DECOMPOSING MODULAR TENSOR PRODUCTS: 'JORDAN PARTITIONS', THEIR PARTS AND P-PARTS

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Abstract. Determining the Jordan canonical form of the tensor product of Jordan blocks has many applications including to the representation theory of algebraic groups, and to tilting modules. Although there are several algorithms for computing this decomposition in literature, it is difficult to predict the output of these algorithms. We call a decomposition of the form $J_r \otimes J_s = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_b}$ a 'Jordan partition'. We prove several deep results concerning the p-parts of the λ_i where p is the characteristic of the underlying field. Our main results include the proof of two conjectures made by McFall in 1980, and the proof that $\mathrm{lcm}(r, s)$ and $\gcd(\lambda_1, \dots, \lambda_b)$ have equal p-parts. Finally, we establish some explicit formulas for Jordan partitions when $p = 2$.

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1. Introduction

Throughout this paper F denotes a field with characteristic $p \geqslant 0$. Given $\alpha \in F$ denote by $J_r(\alpha)$ the $r \times r$ Jordan block with eigenvalue α. Hence $(\alpha I - J_r(\alpha))^k = 0$ holds if and only if $k \geqslant r$. Given $\alpha, \beta \in F$ and $r, s \geqslant 1$ the Jordan canonical form of the tensor product $J_r(\alpha) \otimes J_s(\beta)$ equals $J_{\lambda_1}(\alpha\beta) \oplus \cdots \oplus J_{\lambda_b}(\alpha\beta)$ where $rs = \lambda_1 + \cdots + \lambda_b$. The partition $(\lambda_1, \dots, \lambda_b)$ of rs is easily described when $\alpha\beta = 0$, see for example [9, Prop. 2.1.2]. When $\alpha\beta \neq 0$, a simple change of basis shows that the corresponding partition is the same as that for $J_r(1) \otimes J_s(1)$. We denote it by $\lambda(r, s, p)$ as the Jordan canonical form of $J_r(1) \otimes J_s(1)$ is invariant under field extensions. We call $\lambda(r, s, p) = (\lambda_1, \dots, \lambda_b)$ a 'Jordan partition' and always write its parts in non-increasing order $\lambda_1 \geqslant \ldots \geqslant \lambda_b > 0$. It has been long known that b equals $\min(r, s)$, see [13, Lemma 2.1]. Note that $\lambda(r, s, p) = \lambda(s, r, p)$ since $J_r(1) \otimes J_s(1)$ is similar to $J_s(1) \otimes J_r(1)$.

The partition $\lambda(r, s, p)$ is well known if $\mathrm{char}(F) = 0$, or $\mathrm{char}(F) = p \geqslant r + s - 1$. In these cases, the ith part of $\lambda(r, s, p)$ is $\lambda_i = r + s + 1 - 2i$, see [16, Corollary 1]. Henceforth, we will assume that $\mathrm{char}(F) = p$ is an arbitrary prime, possibly satisfying $p \geqslant r + s - 1$. The *p-part* of a nonzero integer n, denoted by n_p, is the largest p-power dividing n.

There is a well-known link between the partition $\lambda(r, s, p)$ and the modular representation theory of a cyclic group C_{p^n} of order p^n where $\max(r, s) \leqslant p^n$. There are precisely p^n pairwise nonisomorphic indecomposable FC_{p^n}-modules, say $V_1, \dots, V_{p^n}$ where

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$\dim(V_i) = i$. In his pioneering work [7], Green studied a ring, now called the *modular representation ring* or *Green ring*, whose elements are F-linear combinations $\sum_{i=1}^{p^n} \alpha_i[V_i]$ of the isomorphism classes $[V_i]$. Addition and multiplication are given by the direct sum and by tensor product, and denoted $\oplus$ and $\otimes$. It is conventional to write the module V_i instead of the isomorphism class $[V_i]$, and to let V_0 be a 0-dimensional module. As usual mV denotes the direct sum of m copies of V where $m \geqslant 0$ is an integer. Thus $0V$ is just the zero module, and $[V_0] = [0V]$. Given positive integers r, s satisfying $r, s \leqslant p^n$, the module $V_r \otimes V_s$ is a sum of indecomposable modules by the Krull-Schmidt theorem. This gives a Green ring equation

$$V_r \otimes V_s = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_b} \quad \text{where} \quad b := \min(r, s), \tag{1}$$

where the parts λ_i of the partition $\lambda(r, s, p) = (\lambda_1, \ldots, \lambda_b)$ are at most p^n. It is easy to convert between the Green ring decomposition (1) and the partition $\lambda(r, s, p)$, and we shall do so frequently in this paper.

Given positive integers r and s, let p^n be the smallest p-power exceeding $\max(r, s)$. A fundamental question is how to decompose $V_r \otimes V_s$ as $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_b}$. In fact, a majority of papers addressing Jordan partitions in the literature were concerned with this decomposition problem, and there are basically two classes of algorithms. One class of algorithms [10, 12, 14] involves recursive computations to reduce n. Although these algorithms are similar in spirit, the one proposed by Renaud [14] in 1979 is more convenient to apply, and we use it repeatedly in Section 2. The other class of algorithms [9, 11, 13, 16] is related to binomial matrices (matrices of binomial coefficients). Iima and Iwamatsu [9] presented a novel algorithm which, unlike it predecessors, avoided the computation of ranks of binomial matrices over $\mathbb{F}_p$, called p-ranks. In 2009, Iima and Iwamatsu [9] showed that, to compute the parts of $\lambda(r, s, p)$, it suffices to know whether or not the p-ranks of certain binomial matrices are full. This reduces the computation dramatically since the determinants of those binomial matrices can be computed via an explicit formula, and we can study their p-divisibility using number theory. For complementary introductory remarks, see [3, §1].

There are, however, relatively few results on the properties of the decomposition, or the partition in the literature. The following one is due to Green [7], who assumed the λ_i to be positive. It is convenient for us to assume that each part is nonnegative.

Proposition 1. [7, (2.5a)] *Suppose* $1 \leqslant r, s \leqslant p^n$. *If* $V_r \otimes V_s = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_b}$, *then*

$$V_{p^n - r} \otimes V_s = (s - b)V_{p^n} \oplus V_{p^n - \lambda_b} \oplus \cdots \oplus V_{p^n - \lambda_1} \quad \text{where}^{1} \quad b = \min(r, s).$$

Proposition 1 can be a viewed as a 'duality' result on $\lambda(r, s, p)$. For more on this duality and some 'periodicity' results as well as other properties, the reader is referred to [3]. In this paper, the main results are Theorem 2, which was described in the abstract, and Theorems 4 and 5, which were conjectured by McFall [11, p. 87] using different notation.

[1]Surprisingly, the fact that $b = \min(r, s)$ does not appear in [7]. Its proof is easy, see [3, Lemma 9(a)].

We also prove in Section 5 some results about the p-parts of $\lambda_1, \dots, \lambda_b$ when $|r-s| \leqslant 1$, and prove explicit decomposition formulas when $p=2$. Some of these later results were foreshadowed by McFall [10, Theorem 2] who gave an algorithm for computing the Jordan decomposition when $p=2$.

Theorem 2. *Suppose $V_r \otimes V_s = \bigoplus_{i=1}^b V_{\lambda_i}$ where $\mathrm{char}(F)=p$ is prime. Then the p-parts of $\mathrm{lcm}(r,s)$ and $\gcd(\lambda_1,\dots,\lambda_b)$ are equal. That is, $\mathrm{lcm}(r,s)_p = \gcd(\lambda_1,\dots,\lambda_b)_p$.*

Notation 3 (Multiplicity)**.** *Write $V_r \otimes V_s = \bigoplus_{i=1}^b V_{\lambda_i}$ as $V_r \otimes V_s = \bigoplus_{i=1}^t m_i V_{\mu_i}$ where the multiset $\{\lambda_1,\dots,\lambda_b\}$ has distinct parts $\mu_1 > \cdots > \mu_t > 0$, which occur with positive multiplicities $m_1,\dots,m_t$, respectively.*

If t is much smaller than b, it can be helpful to write $\bigoplus_{i=1}^t m_i V_{\mu_i}$ instead of $\bigoplus_{i=1}^b V_{\lambda_i}$. Observe that $\sum_{i=1}^t m_i = b$ and $\sum_{i=1}^t m_i\mu_i = \sum_{i=1}^b \lambda_i = rs$. We will commonly switch between the parts μ_i and the corresponding summand V_{μ_i} with $\dim(V_{\mu_i}) = \mu_i$. Since $\gcd(\lambda_1,\dots,\lambda_b)$ equals $\gcd(\mu_1,\dots,\mu_t)$ and $\mathrm{lcm}(r,s)_p$ equals $\max(r_p,s_p)$, we see that

$$\gcd(\lambda_1,\dots,\lambda_b)_p = \gcd(\mu_1,\dots,\mu_t)_p = \gcd((\mu_1)_p,\dots,(\mu_t)_p) = \min((\mu_1)_p,\dots,(\mu_t)_p). \tag{2}$$

Using multiplicities as described in Notation 3, we paraphrase Theorem 2 as follows:

$$V_r \otimes V_s = \bigoplus_{i=1}^t m_i V_{\mu_i} \quad \text{implies} \quad \max(r_p,s_p) = \min((\mu_1)_p,\dots,(\mu_t)_p). \tag{3}$$

In 1980, McFall made two conjectures, see p. 87 of [11]. His first conjecture is proved by Theorem 4 below. His second conjecture is implied by the formula (5) in Theorem 5.

Theorem 4. *Suppose that $r,s \geqslant 1$ and $V_r \otimes V_s = \bigoplus_{i=1}^t m_i V_{\mu_i}$ where the summands are nonzero and the μ_i are distinct. If a multiplicity satisfies $m_i > 1$, then μ_i is divisible by p.*

Theorem 5. *Suppose that $r,s \geqslant 1$ and $V_r \otimes V_s = \bigoplus_{i=1}^t m_i V_{\mu_i}$ as in Notation 3. Then the multiplicities $m_1,\dots,m_t$ determine the part sizes $\mu_1 > \cdots > \mu_t > 0$, and conversely, via*

$$\mu_i = r+s-m_i-2\sum_{j=1}^{i-1} m_j \qquad \text{for } 1 \leqslant i \leqslant t, \tag{4}$$

$$m_i = (-1)^{i-1}\left[r+s+2\sum_{j=1}^{i-1}\mu_j\right] - \mu_i \qquad \text{for } 1 \leqslant i \leqslant t. \tag{5}$$

These results have several simple consequences. We mention just one. Theorem 4 says $p \nmid \mu_1$ implies $m_1 = 1$, and Theorem 5 says $m_1 = r+s-\mu_1$. Hence $p \nmid \mu_1$ implies $r+s \not\equiv 1 \pmod p$. In many fields, theoretical development precedes and informs algorithmic development. In this field the reverse seems to hold. While algorithms such as those in [9, 10, 12, 14] are helpful for computing Jordan partitions, predicting the output for given input of r,s,p is not at all obvious. Our hope is that the patterns in

Theorems 2, 4, 5 that we prove by appealing to various algorithms may lead, in turn, to simpler, or more efficient, algorithms for computing Jordan partitions.

The layout of this paper is as follows. Renaud's decomposition algorithm is reviewed in Section 2, and it is used to prove Theorems 2 and 4 in Section 3. Section 4 introduces a different decomposition algorithm by Iima and Iwamatsu, and it is used to prove Theorem 5. In the final section 5, we establish some new results when $|r-s|\leqslant 1$.

2. Renaud's Algorithm

It is convenient to view V_r as a module for all cyclic groups C_{p^n} with $p^n\geqslant r$. Renaud's algorithm [14] uses induction on n to decompose $V_r\otimes V_s$ where n is the smallest integer satisfying $\max(r,s)<p^n$ and $\mathrm{char}(F)=p$. The inductive step is achieved by the somewhat complicated reduction formula in Proposition 6. (The base case when $n=1$ is described in Proposition 8.) Note that the summand $V_{(s_0-r_0)p^n+\nu_j}$ in [14, Theorem 2] is incorporated as the $i=0$ summand on the third line of equation (6).

Proposition 6. [14, Theorem 2] *Suppose $1\leqslant r\leqslant s<p^{n+1}$ where $n\geqslant 1$. Write $r=r_0p^n+r_1$ and $s=s_0p^n+s_1$, where $r_0,s_0,r_1,s_1\geqslant 0$ and $r_1,s_1<p^n$. Suppose the decomposition $V_{r_1}\otimes V_{s_1}=\bigoplus_{j=1}^{\ell} n_jV_{\nu_j}$ has $p^n\geqslant\nu_1>\cdots>\nu_\ell>0$ and each $n_j>0$. Then*

$$\begin{aligned} V_r\otimes V_s \;&=\; cV_{p^{n+1}}\oplus|r_1-s_1|\bigoplus_{i=1}^{d_1}V_{(s_0-r_0+2i)p^n}\oplus\max(0,r_1-s_1)V_{(s_0-r_0)p^n}\\ &\quad\oplus(p^n-r_1-s_1)\bigoplus_{i=1}^{d_2}V_{(s_0-r_0+2i-1)p^n}\\ &\quad\oplus\bigoplus_{j=1}^{\ell}n_j\left(\bigoplus_{i=0}^{d_1}V_{(s_0-r_0+2i)p^n+\nu_j}\oplus\bigoplus_{i=1}^{d_1}V_{(s_0-r_0+2i)p^n-\nu_j}\right), \end{aligned} \tag{6}$$

where

$$(c,d_1,d_2)=\begin{cases}(0,r_0,r_0) & \text{if } r_0+s_0<p,\\ (r+s-p^{n+1},p-s_0-1,p-s_0) & \text{if } r_0+s_0\geqslant p.\end{cases}$$

Observe that (6) fails to be a decomposition only when the multiplicity $p^n-r_1-s_1$ on the second line of (6) is negative. However, in this case the whole second line cancels with some terms on the third line; see the remarks following Lemma 7. To see how cancellation occurs in the Green ring to obtain a decomposition, we need a lemma.

Lemma 7. *Suppose r_1,s_1 are positive integers satisfying $r_1,s_1\leqslant p^n$ and $r_1+s_1>p^n$. Then the largest part of $\lambda(r_1,s_1,p)$ is p^n, and it occurs with multiplicity $r_1+s_1-p^n$. That is, if $V_{r_1}\otimes V_{s_1}=\bigoplus_{j=1}^{l}n_jV_{\nu_j}$ using* Notation 3, *then $\nu_1=p^n$ and $n_1=r_1+s_1-p^n$.*

Proof. By our assumption, $p^n - r_1 < s_1$ and $\min(p^n - r_1, s_1) = p^n - r_1$. Suppose that $\lambda(p^n - r_1, s_1, p) = (\lambda_1, \ldots, \lambda_{p^n-r_1})$; equivalently $V_{p^n-r_1} \otimes V_{s_1} = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_{p^n-r_1}}$, where $\lambda_1 \geqslant \ldots \geqslant \lambda_{p^n-r_1} > 0$. Then by Proposition 1,

$$V_{r_1} \otimes V_{s_1} = (s_1 - p^n + r_1)V_{p^n} \oplus V_{p^n-\lambda_{p^n-r_1}} \oplus \cdots \oplus V_{p^n-\lambda_1}.$$

The largest part, and its multiplicity, can now be determined as $p^n > p^n - \lambda_{p^n-r_1}$. ☐

We now establish the way that canceling occurs in (6) when $p^n - r_1 - s_1 < 0$ in order to obtain a decomposition (whose multiplicities are, by definition, always nonnegative). Suppose that $p^n - r_1 - s_1 < 0$. Then Lemma 7 gives

$$V_{r_1} \otimes V_{s_1} = (r_1 + s_1 - p^n)V_{p^n} \oplus \bigoplus_{j=2}^{l} n_j V_{\nu_j}.$$

Thus the summand corresponding to $j = 1$ in the third line of (6) is

$$(r_1 + s_1 - p^n)\left(\bigoplus_{i=0}^{d_1} V_{(s_0-r_0+2i)p^n+p^n} \oplus \bigoplus_{i=1}^{d_1} V_{(s_0-r_0+2i)p^n-p^n}\right). \tag{7}$$

When $r_0 + s_0 < p$, we have from Proposition 6 that $d_2 = d_1$ and the second line of (6) may be written as

$$(p^n - r_1 - s_1)\bigoplus_{i=1}^{d_2} V_{(s_0-r_0+2i-1)p^n} = -(r_1 + s_1 - p^n)\bigoplus_{i=1}^{d_1} V_{(s_0-r_0+2i)p^n-p^n}.$$

This cancels with the second sum in (7). On the other hand, when $r_0 + s_0 \geqslant p$, we have $d_2 = d_1 + 1$ and the second line of (6) may be written as

$$\begin{aligned}(p^n - r_1 - s_1)\bigoplus_{i=1}^{d_2} V_{(s_0-r_0+2i-1)p^n} &= -(r_1 + s_1 - p^n)\bigoplus_{i=1}^{d_1+1} V_{(s_0-r_0+2i-1)p^n} \\ &= -(r_1 + s_1 - p^n)\bigoplus_{j=0}^{d_1} V_{(s_0-r_0+2j)p^n+p^n}.\end{aligned}$$

This cancels with the first sum in (7). Therefore, after canceling in this way, (6) becomes a decomposition for $V_r \otimes V_s$.

In order to complete Renaud's inductive reduction in Proposition 6, we must specify what happens when $n = 1$. This amounts to knowing how $V_r \otimes V_s$ decomposes when $1 \leqslant r \leqslant s < p$. Such a decomposition is given in Proposition 8. It can be deduced easily from [16, Corollary 1, p. 687] and Proposition 1.

Proposition 8. [14, Theorem 1] *If $1 \leqslant r \leqslant s \leqslant p$, then $V_r \otimes V_s$ decomposes as*

$$V_r \otimes V_s = \bigoplus_{i=1}^{e} V_{s-r+2i-1} \oplus (r-e)V_p, \quad \text{where} \quad e = \begin{cases} r & \text{if } r + s \leqslant p, \\ p - s & \text{if } r + s > p. \end{cases} \tag{8}$$

We will need a version of Proposition 8 which works independent of the relative sizes of r and s. This is easy when $r+s\leqslant p$. In the case $r+s\geqslant p$ we have $e=p-s$ in (8). The subscript $s-r+2i-1$ in equation (8) equals $2p-r-s-2j+1$ where $j:=e-i+1$ satisfies $1\leqslant j\leqslant e$. This establishes the following symmetrised version of (8).

Corollary 9. *If $1\leqslant r\leqslant p$ and $1\leqslant s\leqslant p$, then $V_r\otimes V_s$ decomposes as*

$$V_r\otimes V_s=\begin{cases}\bigoplus_{j=1}^{\min(r,s)} V_{r+s-2j+1} & \text{if } r+s\leqslant p,\\ (r+s-p)V_p\oplus\bigoplus_{j=1}^{p-\max(r,s)} V_{2p-r-s-2j+1} & \text{if } r+s>p.\end{cases}\tag{9}$$

Corollary 9 arises in the context of tilting modules of the special linear group $\mathrm{SL}(2,\mathbb{F}_p)$ as we now explain. Brauer and Nesbitt [1] showed that $\mathrm{SL}(2,\mathbb{F}_p)$ has precisely p non-isomorphic indecomposable modules over the field $\mathbb{F}_p$, say $V_1',\dots,V_p'$ where $\dim(V_r')=r$. Indeed, V_r' comprises the homogeneous polynomials in $\mathbb{F}_p[x,y]$ of degree $r-1$ and $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ acts on V_r' via $x^g=ax+by$ and $y^g=cx+dy$. The restriction of V_r' to the subgroup $\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$ of $\mathrm{SL}(2,\mathbb{F}_p)$ gives the familiar module V_r. We thank Martin Liebeck for showing us how to prove Corollary 9 using tilting modules for $\mathrm{SL}(2,\mathbb{F}_p)$; see [8].

3. Proofs of Theorems 2 and 4

Suppose that $r,s\geqslant 1$ and $V_r\otimes V_s=V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_b}$ where $b=\min(r,s)$. In this section we will prove two new results concerning the p-parts $(\lambda_i)_p$ of the λ_i. We begin by proving that the p-parts of $\mathrm{lcm}(r,s)$ and $\gcd(\lambda_1,\dots,\lambda_b)$ are equal, i.e. $\mathrm{lcm}(r,s)_p=\gcd(\lambda_1,\dots,\lambda_b)_p$. It is sometimes more convenient to prove $\max(r_p,s_p)=\min((\mu_1)_p,\dots,(\mu_t)_p)$ by (2).

Proof of Theorem 2. Write $V_r\otimes V_s=V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_b}$, and let p^n be a p-power satisfying $\max(r,s)<p^n$. Write $V_r\otimes V_s=\bigoplus_{i=1}^t m_iV_{\mu_i}$ where $p^n\geqslant\mu_1>\cdots>\mu_t>0$ and each $m_i>0$ as in Notation 3. We use induction on n to prove the statement (3) paraphrasing Theorem 2.

First suppose that $n=1$, and hence $\mathrm{lcm}(r,s)_p=1$. Then $r_p=s_p=1$ and hence $p\nmid\dim(V_r\otimes V_s)$. So if $V_r\otimes V_s=\bigoplus_{i=1}^t m_iV_{\mu_i}$, then $p\nmid\gcd(\mu_1,\dots,\mu_t)_p$. This establishes Theorem 2 when $n=1$.

Suppose by induction that Theorem 2 holds for $\max(r,s)<p^n$ and fixed $n\geqslant 1$. We now show that it also holds for $\max(r,s)<p^{n+1}$. Without loss of generality, assume $r\leqslant s<p^{n+1}$. Write $r=r_0p^n+r_1$ and $s=s_0p^n+s_1$ where $r_0,s_0,r_1,s_1\geqslant 0$ and $r_1,s_1<p^n$. Clearly $r_0\leqslant s_0<p$. The remainder of the proof is divided into four cases.

Case 1. $r_1=s_1=0$. Since $r\leqslant s<p^{n+1}$, we deduce from $r=r_0p^n$ and $s=s_0p^n$ that $1\leqslant r_0\leqslant s_0<p$. Suppose that $V_{r_0}\otimes V_{s_0}=\sum_{j=1}^l n_jV_{\nu_j}$ where $\nu_1>\cdots>\nu_\ell>0$ and each $n_j>0$. It follows by [14, Lemma 2.2] that $V_r\otimes V_s=\sum_{j=1}^l p^nn_jV_{p^n\nu_j}$. Hence $\mu_j=p^n\nu_j$ and $m_j=p^nn_j$ for each j. Now $1=\max((r_0)_p,(s_0)_p)=\min((\nu_1)_p,\dots,(\nu_l)_p)$

by induction. Multiplying this equation by p^n gives

$$p^n = \max(r_p, s_p) = \min(p^n(\nu_1)_p, \dots, p^n(\nu_l)_p) = \min((\mu_1)_p, \dots, (\mu_l)_p).$$

This is equivalent to $\operatorname{lcm}(r,s)_p = \gcd(\lambda_1, \dots, \lambda_b)_p$, as desired.

Case 2. $r_1 = 0$ and $s_1 > 0$. In this case, $\operatorname{lcm}(r,s)_p = r_p = p^n$ since p^n divides r and $1 \leqslant r \leqslant s < p^{n+1}$. Since $V_0 \otimes V_{s_1} = V_0$, the partition $\lambda(r_1, s_1, p)$ has no parts, and the sum on the last line of (6) is empty. Thus Proposition 6 gives

$$V_r \otimes V_s = cV_{p^{n+1}} \oplus s_1 \bigoplus_{i=1}^{d_1} V_{(s_0-r_0+2i)p^n} \oplus (p^n - s_1) \bigoplus_{i=1}^{d_2} V_{(s_0-r_0+2i-1)p^n}, \tag{10}$$

where (c, d_1, d_2) is defined in Proposition 6. It is clear from (10) that p^n divides each of $\lambda_1, \dots, \lambda_b$, and thus divides $\gcd(\lambda_1, \dots, \lambda_b)$. The following paragraph shows that $\gcd(\lambda_1, \dots, \lambda_b)_p$ divides p^n.

If $s_0 = p-1$, then $(c, d_1, d_2) = (r+s-p^{n+1}, 0, 1)$ in Proposition 6 since $r_0+s_0 \geqslant 1+s_0 = p$. Any sum of the form $\bigoplus_{i=1}^{0} W_i$ equals 0, so equation (10) becomes

$$V_r \otimes V_s = (r + s - p^{n+1})V_{p^{n+1}} \oplus (p^n - s_1)V_{(p-r_0)p^n}.$$

Note that $r + s - p^{n+1} \geqslant p^n + s - p^{n+1} = s - s_0 p^n = s_1 > 0$ and $p^n - s_1 > 0$. Hence $\gcd(\lambda_1, \dots, \lambda_b)$ divides $p^{n+1} - (p - r_0)p^n = r_0 p^n$ and so $\gcd(\lambda_1, \dots, \lambda_b)_p$ divides p^n. If $s_0 \leqslant p - 2$, then Proposition 6 shows that $d_2 \geqslant d_1 \geqslant \min(r_0, p - s_0 - 1) \geqslant 1$, and hence $\gcd(\mu_1, \dots, \mu_t)$ divides $(s_0 - r_0 + 2)p^n - (s_0 - r_0 + 1)p^n = p^n$ in light of (10). In summary, $\gcd(\mu_1, \dots, \mu_t)_p$ divides p^n in both cases. Thus $\operatorname{lcm}(r,s)_p = \gcd(\lambda_1, \dots, \lambda_b)_p = p^n$.

Case 3. $r_1 > 0$ and $s_1 = 0$. In this case, $\operatorname{lcm}(r,s)_p = s_p = p^n$ since p^n divides s and $1 \leqslant r \leqslant s < p^{n+1}$. As above, the decomposition of $V_{r_1} \otimes V_0 = V_0$ is empty, so the last line of (6) vanishes. Hence $V_r \otimes V_s$ equals

$$cV_{p^{n+1}} \oplus r_1 \bigoplus_{i=1}^{d_1} V_{(s_0-r_0+2i)p^n} \oplus r_1 V_{(s_0-r_0)p^n} \oplus (p^n - r_1) \bigoplus_{i=1}^{d_2} V_{(s_0-r_0+2i-1)p^n}, \tag{11}$$

where (c, d_1, d_2) is defined in Proposition 6. It is clear from (11) that p^n divides each λ_i and thus divides $\gcd(\lambda_1, \dots, \lambda_b)$. The next paragraph shows that $\gcd(\lambda_1, \dots, \lambda_b)_p$ divides p^n.

If $r_0 = 0$, then $(c, d_1, d_2) = (0, 0, 0)$ and (11) gives $V_r \otimes V_s = r_1 V_{s_0 p^n} = r_1 V_s$. Hence $\gcd(\mu_1, \dots, \mu_t) = \gcd(s) = s$. Thus $\gcd(\mu_1, \dots, \mu_t)_p = s_p = p^n = \operatorname{lcm}(r,s)_p$, as desired. Thus we can assume $0 < r_0 < p$. Then $d_2 \geqslant \min(r_0, p - s_0) \geqslant 1$, and so $\gcd(\mu_1, \dots, \mu_t)$ divides $(s_0-r_0+1)p^n - (s_0-r_0)p^n = p^n$ in light of (11). Consequently $\gcd(\mu_1, \dots, \mu_t) = p^n$, and we conclude that for all values of r_0 that $\operatorname{lcm}(r,s)_p = p^n = \gcd(\mu_1, \dots, \mu_t)_p$ holds.

Case 4. $r_1 > 0$ and $s_1 > 0$. Here $r_p = (r_1)_p$ and $s_p = (s_1)_p$, and it follows that $\max(r_p, s_p) = \max((r_1)_p, (s_1)_p) < p^n$. Suppose that $V_{r_1} \otimes V_{s_1} = \sum_{j=1}^{l} n_j V_{\nu_j}$ where $p^n \geqslant \nu_1 > \cdots > \nu_\ell > 0$ and each $n_j > 0$. Our inductive hypothesis implies that

$$\min((\nu_1)_p, \dots, (\nu_l)_p) = \max((r_1)_p, (s_1)_p) = \max(r_p, s_p) < p^n.$$

Assume $(\nu_k)_p = \min((\nu_1)_p, \ldots, (\nu_l)_p)$, so $(\nu_k)_p$ divides each $(\nu_j)_p$. Since $(\nu_k)_p < p^n$, Proposition 6 implies that $(\nu_k)_p$ divides each $(\mu_i)_p$. Moreover by Proposition 6, one of the μ_i is equal to $(s_0 - r_0)p^n + \nu_k$ which has p-part $(\nu_k)_p$. Hence $\min((\mu_1)_p, \ldots, (\mu_t)_p)$ equals $(\nu_k)_p$ and it follows that

$$\max(r_p, s_p) = \min((\nu_1)_p, \ldots, (\nu_l)_p) = (\nu_k)_p = \min((\mu_1)_p, \ldots, (\mu_t)_p).$$

This is equivalent to $\mathrm{lcm}(r, s)_p = \gcd(\lambda_1, \ldots, \lambda_b)_p$, as desired. □

We now prove Theorem 4 which states that each part of $\lambda(r, s, p)$ with multiplicity greater than 1 must be divisible by p. In other words, if $V_r \otimes V_s = \sum_{i=1}^{t} m_i V_{\mu_i}$ where $\mu_1 > \cdots > \mu_t > 0$ and $m_i > 0$ for each i, then $m_j > 1$ implies p divides μ_j.

Proof of Theorem 4. Our proof uses induction on n where $\max(r, s) < p^n$.

The decomposition when $n = 1$ is described by (9). The only time that $\lambda(r, s, p)$ has a part with multiplicity more than 1 is when $r + s - p > 1$. In this case the part size is p. Thus Theorem 4 is true when $n = 1$.

Next suppose that Theorem 4 is true for $\max(r, s) < p^n$ and fixed $n \geqslant 1$. We will show that it also true when $p^n \leqslant \max(r, s) < p^{n+1}$. Without loss of generality, assume $r \leqslant s$. Set $r = r_0 p^n + r_1$ and $s = s_0 p^n + s_1$ where $r_0, s_0, r_1, s_1 \geqslant 0$ and $r_1, s_1 < p^n$. Suppose that $V_{r_1} \otimes V_{s_1} = \sum_{j=1}^{l} n_j V_{\nu_j}$, where $p^n \geqslant \nu_1 > \cdots > \nu_\ell > 0$ and $n_i > 0$ for each i. By the inductive hypothesis, each ν_j with $n_j > 1$ is a multiple of p. The part sizes, or the dimensions of the indecomposable modules, occurring in the first two lines of (6) are each divisible by p. We show in the next paragraph that the parts occurring in the last line of (6) are either distinct, or are divisible by p. Once this has been established, the inductive hypothesis completes the proof of Theorem 4.

The parts in the first sum $\bigoplus_{j=1}^{\ell} n_j \bigoplus_{i=0}^{d_1} V_{(s_0 - r_0 + 2i)p^n + \nu_j}$ are distinct for distinct (i, j). This is so because

$$(s_0 - r_0 + 2i)p^n + \nu_j = (s_0 - r_0 + 2i')p^n + \nu_{j'} \qquad \text{where } 0 < \nu_j, \nu_{j'} \leqslant p^n$$

implies $\nu_j = \nu_{j'}$ and hence $j = j'$; and then $i = i'$ follows. A similar argument shows that the parts in the second sum $\bigoplus_{j=1}^{\ell} n_j \bigoplus_{i=1}^{d_1} V_{(s_0 - r_0 + 2i)p^n - \nu_j}$ are distinct. If a part from the first sum equals a part from the second sum, then there exist integers i, i', j, j' satisfying

$$(s_0 - r_0 + 2i)p^n + \nu_j = (s_0 - r_0 + 2i')p^n - \nu_{j'}.$$

Hence $2(i' - i)p^n = \nu_j + \nu_{j'}$. However, $0 < \nu_j, \nu_{j'} \leqslant p^n$ implies $0 < \nu_j + \nu_{j'} \leqslant 2p^n$ and hence $\nu_j + \nu_{j'}$ is divisible by $2p^n$ which is possible only when $\nu_j = \nu_{j'} = p^n$. Thus $2(i' - i)p^n = 2p^n$ and $i' - i = 1$. Consequently, a part from the first sum equals a part from the second sum only when the part sizes are divisible by p^n (and hence by p). As remarked above, induction now completes the proof. □

4. Iima and Iwamatsu's Algorithm

Assume $1 \leqslant r \leqslant s$ throughout this section. For $k = 1, \dots, r$, define $D_k = D_k(r, s)$ to be the determinant of the $k \times k$ matrix A_k whose (i, j)th entry is $\binom{r+s-2k}{s+i-j-k}$ for $0 \leqslant i, j < k$. Given nonnegative integers M and N, the matrix $\left(\binom{M}{N+i-j}\right)_{0\leqslant i,j,<k}$ has determinant $\prod_{i=0}^{k-1} \binom{M+i}{N}/\binom{N+i}{N}$, see [15, p. 355]. Setting $M := r + s - 2k$ and $N := s - k$ gives the following closed formula

$$D_k(r, s) = \prod_{i=0}^{k-1} \frac{\binom{r+s-2k+i}{s-k}}{\binom{s-k+i}{s-k}}, \quad \text{where } 1 \leqslant k \leqslant r. \tag{12}$$

Even though the right-hand side of (12) looks like a rational number, $D_k(r, s)$ is an integer (as it is the determinant of a matrix with integer entries). Set $D_0(r, s) := 1$, and note that $D_r(r, s) = 1$. For $k = 0, 1, \dots, r$, define

$$\delta_k = \delta_k(r, s, p) = \begin{cases} 0 & \text{if } D_k(r, s) \equiv 0 \pmod{p}, \\ 1 & \text{if } D_k(r, s) \not\equiv 0 \pmod{p}. \end{cases} \tag{13}$$

Thus $\delta_k = 1$ says that A_k is invertible when viewed as a matrix over $\mathbb{F}_p$. In other words, $\delta_k = 1$ says that A_k has full p-rank. Iima and Iwamatsu [9] found a way to construct $\lambda(r, s, p)$ from the $\{0, 1\}$-sequence $\delta_0(r, s, p), \delta_1(r, s, p), \dots, \delta_r(r, s, p)$. This constrains the number of choices of $\lambda(r, s, p)$ as described in [3]. Note that $\delta_0 = \delta_r = 1$ by our convention that $D_0(r, s) = 1$ and $D_r(r, s) = 1$.

For $1 \leqslant k \leqslant r$, if $\delta_k = 1$, let $\ell(k)$ be the smallest positive integer such that $\delta_{k-\ell(k)} = 1$. Note that $\ell(k)$ is well defined since $\delta_0 = 1$, and $\ell(k) \leqslant k$. The following Proposition is proved by the results in [9] preceding and including Theorem 2.2.9.

Proposition 10. [9, Theorem 2.2.9] *Suppose $1 \leqslant r \leqslant s$, and use the above notation for δ_k and $\ell(k)$ for $1 \leqslant k \leqslant r$. Then the parts of the Jordan partition $\lambda(r, s, p)$ can be computed via the following recurrence where k decreases from r to 1*

$$\lambda_k = \begin{cases} r + s - 2k + \ell(k) & \text{if } \delta_k = 1, \\ \lambda_{k+1} & \text{if } \delta_k = 0. \end{cases}$$

The next proposition is a reformulation of Proposition 10 in the language of Green ring results. While this result essentially appears in [9], its proof is long and somewhat complicated, so we prefer to give our own proof. Recall the definition (13) of δ_k.

Proposition 11. [9, Theorem 2.2.9] *Suppose $1 \leqslant r \leqslant s$, and all the values of k satisfying $\delta_k(r, s, p) = 1$ are $0 = k_0 < k_1 < \cdots < k_t = r$. Then $V_r \otimes V_s$ decomposes as*

$$V_r \otimes V_s = \bigoplus_{i=1}^{t} (k_i - k_{i-1}) V_{r+s-k_i-k_{i-1}}. \tag{14}$$

Proof. Because $k_{i-1} < k_i$ for $1 \leqslant i \leqslant t$, we have

$$\delta_{k_{i-1}} = 1, \quad \delta_{k_{i-1}+1} = \delta_{k_{i-1}+2} = \cdots = \delta_{k_i-1} = 0, \quad \text{and} \quad \delta_{k_i} = 1.$$

Then $\ell(k_i) = k_i - k_{i-1}$. Appealing to second case of the recurrence in Proposition 10 gives

$$\lambda_{k_{i-1}+1} = \cdots = \lambda_{k_i-1} = \lambda_{k_i},$$

and appealing to the first case of Iima and Iwamatsu's recurrence gives

$$\lambda_{k_i} = r + s - 2k_i + \ell(k_i) = r + s - 2k_i + (k_i - k_{i-1}) = r + s - k_i - k_{i-1}.$$

This proves that $V_r \otimes V_s = \bigoplus_{i=1}^{t}(k_i - k_{i-1})V_{r+s-k_i-k_{i-1}}$. Since $0 = k_0 < k_1 < \cdots < k_t = r$, different values of i give different values of $r + s - k_i - k_{i-1}$. Hence (14) is indeed a decomposition, with distinct parts and positive multiplicities, as claimed. □

It follows from Proposition 11 that the multiplicities $m_i = k_i - k_{i-1}$, $1 \leqslant i \leqslant t$, determine the distinct part sizes $\mu_i = r + s - k_i - k_{i-1}$, $1 \leqslant i \leqslant t$, and conversely. Theorem 5 shows how $\mu_1, \dots, \mu_t$ determine $m_1, \dots, m_t$ via explicit formulas.

Proof of Theorem 5. Our strategy is to prove McFall's conjecture [11, Conjecture 2] that

$$m_1 = r + s - \mu_1 \qquad \text{and} \qquad m_i = \mu_{i-1} - \mu_i - m_{i-1} \quad \text{for } 1 < i \leqslant t. \tag{15}$$

A straightforward calculation shows that the formula (5), satisfies this recurrence relation, and hence that (15) implies the equalities in (5). Rearranging (15) gives a recurrence relation for computing the μ_i from the m_j, namely

$$\mu_1 = r + s - m_1 \qquad \text{and} \qquad \mu_i = \mu_{i-1} - m_i - m_{i-1} \quad \text{for } 1 < i \leqslant t. \tag{16}$$

A further simple calculation shows that the formulas (4) are equivalent to the rearranged recurrence relation (16).

As noted above, Proposition 11 shows that $m_i = k_i - k_{i-1}$ and $\mu_i = r + s - k_i - k_{i-1}$ for $1 \leqslant i \leqslant t$. The initial condition of (15) follows from $k_0 = 0$ as

$$r + s - \mu_1 = r + s - (r + s - k_1 - k_0) = k_1 + k_0 = k_1 - k_0 = m_1.$$

For $1 < i \leqslant t$, the inductive step of (15) also follows easily as $\mu_{i-1} - \mu_i - m_{i-1}$ equals

$$(r + s - k_{i-1} - k_{i-2}) - (r + s - k_i - k_{i-1}) - (k_{i-1} - k_{i-2}) = k_i - k_{i-1} = m_i.$$

This establishes the recurrence relation (15), and thereby proves Theorem 5. □

The p-divisibility of the integers $D_0(r,s), D_1(r,s), \dots, D_r(r,s)$ plays a central role in Iima and Iwamatsu's algorithm. Kummer's theorem [6] states that the power of a prime p dividing $\binom{m}{n}$ is the number of 'carries' required to add m and $n-m$ in base-p. This can be used to compute the largest p-power dividing the numerator and denominator of (12). The following lemma gives a more direct approach, and it has a nice application in Section 5.

Lemma 12. *Suppose* $1 \leqslant r \leqslant s$, *and let* $D_k(r,s)$ *be as in* (12) *with* $D_0(r,s) = D_r(r,s) = 1$.

(a) *If* $0 \leqslant k \leqslant r$, *then* $\binom{s}{s-k} D_{k+1}(r+1, s+1) = \binom{r+s-k}{s-k} D_k(r,s)$.

(b) *If* $0 \leqslant k \leqslant r-1$, *then* $\binom{s}{s-k} D_{k+1}(r,s+1) = \binom{r+s-2k-1}{s-k} D_k(r,s)$.
(c) *If* $0 \leqslant k \leqslant r-1$, *then* $\binom{r+s-k-1}{k} D_{k+1}(r,s) = \binom{r+s-2k-2}{s-k-1} D_k(r,s)$.

Proof. The proof is by direct calculation using (12). Part (a) follows from

$$D_{k+1}(r+1,s+1) = \prod_{i=0}^{k} \frac{\binom{r+s-2k+i}{s-k}}{\binom{s-k+i}{s-k}} = \frac{\binom{r+s-k}{s-k}}{\binom{s}{s-k}} \prod_{i=0}^{k-1} \frac{\binom{r+s-2k+i}{s-k}}{\binom{s-k+i}{s-k}} = \frac{\binom{r+s-k}{s-k}}{\binom{s}{s-k}} D_k(r,s).$$

The proof of part (b) follows from the formula (12) and the identities $\binom{m-1}{n} = \frac{m-n}{m}\binom{m}{n}$ and $\binom{m}{n} \prod_{i=0}^{k-1} \frac{m-n-k+i+1}{m-k+i+1} = \binom{m-k}{n}$

$$\begin{aligned} D_{k+1}(r,s+1) &= \frac{\binom{r+s-k-1}{s-k}}{\binom{s}{s-k}} \prod_{i=0}^{k-1} \frac{\binom{r+s-2k-1+i}{s-k}}{\binom{s-k+i}{s-k}} \\ &= \frac{\binom{r+s-k-1}{s-k}}{\binom{s}{s-k}} \prod_{i=0}^{k-1} \frac{\binom{r+s-2k+i}{s-k}(r-k+i)}{\binom{s-k+i}{s-k}(r+s-2k+i)} = \frac{\binom{r+s-2k-1}{s-k}}{\binom{s}{s-k}} D_k(r,s). \end{aligned}$$

To prove part (c), we use the identity

$$D_k(r,s) = \prod_{i=0}^{k-1} \frac{(r+s-2k+i)!i!}{(s-k+i)!(r-k+i)!}.$$

We now write $D_{k+1}(r,s)$ in terms of the above product

$$\begin{aligned} D_{k+1}(r,s) &= \binom{r+s-2k-2}{s-k-1} \prod_{i=1}^{k} \frac{\binom{r+s-2k-2+i}{s-k-1}}{\binom{s-k-1+i}{s-k-1}} \\ &= \binom{r+s-2k-2}{s-k-1} \prod_{i=1}^{k} \frac{(r+s-2k-2+i)!i!}{(s-k-1+i)!(r-k-1+i)!} \\ &= \binom{r+s-2k-2}{s-k-1} \prod_{i=0}^{k-1} \frac{(r+s-2k-1+i)!(i+1)!}{(s-k+i)!(r-k+i)!} \\ &= \binom{r+s-2k-2}{s-k-1} \prod_{i=0}^{k-1} \frac{(r+s-2k+i)!i!(i+1)}{(s-k+i)!(r-k+i)!(r+s-2k+i)} \\ &= \frac{\binom{r+s-2k-2}{s-k-1}}{\binom{r+s-k-1}{k}} \prod_{i=0}^{k-1} \frac{(r+s-2k+i)!i!}{(s-k+i)!(r-k+i)!}. \end{aligned}$$

□

5. Results for $|r-s|$ at most one

In this section, we prove several results when $|r-s| \leqslant 1$. First, we determine the smallest part of $\lambda(r,r,p)$, and its multiplicity. As usual, we denote the p-part of a nonzero integer r by r_p.

Lucas' theorem (see [6]) is a useful number-theoretic result for proving $D_{r-p^k}(r,r) \not\equiv 0 \pmod p$, or $\delta_{r-p^k}(r,r,p) = 1$ as in (13). This theorem says that $\binom{m}{n} \equiv \prod_{i\geqslant 0}\binom{m_i}{n_i} \pmod p$ where $m = \sum_{i\geqslant 0} m_i p^i$ and $n = \sum_{i\geqslant 0} n_i p^i$ are the base-p expansions of m and n, respectively. The base-p 'digits' m_i, n_i satisfy $0 \leqslant m_i, n_i < p$. Note that $\binom{m_i}{n_i} = 0$ if $m_i < n_i$, and $\binom{m_i}{0} = 1$. Thus the infinite product $\prod_{i\geqslant 0}\binom{m_i}{n_i}$ is finite, as $\binom{m_i}{n_i} = 1$ for sufficiently large i.

Theorem 13. *The smallest part of $\lambda(r,r,p)$ is r_p, and it occurs with multiplicity r_p. Using Notation 3 and $b = \min(r,r) = r$, this says that $\lambda_r = r_p = \mu_t$ and $m_t = r_p$.*

Proof. Suppose that $r_p = p^k$. Then $r = ap^k$ with $a_p = 1$. By virtue of Proposition 11 (or by 10), it suffices to show that $D_{r-j}(r,r) \equiv 0 \pmod p$ for $0 < j < p^k$ and $D_{r-p^k}(r,r) \not\equiv 0 \pmod p$, since $D_r(r,r) = 1$.

Using formula (12) and canceling gives

$$D_{r-p^k}(r,r) = \prod_{i=0}^{(a-1)p^k-1} \frac{\binom{2p^k+i}{p^k}}{\binom{p^k+i}{p^k}} = \prod_{i=0}^{p^k-1} \frac{\binom{ap^k+i}{p^k}}{\binom{p^k+i}{p^k}}. \tag{17}$$

For $0 \leqslant i < p^k$, Lucas' theorem shows $\binom{ap^k+i}{p^k} \equiv \binom{a}{1}\binom{i}{0} \equiv a \pmod p$. The numerator in (17) is $\prod_{i=0}^{p^k-1}\binom{ap^k+i}{p^k} \equiv a^{p^k} \not\equiv 0 \pmod p$. Thus $D_{r-p^k}(r,r) \not\equiv 0 \pmod p$, as desired. [Incidentally, $D_{r-p^k}(r,r) \equiv a \pmod p$ as $a^{p-1} \equiv 1 \pmod p$ and $\binom{p^k+i}{p^k} \equiv 1 \pmod p$ holds for $0 \leqslant i \leqslant p^k - 1$.]

One way to prove that $D_{r-j}(r,r) \equiv 0 \pmod p$ is to show that p divides the numerator of (12) to a higher power than the denominator. This requires stronger results than Lucas' theorem. (Kummer proved that the power of p dividing $\binom{m}{n}$ is the number of i for which $m_i < n_i$, see [6].) A simpler approach involves using Lemma 12(a). Suppose $0 < j < p^k$. Then Lemma 12(a) gives

$$\binom{ap^k}{j} D_{r-j+1}(r+1,r+1) = \binom{ap^k+j}{j} D_{r-j}(r,r). \tag{18}$$

Again by Lucas' theorem, $\binom{ap^k}{j} \equiv 0 \pmod p$ and $\binom{ap^k+j}{j} \equiv 1 \pmod p$, so (18) implies that $D_{r-j}(r,r) \equiv 0 \pmod p$. The proof is thus completed. □

For the rest of this section, we establish a decomposition formula for $V_r \otimes V_s$ when $|r-s| \leqslant 1$ and $p=2$. The following proposition shortens the proof of Theorem 15. This result already appears in [2, Corollary 1], albeit in a slightly less general form.

Proposition 14. *Suppose* $1 \leqslant r \leqslant p^n$ *and* $1 \leqslant s \leqslant p^n$. *Then*

$$V_{p^n-r} \otimes V_{p^n-s} = \max(p^n-r-s,0)V_{p^n} \oplus (V_r \otimes V_s).$$

Proof. Let $V_r \otimes V_s = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_b}$ where $b := \min(r,s)$. Proposition 1 yields

$$V_{p^n-r} \otimes V_s = (s-b)V_{p^n} \oplus V_{p^n-\lambda_b} \oplus \cdots \oplus V_{p^n-\lambda_1}.$$

Since $s \leqslant p^n$ and $p^n - r \leqslant p^n$, applying Proposition 1 to $V_s \otimes V_{p^n-r}$ gives

$$V_{p^n-s} \otimes V_{p^n-r} = [(p^n-r) - \min(p^n-r,s)]\, V_{p^n} \oplus V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_b} \oplus (s-b)V_0.$$

Replacing the expression in square brackets with $\max(p^n-r-s,0)$, and omitting the last summand gives the desired decomposition of $V_{p^n-r} \otimes V_{p^n-s}$. □

Our decompositions for $V_r \otimes V_r$ and $V_r \otimes V_{r+1}$ when $p=2$ depend on a *'consecutive-ones-binary-expansion'* which we now define. The binary number $(1\cdots10\cdots0)_2$ with m consecutive ones, and n consecutive zeros, equals $2^{m+n}-2^n$. Thus a binary expansion $r = \sum_{i=1}^{\ell}\sum_{j=b_i}^{a_i-1} 2^j$ with ℓ groups of consecutive ones and $a_1 > b_1 > \cdots > a_\ell > b_\ell \geqslant 0$ simplifies to $r = \sum_{i=1}^{\ell}(2^{a_i}-2^{b_i})$. We call an alternating sum $r = \sum_{i=1}^{k}(-1)^{i-1}2^{e_i}$, with decreasing powers of 2 and minimal length, the *'consecutive-ones-binary-expansion'* of r. Minimal length implies $e_{k-1} > e_k + 1$ when $k > 1$: otherwise $2^{e_k+1} - 2^{e_k}$ can be replaced by 2^{e_k}. Note that $r = \sum_{i=1}^{\ell}(2^{a_i}-2^{b_i})$ is the consecutive-ones-binary-expansion if and only if $a_\ell > b_\ell + 1$. For example, $4 = 2^2, 5 = 2^3 - 2^2 + 2^0, 6 = 2^3 - 2^1$ are consecutive-ones-binary-expansions. The partial sums $r_j = \sum_{i=j+1}^{k}(-1)^{i-j-1}2^{e_i}$, $0 \leqslant j \leqslant k$, associated to the consecutive-ones-binary-expansion $r = \sum_{i=1}^{k}(-1)^{i-1}2^{e_i}$ satisfy $r_0 = r$, $r_k = 0$, $r_i = 2^{e_{i+1}} - r_{i+1}$, and $1 \leqslant r_i \leqslant 2^{e_i}$ for $0 \leqslant i < k$. Also $e_i = \lceil \log_2(r_i) \rceil$ for $1 \leqslant i < k$.

The following theorem originally appeared as Theorems 14 and 16 of [4]. We are grateful to M. J. J. Barry who showed us a simplified proof of Theorem 15, and we thank him for his permission to include (a modified version of) his proof.

Theorem 15. *Suppose* $\operatorname{char}(F) = 2$ *and* $r = \sum_{i=1}^{k}(-1)^{i-1}2^{e_i}$ *is the consecutive-ones-binary-expansion of* r *where* $e_1 > \cdots > e_k \geqslant 0$. *Set* $r_j = \sum_{i=j+1}^{k}(-1)^{i-j-1}2^{e_i}$ *for* $0 \leqslant j \leqslant k$ *where* $r_k = 0$. *Then* $V_r \otimes V_r$ *and* $V_r \otimes V_{r+1}$ *decompose over* F *as*

$$V_r \otimes V_r = \bigoplus_{i=1}^{k}(2^{e_i} - 2r_i)V_{2^{e_i}} \quad \text{and} \quad V_r \otimes V_{r+1} = \bigoplus_{i=1}^{k}(2^{e_i} - 2r_i + (-1)^{i-1})V_{2^{e_i}}. \tag{19}$$

In particular, each part of $\lambda(r,r,2)$ *is a power of* 2. *Furthermore, parts not equal to* 1 *have even multiplicities, and* 1 *has multiplicity at most* 1. *Also each part of* $\lambda(r,r+1,2)$ *is a power of* 2 *greater than* 1.

Proof. We prove (19) using induction on k. The decomposition for $V_r \otimes V_r$ holds when $k = 1$ by [7, (2.7d)]. Suppose now that $k > 1$ and $V_{r_1} \otimes V_{r_1} = \bigoplus_{i=2}^{k}(2^{e_i} - 2r_i)V_{2^{e_i}}$ holds by induction. Observe that $r_1 \leqslant 2^{e_2}$ so $2r_1 \leqslant 2^{e_2+1} \leqslant 2^{e_1}$, and $2^{e_1} - 2r_1 \geqslant 0$. Proposition 14 implies

$$
\begin{aligned}
V_r \otimes V_r &= V_{2^{e_1}-r_1} \otimes V_{2^{e_1}-r_1} && \text{as } r = 2^{e_1} - r_1, \\
&= (2^{e_1} - 2r_1)V_{2^{e_1}} \oplus\ (V_{r_1} \otimes V_{r_1}) && \text{as } 2^{e_1} - 2r_1 \geqslant 0.
\end{aligned} \tag{20}
$$

The decomposition for $V_r \otimes V_{r+1}$ holds when $k = 0$, and when $k = 1$ by [7, (2.7d)]. Suppose $k > 1$, and $V_{r_2} \otimes V_{r_2+1} = \bigoplus_{i=3}^{k}(2^{e_i} - 2r_i + (-1)^{i-1})V_{2^{e_i}}$ is valid by induction. As above, $2^{e_1} - 2r_1 \geqslant 0$ obtains. Moreover, $2^{e_2} - 2r_2 - 1 \geqslant 0$ is true. This is easily seen when $k = 2$, it follows from $r_2 \leqslant 2^{e_3}$ using $e_{k-1} > e_k + 1$ when $k = 3$, and for $k > 3$ it follows from $r_2 < 2^{e_3}$ using $e_3 + 1 \leqslant e_2$. Applying the equations $r = 2^{e_1} - r_1$, $r_1 = 2^{e_2} - r_2$, and Proposition 14 twice, now gives

$$
\begin{aligned}
V_r \otimes V_{r+1} &= V_{r+1} \otimes V_r \\
&= V_{2^{e_1}-(r_1-1)} \otimes V_{2^{e_1}-r_1} \\
&= (2^{e_1} - 2r_1 + 1)V_{2^{e_1}} \oplus\ (V_{r_1-1} \otimes V_{r_1}) \\
&= (2^{e_1} - 2r_1 + 1)V_{2^{e_1}} \oplus\ (V_{2^{e_2}-r_2-1} \otimes V_{2^{e_2}-r_2}) \\
&= (2^{e_1} - 2r_1 + 1)V_{2^{e_1}} \oplus (2^{e_2} - 2r_2 - 1)V_{2^{e_2}} \oplus\ (V_{r_2} \otimes V_{r_2+1}).
\end{aligned} \tag{21}
$$

Thus (19) follows from (20) and (21) by induction on k. As a by-product we have proved that the multiplicities in (19) are nonnegative, and (19) is a valid decomposition. □

To illustrate Theorem 15 take $r = 5$. Then r has consecutive-ones-binary-expansion $5 = 2^3 - 2^2 + 2^0$. Substituting $r_1 = 3$, $r_2 = 1$, $r_3 = 0$ into (19) gives

$$V_5 \otimes V_5 = 2V_8 \oplus 2V_4 \oplus V_1 \quad \text{and} \quad V_5 \otimes V_6 = 3V_8 \oplus V_4 \oplus 2V_1$$

over a field of characteristic 2. The novelty of Theorem 15 is the decomposition (19). The parity of the multiplicities were already known to Gow and Laffey [5, Corollaries 1 and 2].

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