

**A UNIFORM STRUCTURE ON SUBGROUPS OF $GL_n(\mathbb{F}_q)$ AND
ITS APPLICATION TO A CONDITIONAL CONSTRUCTION OF
ARTIN REPRESENTATIONS OF GL_n**

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ABSTRACT. Continuing our investigation in [19], where we associated an Artin representation to a vector-valued real analytic Siegel cusp form of weight $(2, 1)$ under reasonable assumptions, we associate an Artin representation of GL_n to a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with similar assumptions. A main innovation in this paper is to obtain a uniform structure of subgroups in $GL_n(\mathbb{F}_q)$, which enables us to avoid complicated case by case analysis in [19]. We also supplement [19] by showing that we can associate non-holomorphic Siegel modular forms of weight $(2, 1)$ to Maass forms for $GL_2(\mathbb{A}_{\mathbb{Q}})$ and to cuspidal representations of $GL_2(\mathbb{A}_K)$ over imaginary quadratic fields K .

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1. INTRODUCTION

This paper is a continuation of [19], where we associated an irreducible complex Galois representation, called Artin representation, $\rho : G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GSp_4(\mathbb{C})$, to a vector-valued real analytic Siegel cusp form of weight $(2, 1)$, under some reasonable assumptions, by generalizing the result of Deligne and Serre [10] who associated an odd irreducible Artin representation $\rho_f : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$ to any elliptic cusp form f of weight one. Contrary to the case of holomorphic modular forms of weight one, real analytic Siegel cusp forms of weight $(2, 1)$ do not have algebro-geometric structures, and thus several assumptions are needed to carry out Deligne-Serre construction. In this paper, we associate an Artin representation to a (unitary) cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with similar assumptions.

More precisely, let π be a (unitary) cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with the central character ω . Let N be the conductor of π so that π_p is unramified for $p \nmid N$. Let $\{\alpha_1(p), \dots, \alpha_n(p)\}$ be the Satake parameters for $p \nmid N$. Let

$$H_p(T) := (1 - \alpha_1(p)T) \cdots (1 - \alpha_n(p)T) = 1 - a_1(p)T + \cdots + (-1)^n a_n(p)T^n,$$

be the Hecke polynomial for $p \nmid N$. Then

$$\begin{aligned} a_1(p) &= \alpha_1(p) + \cdots + \alpha_n(p), & a_2(p) &= \sum_{1 \leq i < j \leq n} \alpha_i(p)\alpha_j(p), \dots, \\ a_m(p) &= \sum_{i_1 < \cdots < i_m} \alpha_{i_1}(p) \cdots \alpha_{i_m}(p), \dots, & a_n(p) &= \alpha_1(p) \cdots \alpha_n(p) = \omega(p). \end{aligned}$$

Let $\mathbb{Q}_{\pi} = \mathbb{Q}(a_m(p), m = 1, \dots, n, p \nmid N)$ be the Hecke field of π . Let K be the Galois closure of \mathbb{Q}_{π} , and \mathcal{O}_K be the ring of integers of K .

We assume the following:

- (1) (Finiteness of Hecke fields) \mathbb{Q}_{π} is a finite extension of \mathbb{Q} , i.e., K is a finite extension of \mathbb{Q} ;
- (2) (Integrality of Hecke polynomials) There exists an integer $M > 0$ such that $Ma_m(p) \in \mathcal{O}_K$ for $m = 1, \dots, n$ and $p \nmid N$;
- (3) (Existence of Galois conjugates) For each $\sigma \in \text{Gal}(K/\mathbb{Q})$, ${}^{\sigma}\pi$ is a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with conductor N_{σ} such that for $p \nmid NN_{\sigma}$, the Satake parameters are $\{\sigma(\alpha_1(p)), \dots, \sigma(\alpha_n(p))\}$;
- (4) Existence of mod ℓ Galois representation attached to π ;

(5) (Rankin-Selberg L -functions) For each $m = 1, \dots, n$, and each $\sigma \in \text{Gal}(K/\mathbb{Q})$,

$$\sum_{p \nmid NN_\sigma} \frac{|\sigma(a_m(p))|^2}{p^s} \leq C_{n,m}^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+,$$

$$\text{where } C_{n,m} = \binom{n}{m} := \frac{n!}{m!(n-m)!}.$$

These assumptions (1)-(4) are very natural, and are valid for cuspidal representations attached to elliptic cusp forms of weight one. Note that we assume that the Satake parameters $\alpha_i(p)$'s themselves are integral. It enables us to show that the Satake parameters take only finitely many values by Assumption (5) (Proposition 3.1). Recall that in the case of holomorphic Siegel cusp forms of weight (k_1, k_2) , $k_1 \geq k_2 \geq 2$, Hecke eigenvalues are algebraic integers, but Satake parameters are twisted by $p^{-\frac{k_1+k_2-3}{2}}$. Assumption (5) is also natural. The L -function $\sum_p \frac{|\sigma(a_m(p))|^2}{p^s}$ is closely related to the Rankin-Selberg L -function of the exterior m -th power $\wedge^m(\sigma\pi)$. In the appendix, we describe the relationship, and show that the Langlands functoriality of the exterior m -th power $\wedge^m(\pi)$ as an automorphic representation of $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$ implies Assumption (5).

Then we prove the following main theorem:

Theorem 1.1 (Main Theorem). *Let π satisfy the above five assumptions. Then there exists the Artin representation $\rho_\pi : G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{C})$ which is unramified for $p \nmid N$ such that*

$$\det(I_n - \rho_\pi(\text{Frob}_p)T) = H_p(T)$$

for all $p \nmid N$. Furthermore, $\rho_\pi(c) \stackrel{GL_n(\mathbb{C})}{\sim} \text{diag}(\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{\pm 1\}$ for the complex conjugate c , and $\pi_\infty \simeq \pi(\epsilon'_1, \dots, \epsilon'_n)$ where $\epsilon'_i = \begin{cases} 1, & \text{if } \epsilon_i = 1 \\ \text{sgn}, & \text{if } \epsilon_i = -1 \end{cases}$.

As a corollary, we obtain that the above assumptions on π imply the Ramanujan conjecture for π , namely, π_p is tempered for all p .

The assumptions force π_∞ to be a principal series representation of the form $\pi(\epsilon'_1, \dots, \epsilon'_n)$. Naively we hope that this kind of automorphic representation π should satisfy the above assumptions (1)-(5). However, unlike holomorphic modular forms in $GL_2(\mathbb{A}_{\mathbb{Q}})$ case where one can use algebraic geometry as Deligne and Serre did, it seems difficult in general GL_n case to verify whether a given π satisfies these strong assumptions. Note that in holomorphic modular forms in GL_2 case, π_∞ is a limit of discrete series. But in the case of GL_n , $n \geq 3$, π_∞ is not a limit of discrete series (cf. [20]).

One key ingredient in Deligne-Serre construction is to obtain bounds of orders of certain subgroups of $GL_2(\mathbb{F}_{\ell^n})$ for any odd prime ℓ . In [10], it was done by classification of semisimple subgroups of $GL_2(\mathbb{F}_{\ell^n})$ and case by case analysis. In [19], we carried out the same thing by using the classification of semisimple subgroups of $GSp_4(\mathbb{F}_{\ell^n})$ and case by case analysis. A main innovation in this paper is to prove some structure theorem (see Section 4) for semisimple subgroups of $GL_n(\mathbb{F}_q)$ for any finite field \mathbb{F}_q by using result of Larsen-Pink [21] combined with the appendix in [15]. This enables us to generalize Proposition 7.2 of [10] to general linear groups without explicit forms of subgroups in question at hand. So we can avoid complicated case by case analysis as we have done in [19]. We remark that this is not at all obvious from the existing results in the literature (cf. [21] and [26]).

The organization of this paper is as follows. In Section 2, we investigate the infinity type of an automorphic representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ which gives rise to an Artin representation. In Section 3, we apply the Rankin-Selberg method to prove that the number of Satake parameters outside a certain infinite set of primes is finite. By using the results in Section 4 with the finiteness of Satake parameters, we bound the size of the image of mod ℓ Galois representations, provided that they exist. The formulation of the conjecture for the existence of such mod ℓ Galois representations is given in Section 5. The proof of the main theorem is given in Section 6 following Deligne-Serre.

In Sections 7 and 8, we recall our previous work [19] for the case of GSp_4/\mathbb{Q} . We discuss a relation between non-holomorphic Siegel modular forms and holomorphic Siegel modular forms. After the completion of the previous work, we realized that we do not need to use the unproven hypothesis on the existence of the weak transfer from GSp_4 to GL_4 as in [2]. We explain how to get around this. Finally in Sections 9 and 10, we associate non-holomorphic Siegel modular forms of weight $(2, 1)$ to automorphic representations of GL_2 over imaginary quadratic fields, and Maass forms for $GL_2(\mathbb{A}_{\mathbb{Q}})$.

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2. INFINITY TYPE OF ARTIN REPRESENTATION

We show that the cuspidal representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ which we are considering has a very special infinity type π_{∞} .

Proposition 2.1. *Let $\rho : G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{C})$ be an irreducible continuous Galois representation, and let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ such that*

$$\det(I_n - \rho_F(\text{Frob}_p)T) = H_p(T)$$

for all $p \nmid N$, where N is the conductor of π . Then π_{∞} is the full induced representation $\pi(\epsilon'_1, \dots, \epsilon'_n)$, where $\rho_{\pi}(c) \stackrel{GL_n(\mathbb{C})}{\sim} \text{diag}(\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{\pm 1\}$ for the complex conjugate c , and $\epsilon'_i = \begin{cases} 1, & \text{if } \epsilon_i = 1 \\ \text{sgn}, & \text{if } \epsilon_i = -1 \end{cases}$.

Proof. From the assumption, it is clear that $L(s, \pi_p) = L(s, \rho_p)$ for almost all p . Then by Proposition A.1 of [25], $L(s, \pi_p) = L(s, \rho_p)$ for all p , and $L(s, \pi_{\infty}) = L(s, \rho_{\infty})$. Since $\rho_{\pi}(c) \stackrel{GL_n(\mathbb{C})}{\sim} \text{diag}(\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{\pm 1\}$ for the complex conjugate c , the Langlands parameter of π_{∞} is

$$\phi : W_{\mathbb{R}} = \mathbb{C}^{\times} \cup j\mathbb{C}^{\times} \rightarrow GL_n(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(\epsilon_1, \dots, \epsilon_n),$$

where $\epsilon_i \in \{\pm 1\}$. Our result follows from this observation. \square

3. APPLICATION OF THE RANKIN-SELBERG METHOD

We prove that Satake parameters take only finitely many values under Assumptions (1)-(3) and (5) in the introduction.

Proposition 3.1. *Suppose $\pi = \otimes'_p \pi_p$ is a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with conductor N which satisfies Assumptions (1)-(3) and (5) in the introduction. Then for any positive integer η , there exists a set X_{η} of rational primes such that $\text{den. sup } X_{\eta} \leq \eta$, and the set $\{(a_1(p), \dots, a_n(p)) \mid p \notin X_{\eta}\}$ is a finite set, or equivalently, $\{\text{Satake parameters at } p \mid p \notin X_{\eta}\}$ is finite.*

Here $\text{den. sup } X_{\eta}$ is defined by

$$\limsup_{s \rightarrow 1^+} \frac{\sum_{p \in X_{\eta}} p^{-s}}{\log \frac{1}{s-1}}.$$

We also define the Dirichlet density $\text{den}(X_{\eta})$ by

$$\lim_{s \rightarrow 1^+} \frac{\sum_{p \in X_{\eta}} p^{-s}}{\log \frac{1}{s-1}}.$$

Proof. By Assumption (3), for each $\sigma \in \text{Gal}(K/\mathbb{Q})$, $\sigma\pi$ is a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with the conductor N_σ such that for $p \nmid NN_\sigma$, the Satake parameters are $\{\sigma(\alpha_1(p)), \dots, \sigma(\alpha_n(p))\}$. Hence for each $m = 1, \dots, n$ and each σ , by Assumption (5),

$$\sum_{p \nmid NN_\sigma} \frac{|\sigma(a_m(p))|^2}{p^s} \leq C_{n,m}^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Let $N_K = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} N_\sigma$.

By Assumption (2), there exists an integer $M > 0$ so that $Ma_m(p) \in \mathcal{O}_K$ if $p \nmid N$. For $c > 0$, consider two sets:

$$Y(c) = \{a \in \mathcal{O}_K \mid |\sigma(a)|^2 \leq c \text{ for any } \sigma \in \text{Gal}(K/\mathbb{Q})\},$$

$$X(c) = \{p \mid \text{at least one of } Ma_m(p), m = 1, \dots, n, \text{ does not belong to } Y(c), \text{ or } p \mid N_K\}.$$

Note that since \mathcal{O}_K is a lattice in K , $Y(c)$ is a finite set for any $c > 0$. Hence the set $\{(Ma_1(p), \dots, Ma_n(p)) \mid p \notin X(c)\}$ is finite, and so the set $\{(a_1(p), \dots, a_n(p)) \mid p \notin X(c)\}$ is finite.

Let $r = [K : \mathbb{Q}]$. If $p \in X(c)$ and $p \nmid N_K$, there exists m such that $|\sigma_m(Ma_m(p))|^2 > c$ for some $\sigma_m \in \text{Gal}(K/\mathbb{Q})$. Hence

$$c \sum_{p \in X(c)} p^{-s} \leq \sum_{m=1}^n \sum_{\sigma} \sum_{p \nmid N_K} \frac{|\sigma(Ma_m(p))|^2}{p^s} + O(1) \leq \left(\sum_{m=1}^n C_{n,m}^2 \right) rM^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Therefore, $\text{den. sup } X(c) \leq \frac{rM^2}{c} (\sum_{m=1}^n C_{n,m}^2)$. Take c such that $c \geq \frac{rM^2 \sum_{m=1}^n C_{n,m}^2}{\eta}$, and let $X_\eta = X(c)$. \square

4. BOUNDS FOR THE ORDERS OF CERTAIN SUBGROUPS OF $GL_n(\mathbb{F}_q)$

Fix a positive integer $n \geq 1$. Let q be a power of a rational prime p and \mathbb{F}_q be the finite field with q elements. Let G be a subgroup of $GL_n(\mathbb{F}_q)$. We say G is semisimple if the natural action of G on $V := \mathbb{F}_q^{\oplus n}$ is semisimple or equivalently V is a semisimple G -module. We say G is an irreducible (resp. absolutely irreducible) subgroup of $GL_n(\mathbb{F}_q)$ if V (resp. $V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$) is an irreducible G -module. As in [10], for positive constants N and η , ($0 < \eta < 1$), we introduce the following property $C(\eta, N)$ for G :

$$C(\eta, N) : \text{there exists a subset } H \text{ of } G \text{ such that } \begin{cases} (i) & (1 - \eta)|G| \leq |H|, \\ (ii) & |\{\det(1 - hT) \in \mathbb{F}_q[T] \mid h \in H\}| \leq N. \end{cases}$$

Theorem 4.1. [Theorem 0.2 of [21]] *There exists a constant $J_1(n)$, depending only on n such that any finite subgroup G of $GL_n(\mathbb{F}_q)$ possesses a series $G = G_0 \supset G_1 \supset G_2 \supset G_3$ of subgroups such that G_i is normal in G_{i-1} for each $1 \leq i \leq 3$ and it satisfies*

- (1) $[G : G_1] \leq J_1(n)$;
- (2) G_1/G_2 is a direct product of finite simple groups of Lie type in characteristic p and the number of direct factors is bounded uniformly in n ;
- (3) G_2/G_3 is abelian of order not divisible by p ;
- (4) G_3 is a p -group.

Remark 4.2. *In the above theorem, the boundedness of the number of direct factors is not in the statement of Theorem 0.2 of [21]. However, it is implicit in the proof of the theorem. It is important for our purpose.*

Theorem 4.3. [Chap. V, Section 19, Th. 7, of [36]] *There exists a constant $J_2(n)$, depending only on n such that any solvable subgroup G of $GL_n(\mathbb{F}_q)$ possesses a normal subgroup N such that*

- (1) $[G : N] \leq J_2(n)$,
- (2) N is conjugate to a subgroup of the group of upper triangular matrices in $GL_n(\mathbb{F}_q)$.

Corollary 4.4. *Assume that $p > J_2(n)$. Let G be any semisimple, solvable subgroup of $GL_n(\mathbb{F}_q)$. If G' is a normal p -subgroup of G , then G' is trivial.*

Proof. By Theorem 4.3, there exists a normal subgroup N such that $[G : N] < p$. Hence the p -group G' is a subgroup of N . By Theorem 4.3, N is conjugate to a subgroup of the group of upper triangular matrices and in particular we may assume that G' is a subgroup consisting of unipotent, upper triangular matrices. By Clifford's theorem [1], p. 17, G' is semisimple. Hence $G' = 1$. □

Corollary 4.5. *Assume that $p > J_2(n)$. Let G, G_1, G_2, G_3 be as in Theorem 4.1. Then $G_3 = 1$.*

Proof. Since any p -group is solvable, so is G_2 . Then Corollary 4.4 implies the assertion. □

Theorem 4.6. [cf. Theorem 0.1 of [21]] *Let k be an algebraically closed field of characteristic zero. For every n , there exists a constant $J_3(n)$, depending only on n such that any finite subgroup G of $GL_n(k)$ possesses an abelian normal subgroup A such that $[G : A] \leq J_3(n)$.*

Henceforth we fix $n \geq 1$ and a positive number $C(n)$ so that

$$C(n) > \max\{n + 3, J_1(n), J_2(n), J_3(n)\}.$$

We will implicitly use the assumption $C(n) > n + 3$ in the proof of Proposition 4.10 later to apply a result of [14]. (In Theorem B of [14], one needs $p \geq n + 3$, not $n - 3$. It was pointed out by F. Herzig.)

The following lemma is easy to prove:

Lemma 4.7. *Let G be a subgroup of $GL_n(\mathbb{F}_q)$, and G' a subgroup of G . Let $[G : G'] < d$. Then if G satisfies $C(\eta, N)$ for $(0 < \eta < 1/d)$, then G' satisfies $C(d\eta, N)$.*

Proof. Set $M = [G : G']$. Take a subset H from the first property of $C(\eta, N)$ for G . Then one can see that

$$|H| \geq (1 - \eta)|G| = (M - M\eta)|G'| \geq (1 - d\eta)|G'|$$

giving the claim. □

Proposition 4.8. *Let G be a semisimple subgroup of $GL_n(\mathbb{F}_q)$ with the order not divisible by p . If G satisfies $C(\eta, N)$ for $(0 < \eta < \frac{1}{C(n)})$, then $|G| \leq B$, where $B = B(\eta, N, n)$ is a constant depending only on η, N , and n .*

Proof. By assumption, we may assume that G is a subgroup of $GL_n(\mathbb{C})$. For example, we apply Schur-Zassenhaus' theorem (cf. [11], page 829) to the natural projection $GL_n(W(\mathbb{F}_q)) \rightarrow GL_n(\mathbb{F}_q)$ where $W(\mathbb{F}_q)$ the ring of Witt vectors and then get a lift G to $GL_n(W(\mathbb{F}_q))$. Then we have only to compose this with an embedding $GL_n(W(\mathbb{F}_q)) \hookrightarrow GL_n(\mathbb{C})$.

By Theorem 4.6 there exists an abelian normal subgroup A of G such that $[G : A] \leq J_3(n)$. By Lemma 4.7, A satisfies $C(C(n)\eta, N)$. We may assume that A is a subgroup consisting of diagonal matrices in $GL_n(\mathbb{F}_q)$. Then one has

$$(1 - C(n)\eta)|A| \leq |H| \leq n!N$$

giving a bound of $|A|$. Since $[G : A]$ is bounded, so is $|G| = [G : A] \times |A|$. □

Proposition 4.9. *Let G be an irreducible subgroup of $GL_n(\mathbb{F}_q)$. Then the following properties hold:*

- (1) there exists a finite extension field \mathbb{F}_{q^r} and an absolutely irreducible subgroup $G' \subset GL_m(\mathbb{F}_{q^r})$ with $n = rm$ such that G is isomorphic to G' . Furthermore, for any $g \in G$ and the corresponding $g' \in G'$ under this isomorphism,

$$f_g(T) = \prod_{\sigma \in \text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)} f_{g'}(T)^\sigma,$$

where $f_g(T), f_{g'}(T)$ stand for characteristic polynomials of g, g' , resp.

- (2) the center of G' is a cyclic subgroup of $\mathbb{F}_{q^r}^\times I_m \subset GL_m(\mathbb{F}_{q^r})$.

Proof. By Schur's lemma, the centralizer $Z = Z_{M_n(\mathbb{F}_q)}(G)$ is a finite division ring. By Wedderburn's theorem, Z is a finite field over \mathbb{F}_q , say \mathbb{F}_{q^r} , since Z contains $\mathbb{F}_q I_n$. Since Z^\times acts on $V := \mathbb{F}_q^{\oplus n}$ faithfully, $\dim_{\mathbb{F}_q} \mathbb{F}_{q^r} = r$ has to divide n . Put $m = \frac{n}{r}$. We view V as a \mathbb{F}_{q^r} -module. Then one has a faithful representation $G \longrightarrow GL(V) \simeq GL_m(\mathbb{F}_{q^r})$. To be more precise, if we take a basis $\{e_1, \dots, e_m\}$ of V as a \mathbb{F}_{q^r} -module and a generator $\alpha \in \mathbb{F}_{q^r}$ over \mathbb{F}_q , then a basis of V is given by $\{\alpha^{p^i} e_j \mid 0 \leq i \leq r-1, 1 \leq j \leq m\}$. Denote by G' the image of G under this representation. Then G' is absolutely irreducible since $\mathbb{F}_{q^r} = Z = Z_{M_n(\mathbb{F}_q)}(G) = Z_{M_m(\mathbb{F}_{q^r})}(G')$. The last claim follows from the direct calculation in this explicit basis.

For the claim (2), let g be an element in the center $Z(G')$. Since g commutes with the action of G' , it belongs to $Z_{M_m(\mathbb{F}_{q^r})}(G')^\times = \mathbb{F}_{q^r}^\times$. \square

Let T be the group of all diagonal matrices in $GL_n(\mathbb{F}_q)$. Let G be a semisimple subgroup of $GL_n(\mathbb{F}_q)$. Assume $p > C(n)$. Fix a series of normal subgroups $G \supset G_1 \supset G_2 \supset G_3$ in Theorem 4.1 for G . By Corollary 4.5, $G_3 = 1$. Assume $G_1 \neq G_2$ until the end of the proof of the following proposition.

By Clifford's theorem, $V = \mathbb{F}_q^{\oplus n}$ is a semisimple G_1 -module. Therefore we have a decomposition $V = \bigoplus_{1 \leq i \leq m} W_i$ into irreducible components as a G_1 -module, where $\dim_{\mathbb{F}_q} W_i = n_i r_i$ for each i . By Proposition 4.9, we may assume that each (G_1, W_i) is an absolutely irreducible module over the field extension $\mathbb{F}_{q^{r_i}}$ of \mathbb{F}_q , and we have a faithful representation $\pi_i : G_1 \longrightarrow GL(W_i) \simeq GL_{n_i}(\mathbb{F}_{q^{r_i}})$. We denote by $G^{(i)}$ the image of G_1 under π_i . Then we get an injection

$$G_1 \hookrightarrow \prod_{i=1}^m G^{(i)}$$

which is not necessarily surjective, but we will see this map will be an isomorphism under the natural quotients. Note that clearly $r_i < n$.

Proposition 4.10. *Under the above setting, the following properties hold:*

- (1) for each $i = 1, \dots, m$, the center $Z(G^{(i)})$ is a subgroup of $\mathbb{F}_{q^{r_i}}^\times \text{id}_{W_i}$, and G_2 is a subgroup of T so that $\pi_i(G_2) \subset Z(G^{(i)})$. Further $G_2 \hookrightarrow \prod_{1 \leq i \leq m} \pi_i(G_2)$.
- (2) For each $i = 1, \dots, m$, there exists a simple and simply connected linear algebraic group \mathcal{G}_i over $\mathbb{F}_{q^{r_i}}$ realized inside $GL(W_i)$ such that $G^{(i)} = Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$ and $Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}})) \subset Z(G^{(i)})$. In particular, the natural map

$$G_1/G_2 \longrightarrow \prod_{1 \leq i \leq m} G^{(i)}/Z(G^{(i)}) \simeq \prod_{1 \leq i \leq m} \mathcal{G}_i(\mathbb{F}_{q^{r_i}})/Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))$$

is isomorphic where each component of the right hand side is a simple Chevalley group (cf. [22]). Further $G_1 \hookrightarrow \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$ with respect to the decomposition $V = \bigoplus_{1 \leq i \leq m} W_i$, and there exist a constant $C_1(n)$ depending only on n so that

$$|G_1| \geq C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|.$$

Proof. The first part of (1) follows from Proposition 4.9-(2). Since (G_1, W_i) is (absolutely) irreducible, by Clifford's theorem, $W_i|_{G_2}$ decomposes into isotypical representations of 1-dimensional representations. Hence $\pi_i(G_2)$ are scalar matrices. Hence it clearly commutes with $G^{(i)}$. The latter claim is clear from the injectivity of $G_1 \hookrightarrow \prod_{i=1}^m G^{(i)}$.

We now prove the second claim. Let Γ_i^0 be the group generated by all elements of p -th power order in $G^{(i)}$. Then by Theorem B of [14] (see also step 1 in the proof of Proposition A.7 of [15]), $\Gamma_i^0/Z(\Gamma_i^0)$ is a simple Chevalley group. Since Γ_i^0 is a normal subgroup of $G^{(i)}$, so is $Z(G^{(i)}) \cdot \Gamma_i^0/Z(G^{(i)})$ in $G^{(i)}/Z(G^{(i)})$. However $G^{(i)}/Z(G^{(i)})$ is by construction (see Theorem 4.1-(b)), a simple group. Then one has $Z(G^{(i)}) \cdot \Gamma_i^0/Z(G^{(i)}) = G^{(i)}/Z(G^{(i)})$. Hence $Z(G^{(i)})\Gamma_i^0 = G^{(i)}$. The surjective map $\Gamma_i^0 \longrightarrow G^{(i)}/Z(G^{(i)})$ induces an isomorphism $\Gamma_i^0/Z(\Gamma_i^0) \xrightarrow{\sim} G^{(i)}/Z(G^{(i)})$. If an element $g \in Z(\Gamma_i^0)$ does not belong to $Z(G^{(i)})$, then the previous isomorphism never be isomorphic, hence it gives a contradiction. Hence one has $Z(\Gamma_i^0) \subset Z(G^{(i)}) \subset \mathbb{F}_{q^{r_i}}^\times \text{id}_{W_i}$. This means that the image of Γ_i^0 under the projective map $GL(W_i) \longrightarrow PGL(W_i)$ is $\Gamma_i^0/Z(\Gamma_i^0)$ and it is a simple Chevalley group. Then the claim follows by looking for any simple Chevalley group which appears in this way (see [22]). Therefore there exists a simple and simply connected algebraic group \mathcal{G}_i over $\mathbb{F}_{q^{r_i}}$ such that $G^{(i)} = Z(G^{(i)})\Gamma_i^0 = Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$.

To prove the last claim, let us consider the following commutative diagram

$$\begin{array}{ccc} G_1 & \longrightarrow & \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ G_1/G_2 & \xrightarrow{\sim} & \prod_{1 \leq i \leq m} \mathcal{G}_i(\mathbb{F}_{q^{r_i}})/Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}})). \end{array}$$

where the top arrow is an injective map which is defined by the decomposition $V = \bigoplus_{1 \leq i \leq m} W_i$.

Here π_1 and π_2 stand for natural projections. Then one has

$$|G_1| \geq C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|, \quad C_1(n) := \frac{1}{\prod_{1 \leq i \leq m} |Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))|}.$$

Since \mathcal{G}_i is simple and simply connected algebraic group, the cardinality of $Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))$ is depending only on the rank of \mathcal{G}_i , hence on n . This completes the proof. \square

For any subgroup $D \subset GL_n(\mathbb{F}_q)$ and each $g \in D$, we define by $M_D(g)$ the number of elements of D which have the same characteristic polynomial as g and put $M_D = \max_{g \in D} \{M_D(g)\}$. Note that for any subgroup A of the center $\mathbb{F}_q^\times I_n$, $M_{A \cdot D} = M_D$ and $M_{D_1} \leq M_{D_2}$ for subgroups $D_1 \subset D_2 \subset GL_n(\mathbb{F}_q)$. This simple observations will be used in the proof of Theorem 4.14 below.

Lemma 4.11. *For each i ($1 \leq i \leq m$), let D_i be a subgroup of $GL_{n_i}(\mathbb{F}_{q^{r_i}})$. We identify the product $D := \prod_{1 \leq i \leq m} D_i$ with the Levi subgroup of the parabolic subgroup $P_{(n_1, \dots, n_m)}$ in $GL_n(\overline{\mathbb{F}}_q)$ with respect to the partition $n = n_1 + \dots + n_m$. Then there exists a constant $C_2(n)$ depending only on n so that*

$$M_D \leq C_2(n) \prod_{1 \leq i \leq m} M_{D_i}.$$

Proof. We will give a very rough estimation for $C_2(n)$. For any $g = (g_1, \dots, g_m)$, $f_g(T) = \prod_{1 \leq i \leq m} f_{g_i}(T)$. We denote the eigenvalues by $\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}, \dots, \alpha_1^{(m)}, \dots, \alpha_{n_m}^{(m)}$ which are not necessarily different from each other. Then the number of all permutations which preserve the type (n_1, \dots, n_m) is $\frac{n!}{n_1! \dots n_m!}$. We may take this as $C_2(n)$. \square

Proposition 4.12. *[Proposition 3.1 of [21] or Lemma 3.5 of [26]] For any connected algebraic group G over \mathbb{F}_q , we have*

$$(\sqrt{q} - 1)^{2\dim G} \leq |G(\mathbb{F}_q)| \leq (\sqrt{q} + 1)^{2\dim G}.$$

For connected linear algebraic groups, one has a stronger estimate

$$(q-1)^{\dim G} \leq |G(\mathbb{F}_q)| \leq (q+1)^{\dim G}.$$

Let G be a simple and simply connected algebraic group over a finite field \mathbb{F}_q . We follow [8] for the following proposition.

Proposition 4.13. *Let $l = \text{rank}(G)$, and let A be a semisimple element in G and $C(A)$ be the centralizer of A in $G(\mathbb{F}_q)$, and $d = \dim C(A)$. Then*

$$\frac{q^d}{(q+1)^d} \frac{|G(\mathbb{F}_q)|}{q^l} \leq M_G(A) \leq \frac{q^d}{(q-1)^d} \frac{|G(\mathbb{F}_q)|}{q^l}.$$

Proof. Let $\Delta_G(A)$ be the set of $g \in G$ which has the same characteristic polynomial as A so that $M_G(A) = |\Delta_G(A)|$. Suppose $g \in \Delta_G(A)$. Then $g = g_s g_u$ with g_s semisimple, g_u unipotent. Then $\det(1 - Tg_s) = \det(1 - TA)$.

Since g_s and A are conjugate in $G(\overline{\mathbb{F}}_q)$, they are conjugate in $G(\mathbb{F}_q)$ [35]. Over $\overline{\mathbb{F}}_q$, the algebraic group $C(A)$ is the centralizer in $G(\overline{\mathbb{F}}_q)$ of A . Since A is semisimple and G is simply connected, $C(A)$ is a connected reductive group [34]. Since $C(A)$ contains any maximal torus of $G(\mathbb{F}_p)$ and $C(A) \subset G(\mathbb{F}_p)$, $\text{rank}(C(A)) = l$. By Steinberg [34],

$$\#\{\text{unipotent elements in } C(A)(\mathbb{F}_q)\} = q^{d-l}.$$

Therefore,

$$\begin{aligned} M_G(A) &= \#\{\text{pairs } (g_s, g_u) \mid g_s \text{ is } G(\mathbb{F}_p)\text{-conjugate to } A \text{ and } g_u \in (C(g_s))_u(\mathbb{F}_q)\} \\ &= q^{d-l} \#\{g_s \text{ which is } G(\mathbb{F}_q)\text{-conjugate to } A\} = q^{d-l} \frac{\#G(\mathbb{F}_q)}{\#C(A)(\mathbb{F}_q)}. \end{aligned}$$

Since $C(A)$ is connected, by Proposition 4.12, $(q-1)^d \leq \#C(A)(\mathbb{F}_p) \leq (q+1)^d$. Hence our assertion follows. \square

Here $M_G(A) \leq K \frac{|G(\mathbb{F}_q)|}{q^l}$ for a constant K depending only on $\dim G$.

Theorem 4.14. *Let G be a semisimple subgroup of $GL_n(\mathbb{F}_q)$. Assume that $p > C(n)$. If G satisfies the property $C(\eta, N)$ for $0 < \eta < \frac{1}{C(n)}$, then either $|G|$ or q is bounded by a constant depending only on n . Hence there exists a constant $B = B(\eta, N, n)$ depending only on η, N, n such that $|G| \leq B$.*

Proof. Take a series of normal subgroups $G \supset G_1 \supset G_2 \supset G_3$ in Theorem 4.1. By Corollary 4.5, $G_3 = \{1\}$. If $G_1 = G_2$, then the claim follows from Proposition 4.8.

Henceforth we assume $G_1 \neq G_2$. Then by Proposition 4.10, there exists a injective map

$$G_1 \longrightarrow \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}), \quad \mathcal{G}_i(\mathbb{F}_{q^{r_i}}) \subset GL_{n_i}(\mathbb{F}_{q^{r_i}}), \quad 1 \leq i \leq m.$$

Then one has

$$M_{G_1} \leq M_D, \quad D := \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}).$$

Since G satisfies $C(\eta, N)$, by Lemma 4.7, G_1 satisfies $C(C(n)\eta, N)$. This means that

$$(1 - C(n)\eta)|G_1| \leq |H|.$$

Then applying Proposition 4.13 to $D = \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$, and by Lemma 4.11, one has

$$\begin{aligned} (1 - C(n)\eta)C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})| &\leq (1 - C(n)\eta)|G_1| \leq |H| \leq NM_{G_1} \\ &\leq NM_D \leq NC_2(n) \prod_{1 \leq i \leq m} M_{Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})} = NC_2(n) \prod_{1 \leq i \leq m} M_{\mathcal{G}_i(\mathbb{F}_{q^{r_i}})} \\ &\leq NK(n)C_2(n) \prod_{1 \leq i \leq m} \frac{|\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|}{q^{r_i l_i}} \end{aligned}$$

with a constant $K(n)$, where $l_i = \text{rank } \mathcal{G}_i$. This gives us the bound

$$q \leq \prod_{1 \leq i \leq m} q^{r_i l_i} \leq \frac{NK(n)C_2(n)}{(1 - C(n)\eta)C_1(n)}.$$

Hence the claim follows. \square

Corollary 4.15. *Let S be an infinite set of rational primes. Suppose for each prime $\ell \in S$, the image of a mod ℓ semisimple Galois representation $\rho_\ell : G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{F}_\ell)$ satisfies $C(\eta, N)$ for $0 < \eta < \frac{1}{C(n)}$. Then there exists a constant $A = A(\eta, N, n)$ such that $|\text{Im } \rho_\ell| \leq A$.*

5. MOD ℓ REPRESENTATIONS

We state the assumption on the existence of mod ℓ representations.

Conjecture 5.1. *Let π be a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ satisfying Assumptions (1) and (2) in the introduction, namely, the finiteness of the Hecke field \mathbb{Q}_π , and the integrability of the Hecke polynomial $H_p(T)$.*

Then for all but finitely many ℓ coprime to N and each finite place λ of \mathbb{Q}_π above ℓ with the residue field \mathbb{F}_λ , there exists a continuous semi-simple representation

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{F}}_\lambda)$$

which is unramified outside of ℓN , so that

$$\det(I_n - \rho_\lambda(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda},$$

for any $p \nmid \ell N$.

6. ARTIN REPRESENTATIONS ASSOCIATED TO CUSPIDAL REPRESENTATIONS

In this section we give a proof of the main theorem (Theorem 1.1). Let π satisfy Assumptions (1)-(5) in the introduction. Let Σ_π be the set of primes ℓ which are excluded in the statement of Conjecture 5.1 for the existence of mod ℓ representation for π . We denote by S_π the union of Σ_π and the set of rational primes consisting of primes p so that π_p is ramified. Let K be a Galois closure of \mathbb{Q}_π . By assumptions on π , this is a finite extension of \mathbb{Q} . Let P_K be the set prime numbers ℓ which splits completely in K . For each $\ell \in P_K$, choose a finite place λ_ℓ of K dividing ℓ . By Conjecture 5.1, there exists a continuous semi-simple representation

$$\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{F}}_\ell)$$

which is unramified outside $S_\pi \cup \{\ell\}$, and

$$\det(I_n - \rho_\ell(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda_\ell}.$$

By Lemma 6.13 of [10], we may assume that the image of ρ_ℓ takes the values in $GL_n(\mathbb{F}_\ell)$. Let $G_\ell := \text{Im } \rho_\ell$.

Lemma 6.1. *For any η , $0 < \eta < 1$, there exists a constant M such that G_ℓ satisfies $C(\eta, M)$ for every $\ell \in P_K$.*

Proof. By Proposition 3.1, if we let $\mathcal{M} := \{H_p(T) \mid p \notin X_\eta\}$, then \mathcal{M} is a finite set. Let $M := |\mathcal{M}|$ which will be a desired constant as below. Let us consider the subset of G_ℓ defined by

$$H_\ell := \{g \in G_\ell \mid g \stackrel{G_\ell}{\sim} \rho_\ell(\text{Frob}_p) \text{ for some } p \notin X_\eta\}.$$

By Chebotarev density theorem, one has

$$1 = \frac{|H_\ell|}{|G_\ell|} + \text{den}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \text{den.sup}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \eta,$$

giving $(1 - \eta)|G_\ell| \leq |H_\ell|$.

The characteristic polynomial of each element of H_ℓ is the reduction of some element of \mathcal{M} . Therefore one has

$$|\{\det(I_n - hT) \mid h \in H_\ell\}| \leq M.$$

□

By Lemma 6.1 together with Corollary 4.15, there exists a constant A such that $|G_\ell| \leq A$ for any $\ell \in P_K$. Let Y be the set of polynomials $\prod_{i=1}^n (1 - \alpha_i T)$, where α_i 's are roots of unity of order less than A .

If $p \notin S_\pi$, for all $\ell \in P_K$ with $\ell \neq p$, there exists $R(T) \in Y$ such that

$$H_p(T) \equiv R(T) \pmod{\lambda_\ell}.$$

Since Y is finite and P_L is infinite,

$$H_p(T) = R(T).$$

Let P'_K be the set of $\ell \in P_K$ such that $\ell > A$ and for $R, S \in Y$ with $R \neq S$, $R \not\equiv S \pmod{\lambda_\ell}$. Then it is easy to see that P'_K is infinite. For each $\ell \in P'_K$, ℓ does not divide $|G_\ell|$, since $\ell > A \geq |G_\ell|$. Let $\pi_\ell : GL_n(\mathcal{O}_{\lambda_\ell}) \rightarrow GL_n(\mathbb{F}_\ell)$ be the reduction modulo λ . Applying a profinite version of Schur-Zassenhaus' theorem (cf. [28], page 40, Theorem 2.3.15) to $\pi^{-1}(G_\ell)$ and $\pi^{-1}(G_\ell) \cap \text{Ker}(\pi)$ (note that the latter group is a Hall subgroup of $\pi^{-1}(G_\ell)$ in the sense of [28]), there exists a subgroup $H \subset \pi^{-1}(G_\ell)$ such that $\pi^{-1}(G_\ell) = H \cdot (\pi^{-1}(G_\ell) \cap \text{Ker}(\pi))$ and $H \cap (\pi^{-1}(G_\ell) \cap \text{Ker}(\pi)) = 1$. Then the composition of the inclusion $H \hookrightarrow \pi^{-1}(G_\ell)$ and π induces an isomorphism

$$H \xrightarrow{\sim} G_\ell = \text{Im } \rho_\ell.$$

Hence we have a lift $\rho'_\ell : G_\mathbb{Q} \rightarrow GL_n(\mathcal{O}_{\lambda_\ell})$ of ρ_ℓ . Since the coefficient of ρ'_ℓ is of characteristic zero and its image is finite, for $p \nmid N\ell$, one has $\det(I_n - \rho'_\ell(\text{Frob}_p)T) \in Y$. On the other hand, we have

$$\det(I_n - \rho'_\ell(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda_\ell}.$$

Since $\ell \in P'_K$, the above congruence relation implies the equality

$$\det(I_n - \rho'_\ell(\text{Frob}_p)T) = H_p(T).$$

for all $p \nmid N\ell$. Now we replace ℓ with another prime $\ell' \in P'_K$. Then one has $\rho'_{\ell'} : G_\mathbb{Q} \rightarrow GL_n(\mathcal{O}_{\lambda_{\ell'}})$ such that

$$\det(I_n - \rho'_\ell(\text{Frob}_p)T) = \det(I_n - \rho'_{\ell'}(\text{Frob}_p)T)$$

for all $p \nmid N\ell\ell'$. By Chebotarev density theorem, one has $\rho'_{\ell'} \sim \rho'_\ell$ and this means that ρ'_ℓ is unramified at ℓ . Hence we have the desired representation

$$\rho_\pi := \rho'_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathcal{O}_{\lambda_\ell}) \hookrightarrow GL_n(\mathbb{C}),$$

where the second map comes from a fixed embedding $\mathcal{O}_{\lambda_\ell} \hookrightarrow \mathbb{C}$. This representation is independent of any choice of such a embedding by Chebotarev density theorem. The infinity type π_∞ was determined in Proposition 2.1.

Corollary 6.2. *Let π be a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ which satisfies Assumptions (1)-(5) in the introduction. Then π_p is tempered for all p .*

Proof. By Theorem 1.1, there exists the Artin representation $\rho_\pi : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ such that for almost all q ,

$$\det(I_n - \rho_\pi(\text{Frob}_q)T) = H_q(T).$$

This shows that π_q is tempered for almost all q .

Suppose π_p is non-tempered. We apply Proposition A.1 of [25] to the Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi})$: Since $L(s, \pi_q \times \tilde{\pi}_q) = L(s, \rho_q \times \tilde{\rho}_q)$ for almost all q , in particular, we have $L(s, \pi_p \times \tilde{\pi}_p) = L(s, \rho_p \times \tilde{\rho}_p)$. Suppose π_p is of the form (11.1). Then by (11.2), the left hand side has a factor $L(s - 2r_1, \eta_1 \times \tilde{\eta}_1)$ which has a pole at $s = 2r_1 > 0$. On the other hand, the right hand side is holomorphic for $\text{Re}(s) > 0$. Contradiction. Hence π_p is tempered for all p . \square

7. NON-HOLOMORPHIC SIEGEL MODULAR FORMS AND HOLOMORPHIC SIEGEL MODULAR FORMS VIA THE CONGRUENCE METHOD

In this section we follow the notation of [19]. Let us first recall the existence of a Galois representation for any holomorphic Siegel modular form of weight (k_1, k_2) , $k_1 \geq k_2 \geq 2$, for $GS p_4$. Thanks to the works of [23], [40] and [39] with the classification of CAP forms ([27], [33], [32]) and endoscopic representations for $GS p_4$ ([29]), we can associate a Galois representation to F .

Theorem 7.1. *For any prime ℓ , there exists a number field E including \mathbb{Q}_F , such that for each rational prime ℓ and a finite place $\lambda|\ell$ of \mathbb{Q}_F , there exists a continuous representation $\rho_{F,\ell} : G_{\mathbb{Q}} \longrightarrow GL_4(E_\lambda)$ which is unramified outside of ℓN so that*

$$\det(I_4 - \rho_{F,\ell}(\text{Frob}_p)p^{-s})^{-1} = L_p(s, F) = L_p(s - \frac{k_1+k_2-3}{2}, \pi_F)$$

for any $p \nmid \ell N$. Furthermore, if $k_1 \geq k_2 \geq 3$ and π_F is neither endoscopic nor CAP, then the image of $\rho_{F,\ell}$ can be taken in $GSp_4(E_\lambda)$.

Let us denote by $\bar{\rho}_{F,\ell} : G_{\mathbb{Q}} \rightarrow GL_4(\mathbb{F})$ the reduction modulo $\lambda(\lambda|\ell)$ where \mathbb{F} is the residue field of λ .

Let f be an elliptic newform of weight one which is neither of dihedral nor of tetrahedral type. Then this gives rise to a unique Artin representation $\rho_f : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$. Since the image is finite, we can take a finite extension K of \mathbb{Q} so that $\text{Im}(\rho_f) \subset GL_2(\mathcal{O})$ where \mathcal{O} is the ring of integers of K . Then taking the reduction modulo a prime ideal above a rational prime ℓ , we obtain a mod ℓ representation $\bar{\rho}_{f,\ell} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F})$.

By Theorem 10.1 of [19], there exists a real analytic Siegel modular form F of weight $(2, 1)$ with eigenvalues $-\frac{5}{12}$ (resp. 0) for Δ_1 (resp. Δ_2) (see [19] for Δ_i) such that $\pi_F \sim \text{Sym}^3(\pi_f)$. By using this, we obtain a mod ℓ representation $\pi_{F,\ell} : G_{\mathbb{Q}} \rightarrow GL_4(\mathbb{F})$ for F .

On the other hand, by multiplying Hasse invariant of weight $\ell - 1$, we obtain an eigenform g of weight $1 + a(p - 1)$ for any positive integer a such that g is congruent to f modulo ℓ hence $\bar{\rho}_{g,\ell} \simeq \bar{\rho}_{f,\ell}$. By using symmetric cube lift and generic transfer from GSp_4 to GL_4 , one can show the existence of a holomorphic Siegel cusp form G of weight $(k_1, k_2) = (2a(\ell - 1) + 2, a(\ell - 1) + 1)$ such that $\rho_{G,\ell} = \text{Sym}^3(\rho_{g,\ell})$. From this one concludes that there exist a non-holomorphic Siegel modular form F of weight $(2, 1)$ and a holomorphic Siegel modular form G of weight $(2a(\ell - 1) + 2, a(\ell - 1) + 1)$ such that

$$\bar{\rho}_{F,\ell} \simeq \bar{\rho}_{G,\ell}.$$

We denote by this property $F \equiv G \pmod{\ell}$ provided if the existence of and mod ℓ representations of F and G is guaranteed.

We can also construct such F and G by using endoscopic lift from a pair $(f_1, f_2), \pi_{f_1} \not\sim \pi_{f_2}$ of elliptic newform of weight one whose central characters are same as follows. By using theta lift (cf [29]) and Section 5 of [19], there exists a real analytic Siegel modular form of weight $(2, 1)$ as above such that $\rho_{F,\ell} \simeq \rho_{f_1} \oplus \rho_{f_2}$. By multiplying Hasse invariant again, one has a pair of elliptic modular forms $(g_1, g_2), \pi_{g_1} \not\sim \pi_{g_2}$ of elliptic newform g_1 (resp. g_2) of weight $r_1 = 1 + a(\ell - 1)$ (resp. $r_2 = 1 + b(\ell - 1)$) with the same central character. Then by using theta lift (cf [29]), one can construct a holomorphic Siegel cusp form G of weight $(k_1, k_2) = (\frac{(\ell-1)(a+b)}{2} + 1, \frac{(\ell-1)(a-b)}{2} + 2)$ such that $\rho_{F,\ell} \simeq \rho_{g_1,\ell} \oplus \rho_{g_2,\ell}$. Taking reduction modulo ℓ , one concludes $F \equiv G \pmod{\ell}$.

For such F (and G), the mod ℓ representation $\bar{\rho}_{F,\ell}$ has a remarkable property that $\bar{\rho}_{F,\ell}$ is unramified at ℓ . In case elliptic newform, this property characterizes a weight ℓ form so that it

comes from a weight one form by multiplying Hasse invariant of weight $\ell - 1$. This principle is discussed in Proposition 2.7 of [12] which plays an important role for proving Serre conjecture. So this gives rise to the following natural question:

Question 7.2. *Let G be a holomorphic Siegel cusp form of weight (k_1, k_2) so that $k_1 - 1$ and $k_2 - 2$ are both divided by $\ell - 1$, where ℓ is a rational prime. Assume that $\bar{\rho}_{G, \ell}$ is unramified at ℓ . Can one associate a non-holomorphic Siegel cusp form F of weight $(2, 1)$ with a mod ℓ representation such that $F \equiv G \pmod{\ell}$?*

8. SUPPLEMENT TO OUR PAPER [19]

In [19], we used Arthur's conjectural result on the correspondence between cuspidal representations of GSp_4 and GL_4 [2]. It depends on the stabilization of the trace formula, which is not proved yet. In this section, we explain how to get around this by using the transfer from Sp_4 to GL_5 in [3]. The result depends on the twisted fundamental lemma which may have been resolved by now.

Let $\pi = \pi_F$ be the cuspidal representation of $GSp_4(\mathbb{A}_{\mathbb{Q}})$ attached to the Siegel cusp form F of weight $(2, 1)$. We showed in [19] that π_F is not a CAP representation. Let π' be one of components of $\pi|_{Sp_4(\mathbb{A})}$. Then it is a cuspidal representation of $Sp_4(\mathbb{A}_{\mathbb{Q}})$. By [3], π' corresponds to an automorphic representation Π_5 of GL_5 . Since π' is not a CAP representation, Π_5 is either cuspidal or an isobaric representation.

By using the descent construction [13], we can find a globally generic cuspidal representation τ' of $Sp_4(\mathbb{A}_{\mathbb{Q}})$ which is in the same L -packet as π' . Now let τ be a globally generic cuspidal representation of $GSp_4(\mathbb{A}_{\mathbb{Q}})$ such that τ' occurs in the restriction $\tau|_{Sp_4(\mathbb{A})}$. By [4], we have a functorial lift Π of τ as an automorphic representation of GL_4 . This Π is the transfer of π . We can see easily that $\wedge^2(\Pi) = \Pi_5 \otimes \omega_{\pi} \boxplus \omega_{\pi}$, i.e., Π_5 is the transfer of π to GL_5 corresponding to the L -group homomorphism $GSp_4(\mathbb{C}) \rightarrow GL_5(\mathbb{C})$. Hence we do not need the exterior square lift of Π in [18] in order to obtain Π_5 .

9. NON-HOLOMORPHIC SIEGEL CUSP FORMS OF WEIGHT $(2, 1)$ ATTACHED TO CUSP FORMS ON IMAGINARY QUADRATIC FIELDS

In this section, as a supplement to our paper [19], we use the idea of [6] to construct a non-holomorphic Siegel cusp form of weight $(2, 1)$ attached to Maass forms for GL_2/\mathbb{Q} and cuspidal representations of GL_2 over imaginary quadratic fields. This idea was used by Harris, Soudry,

and Taylor [16] to construct holomorphic Siegel cusp forms from certain modular forms over imaginary quadratic fields.

Let $K = \mathbb{Q}[\sqrt{-D}]$ be an imaginary quadratic field. Let $\text{Gal}(K/\mathbb{Q}) = \{1, \theta\}$, and $\omega_{K/\mathbb{Q}}$ be the quadratic character attached to K/\mathbb{Q} i.e., $\omega_{K/\mathbb{Q}}(p) = (\frac{-D}{p})$.

Let $\mathbf{G} = R_{K/\mathbb{Q}}GL_2$ be the quasi-split group obtained by the restriction of scalars. Then ${}^L\mathbf{G} = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \text{Gal}(K/\mathbb{Q})$, and $\mathbf{G}(\mathbb{A}) = GL_2(\mathbb{A}_K)$. Let $\pi = \pi_\infty \otimes \otimes'_p \pi_p$ be a cuspidal representation of $\mathbf{G}(\mathbb{A})$. Here π_∞ is a unitary representation of $GL_2(\mathbb{C})$. If p splits in K into (v_1, v_2) , then $\pi_p = \pi_{v_1} \otimes \pi_{v_2}$. We make the following assumption on π .

Assumption 9.1. ω_π factors through $N_{K/\mathbb{Q}}$, i.e., $\omega_\pi = \omega \circ N_{K/\mathbb{Q}}$ with a grössencharacter ω .

The automorphic induction corresponds to the L -group homomorphism

$$I_K^{\mathbb{Q}} : {}^L\mathbf{G} \longrightarrow GL(\mathbb{C}^2 \oplus \mathbb{C}^2) \simeq GL_4(\mathbb{C}), \quad I_K^{\mathbb{Q}}(g, g'; 1)(x \oplus y) = g(x) \oplus g'(y), \quad I_K^{\mathbb{Q}}(1, 1; \theta)(x \oplus y) = y \oplus x.$$

Let $I_K^{\mathbb{Q}}\pi$ be the automorphic induction. It is automorphic representation of GL_4/\mathbb{Q} , and it is not cuspidal if and only if $\pi \simeq \pi \circ \theta$. In that case, π is a base change of a cuspidal representation π_0 of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and $I_K^{\mathbb{Q}}\pi = \pi_0 \boxplus (\pi_0 \otimes \omega_{K/\mathbb{Q}})$.

The Asai lift corresponds to the L -group homomorphism

$$As : {}^L\mathbf{G} \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2) \simeq GL_4(\mathbb{C}), \quad As(g, g'; 1)(x \otimes y) = g(x) \otimes g'(y), \quad As(1, 1; \theta)(x \otimes y) = y \otimes x.$$

If $\rho : G_K \longrightarrow GL_2(\mathbb{C})$, we have [17]

$$\wedge^2(\text{Ind}_K^{\mathbb{Q}}(\rho)) = (As(\rho) \otimes \omega_{K/\mathbb{Q}}) \oplus \text{Ind}_K^{\mathbb{Q}}(\det \rho).$$

Hence if ρ corresponds to π , $\det(\rho)$ corresponds to ω_π . If π satisfies Assumption 9.1, $I_K^{\mathbb{Q}}\omega_\pi = \omega \oplus \omega\omega_{K/\mathbb{Q}}$. Hence we can $L(s, \wedge^2(I_K^{\mathbb{Q}}) \otimes \chi^{-1})$ with $\chi = \omega$ or $\omega\omega_{K/\mathbb{Q}}$, has a pole at $s = 1$, and $I_K^{\mathbb{Q}}\pi$ descends to a cuspidal representation of $GSp_4(\mathbb{A}_{\mathbb{Q}})$ with the central character χ (cf. [5]).

Let $\pi_\infty = \pi(1, 1)$. Then the Langlands' parameter of π_∞ is

$$\phi : W_{\mathbb{C}} = \mathbb{C}^\times \longrightarrow (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \text{Gal}(K/\mathbb{Q}), \quad \phi(z) = (I, I; \theta).$$

So the Langlands' parameter of $I_K^{\mathbb{Q}}(\pi_\infty)$ is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

$$\text{Here if } P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$P^{-1} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} P = \text{diag}(1, -1, -1, 1), \quad {}^t P \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} P = -\frac{1}{2} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

So $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ is conjugate to $\text{diag}(1, -1, -1, 1)$ in $GS p_4(\mathbb{C})$, and up to conjugacy, we have

$$\phi : W_{\mathbb{R}} \longrightarrow GS p_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, -1, -1, 1).$$

Then ϕ is the Langlands' parameter for $\text{Ind}_B^{GS p_4} 1 \otimes sgn \otimes sgn$, and as in [19], we can show that there exists a Siegel cusp form F of weight $(2, 1)$ corresponding to $I_K^{\mathbb{Q}} \pi$. We have proved

Theorem 9.1. *Let π be a cuspidal representation of $GL_2(\mathbb{A}_K)$, $K = \mathbb{Q}[\sqrt{-D}]$ which satisfies Assumption 9.1, and $\pi_{\infty} = \pi(1, 1)$. Then there exists a non-holomorphic Siegel cusp form F of weight $(2, 1)$ such that $L(s, \pi_F) = L(s, \pi)$.*

10. NON-HOLOMORPHIC SIEGEL CUSP FORMS OF WEIGHT $(2, 1)$ ATTACHED TO MAASS FORMS OVER \mathbb{Q}

Let π be a cuspidal representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ such that $\pi_{\infty} = \pi(1, 1)$, i.e., Maass cusp form. The Langlands parameter of π_{∞} is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_2(\mathbb{C}), \quad \phi(z) = I_2, \quad \phi(j) = I_2.$$

Let $BC(\pi)$ be the base change to $K = \mathbb{Q}[\sqrt{-D}]$, and consider

$$\Pi = I_K^{\mathbb{Q}}(BC(\pi)) = \pi \boxplus (\pi \otimes \omega_{K/\mathbb{Q}}).$$

Then Π descends to a generic cuspidal representation τ of $GS p_4(\mathbb{A}_{\mathbb{Q}})$ (cf. [5]). The Langlands parameter of Π_{∞} is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, 1, -1, -1).$$

Then we can show easily that for $s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, which is the Weyl group element corresponding to the long simple root,

$$s_2^{-1} \text{diag}(1, 1, -1, -1) s_2 = \text{diag}(1, -1, -1, 1).$$

Hence $\text{diag}(1, 1, -1, -1)$ and $\text{diag}(1, -1, -1, 1)$ are conjugate in $GS_{p_4}(\mathbb{C})$, and up to conjugacy, we have

$$\phi : W_{\mathbb{R}} \longrightarrow GS_{p_4}(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, -1, -1, 1).$$

Since ϕ is the Langlands' parameter for $\text{Ind}_B^{GS_{p_4}} 1 \otimes sgn \otimes sgn$, as in [19], we can show that there exists a Siegel cusp form F of weight $(2, 1)$ corresponding to $I_K^{\mathbb{Q}}(BC(\pi))$. We have proved

Theorem 10.1. *Let π be a cuspidal representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ such that $\pi_{\infty} = \pi(1, 1)$. Then there exists a non-holomorphic Siegel cusp form F of weight $(2, 1)$ such that $L(s, \pi_F) = L(s, \pi)L(s, \pi \otimes \omega_{K/\mathbb{Q}})$.*

11. APPENDIX

Let $\pi = \otimes'_p \pi_p$ be a cuspidal representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with conductor N . We show that the Langlands functoriality of the exterior m -th power $\wedge^m(\pi)$ as an automorphic representation of $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$ implies Assumption (5).

First, recall the following fact on the local L -factors of the ramified places: Let $\Pi = \otimes'_p \Pi_p$ be a cuspidal representation of $GL_M(\mathbb{A}_{\mathbb{Q}})$. Then for each prime p , Π_p is unitary and generic. Recall that a non-tempered, unitary and generic representation of $GL_N(\mathbb{Q}_p)$ can be written as a full induced representation

$$(11.1) \quad \text{Ind } \eta_1 | \det |^{r_1} \otimes \cdots \otimes \eta_k | \det |^{r_k} \otimes \eta_0 \otimes \eta_k | \det |^{-r_k} \otimes \cdots \otimes \eta_1 | \det |^{-r_1},$$

where η_1, \dots, η_k are unitary square-integrable representations of $GL_{n_1}(\mathbb{Q}_p), \dots, GL_{n_k}(\mathbb{Q}_p)$, resp. and η_0 is a tempered representation of $GL_{n_0}(\mathbb{Q}_p)$ such that $n_0 + n_1 + \cdots + n_k = M$, and $0 <$

$r_k \leq \cdots \leq r_1 \leq \frac{1}{2} - \frac{1}{M^2+1}$ ([38]). (See [24] for the bound.) Then by [31],

$$(11.2) \quad \begin{aligned} L(s, \Pi_p) &= L(s, \eta_0) \prod_{i=1}^k L(s \pm r_i, \eta_i), \\ L(s, \Pi_p \times \tilde{\Pi}_p) &= \prod_{i,j=1}^k L(s \pm r_i \pm r_j, \eta_i \times \tilde{\eta}_j) \prod_{i=1}^k L(s \pm r_i, \eta_i \times \tilde{\eta}_0) L(s \pm r_i, \eta_0 \times \tilde{\eta}_i). \end{aligned}$$

Note that for any i, j , $L(s, \eta_i)$ and $L(s, \eta_i \times \tilde{\eta}_j)$ are holomorphic for $\operatorname{Re}(s) > 0$.

For $1 \leq m \leq n$, let $\wedge^m : GL_n(\mathbb{C}) \rightarrow GL_{C_{n,m}}(\mathbb{C})$ be the exterior m -th power, and let

$$L(s, \pi, \wedge^m) = \sum_{n=1}^{\infty} \frac{\lambda_m(n)}{n^s}$$

be the exterior m -th power L -function. Then it is easy to see that, for a prime $p \nmid N$,

$$\lambda_m(p) = a_m(p), \quad \text{for all } m = 1, \dots, n.$$

For each p , by the local Langlands' correspondence, $\wedge^m(\pi_p)$ is a well-defined representation of $GL_{C_{n,m}}(\mathbb{Q}_p)$: Let $\phi_p : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$ be the parametrization of π_p . Then we have a map $\wedge^m(\phi_p) : W_{\mathbb{Q}_p} \times SL_2(\mathbb{C}) \rightarrow GL_{C_{n,m}}(\mathbb{C})$. Then $\wedge^m(\pi_p)$ is the representation of $GL_{C_{n,m}}(\mathbb{Q}_p)$, corresponding to $\wedge^m(\phi_p)$. Let $\wedge^m(\pi) = \otimes_p \wedge^m(\pi_p)$. It is an irreducible admissible representation of $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$.

Conjecture 11.1. (*Langlands functoriality conjecture*) *The exterior m -th power $\Pi_m = \wedge^m(\pi)$ is an automorphic representation of $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$.*

Since $\wedge^m(\pi) = \wedge^{n-m} \tilde{\pi} \otimes \omega$, it is enough to consider it for $m \leq \lfloor \frac{n}{2} \rfloor$. If $n \leq 3$, it is trivial. When $n = 4$, it is proved in [18].

Suppose Π_m is an automorphic representation of $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$. Consider the Rankin-Selberg L -function $L(s, \Pi_m \times \tilde{\Pi}_m)$.

Proposition 11.2. *There exists a holomorphic function $g(s)$ near $s = 1$ such that*

$$\log L(s, \Pi_m \times \tilde{\Pi}_m) = \sum_{p \nmid N} \frac{|a_m(p)|^2}{p^s} + g(s),$$

and if $s > 1$,

$$\sum_{p \nmid N} \frac{|a_m(p)|^2}{p^s} \leq C_{n,m}^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Proof. Let $r_m = C_{n,m}$, and $L(s, \Pi_m) = \prod_{p \nmid N} \prod_{i=1}^{r_m} (1 - \beta_i(p)p^{-s})^{-1} \prod_{p \mid N} L_p(s)$, where $L_p(s)$ is a ramified factor as in (11.2). Then $\log L(s, \Pi_m) = \sum_{l=1}^{\infty} \sum_{p \nmid N} \frac{b(p^l)}{lp^{ls}} + g'(s)$, where for $p \nmid N$, $b(p^l) = \beta_1(p)^l + \cdots + \beta_{r_m}(p)^l$. Hence $b(p) = a_m(p)$ for $p \nmid N$. Here $g'(s)$ is a holomorphic function for $\operatorname{Re}(s) > \frac{1}{2}$ by (11.2) and the fact that the local factors are non-vanishing. Then we can easily see that

$$\log L(s, \Pi_m \times \tilde{\Pi}_m) = \sum_{l=1}^{\infty} \sum_{p \nmid N} \frac{|b(p^l)|^2}{lp^{ls}} + g''(s) = \sum_{p \nmid N} \frac{|a_m(p)|^2}{p^s} + \sum_{l=2}^{\infty} \sum_{p \nmid N} \frac{|b(p^l)|^2}{lp^{ls}} + g''(s),$$

where $g''(s)$ is a holomorphic function for $\operatorname{Re}(s) > 1 - \frac{2}{r_m^2+1}$ by (11.2).

Now we show that $\sum_{l=2}^{\infty} \sum_{p \nmid N} \frac{|b(p^l)|^2}{lp^{ls}}$ converges for $\operatorname{Re}(s) > 1 - \frac{2}{r_{[\frac{n}{2}]^2+1}}$. By the classification of spherical unitary generic representation of $GL_n(\mathbb{Q}_p)$ [37], if $p \nmid N$, $\alpha_1(p), \dots, \alpha_n(p)$ is of the form

$$u_1 p^{a_1}, u_2 p^{a_2}, \dots, u_{k_p} p^{a_{k_p}}, u_{k_p+1}, \dots, u_{k_p+k'_p}, u_{k_p} p^{-a_{k_p}}, \dots, u_1 p^{-a_1},$$

where $u_i \in \mathbb{C}$ and $|u_i| = 1$ for all i , and $0 < a_{k_p} \leq \cdots \leq a_1 \leq \frac{1}{2} - \frac{1}{n^2+1}$. Now let S_0 be the set of primes where π_p is tempered, and for $0 < k \leq [\frac{n}{2}]$, let S_k be the set of primes where $k_p = k$. Then $\sum_{l=2}^{\infty} \sum_{p \in S_0} \frac{|b(p^l)|^2}{lp^{ls}}$ converges for $\operatorname{Re}(s) > \frac{1}{2}$.

Now for each $0 < k \leq [\frac{n}{2}]$, consider $\sum_{l=2}^{\infty} \sum_{p \in S_k} \frac{|b(p^l)|^2}{lp^{ls}}$. Recall that $a_m(p) = \sum_{i_1 < \cdots < i_m} \alpha_{i_1}(p) \cdots \alpha_{i_m}(p)$.

Hence $|b(p^l)| \ll p^{l(a_1 + \cdots + a_m)}$, where we let $a_i = 0$ if $i > k$. Also we have $|a_k(p)| \gg p^{a_1 + \cdots + a_k}$. Therefore,

$$|b(p^l)| \ll |a_k(p)|^l.$$

By assumption, $\wedge^k \pi$ is an automorphic representation of GL_{r_k}/\mathbb{Q} . Hence by [24], $|a_k(p)| \leq r_k p^{\frac{1}{2} - \frac{1}{r_k^2+1}}$. Hence

$$\sum_{p \in S_k} \frac{|b(p^l)|^2}{lp^{ls}} \ll \sum_{p \in S_k} \frac{|a_k(p)|^2}{p^{l \operatorname{Re}(s) - l + 1 + \frac{2(l-1)}{r_k^2+1}}}.$$

Since $\wedge^k \pi$ is automorphic, by considering the Rankin-Selberg L -function $L(s, \wedge^k \pi \times \widetilde{\wedge^k \pi})$, we have $\sum_p |a_k(p)|^2 \ll x$. Hence by partial summation,

$$\sum_{p \in S_k} \frac{|a_k(p)|^2}{p^{l \operatorname{Re}(s) - l + 1 + \frac{2(l-1)}{r_k^2+1}}} \ll 2^{-l \operatorname{Re}(s) + l - \frac{2(l-1)}{r_k^2+1}}.$$

Now $\sum_{l=2}^{\infty} 2^{-l\operatorname{Re}(s)+l-\frac{2(l-1)}{r_k^2+1}}$ converges for $\operatorname{Re}(s) > 1 - \frac{2}{r_k^2+1}$. Hence our result follows.

Now $L(s, \Pi_m \times \tilde{\Pi}_m)$ has a pole at $s = 1$, of order at least 1, and at most r_m^2 . Also $L(s, \Pi_m \times \tilde{\Pi}_m)$ is zero free for $\operatorname{Re}(s) > 1 - \frac{c}{(1+|t|)^n}$ for some constant $c > 0$ (for example, [7]). Hence as $s \rightarrow 1^+$,

$$\log L(s, \Pi_m \times \tilde{\Pi}_m) \leq r_m^2 \log \frac{1}{s-1} + O(1).$$

This completes the proof. \square

The above proof shows the following. (A different proof was given in [9].)

Corollary 11.3. *Let $\log L(s, \pi) = \sum_{l=1}^{\infty} \sum_{p \nmid N} \frac{a(p^l)}{lp^{ls}} + g(s)$, where $a(p^l) = \alpha_1(p)^l + \cdots + \alpha_n(p)^l$ for $p \nmid N$, and $g(s)$ is a holomorphic function for $\operatorname{Re}(s) > \frac{1}{2}$. Then the functoriality of $\wedge^m \pi$ for all m implies Hypothesis H in [31], namely, for each $l \geq 2$,*

$$\sum_{p \nmid N} \frac{|a(p^l)|^2 (\log p)^2}{p^l}$$

converges.

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