; q, w_α are continuous, $w_\alpha(t) > 0$ and $(\sqrt{w_\alpha})'$ is Hölderian of order $\beta \leq \alpha - \frac{1}{2}$. Consider the fractional differential equation

$${}_t^C D_b^\alpha [p(\tau) {}_a^C D_\tau^\alpha[y](\tau)](t) + q(t)y(t) = \lambda w_\alpha(t)y(t), \quad (5.16)$$

that will be called the fractional Sturm–Liouville equation, subject to the boundary conditions

$$y(a) = y(b) = 0. \quad (5.17)$$

Theorem 23. *Under assumptions (H1), the fractional Sturm–Liouville problem (FSLP) (5.16)–(5.17) has an infinite increasing sequence of eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \dots$, and to each eigenvalue $\lambda^{(n)}$ there corresponds an eigenfunction $y^{(n)}$ which is unique up to a constant factor. Furthermore, eigenfunctions $y^{(n)}$ form an orthogonal set of solutions.*

Proof. The proof is similar in spirit to [44] and will be divided into 6 steps. As in [44] at the same time we shall derive a method for approximating the eigenvalues and eigenfunctions.

Step 1. We shall consider problem of minimizing the functional

$$\mathcal{I}(y) = \int_a^b [p(t)({}_a^C D_t^\alpha[y](t))^2 + q(t)y^2(t)] dt \quad (5.18)$$

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subject to an isoperimetric constraint

$$\mathcal{J}(y) = \int_a^b w_\alpha(t) y^2(t) dt = 1 \quad (5.19)$$

and boundary conditions (5.17). First, let us point out that functional (5.18) is bounded from below. Indeed, as $p(t) > 0$ we have

$$\begin{aligned} \mathcal{I}(y) &= \int_a^b [p(t)({}^C D_t^\alpha[y](t))^2 + q(t)y^2(t)] dt \\ &\geq \min_{t \in [a,b]} \frac{q(t)}{w_\alpha(t)} \cdot \int_a^b w_\alpha(t) y^2(t) dt = \min_{t \in [a,b]} \frac{q(t)}{w_\alpha(t)} =: M_0 > -\infty. \end{aligned}$$

From now on, for simplicity, we assume that $a = 0$ and $b = \pi$. According to the Ritz method, we approximate solution of (5.17)–(5.19) using the following trigonometric function with coefficient depending on w_α :

$$y_m(t) = \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \beta_k \sin(kt). \quad (5.20)$$

Observe that $y_m(0) = y_m(\pi) = 0$. Substituting (5.20) into (5.18) and (5.19) we obtain the problem of minimizing the function

$$\begin{aligned} I(\beta_1, \dots, \beta_m) &= I([\beta]) \\ &= \sum_{k,j=1}^m \beta_k \beta_j \int_0^\pi \left[p(t) \left({}^C D_t^\alpha \left[\frac{\sin(k\tau)}{\sqrt{w_\alpha}} \right] (t) \cdot {}^C D_t^\alpha \left[\frac{\sin(j\tau)}{\sqrt{w_\alpha}} \right] (t) \right) + \frac{q(t)}{w_\alpha(t)} \sin(kt) \sin(jt) \right] dt \end{aligned} \quad (5.21)$$

subject to the condition

$$J(\beta_1, \dots, \beta_m) = J([\beta]) = \frac{\pi}{2} \sum_{k=1}^m (\beta_k)^2 = 1. \quad (5.22)$$

Since $I([\beta])$ is continuous and the set given by (5.22) is compact, function $I([\beta])$ attains minimum, denoted by $\lambda_m^{(1)}$, at some point $[\beta^{(1)}] = (\beta_1^{(1)}, \dots, \beta_m^{(1)})$. If this procedure is carried out for $m = 1, 2, \dots$, we obtain a sequence of numbers $\lambda_1^{(1)}, \lambda_2^{(1)}, \dots$. Because $\lambda_{m+1}^{(1)} \leq \lambda_m^{(1)}$ and $\mathcal{I}(y)$ is bounded from below, we can find the limit

$$\lim_{m \rightarrow \infty} \lambda_m^{(1)} = \lambda^{(1)}.$$

Step 2. Let

$$y_m^{(1)}(t) = \frac{1}{\sqrt{w_\alpha}} \sum_{k=1}^m \beta_k^{(1)} \sin(kt)$$

denote the linear combination (5.20) achieving the minimum $\lambda_m^{(1)}$. We shall prove that sequence $(y_m^{(1)})_{m \in \mathbb{N}}$ contains a uniformly convergent subsequence. From now on, for simplicity, we will write y_m instead of $y_m^{(1)}$. Recall that

$$\lambda_m^{(1)} = \int_0^\pi \left[p(t) ({}_0^C D_t^\alpha [y_m](t))^2 + q(t) y_m^2(t) \right] dt$$

is convergent, so it must be bounded, i.e., there exists constant $M > 0$ such that

$$\int_0^\pi \left[p(t) ({}_0^C D_t^\alpha [y_m](t))^2 + q(t) y_m^2(t) \right] dt \leq M, \quad m \in \mathbb{N}.$$

Therefore, for all $m \in \mathbb{N}$ it hold the following

$$\begin{aligned} \int_0^\pi p(t) ({}_0^C D_t^\alpha [y_m](t))^2 dt &\leq M + \left| \int_0^\pi q(t) y_m^2(t) dt \right| \\ &\leq M + \max_{t \in [0, \pi]} \left| \frac{q(t)}{w_\alpha(t)} \right| \int_0^\pi w_\alpha(t) y_m^2(t) dt = M + \max_{t \in [0, \pi]} \left| \frac{q(t)}{w_\alpha(t)} \right| =: M_1. \end{aligned}$$

Moreover, since $p(t) > 0$ one has

$$\min_{t \in [0, \pi]} p(t) \int_0^\pi ({}_0^C D_t^\alpha [y_m](t))^2 dt \leq \int_0^\pi p(t) ({}_0^C D_t^\alpha [y_m](t))^2 dt \leq M_1,$$

and hence

$$\int_0^\pi ({}_0^C D_t^\alpha [y_m](t))^2 dt \leq \frac{M_1}{\min_{t \in [0, \pi]} p(t)} =: M_2. \quad (5.23)$$

Using (3.68), (5.23), condition $y_m(0) = 0$ and Schwartz inequality, one has

$$\begin{aligned} |y_m(t)|^2 &= |({}_0 I_t^\alpha \circ {}_0^C D_t^\alpha) [y_m](t)|^2 = \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-\tau)^{\alpha-1} {}_0^C D_\tau^\alpha y_m(\tau) d\tau \right|^2 \\ &\leq \frac{1}{\Gamma(\alpha)^2} \left(\int_0^\pi |{}_0^C D_\tau^\alpha [y_m](\tau)|^2 d\tau \right) \left(\int_0^t (t-\tau)^{2(\alpha-1)} d\tau \right) \\ &\leq \frac{1}{\Gamma(\alpha)^2} M_2 \int_0^t (t-\tau)^{2(\alpha-1)} d\tau < \frac{1}{\Gamma(\alpha)} M_2 \frac{1}{2\alpha-1} \pi^{2\alpha-1}, \end{aligned}$$

so that $(y_m)_{m \in \mathbb{N}}$ is uniformly bounded. Now, using Schwartz inequality, equation (5.23) and the fact that the following inequality holds

$$\forall x_1 \geq x_2 \geq 0, \quad (x_1 - x_2)^2 \leq x_1^2 - x_2^2,$$

we have for any $0 < t_1 < t_2 \leq \pi$

$$\begin{aligned}
 |y_m(t_2) - y_m(t_1)| &= |({}_0I_t^\alpha \circ {}_0^C D_t^\alpha)[y_m](t_2) - ({}_0I_t^\alpha \circ {}_0^C D_t^\alpha)[y_m](t_1)| \\
 &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - \tau)^{\alpha-1} {}_0^C D_t^\alpha[y_m](\tau) d\tau - \int_0^{t_1} (t_1 - \tau)^{\alpha-1} {}_0^C D_t^\alpha[y_m](\tau) d\tau \right| \\
 &= \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} {}_0^C D_t^\alpha[y_m](\tau) d\tau - \int_0^{t_1} ((t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}) {}_0^C D_t^\alpha[y_m](\tau) d\tau \right| \\
 &\leq \frac{1}{\Gamma(\alpha)} \left[\left(\int_{t_1}^{t_2} (t_2 - \tau)^{2(\alpha-1)} d\tau \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} [({}_0^C D_t^\alpha[y_m](\tau))^2] d\tau \right)^{\frac{1}{2}} \right. \\
 &\quad \left. + \left(\int_0^{t_1} ((t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1})^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} [({}_0^C D_t^\alpha[y_m](\tau))^2] d\tau \right)^{\frac{1}{2}} \right] \\
 &\leq \frac{\sqrt{M_2}}{\Gamma(\alpha)} \left[\left(\int_{t_1}^{t_2} (t_2 - \tau)^{2(\alpha-1)} d\tau \right)^{\frac{1}{2}} + \left(\int_0^{t_1} ((t_1 - \tau)^{2(\alpha-1)} - (t_2 - \tau)^{2(\alpha-1)}) d\tau \right)^{\frac{1}{2}} \right] \\
 &= \frac{\sqrt{M_2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left[(t_2 - t_1)^{\alpha-\frac{1}{2}} + [(t_2 - t_1)^{2\alpha-1} - t_2^{2\alpha-1} + t_1^{2\alpha-1}]^{\frac{1}{2}} \right] \\
 &\leq \frac{2\sqrt{M_2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} (t_2 - t_1)^{\alpha-\frac{1}{2}}.
 \end{aligned}$$

Therefore, by Ascoli's theorem, there exists a uniformly convergent subsequence $(y_{m_n})_{n \in \mathbb{N}}$ of sequence $(y_m)_{m \in \mathbb{N}}$. It means that we can find $y^{(1)} \in C([a, b]; \mathbb{R})$ such that

$$y^{(1)} = \lim_{n \rightarrow \infty} y_{m_n}.$$

Step 3. Observe that by the Lagrange multiplier rule at $[\beta] = [\beta^{(1)}]$ we have

$$0 = \frac{\partial}{\partial \beta_j} [I([\beta]) - \lambda_m^{(1)} J([\beta])] |_{[\beta]=[\beta^{(1)}]}, \quad j = 1, \dots, m.$$

Multiplying equations by an arbitrary constant C^j and summing from 1 to m we obtain

$$0 = \sum_{j=1}^m C^j \frac{\partial}{\partial \beta_j} [I([\beta]) - \lambda_m^{(1)} J([\beta])] |_{[\beta]=[\beta^{(1)}]}. \quad (5.24)$$

Introducing

$$h_m(x) = \frac{1}{\sqrt{w_\alpha}} \sum_{j=1}^m C^j \sin(jt)$$

we can rewrite (5.24) in the form

$$0 = \int_0^\pi [p(t) {}_0^C D_t^\alpha[y_m](t) {}_0^C D_t^\alpha[h_m](t) + [q(t) - \lambda_m^{(1)} w_\alpha(t)] y_m(t) h_m(t)] dt, \quad (5.25)$$

Using the differentiation properties and formula ${}_0^C D_t^\alpha [y_m] = \frac{d}{dt} {}_0 I_t^{1-\alpha} [y_m]$ we write (5.25) as

$$0 = \int_0^\pi \left[-p'(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) - p(t) {}_0 I_t^{1-\alpha} [y_m](t) \frac{d}{dt} {}_0^C D_t^\alpha [h_m](t) \right] dt \\ + p(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) \Big|_{t=0}^{t=\pi} + \int_0^\pi [q(t) - \lambda_m^{(1)} w_\alpha(t)] y_m(t) h_m(t) dt := I_m. \quad (5.26)$$

By Lemma 7 (with $w = 1/\sqrt{w_\alpha}$) and Lemma 8 (Appendix), for function h fulfilling assumptions of Lemma 5, we shall obtain in the limit (at least for the convergent subsequence $(y_{m_n})_{n \in \mathbb{N}}$) the relation

$$0 = \int_0^\pi \left[-p'(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) \frac{d}{dt} {}_0^C D_t^\alpha [h](t) \right] dt \\ + p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) \Big|_{t=0}^{t=\pi} + \int_0^\pi [q(t) - \lambda^{(1)} w_\alpha(t)] y^{(1)}(t) h(t) dt := I. \quad (5.27)$$

Let us check the convergence of integrals (5.26) explicitly

$$|I_m - I| \leq \int_0^\pi \left| -p'(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) + p'(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) \right| dt \\ + \int_0^\pi \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) \frac{d}{dt} {}_0^C D_t^\alpha [h_m](t) - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) \frac{d}{dt} {}_0^C D_t^\alpha [h](t) \right| dt \\ + \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) \Big|_{t=0} - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) \Big|_{t=0} \right| \\ + \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) \Big|_{t=\pi} - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) \Big|_{t=\pi} \right| \\ + \int_0^\pi \left| [q(t) - \lambda_m^{(1)} w_\alpha(t)] y_m(t) h_m(t) - [q(t) - \lambda^{(1)} w_\alpha(t)] y^{(1)}(t) h(t) \right| dt. \quad (5.28)$$

For the first integral we get

$$\int_0^\pi \left| -p'(t) {}_0 I_t^{1-\alpha} [y_m](t) {}_0^C D_t^\alpha [h_m](t) + p'(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}_0^C D_t^\alpha [h](t) \right| dt \\ \leq \|p'\| \cdot \left[\|{}_0^C D_t^\alpha [h]\| \cdot \|{}_0 I_t^{1-\alpha} [y_m - y^{(1)}]\|_{L^1} + M_3 K_{1-\alpha} \sqrt{\pi} \|{}_0^C D_t^\alpha [h_m - h]\|_{L^2} \right],$$

where constant $M_3 = \sup_{m \in \mathbb{N}} \|y_m\|$ and $\|\cdot\|$ denotes the supremum norm in the $C([0, \pi]; \mathbb{R})$ space. Now, we estimate the second integral

$$\int_0^\pi \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) \frac{d}{dt} {}_0^C D_t^\alpha [h_m](t) - p(t) {}_0 I_{0+}^{1-\alpha} y^{(1)}(t) \frac{d}{dt} {}_0^C D_t^\alpha [h](t) \right| dt$$

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$$\leq \|p\| \cdot \left[\left\| \frac{d}{dt} {}^C D_t^\alpha [h] \right\|_{L^2} \cdot \| {}_0 I_t^{1-\alpha} [y_m - y^{(1)}] \|_{L^2} + M_3 K_{1-\alpha} \cdot \left\| \frac{d}{dt} {}^C D_t^\alpha [h_m - h] \right\|_{L^1} \right].$$

For the next two terms we have

$${}_0 I_t^{1-\alpha} [y_m](0) \longrightarrow {}_0 I_t^{1-\alpha} [y](0), \quad {}_0 I_t^{1-\alpha} [y_m](\pi) \longrightarrow {}_0 I_t^{1-\alpha} [y](\pi) \quad (5.29)$$

resulting from the convergence of sequence $y_m \xrightarrow{C} y$. For sequence $h_m = g_m / \sqrt{w_\alpha}$, we infer from Lemma 8 that

$$h'_m \xrightarrow{C} h'.$$

Hence, also

$${}^C D_t^\alpha [h_m] \xrightarrow{C} {}^C D_t^\alpha [h], \quad {}_0 I_t^{1-\alpha} [h'_m] \xrightarrow{C} {}_0 I_t^{1-\alpha} [h']$$

and at points $t = 0, \pi$ we obtain

$${}^C D_t^\alpha [h_m](0) \longrightarrow {}^C D_t^\alpha [h](0), \quad {}^C D_t^\alpha [h_m](\pi) \longrightarrow {}^C D_t^\alpha [h](\pi). \quad (5.30)$$

The above pointwise convergence (5.29) and (5.30) imply that

$$\begin{aligned} \lim_{m \rightarrow \infty} \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) {}^C D_t^\alpha [h_m](t)|_{t=0} - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}^C D_t^\alpha [h](t)|_{t=0} \right| &= 0 \\ \lim_{m \rightarrow \infty} \left| p(t) {}_0 I_t^{1-\alpha} [y_m](t) {}^C D_t^\alpha [h_m](t)|_{t=\pi} - p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t) {}^C D_t^\alpha [h](t)|_{t=\pi} \right| &= 0. \end{aligned}$$

Finally, for the last term in estimation (5.28) we get

$$\begin{aligned} & \int_0^\pi |[q(t) - \lambda_m^{(1)} w_\alpha(t)] y_m(t) h_m(t) - [q(t) - \lambda^{(1)} w_\alpha(t)] y^{(1)}(t) h(t)| dt \\ & \leq \int_0^\pi |q(t)(y_m(t) h_m(t) - y^{(1)}(t) h(t))| dt + \int_0^\pi |w_\alpha(t)(\lambda_m^{(1)} y_m(t) h_m(t) - \lambda^{(1)} y^{(1)}(t) h(t))| dt \\ & \leq \pi \cdot \|q\| \cdot \left[M_3 \cdot \|h_m - h\| + \|h\| \cdot \|y_m - y^{(1)}\| \right] \\ & \quad + \pi \cdot \|w_\alpha\| \cdot \left[\Lambda \left(M_3 \cdot \|h_m - h\| + \|h\| \cdot \|y_m - y^{(1)}\| \right) + \|y^{(1)} h\| \cdot |\lambda_m^{(1)} - \lambda^{(1)}| \right], \end{aligned}$$

where constants $M_3 = \sup_{m \in \mathbb{N}} \|y_m\|$ and $\Lambda = \sup_{m \in \mathbb{N}} |\lambda_m^{(1)}|$. We conclude that

$$0 = \lim_{m \rightarrow \infty} I_m = I$$

and (5.27) is fulfilled for function $y^{(1)}$ being the limit of subsequence (y_{m_n}) of the sequence $(y_m)_{m \in \mathbb{N}}$.

Step 4. Let us denote in relation (5.27):

$$\begin{aligned} \gamma_1(t) &:= (q(t) - \lambda^{(1)} w_\alpha(t)) y^{(1)}(t), \\ \gamma_2(t) &:= -p'(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t), \\ \gamma_3(t) &:= -p(t) {}_0 I_t^{1-\alpha} [y^{(1)}](t). \end{aligned}$$

We observe that $\gamma_j \in C([0, \pi]; \mathbb{R})$, $j = 1, 2, 3$ and ${}_0D_t^{1-\alpha}[\gamma_3] \in L^2(0, \pi; \mathbb{R})$ because

$$\begin{aligned} {}_0D_t^{1-\alpha}[\gamma_3] &= {}_0D_t^{1-\alpha} \left[p \cdot {}_0I_t^{1-\alpha}[y^{(1)}] \right] = {}_0I_t^\alpha \left[\frac{d}{dt} \left(p \cdot {}_0I_t^{1-\alpha}[y^{(1)}] \right) \right] \\ &= {}_0I_t^\alpha \left[p' \cdot {}_0I_t^{1-\alpha}[y^{(1)}] + p \cdot {}_0^C D_t^\alpha[y^{(1)}] \right]. \end{aligned}$$

Both parts of the above function belong to the $L^2(0, \pi; \mathbb{R})$ space.

Assuming that function h in (5.27) is an arbitrary function fulfilling assumptions of Lemma 6 and applying Lemma 6 part (a), we conclude that $\gamma_3 = -p \cdot {}_0I_t^{1-\alpha}[y^{(1)}] \in C^1([0, \pi]; \mathbb{R})$. From this fact it follows that $p \cdot \frac{d}{dt} {}_0I_t^\alpha[y^{(1)}] \in C([0, \pi]; \mathbb{R})$ and integral (5.27) can be rewritten as

$$0 = \int_0^\pi \left[p(t) {}_0^C D_t^\alpha[y^{(1)}](t) {}_0^C D_t^\alpha[h](t) + (q(t) - \lambda^{(1)} w_\alpha(t)) y^{(1)}(t) h(t) \right] dt.$$

Now, we apply Lemma 6 part (b) defining

$$\begin{aligned} \bar{\gamma}_1(t) &:= \gamma_1(t) = (q(t) - \lambda^{(1)} w_\alpha(t)) y^{(1)}(t), \\ \bar{\gamma}_2(t) &:= p(t) \frac{d}{dt} {}_0I_t^{1-\alpha}[y^{(1)}](t). \end{aligned}$$

This time $\bar{\gamma}_1, \bar{\gamma}_2 \in C([0, \pi]; \mathbb{R})$ and from Lemma 6 part (b) it follows that

$${}_t^C D_\pi^\alpha \left[p(\tau) {}_0^C D_\tau^\alpha[y^{(1)}](\tau) \right] (t) + q(t) y^{(1)}(t) = \lambda^{(1)} w_\alpha(t) y^{(1)}(t).$$

By construction this solution fulfills the Dirichlet boundary conditions

$$y^{(1)}(0) = y^{(1)}(\pi) = 0$$

and is nontrivial because

$$\mathcal{J}(y^{(1)}) = \int_0^\pi w_\alpha(t) \left(y^{(1)}(t) \right)^2 dt = 1.$$

In addition, we also have for the solution

$${}_0D_t^\alpha[y^{(1)}] = {}_0^C D_t^\alpha[y^{(1)}] \in C([0, \pi]; \mathbb{R}).$$

Let us observe that from the Dirichlet boundary conditions it follows that $y^{(1)}$ also solves the FSLP (5.16)-(5.17) in $[0, \pi]$.

Step 5. Now, let us restore the superscript on $y_m^{(1)}$ and show that $\left(y_m^{(1)} \right)_{m \in \mathbb{N}}$ itself converges to $y^{(1)}$. First, let us point out that for given λ the solution of

$${}_t^C D_\pi^\alpha [p(\tau) {}_0^C D_\tau^\alpha[y](\tau)] (t) + q(t) y(t) = \lambda^{(1)} w_\alpha(t) y(t), \quad (5.31)$$

subject to the boundary conditions

$$y(a) = y(b) = 0 \quad (5.32)$$

and the normalization condition

$$\int_0^{\pi} w_{\alpha}(t)y^2(t) dt = 1 \quad (5.33)$$

is unique except for a sign. Next, let us assume that $y^{(1)}$ solves Sturm–Liouville equation (5.31) and that corresponding eigenvalue is $\lambda = \lambda^{(1)}$. In addition, suppose that $y^{(1)}$ is nontrivial, i.e., we can find $t_0 \in [0, \pi]$ such that $y^{(1)}(t_0) \neq 0$ and choose the sign so that $y^{(1)}(t_0) > 0$. Similarly, for all $m \in \mathbb{N}$, let $y_m^{(1)}$ solve (5.31) with corresponding eigenvalue $\lambda = \lambda_m^{(1)}$ and let us choose the signs so that $y_m^{(1)}(t_0) \geq 0$. Now, suppose that $(y_m^{(1)})_{m \in \mathbb{N}}$ does not converge to $y^{(1)}$. It means that we can find another subsequence of $(y_m^{(1)})_{m \in \mathbb{N}}$ such that it converges to another solution $\bar{y}^{(1)}$ of (5.31) with $\lambda = \lambda^{(1)}$. We know that for $\lambda = \lambda^{(1)}$ solution of (5.31) subject to (5.32) and (5.33) must be unique except for a sign, thence

$$\bar{y}^{(1)} = -y^{(1)}$$

and we must have $\bar{y}^{(1)}(t_0) < 0$. However, it is impossible because for all $m \in \mathbb{N}$ value of $y_m^{(1)}$ in t_0 is greater or equal zero. It means that we have contradiction and hence, choosing each $y_m^{(1)}$ with adequate sign, we obtain $y_m^{(1)} \rightarrow y^{(1)}$.

Step 6. In order to find eigenfunction $y^{(2)}$ and the corresponding eigenvalue $\lambda^{(2)}$, we again minimize functional (5.18) subject to (5.19) and (5.17), but now with an extra orthogonality condition

$$\int_0^{\pi} w_{\alpha}(t)y(t)y^{(1)}(t) dt = 0. \quad (5.34)$$

If we approximate solution by

$$y_m(t) = \frac{1}{\sqrt{w_{\alpha}}} \sum_{k=1}^m \beta_k \sin(kt), \quad y_m(0) = y_m(\pi) = 0,$$

then we again receive quadratic form (5.21). However, in this case admissible solutions are points satisfying (5.22) together with

$$\frac{\pi}{2} \sum_{k=1}^m \beta_k \beta_k^{(1)} = 0, \quad (5.35)$$

i.e., they lay in $(m - 1)$ -dimensional sphere. As before, we find that function $I([\beta])$ has a minimum $\lambda_m^{(2)}$ and there exists $\lambda^{(2)}$ such that

$$\lambda^{(2)} = \lim_{m \rightarrow \infty} \lambda_m^{(2)},$$

because $J(y)$ is bounded from below. Moreover, it is clear that the following relation:

$$\lambda^{(1)} \leq \lambda^{(2)} \quad (5.36)$$

holds. Now, let us denote by

$$y_m^{(2)}(t) = \frac{1}{\sqrt{w_{\alpha}}} \sum_{k=1}^m \beta_k^{(2)} \sin(kt),$$

the linear combination achieving the minimum $\lambda_m^{(2)}$, where $\beta^{(2)} = (\beta_1^{(2)}, \dots, \beta_m^{(2)})$ is the point satisfying (5.22) and (5.35). By the same argument as before, we can prove that the sequence $(y_m^{(2)})_{m \in \mathbb{N}}$ converges uniformly to a limit function $y^{(2)}$, which satisfies the Sturm-Liouville equation (5.16) with $\lambda^{(2)}$, the boundary conditions (5.17), normalization condition (5.19) and the orthogonality condition (5.34). Therefore, solution $y^{(2)}$ of the FSLP corresponding to the eigenvalue $\lambda^{(2)}$ exists. Furthermore, because orthogonal functions cannot be linearly dependent, and since only one eigenfunction corresponds to each eigenvalue (except for a constant factor), we have the strict inequality

$$\lambda^{(1)} < \lambda^{(2)}$$

instead of (5.36). Finally, if we repeat the above procedure, with similar modifications, we can obtain eigenvalues $\lambda^{(3)}, \lambda^{(4)}, \dots$ and corresponding eigenfunctions $y^{(3)}, y^{(4)}, \dots$ \square

5.2.2 The First Eigenvalue

In this section we prove two theorems showing that the first eigenvalue of problem (5.16)–(5.17) is a minimum value of certain functionals. As in the proof of Theorem 23 in the sequel, for simplicity, we assume that $a = 0$ and $b = \pi$ in the problem (5.16)–(5.17).

Theorem 24. *Let $y^{(1)}$ denote the eigenfunction, normalized to satisfy the isoperimetric constraint*

$$\mathcal{J}(y) = \int_0^\pi w_\alpha(t) y^2(t) dt = 1, \quad (5.37)$$

associated to the first eigenvalue $\lambda^{(1)}$ of problem (5.16)–(5.17) and assume that function ${}_t D_\pi^\alpha [p \cdot {}_0^C D_t^\alpha [y]]$ is continuous. Then, $y^{(1)}$ is a minimizer of the following variational functional

$$\mathcal{I}(y) = \int_0^\pi [p(t)({}_0^C D_t^\alpha [y](t))^2 + q(t)y^2(t)] dt, \quad (5.38)$$

in the class $C([0, \pi]; \mathbb{R})$ with ${}_0^C D_t^\alpha [y] \in C([0, \pi]; \mathbb{R})$ subject to the boundary conditions

$$y(0) = y(\pi) = 0 \quad (5.39)$$

and an isoperimetric constraint (5.37). Moreover,

$$\mathcal{I}(y^{(1)}) = \lambda^{(1)}.$$

Proof. Suppose that $y \in C([0, \pi]; \mathbb{R})$ is a minimizer of \mathcal{I} and ${}_0^C D_t^\alpha [y] \in C([0, \pi]; \mathbb{R})$. Then, by Theorem 22, there is number λ such that y satisfies equation

$${}_t D_\pi^\alpha [p(\tau) {}_0^C D_\tau^\alpha [y](\tau)](t) + q(t)y(t) = \lambda w_\alpha(t)y(t), \quad (5.40)$$

and conditions (5.37), (5.39). Since ${}_t D_\pi^\alpha [p \cdot {}_0^C D_t^\alpha [y]]$ and ${}_t^C D_\pi^\alpha [p \cdot {}_0^C D_t^\alpha [y]]$ are continuous, it follows that $p(t) \cdot {}_0^C D_t^\alpha [y](t)|_{t=\pi} = 0$. Therefore, equation (5.40) is equivalent to

$${}_t^C D_\pi^\alpha [p(\tau) {}_0^C D_\tau^\alpha [y](\tau)](t) + q(t)y(t) = \lambda w_\alpha(t)y(t). \quad (5.41)$$

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Let us multiply (5.40) by y and integrate it on the interval $[0, \pi]$, then

$$\int_0^{\pi} (y(t) \cdot {}_t D_{\pi}^{\alpha} [p(\tau) {}_0^C D_{\tau}^{\alpha} [y](\tau)](t) + q(t)y^2(t)) dt = \lambda \int_0^{\pi} w_{\alpha}(t)y^2(t) dt.$$

Applying the integration by the parts formula for fractional derivatives (cf. (1.9)) and having in mind that conditions (5.39), (5.37) and $p(t) \cdot {}_0^C D_t^{\alpha} [y](t)|_{t=\pi} = 0$ hold, one has

$$\int_0^{\pi} \left(({}_0^C D_t^{\alpha} [y](t))^2 p(t) + q(t)y^2(t) \right) dt = \lambda.$$

Hence

$$\mathcal{I}(y) = \lambda.$$

Any solution to problem (5.37)–(5.39) satisfies equation (5.41) must be nontrivial since (5.37) holds, so λ must be an eigenvalue. Moreover, according to Theorem 23 there is the least element in the spectrum being eigenvalue $\lambda^{(1)}$ and the corresponding eigenfunction $y^{(1)}$ normalized to meet the isoperimetric condition. Therefore $J(y^{(1)}) = \lambda^{(1)}$. \square

Definition 30. We call to functional \mathcal{R} defined by

$$\mathcal{R}(y) = \frac{\mathcal{I}(y)}{\mathcal{J}(y)},$$

where $\mathcal{I}(y)$ is given by (5.38) and $\mathcal{J}(y)$ by (5.37), the Rayleigh quotient for the fractional Sturm–Liouville problem (5.16)–(5.17).

Theorem 25. Let us assume that function $y \in C([0, \pi]; \mathbb{R})$ with ${}_0^C D_t^{\alpha} [y] \in C([0, \pi]; \mathbb{R})$, satisfying boundary conditions $y(0) = y(\pi) = 0$ and being nontrivial, is a minimizer of Rayleigh quotient \mathcal{R} for the Sturm–Liouville problem (5.16)–(5.17). Moreover, assume that function ${}_t D_{\pi}^{\alpha} [p \cdot {}_0^C D_t^{\alpha} [y]]$ is continuous. Then, value of \mathcal{R} in y is equal to the first eigenvalue $\lambda^{(1)}$, i.e., $\mathcal{R}(y) = \lambda^{(1)}$.

Proof. Suppose that function $y \in C([0, \pi]; \mathbb{R})$ with ${}_0^C D_t^{\alpha} [y] \in C([0, \pi]; \mathbb{R})$, satisfying $y(0) = y(\pi) = 0$ and nontrivial, is a minimizer of Rayleigh quotient \mathcal{R} and that value of \mathcal{R} in y is equal to λ , i.e.,

$$\mathcal{R}(y) = \frac{\mathcal{I}(y)}{\mathcal{J}(y)} = \lambda.$$

Consider one-parameter family of curves

$$\hat{y} = y + h\eta, \quad |h| \leq \varepsilon,$$

where $\eta \in C([0, \pi]; \mathbb{R})$ with ${}_0^C D_t^{\alpha} [\eta] \in C([0, \pi]; \mathbb{R})$ is such that $\eta(0) = \eta(\pi) = 0$, $\eta \neq 0$ and define the following functions

$$\begin{aligned} \phi : [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{J}(y + h\eta) = \int_0^{\pi} w_{\alpha}(t)(y(t) + h\eta(t))^2 dt, \end{aligned}$$

$$\begin{aligned} \psi : [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{I}(y + h\eta) = \int_0^\pi [p(t)({}_0^C D_t^\alpha [y + h\eta](t))^2 + q(t)(y(t) + h\eta(t))^2] dt \end{aligned}$$

and

$$\begin{aligned} \zeta : [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{R}(y + h\eta) = \frac{\mathcal{I}(y+h\eta)}{\mathcal{J}(y+h\eta)}. \end{aligned}$$

Since ζ is of class C^1 on $[-\varepsilon, \varepsilon]$ and

$$\zeta(0) \leq \zeta(h), \quad |h| \leq \varepsilon,$$

we deduce that

$$\zeta'(0) = \left. \frac{d}{dh} \mathcal{R}(y + h\eta) \right|_{h=0} = 0.$$

Moreover, notice that

$$\zeta'(h) = \frac{1}{\phi(h)} \left(\psi'(h) - \frac{\psi(h)}{\phi(h)} \phi'(h) \right)$$

and that

$$\psi'(0) = \left. \frac{d}{dh} \mathcal{I}(y + h\eta) \right|_{h=0} = 2 \int_0^\pi [p(t) \cdot {}_0^C D_t^\alpha [y](t) \cdot {}_0^C D_t^\alpha [\eta](t) + q(t)y(t)\eta(t)] dt,$$

$$\phi'(0) = \left. \frac{d}{dh} \mathcal{J}(y + h\eta) \right|_{h=0} = 2 \int_0^\pi [w_\alpha(t)y(t)\eta(t)] dt.$$

Therefore

$$\begin{aligned} \zeta'(0) &= \left. \frac{d}{dh} \mathcal{R}(y + h\eta) \right|_{h=0} \\ &= \frac{2}{\mathcal{J}(y)} \left(\int_0^\pi p(t) \cdot {}_0^C D_t^\alpha [y](t) \cdot {}_0^C D_t^\alpha [\eta](t) + q(t)y(t)\eta(t) dt - \frac{\mathcal{I}(y)}{\mathcal{J}(y)} \int_0^\pi w_\alpha(t)y(t)\eta(t) dt \right) = 0. \end{aligned}$$

Having in mind that $\frac{\mathcal{I}(y)}{\mathcal{J}(y)} = \lambda$ and $\eta(0) = \eta(\pi) = 0$, using the integration by parts formula (1.9) we obtain

$$\int_0^\pi ({}_t D_\pi^\alpha [p({}_0^C D_\tau^\alpha [y])](t) + q(t)y(t) - \lambda w_\alpha(t)y(t)) \eta(t) dt = 0.$$

Now, applying the fundamental lemma of the calculus of variations, we arrive at

$${}_t D_\pi^\alpha [p(\tau) \cdot {}_0^C D_\tau^\alpha [y](\tau)](t) + q(t)y(t) = \lambda w_\alpha(t)y(t). \quad (5.42)$$

Under our assumptions, $p(t) \cdot {}_0^C D_t^\alpha [y](t)|_{t=\pi} = 0$ and therefore equation (5.42) is equivalent to

$${}_t^C D_\pi^\alpha [p(\tau) \cdot {}_0^C D_\tau^\alpha [y](\tau)](t) + q(t)y(t) = \lambda w_\alpha(t)y(t). \quad (5.43)$$

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Since $y \neq 0$ we deduce that number λ is an eigenvalue of (5.43). On the other hand, let $\lambda^{(m)}$ be an eigenvalue and $y^{(m)}$ the corresponding eigenfunction, then

$${}_t^C D_\pi^\alpha \left[p(\tau) {}_0^C D_\tau^\alpha [y^{(m)}](\tau) \right] (t) + q(t)y^{(m)}(t) = \lambda^{(m)}w_\alpha(t)y^{(m)}(t). \quad (5.44)$$

Similarly to the proof of Theorem 24, we can obtain

$$\frac{\int_0^\pi \left(({}_0^C D_t^\alpha [y^{(m)}](t))^2 p(t) + q(t)(y^{(m)}(t))^2 \right) dt}{\int_0^\pi \lambda^{(m)}w_\alpha(t)(y^{(m)}(t))^2 dt} = \lambda^{(m)},$$

for any $m \in \mathbb{N}$. That is $\mathcal{R}(y^{(m)}) = \frac{\mathcal{I}(y^{(m)})}{\mathcal{J}(y^{(m)})} = \lambda^{(m)}$. Finally, since minimum value of \mathcal{R} at y is equal to λ , i.e.,

$$\lambda \leq \mathcal{R}(y^{(m)}) = \lambda^{(m)} \quad \forall m \in \mathbb{N},$$

we have $\lambda = \lambda^{(1)}$. □

5.2.3 An Illustrative Example

Let us consider the following fractional oscillator equation:

$${}_t D_b^\alpha [{}_a^C D_\tau^\alpha [y](\tau)](t) - \lambda y(t) = 0, \quad (5.45)$$

where $y(a) = y(b) = 0$. One can easily check that problem of finding nontrivial solutions to equation (5.45) and corresponding values of parameter λ is a particular case of problem (5.16)–(5.17) with $p(t) \equiv 1$, $q(t) \equiv 0$ and $w_\alpha(t) \equiv 1$. The corresponding variational functional is

$$\mathcal{I}_\alpha(y) = \int_a^b p(t) \cdot ({}_a^C D_t^\alpha [y](t))^2 dt = \|\sqrt{p} {}_a^C D_t^\alpha [y]\|_{L^2}^2$$

with the isoperimetric condition

$$\int_a^b y^2(t) dt = 1.$$

Let us fix the value of parameter p and assume that orders α_1, α_2 fulfill the condition $\frac{1}{2} < \alpha_1 < \alpha_2 < 1$. Then, we obtain for functionals $\mathcal{I}_{\alpha_1}, \mathcal{I}_{\alpha_2}$ the following relation

$$\begin{aligned} \mathcal{I}_{\alpha_1}(y) &= \|\sqrt{p} {}_a^C D_t^{\alpha_1} [y]\|_{L^2}^2 = \|\sqrt{p} {}_a I_t^{1-\alpha_1} \left[\frac{d}{dt} y \right]\|_{L^2}^2 = \|\sqrt{p} {}_a I_t^{\alpha_2-\alpha_1} {}_a I_t^{1-\alpha_2} \left[\frac{d}{dt} y \right]\|_{L^2}^2 \\ &\leq K_{\alpha_2-\alpha_1}^2 \cdot \|\sqrt{p} {}_a^C D_t^{\alpha_2} [y]\|_{L^2}^2 = K_{\alpha_2-\alpha_1}^2 \mathcal{I}_{\alpha_2}(y), \end{aligned}$$

where we denoted

$$K_{\alpha_2-\alpha_1} := \frac{(b-a)^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2-\alpha_1+1)}.$$

We observe that in the above estimation two cases occur:

$$\begin{aligned} \text{if } K_{\alpha_2-\alpha_1} \leq 1, \text{ then } \mathcal{I}_{\alpha_1}(y) &\leq \mathcal{I}_{\alpha_2}(y); \\ \text{if } K_{\alpha_2-\alpha_1} > 1, \text{ then } \mathcal{I}_{\alpha_1}(y) &\leq K_{\alpha_2-\alpha_1}^2 \cdot \mathcal{I}_{\alpha_2}(y). \end{aligned}$$

The relations between functionals for different values of fractional order lead to the set of inequalities for eigenvalues λ_j valid for any $j \in \mathbb{N}$:

$$\begin{aligned} \text{if } K_{\alpha_2-\alpha_1} \leq 1, \text{ then } \lambda_j(\alpha_1) &\leq \lambda_j(\alpha_2); \\ \text{if } K_{\alpha_2-\alpha_1} > 1, \text{ then } \lambda_j(\alpha_1) &\leq K_{\alpha_2-\alpha_1}^2 \cdot \lambda_j(\alpha_2). \end{aligned}$$

In particular when order $\alpha_2 = 1$ we get

$$\begin{aligned} \mathcal{I}_{\alpha_1}(y) &= \|\sqrt{p_a} D_t^{\alpha_1} [y]\|_{L^2}^2 = \|\sqrt{p_a} I_t^{1-\alpha_1} [Dy]\|_{L^2}^2 \\ &\leq K_{1-\alpha_1}^2 \cdot \|\sqrt{p} Dy\|_{L^2}^2 = K_{1-\alpha_1}^2 \mathcal{I}_1(y) \end{aligned}$$

and the following relations dependent on the value of constant $K_{1-\alpha_1}$

$$\begin{aligned} \text{if } K_{1-\alpha_1} \leq 1, \text{ then } \mathcal{I}_{\alpha_1}(y) &\leq \mathcal{I}_1(y); \\ \text{if } K_{1-\alpha_1} > 1, \text{ then } \mathcal{I}_{\alpha_1}(y) &\leq K_{1-\alpha_1}^2 \cdot \mathcal{I}_1(y). \end{aligned}$$

Thus, comparing the eigenvalues for the fractional and the classical harmonic oscillator equation for boundary conditions $y(a) = y(b) = 0$, we conclude that the respective classical eigenvalues are higher than the ones resulting from the fractional problem for any $j \in \mathbb{N}$. Namely,

$$\begin{aligned} \text{if } K_{1-\alpha_1} \leq 1, \text{ then } \lambda_j(\alpha_1) &\leq \lambda_j(1) = p \left(\frac{j\pi}{b-a} \right)^2; \\ \text{if } K_{1-\alpha_1} > 1, \text{ then } \lambda_j(\alpha_1) &\leq K_{1-\alpha_1}^2 \cdot \lambda_j(1) = p \left(\frac{j\pi}{(b-a)^{\alpha_1} \Gamma(2-\alpha_1)} \right)^2. \end{aligned}$$

5.3 State of the Art

The results of this chapter can be found in the paper [58].

Appendix

In this appendix we prove two lemmas, concerning certain convergence properties of fractional and classical derivatives, that are important in the proof of Theorem 23. Let us begin with the following definition of Hölder continuous functions.

Definition 31. *Function g is Hölder continuous in the interval $[a, b]$ with coefficient $0 < \beta \leq 1$ if*

$$\sup_{x, y \in [a, b], x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\beta} < \infty. \quad (5.46)$$

We denote this class of Hölder continuous functions as $C_H^\beta([a, b]; \mathbb{R})$.

Lemma 7. *Let $\alpha \in (0, 1)$, functions $w, g \in C^1([0, \pi]; \mathbb{R}) \cap C_H^1([-\pi, \pi]; \mathbb{R})$ be odd functions in $[-\pi, \pi]$ such that $w'', g'' \in L^2(0, \pi; \mathbb{R})$. If we denote as g_m the m -th sum of the Fourier series of function g , then the following convergences are valid in $[0, \pi]$*

$${}_0^C D_t^\alpha [g_m] \xrightarrow{L^2} {}_0^C D_t^\alpha [g] \quad (5.47)$$

$$\frac{d}{dt} {}_0^C D_t^\alpha [g_m] \xrightarrow{L^1} \frac{d}{dt} {}_0^C D_t^\alpha [g] \quad (5.48)$$

$${}_0^C D_t^\alpha [wg_m] \xrightarrow{L^2} {}_0^C D_t^\alpha [wg] \quad (5.49)$$

$$\frac{d}{dt} {}_0^C D_t^\alpha [wg_m] \xrightarrow{L^1} \frac{d}{dt} {}_0^C D_t^\alpha [wg]. \quad (5.50)$$

Proof. We can apply Property 7 and estimate the $\|{}_0^C D_t^\alpha [g_m] - {}_0^C D_t^\alpha [g]\|_{L^2}$ norm in $[0, \pi]$ as follows:

$$\|{}_0^C D_t^\alpha [g_m] - {}_0^C D_t^\alpha [g]\|_{L^2} \leq \|{}_0 I_t^{1-\alpha} [g'_m - g']\|_{L^2} \leq K_{1-\alpha} \cdot \|g'_m - g'\|_{L^2}.$$

For even functions from the $C^1([0, \pi]; \mathbb{R}) \cap C_H^1([-\pi, \pi]; \mathbb{R})$ space, g'_m is the m th sum of the Fourier series of the derivative g' . Hence, in interval $[0, \pi]$,

$$g'_m \xrightarrow{L^2} g'$$

and from the above inequalities it follows that (5.47) is valid on $[0, \pi]$:

$${}_0^C D_t^\alpha [g_m] \xrightarrow{L^2} {}_0^C D_t^\alpha [g].$$

Let us observe that for $t > 0$

$$\frac{d}{dt} {}_0^C D_t^\alpha [g_m](t) = {}_0 D_t^\alpha [g'_m](t) = {}_0^C D_t^\alpha [g'_m](t) + \frac{g'_m(0) \cdot t^{-\alpha}}{\Gamma(1-\alpha)}$$

$$\frac{d}{dt} {}^C D_t^\alpha [g](t) = {}_0 D_t^\alpha [g'](t) = {}^C D_t^\alpha [g'](t) + \frac{g'(0) \cdot t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Therefore, we can estimate the distance between $\frac{d}{dt} {}^C D_t^\alpha [g_m]$ and $\frac{d}{dt} {}^C D_t^\alpha [g]$ in interval $[0, \pi]$ using (1.10) for $\beta = 1 - \alpha$ and $p = 1$

$$\begin{aligned} & \left\| \frac{d}{dt} {}^C D_t^\alpha [g_m] - \frac{d}{dt} {}^C D_t^\alpha [g] \right\|_{L^1} \\ & \leq \left\| {}^C D_t^\alpha [g'_m - g'] \right\|_{L^1} + \left\| (g'_m(0) - g'(0)) \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right\|_{L^1} \\ & = \left\| {}_0 I_t^{1-\alpha} [g''_m - g''] \right\|_{L^1} + |g'_m(0) - g'(0)| \cdot \left\| \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right\|_{L^1} \\ & \leq K_{1-\alpha} \cdot \|g''_m - g''\|_{L^1} + |g'_m(0) - g'(0)| \cdot \frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)} \\ & \leq K_{1-\alpha} \cdot \sqrt{\pi} \cdot \|g''_m - g''\|_{L^2} + |g'_m(0) - g'(0)| \cdot \frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)}. \end{aligned}$$

By assumptions we have in $[-\pi, \pi]$ (thence also in $[0, \pi]$)

$$g''_m \xrightarrow{L^2} g'', \quad g'_m(0) \longrightarrow g'(0).$$

Hence we conclude that (5.48) is valid.

The convergence given in (5.49) follows from (5.47), namely

$$\begin{aligned} & \left\| {}^C D_t^\alpha [wg_m] - {}^C D_t^\alpha [wg] \right\|_{L^2} \\ & = \left\| {}_0 I_t^{1-\alpha} [(wg_m)' - (wg)'] \right\|_{L^2} \\ & \leq \left\| {}_0 I_t^{1-\alpha} [w(g'_m - g')] \right\|_{L^2} + \left\| {}_0 I_t^{1-\alpha} [w'(g_m - g)] \right\|_{L^2} \\ & \leq K_{1-\alpha} \cdot \|w(g'_m - g')\|_{L^2} + K_{1-\alpha} \cdot \|w'(g_m - g)\|_{L^2} \\ & \leq K_{1-\alpha} (\|w\| \cdot \|g'_m - g'\|_{L^2} + \|w'\| \cdot \|g_m - g\|_{L^2}), \end{aligned}$$

where $\|\cdot\|$ denotes the supremum norm in the $C([0, \pi]; \mathbb{R})$ space. From assumptions of our lemma, it follows that in $[0, \pi]$

$$g'_m \xrightarrow{L^2} g', \quad g_m \xrightarrow{L^2} g.$$

Thus convergence (5.49) is valid.

To prove convergence (5.50) we start by observing that for $t > 0$

$$\begin{aligned} \frac{d}{dt} {}^C D_t^\alpha [wg_m](t) &= {}_0 D_t^\alpha [(wg_m)'](t) = {}^C D_t^\alpha [(wg_m)'](t) + \frac{(wg_m)'(0) \cdot t^{-\alpha}}{\Gamma(1-\alpha)} \\ \frac{d}{dt} {}^C D_t^\alpha [wg](t) &= {}_0 D_t^\alpha [(wg)'](t) = {}^C D_t^\alpha [(wg)'](t) + \frac{(wg)'(0) \cdot t^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

For the L^1 -distance between $\frac{d}{dt} {}^C D_t^\alpha [wg_m]$ and $\frac{d}{dt} {}^C D_t^\alpha [wg]$ in interval $[0, \pi]$ we have

$$\begin{aligned}
& \left\| \frac{d}{dt} {}^C D_t^\alpha [wg_m] - \frac{d}{dt} {}^C D_t^\alpha [wg] \right\|_{L^1} \tag{5.51} \\
& \leq \| {}^C D_t^\alpha [(wg_m)' - (wg)'] \|_{L^1} + \left\| [(wg_m)'(0) - (wg)'(0)] \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right\|_{L^1} \\
& \leq \| {}^I_t^{1-\alpha} [(wg_m)'' - (wg)'] \|_{L^1} + |(wg_m)'(0) - (wg)'(0)| \cdot \left\| \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right\|_{L^1} \\
& \leq K_{1-\alpha} \cdot \| (wg_m)'' - (wg)'' \|_{L^1} + \|(wg_m)'(0) - (wg)'(0)\| \cdot \frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)} \\
& \leq K_{1-\alpha} \cdot \sqrt{\pi} \cdot \| (wg_m)'' - (wg)'' \|_{L^2} + \|(wg_m)'(0) - (wg)'(0)\| \cdot \frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Because

$$\begin{aligned}
& (wg_m)'' - (wg)'' \\
& = w(g_m'' - g'') + 2(w)' \cdot (g_m' - g') + (w)'' \cdot (g_m - g)
\end{aligned}$$

we have

$$\begin{aligned}
& \| (wg_m)'' - (wg)'' \|_{L^2} \\
& \leq \|w\| \cdot \|g_m'' - g''\|_{L^2} + 2 \cdot \|(w)'\| \cdot \|g_m' - g'\|_{L^2} + \|(w)''\|_{L^2} \cdot \|g_m - g\|_{L^2}.
\end{aligned}$$

From the assumptions of the lemma it follows that for $j = 0, 1, 2$

$$\lim_{m \rightarrow \infty} \|g_m^{(j)} - g^{(j)}\|_{L^2} = 0.$$

Hence,

$$\lim_{m \rightarrow \infty} \| (wg_m)'' - (wg)'' \|_{L^2} = 0.$$

In addition,

$$\begin{aligned}
& \lim_{m \rightarrow \infty} |(wg_m)'(0) - (wg)'(0)| \\
& = \lim_{m \rightarrow \infty} |(w)'(0)(g_m(0) - g(0)) + w(0)(g_m'(0) - g'(0))| \\
& \leq \lim_{m \rightarrow \infty} |(w)'(0)(g_m(0) - g(0))| + \lim_{m \rightarrow \infty} |w(0)(g_m'(0) - g'(0))| = 0.
\end{aligned}$$

Taking into account estimation (5.51) and the above inequalities, we conclude that (5.50) is valid. \square

Lemma 8. Let $\alpha \in (\frac{1}{2}, 1)$, $\beta \leq \alpha - \frac{1}{2}$, function w be positive, even function in $[-\pi, \pi]$ and $w' \in C_H^\beta([-\pi, \pi]; \mathbb{R})$. Function h' is the derivative of h defined by assumptions of Lemma 5 and formula (5.7), function g is defined as

$$g(t) := h(t)w(t).$$

If we denote as g_m the m th sum of the Fourier series of function g , then the following convergences are valid in interval $[0, \pi]$:

$$g_m' \xrightarrow{C} g', \tag{5.52}$$

$$g_m'(0) \longrightarrow g'(0), \tag{5.53}$$

$$g_m'(\pi) \longrightarrow g'(\pi). \tag{5.54}$$

Proof. Definition (5.7) in interval $[0, \pi]$ implies for derivative h'

$$h'(t) = {}_0I_t^\alpha[\gamma](t) + At^\alpha + Bt^{1+\alpha}, \quad (5.55)$$

where $\gamma \in C[0, \pi]$ and constants $A, B \in \mathbb{R}$ are specified by conditions (5.3), (5.4) in the proof of Lemma 4. Let us observe that $t^{1+\alpha} \in C^1([0, \pi]; \mathbb{R})$, function t^α is Hölder continuous in $[0, \pi]$ with coefficient $\beta \leq \alpha$, thus it can be extended to an odd/even, Hölder continuous function in interval $[-\pi, \pi]$. In addition ${}_0I_t^\alpha[\gamma](t)$ is Hölder continuous function in $[0, \pi]$ with coefficient $\beta \leq \alpha - \frac{1}{2}$ because:

$$\frac{|{}_0I_t^\alpha[\gamma](t) - {}_0I_t^\alpha[\gamma](s)|}{|t - s|^\beta} \leq \frac{2 \cdot \|\gamma\|_{L^2}}{\Gamma(\alpha)\sqrt{2\alpha - 1}} \cdot |t - s|^{\alpha - \frac{1}{2} - \beta} \leq \frac{2 \cdot \|\gamma\|_{L^2}}{\Gamma(\alpha)\sqrt{2\alpha - 1}} \cdot \pi^{\alpha - \frac{1}{2} - \beta} < \infty$$

and can be extended to an odd/even, Hölder continuous function in interval $[-\pi, \pi]$. Observe that for Hölder continuous functions in $[-\pi, \pi]$ we have the absolute convergence of their Fourier series. For function g' we obtain in $[0, \pi]$

$$g'(t) = h'(t)w(t) + h(t)w'(t).$$

Both terms on the right-hand side are, by assumption, functions from the $C_H^\beta([0, \pi]; \mathbb{R})$ -space and can be extended to odd/even functions in the $C_H^\beta([-\pi, \pi]; \mathbb{R})$ space. Hence, their Fourier series are absolutely convergent in $[-\pi, \pi]$. Concluding, we have for function g' the convergence in interval $[-\pi, \pi]$

$$g'_m \xrightarrow{C} g'.$$

Thus the sequence g'_m of partial sums is also absolutely convergent in interval $[0, \pi]$. Formulas (5.53), (5.54) are a straightforward consequence of this fact. \square

Conclusions and Future Work

This thesis was dedicated to generalized fractional calculus of variations. We extended standard fractional variational calculus, by considering problems with generalized fractional operators, that by choosing special kernels reduce, e.g., to fractional operators of Riemann–Liouville, Caputo, Hadamard, Riesz or Katugampola types. First, we proved several properties of generalized fractional operators, including boundedness in the space of p -Lebesgue integrable functions, and generalized fractional integration by parts formula. Next, we applied standard methods of fractional variational calculus to find admissible functions giving minima to certain functionals. We considered cases of one and several variables. However, because in standard methods it is assumed that Euler–Lagrange equations are solvable, we presented certain results according to direct methods, where it is not the case. We proved a Tonelli type theorem ensuring existence of minimizers and then obtained necessary optimality condition giving candidates for solutions. The last chapter was devoted to the fractional Sturm–Liouville problem. Applying methods of fractional variational calculus we proved that there exists an infinite increasing sequence of eigenvalues, to each eigenvalue corresponds an eigenfunction and all of them are orthogonal. Moreover, we presented two theorems concerning the first eigenvalue.

Concluding, our results cover several variational problems with particular fractional operators and give a compact and transparent view for the fractional calculus of variations. We trust that our work will provide new insights to further research on the subject, where still much remains to be done.

This research can choose several directions. Here, we find important to mention the following ones. We can

- consider variational problems with higher order derivatives;
- consider Lagrangians with different operators K_{p_1}, K_{p_2}, \dots and B_{p_1}, B_{p_2}, \dots ;
- apply direct methods to multidimensional variational problems;
- explore problems of generalized fractional calculus of variations with holonomic or non holonomic constraints;
- with the help of fractional variational calculus show fractional counterpart of isoperimetric inequality;
- continue with variational methods in problems of generalized fractional optimal control including necessary conditions of optimality or maximum principle;

The results mentioned in this thesis were published to peer reviewed international journals (see [23, 24, 58, 75, 77–80, 82, 83, 87]).

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