String geometry vs. spin geometry on loop spaces

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Abstract

We introduce various versions of spin structures on free loop spaces of smooth manifolds, based on a classical notion due to Killingback, and additionally coupled to two relations between loops: thin homotopies and loop fusion. The central result of this article is an equivalence between these enhanced versions of spin structures on the loop space and string structures on the manifold itself. The equivalence exists in two settings: in a purely topological one and a in geometrical one that includes spin connections and string connections. Our results provide a consistent, functorial, one-to-one dictionary between string geometry and spin geometry on loop spaces.

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1 Introduction

One perspective to classical two-dimensional field theories on a Riemannian manifold M, also known as sigma models, is to regard them as one-dimensional field theories on the free loop space LM: the points of LM are the "closed strings" in M. For example, if we want to understand the coupling of strings to gauge fields, this perspective makes us study principal bundles with connections over LM. And if we want to understand fermions, it lets us ask for spin structures on loop spaces.

In order to study fermions on an oriented, n-dimensional Riemannian manifold, one has to lift the structure group of the frame bundle of M from $\mathrm{SO}(n)$ to a covering group that admits appropriate unitary representations, $\mathrm{Spin}(n)$. Analogous steps on the loop space require, in the first place, to choose an orientation: a reduction of the structure group of LM, namely $L\mathrm{SO}(n)$, to the connected component of the identity, $L\mathrm{Spin}(n)$, see [Ati85, McL92]. Such a reduction can, for instance, be induced from a spin structure on M. In the next step, one observes that $L\mathrm{Spin}(n)$ has no appropriate unitary representations. It only has projective ones, i.e. representations of its universal central extension,

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widetilde{L\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1. \tag{1.1}$$

Thus, we require a lift of the structure group of LM from $L\mathrm{Spin}(n)$ to this central extension; such a lift is called a *spin structure* on LM [Kil87]. We remark that an important difference to ordinary spin structures is that the central subgroup of the extension (1.1) is the *continuous* group $\mathrm{U}(1)$ instead of the *discrete* group $\mathbb{Z}/2\mathbb{Z}$. One effect of this difference is that it is non-trivial to lift a given connection on the frame bundle of LM to a connection on the lifted bundle, a *spin connection* [CP98]. Every spin structure on LM admits a spin connection [Man02], but there might be non-equivalent choices.

Deficits of the loop space theory

Returning to the attempt to understand the coupling of strings to gauge fields via, say, principal $\mathrm{U}(1)$-bundles with connection over loop spaces, one soon encounters the problem that not all aspects of the two-dimensional theory can be described in terms of such bundles. For example, if two strings join in form of a pair of pants, there is no sensible way to describe the gauge field coupling of this process solely in terms of a bundle over LM. This deficit of the loop space theory has lead to the development of *B-fields*, structure defined on the manifold itself that fulfills all requirements for a gauge field for strings. Nowadays it is well understood that a B-field is a $\mathrm{U}(1)$*-gerbe with connection* [Gaw88, Bry93]. The relation between gerbes over M and bundles over LM can be understood on a cohomological level in terms of a transgression homomorphism

$$\tau : \mathrm{H}^n(M, \mathbb{Z}) \longrightarrow \mathrm{H}^{n-1}(LM, \mathbb{Z}), \tag{1.2}$$

which for $n = 3$ takes the Dixmier-Douady class of a gerbe over M to the first Chern class of a principal $\mathrm{U}(1)$-bundle over LM. Various differential-geometric versions of the transgression homomorphism have been developed that also include connections on both sides [Bry93, GR02, Wal10]. A general fact is that transgression is not injective; this loss of information explains precisely the above-mentioned deficit of the loop space theory for gauge fields.

A similar phenomenon has been observed for *geometric spin structures* on loop spaces, i.e. spin structures with spin connections. For the consistency of the fermionic theory (a version of the supersymmetric sigma model) it is necessary to trivialize a certain *Pfaffian line bundle* over the mapping spaces of closed spin surfaces into M [Fre87, FM06]. A spin structure on the loop space only provides such trivializations over the mapping space of *genus one* surfaces. In order to remedy this deficit (among other issues) Stolz and Teichner have proposed a notion of a *geometric string structure* on M, consisting of a *string structure* and a *string connection* [ST04]. Spin manifolds that admit such structures are called *string manifolds*; they are characterized by the vanishing of the first fractional Pontryagin class $\frac{1}{2}p_1(M)$. The proof that a geometric string structure indeed provides trivializations of the Pfaffian line bundle over mapping spaces of *arbitrary* surfaces was provided later by Bunke [Bun11] based on a gerbe-theoretical formulation of geometric string structures introduced in [Wal13]. Additionally, that formulation allows to define a transgression procedure for geometric string structures on M, analogous to the homomorphism (1.2), that results in spin structures on LM [Wala]. This transgression procedure is again afflicted with a loss of information [PW88], explaining the limitation of the loop space theory to genus one surfaces.

We remark that several other aspects are not yet understood, neither in terms of spin geometry on LM nor in terms of string geometry on M. Examples are the Dirac operator on LM postulated by Witten [Wit86], or the Höhn-Stolz conjecture [Sto96]. The quest for methods to attack problems like these is the motivation for studying relations between geometry on M and geometry on LM. The purpose of the present article is to contribute a new instance of such relations: an equivalence between (geometric) string structures on M and a version of (geometric) spin structures on LM.

Thin homotopy and loop fusion

We return to the above-mentioned transgression of gerbes (with connection) over M to principal U(1)-bundles (with connection) over LM, suffering from a loss of information. It turns out that one can equip U(1)-bundles over loop spaces with additional structures, in such a way that an inverse of transgression can be defined, and an equivalence between gerbes over M and versions of U(1)-bundles over LM is achieved; see [Wal12b, Walb, Wal12c] or [KMa, KMb] for an alternative approach. The relevant additional structures couple U(1)-bundles over LM to two operations that only exist in loop spaces (rather than in general manifolds): thin homotopies and loop fusion. Roughly speaking, *thin*

(a) A homotopy between loops, regarded as a tube.

(b) Two loops with a common segment.

homotopies are homotopies between loops that have "zero area" when regarded as tubes in M as shown in Figure (a). The relevance of thin homotopies has been noticed in axiomatic approaches to the parallel transport of connections on bundles and gerbes [Bar91, SW09]. A *thin structure* on a principal U(1)-bundle P over LM is a way to identify consistently the fibres of P over thin homotopic loops. A connection on P is called *superficial*, if such a thin structure can be induced by parallel transport along a thin homotopy (independently of the choice of the thin homotopy).

The second operation, *loop fusion*, joins two loops along a common segment, see Figure (b). A *fusion product* on a principal U(1)-bundle P over LM is a structure that lifts loop fusion to the fibres of P. These additional structures furnish a category $\mathcal{F}us\mathcal{B}un^{th}(LM)$ of principal U(1)-bundles over LM equipped with fusion products and thin structures, and another category $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM)$ of principal U(1)-bundles over LM equipped with fusion products and superficial connections. These two categories are "loop space duals" of the bicategories $\mathcal{G}rb(M)$ of gerbes over M and $\mathcal{G}rb^{\nabla}(M)$ of gerbes with connection over M, respectively. These dualities can be expressed in terms of a commutative diagram

$$\begin{array}{ccc} \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM) & \longrightarrow & \mathrm{h}_1\mathcal{G}rb^{\nabla}(M) \\ \downarrow & & \downarrow \\ h\mathcal{F}us\mathcal{B}un^{th}(LM) & \longrightarrow & \mathrm{h}_1\mathcal{G}rb(M) \end{array} \tag{1.3}$$

of monoidal categories and functors, in which the horizontal arrows are equivalences and the vertical arrows describe the passage from the setting "with connections" to the one "without connections". The symbol h_1 stands for the truncation of a bicategory down to a category and the symbol h stands for the homotopy category (where bundle morphisms become identified if they are homotopic). The horizontal functors in the diagram are called *regression* as they are inverse to transgression; we refer to [Wal12c] for a more detailed exposition.

The equivalence on top of the diagram explains how the deficit of the loop space theory of gauge fields for strings has to be compensated, namely by the addition of a fusion product and the requirement that the connection be superficial. Indeed, a fusion product provides exactly the structure needed in order to account for the joining of two strings in form of a pair of pants, see the discussion in [Walb, Section 5.3].

Results of the present article

In the present article we discuss an equivalence between the string geometry on M and versions of spin geometry on the loop space LM. The first part of this article is concerned with determining how exactly these versions have to be defined, and the second part is concerned with the proof that they serve their purpose and yield the claimed equivalence.

We introduce two versions of spin structures on loop spaces: a category $\mathcal{S}pin^{th}_{fus}(LM)$ of *thin fusion spin structures* (Definition 3.1.5) and another category $\mathcal{S}pin^{\nabla_{sf}}_{fus}(LM)$ of *superficial geometric fusion spin structures* (Definition 4.1.9). As the terminology suggests, our strategy is to equip Killingback's original spin structures with structures that have already proved themselves: fusion products, thin structures, and superficial connections. The main issue is to connect these structures correctly to action of the central extension $\widetilde{L\mathrm{Spin}(n)}$ on the spin structure. Therefore, we start the first part of the present article by revisiting loop group geometry through transgression of multiplicative gerbes, bringing fusion products, thin structures, and superficial connections in context with central extensions of loop groups.

The categories $\mathcal{S}pin^{th}_{fus}(LM)$ and $\mathcal{S}pin^{\nabla_{sf}}_{fus}(LM)$ are related, respectively, to the bicategories $\mathcal{S}tring(M)$ and $\mathcal{S}tring^{\nabla}(M)$ of string structures and geometric string structures introduced in [Wal13] as bicategory of trivializations of the Chern-Simons 2-gerbe. The relation is established by regression functors that are inverse to the above-mentioned transgression procedure for geometric string structures. The main result of this article is the following.

Theorem A. *Let M be a connected spin manifold of dimension $n = 3$ or $n > 4$. There is a commutative diagram of categories and functors,*

$$\begin{array}{ccc} \mathcal{S}pin^{\nabla_{sf}}_{fus}(LM) & \longrightarrow & \mathrm{h}_1\mathcal{S}tring^{\nabla}(M) \\ \big\downarrow & & \big\downarrow \\ h\mathcal{S}pin^{th}_{fus}(LM) & \longrightarrow & \mathrm{h}_1\mathcal{S}tring(M). \end{array}$$

If M is string, all categories in the diagram are non-empty, and the following results hold:

(i) The horizontal functors are equivalences of categories, and the vertical functors are essentially surjective.

(ii) The diagram is a torsor over the diagram (1.3) in the sense that each category is a torsor over the monoidal category in the corresponding corner of (1.3), and each functor is equivariant along the corresponding functor in (1.3).

If M is not string, then all four categories in the diagram are empty.

Here, a category is a torsor over a monoidal category if it is a module for that monoidal category and the action is free and transitive in a sense explained later.

We spell out explicitly what Theorem A implies upon passing to isomorphism classes of objects, an operation that we denote by the symbol h_0. The set $\mathrm{h}_0\mathcal{S}tring(M)$ can be identified with a set $\mathcal{S}tr\mathcal{C}l(M)$ of *string classes* [Red06, Wal13]; these can easily be described as cohomology classes $\xi \in \mathrm{H}^3(FM, \mathbb{Z})$ on the total space of the spin-oriented frame bundle FM of M that restrict over each fibre to a generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$. We introduce an analogous description of *geometric* string structures in terms of *differential* cohomology, which we call *differential string classes* (Definition 6.3.1). A differential string class is a differential cohomology class $\hat{\xi} \in \hat{\mathrm{H}}^3(FM)$, subject to a condition in the differential cohomology of $FM \times \mathrm{Spin}(n)$ that involves a certain 2-form known from classical Chern-Simons theory. We prove that the set $\mathcal{S}tr\mathcal{C}l^{\nabla}(M)$ of differential string classes can be identified with the set $\mathrm{h}_0\mathcal{S}tring^{\nabla}(M)$ of isomorphism classes of geometric string structures (Theorem 6.3.3). Under these identifications, Theorem A implies the following statement.

Corollary B. *Let M be a connected string manifold of dimension $n = 3$ or $n > 4$. There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{h}_0\mathcal{S}pin^{\nabla_{sf}}_{fus}(LM) & \longrightarrow & \mathcal{S}tr\mathcal{C}l^{\nabla}(M) \\ \big\downarrow & & \big\downarrow \\ \mathrm{h}_0\mathcal{S}pin^{th}_{fus}(LM) & \longrightarrow & \mathcal{S}tr\mathcal{C}l(M). \end{array}$$

The map in the first row is an equivariant bijection between torsors over the differential cohomology group $\hat{\mathrm{H}}^3(M)$, and the map in the second row is an equivariant bijection between torsors over the ordinary cohomology group $\mathrm{H}^3(M, \mathbb{Z})$. Moreover, the vertical maps are surjective and equivariant along the projection $\hat{\mathrm{H}}^3(M) \longrightarrow \mathrm{H}^3(M, \mathbb{Z})$. In particular, the fibres of the vertical maps are torsors over the group $\Omega^2(M)/\Omega^2_{cl,\mathbb{Z}}(M)$ of 2-forms modulo closed 2-forms with integral periods.

The last statement follows because the group $\Omega^2(M)/\Omega^2_{cl,\mathbb{Z}}(M)$ is precisely the kernel of the projection $\hat{\mathrm{H}}^3(M) \longrightarrow \mathrm{H}^3(M)$ from differential to ordinary cohomology.

Summarizing, either in the categorical or in the set-theoretical setting, we provide a consistent dictionary between string geometry and spin geometry on loop spaces. We remark that in a first approximation of such an equivalence, Witten proposed to impose that spin structures be equivariant under the rotation action of U(1) on LM [Wit86]. In a previous article [Wala] I have considered a version of spin structures with fusion products (but without thin structures), and proved that such spin structures exist if and only if M is a string manifold. Recently, Kottke and Melrose introduced a version of spin structures that combines fusion products and equivariance under a group of reparameterizations of S^1 (including rotations) [KMa]. This version achieved, on the level of equivalence classes, a bijection with the set of string classes. The results of the present article improve that bijection in two aspects: we upgrade it to an equivalence of categories and amend it by a second equivalence in the setting "with connections".

Method of proof and organization of the paper

For the proof of Theorem A we will collect various partial results throughout this article; in the final Section 8 we summarize these and show that the theorem is fully proved. The main tool in the proof is *lifting gerbe theory* over the loop space, which allows us to split the work into two parts. The first part (Sections 2, 3, 4) is to reformulate spin structures and all additional structures in terms of trivializations of the *spin lifting gerbe* over LM (Proposition 3.2.6 and Corollary 4.2.12). This reformulation is based on work of Murray [Mur96], Gomi [Gom03], and previous work [Wala]. A crucial new aspect we encounter here is that the standard theory for connections on lifting gerbes must be refined in a certain way in order to take thin homotopies into account (Proposition 4.2.4).

The second part (Sections 5, 6, 7) is concerned with the problem to tailor the bicategories $\mathcal{S}tring^{\nabla}(M)$ and $\mathcal{S}tring(M)$ of (geometric) string structures into a form that allows a direct application of the duality between gerbes and bundles over loop spaces. The resulting loop space structure can then be identified with exactly those trivializations of the spin lifting gerbe that we identified in the first part as reformulations of the categories $\mathcal{S}pin_{fus}^{th}(LM)$ and $\mathcal{S}pin_{fus}^{\nabla_{sf}}(LM)$, see Theorems 7.3 and 7.4. The tailoring of the bicategories involves a general decategorification procedure for trivializations of bundle 2-gerbes. A key result that we prove is that in case of the Chern-Simons 2-gerbe, this decategorification procedure is an equivalence of categories (Theorems 5.3.1 and 6.2.2).

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2 Loop group geometry via multiplicative gerbes

In this section we explore the geometry of central extensions of the loop group LG of a Lie group G via multiplicative bundle gerbes over G. The goal is to construct models for central extensions with specific additional structures: superficial connections, thin structures, and fusion products. The results of this section will be applied in the sequel to $G = \mathrm{Spin}(n)$.

2.1 Transgression and central extensions

We use the theory of bundle gerbes (with structure group U(1)) and connections on those. Introductions can be found in [Mur96, CJM02, Mur10, Wal07]. We denote by $\mathcal{G}rb(X)$ and $\mathcal{G}rb^{\nabla}(X)$ the bicategories of bundle gerbes and bundle gerbes with connection over a smooth manifold X, respectively. The 1-morphisms are called (connection-preserving) *isomorphisms*, and the 2-morphisms are called (connection-preserving) *transformations*. The operation of "forgetting the connection" is a surjective, but neither full nor faithful 2-functor

$$\mathcal{G}rb^{\nabla}(X) \longrightarrow \mathcal{G}rb(X). \tag{2.1.1}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\langle -,-\rangle$ be a symmetric invariant bilinear form on $\mathfrak{g}$. There is a canonical, left-invariant closed 3-form $H \in \Omega^3(G)$ whose value at the identity is given by $H_1(X,Y,Z) = \langle X,[Y,Z]\rangle$. In terms of the left-invariant Maurer-Cartan form θ on G it is given by

$$H = \frac{1}{6}\,\langle \theta \wedge [\theta \wedge \theta]\rangle\,. \tag{2.1.2}$$

We fix a bundle gerbe $\mathcal{G}$ over G with connection of curvature H. Such a bundle gerbe exists if and only if H has integral periods, in which case H represents the Dixmier-Douady class $\mathrm{DD}(\mathcal{G}) \in \mathrm{H}^3(G,\mathbb{Z})$ in real cohomology. Different choices of possible bundle gerbes with connection (up to connection-preserving isomorphisms) are parameterized by $\mathrm{H}^2(G,\mathrm{U}(1))$.

Example 2.1.1. Suppose G is compact, simple and simply-connected, for example, $G = \mathrm{Spin}(n)$ for $n = 3$ or $n > 4$. Then, $\langle -,-\rangle$ is a multiple of the Killing form, and it can be normalized such that H has integral periods and represents a generator $\gamma \in \mathrm{H}^3(G,\mathbb{Z}) \cong \mathbb{Z}$. We have $\mathrm{H}^2(G,\mathrm{U}(1)) = 0$. Thus, there exists a (up to connection-preserving isomorphisms) unique bundle gerbe $\mathcal{G}$ with connection of curvature H. Its Dixmier-Douady class is the generator γ. This bundle gerbe $\mathcal{G}$ is called the *basic gerbe* over G, and it will be denoted by $\mathcal{G}_{bas}$. There exist Lie-theoretical models for $\mathcal{G}_{bas}$ [GR02, Mei02].

The double group G^2 carries a canonical 2-form

$$\rho := \left\langle \mathrm{pr}_1^*\theta \wedge \mathrm{pr}_2^*\bar\theta \right\rangle \in \Omega^2(G^2), \tag{2.1.3}$$

with θ, $\bar\theta$ the left- and right-invariant Maurer-Cartan forms on G, respectively. Let $m, \mathrm{pr}_1, \mathrm{pr}_2 \colon G^2 \longrightarrow G$ denote the multiplication and the two projections, respectively. Then

$$\mathrm{pr}_1^*H + \mathrm{pr}_2^*H = m^*H + \mathrm{d}\rho \tag{2.1.4}$$

as 3-forms on G^2. We have four maps $\mathrm{pr}_{12}, \mathrm{pr}_{23}, m_{23}, m_{12} : G^3 \longrightarrow G^2$, where pr_{12} and pr_{23} project to the indexed components, and m_{23} and m_{12} multiply the indexed components. Then,

$$\mathrm{pr}_{23}^*\rho + m_{23}^*\rho = \mathrm{pr}_{12}^*\rho + m_{12}^*\rho. \tag{2.1.5}$$

We may re-interpret Equations (2.1.4) and (2.1.5) by considering the simplicial manifold BG, see [Wal10]. Denoting by $\Delta : \Omega^*(G^{q-1}) \longrightarrow \Omega^*(G^q)$ the alternating sum over the pullbacks along the face maps, Equations (2.1.4) and (2.1.5) become

$$\Delta H = \mathrm{d}\rho \quad \text{ and } \quad \Delta\rho = 0. \tag{2.1.6}$$

A bundle gerbe $\mathcal{G}$ with connection of curvature H can be seen as a lift from a differential form setting to a cohomological setting. A corresponding lift of equations (2.1.4) and (2.1.5) is called a *multiplicative structure* on $\mathcal{G}$. We recall that 2-forms are connections on trivial gerbes; thus, we have a bundle gerbe $\mathcal{I}_\rho$ over G^2 – it has vanishing Dixmier-Douady class and curvature $\mathrm{d}\rho$. A multiplicative structure on $\mathcal{G}$ consists of a connection-preserving isomorphism

$$\mathcal{M} : \mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G} \longrightarrow m^*\mathcal{G} \otimes \mathcal{I}_\rho$$

of bundle gerbes over G^2, and of a connection-preserving transformation

$$\begin{array}{ccc} \mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G} \otimes \mathrm{pr}_3^*\mathcal{G} & \xrightarrow{\mathrm{pr}_{12}^*\mathcal{M}\otimes\mathrm{id}} & \mathrm{pr}_{12}^*\mathcal{G} \otimes \mathrm{pr}_3^*\mathcal{G} \otimes \mathcal{I}_{\mathrm{pr}_{12}^*\rho} \\ {\scriptstyle \mathrm{id}\otimes\mathrm{pr}_{23}^*\mathcal{M}}\big\downarrow & \swarrow\!\!\!\!\swarrow \; \alpha & \big\downarrow{\scriptstyle m_{12}^*\mathcal{M}\otimes\mathrm{id}} \\ \mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_{23}^*\mathcal{G} \otimes \mathcal{I}_{\mathrm{pr}_{23}^*\rho} & \xrightarrow[m_{23}^*\mathcal{M}\otimes\mathrm{id}]{} & \mathcal{G}_{123} \otimes \mathcal{I}_{\rho_\Delta} \end{array}$$

between isomorphisms over G^3, where ρ_Δ is either side of (2.1.5). The transformation α has to satisfy a pentagon axiom over G^4. Multiplicative bundles gerbes (without connections) have been introduced in [CJM$^+$05], the theory of connections is developed in [Wal10].

The quadruple $(H, \rho, 0, 0)$ is a degree 4 chain in the de Rham complex of the simplicial manifold BG. Closedness of H together with Equations (2.1.6) show that it is a cocycle, and thus represents an element in $\mathrm{H}^4(BG, \mathbb{R})$. Multiplicative bundle gerbes with connection relative to the differential forms H and ρ exist if and only if that class is integral. Different choices are parameterized by $\mathrm{H}^3(BG, \mathrm{U}(1))$, see [Wal10, Proposition 2.4].

Example 2.1.2. If G is compact and simple, then $\mathrm{H}^3(BG, \mathrm{U}(1)) = 0$, so that multiplicative gerbes with connection are (up to connection-preserving isomorphisms compatible with the multiplicative structure) uniquely determined by H and ρ, hence by $\langle -, - \rangle$. If G is in addition simply connected, every bundle gerbe with connection of curvature H admits a multiplicative structure relative to the 2-form ρ [Wal10, Example 1.5]. In particular, the basic gerbe $\mathcal{G}_{bas}$ over a compact, simple and simply-connected Lie group has a unique multiplicative structure. Based on explicit models for the basic gerbe over such groups, it is possible to construct this unique multiplicative structure [Wal12a].

In the following we continue with a fixed multiplicative bundle gerbe $\mathcal{G}$ with connection over a general Lie group G, relative to the differential forms H and ρ of Equations (2.1.2) and (2.1.3).

For every smooth manifold X, there is a transgression functor

$$\mathrm{h}_1\mathcal{G}rb^{\nabla}(X) \longrightarrow \mathcal{B}un(LX) : \mathcal{G} \longmapsto \mathscr{T}_{\mathcal{G}} \tag{2.1.7}$$

with target the category of Fréchet principal U(1)-bundles over the free loop space $LX = C^\infty(S^1, X)$. The category $\mathrm{h}_1\mathcal{G}rb^{\nabla}(X)$ is obtained from the bicategory $\mathcal{G}rb^{\nabla}(X)$ by identifying 2-isomorphic isomorphisms.

Transgression for gerbes has first been defined by Gawędzki in terms of cocycles for Deligne cohomology [Gaw88], and by Gawędzki-Reis for bundle gerbes [GR02]. Brylinski has defined transgression in terms of Dixmier-Douady sheaves of categories [Bry93]. The functor (2.1.7) that we use here is defined in [Wal10]. It is monoidal with respect to the tensor product of bundle gerbes and principal U(1)-bundles, it is natural with respect to smooth maps $f\colon X \longrightarrow X'$ between smooth manifolds and the induced maps $Lf\colon LX \longrightarrow LX'$ between their loop spaces, and it sends trivial bundle gerbes $\mathcal{I}_\rho$ to canonically trivializable bundles. Furthermore, it satisfies

$$\mathrm{c}_1(\mathscr{T}_{\mathcal{G}}) = -\tau(\mathrm{DD}(\mathcal{G})) \tag{2.1.8}$$

for all bundle gerbes $\mathcal{G}$ with connection over X, where c_1 denotes the first Chern class of a principal U(1)-bundle, and τ is the transgression homomorphism (1.2), see [Wal10].

Applying the transgression functor to the bundle gerbe $\mathcal{G}$ over G, we obtain a Fréchet principal U(1)-bundle $\mathscr{T}_{\mathcal{G}}$ over the loop group LG. Because transgression is functorial and monoidal, the multiplicative structure $\mathcal{M}$ on $\mathcal{G}$ transgresses to a bundle isomorphism

$$\mathrm{pr}_1^*\mathscr{T}_{\mathcal{G}} \otimes \mathrm{pr}_2^*\mathscr{T}_{\mathcal{G}} \xrightarrow{\mathscr{T}_{\mathcal{M}}} m^*\mathscr{T}_{\mathcal{G}} \otimes \mathscr{T}_{\mathcal{I}_\rho} \cong m^*\mathscr{T}_{\mathcal{G}}$$

over $LG \times LG$, inducing a binary operation on the total space $\mathscr{T}_{\mathcal{G}}$ that covers the group structure of LG. The mere existence of the associator α for the multiplicative structure $\mathcal{M}$ implies the commutativity of a diagram in the category $\mathrm{h}_1\mathcal{G}rb^\nabla(G^3)$, which implies under transgression the associativity of the binary operation $\mathscr{T}_{\mathcal{M}}$.

Theorem 2.1.3 ([Wal10, Theorem 3.1.7])**.** *The associative binary operation $\mathscr{T}_{\mathcal{M}}$ equips $\mathscr{T}_{\mathcal{G}}$ with the structure of a Fréchet Lie group, making up a central extension*

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathscr{T}_{\mathcal{G}} \longrightarrow LG \longrightarrow 1.$$

Example 2.1.4. Consider again a compact, simple and simply-connected Lie group G, equipped with the basic gerbe $\mathcal{G}_{bas}$ and its unique multiplicative structure. We get from Equation (2.1.8)

$$\mathrm{c}_1(\mathscr{T}_{\mathcal{G}_{bas}}) = -\tau(\mathrm{DD}(\mathcal{G}_{bas})) = -\tau(\gamma).$$

This means that $\mathscr{T}_{\mathcal{G}_{bas}}$ is the universal central extension of LG, see [PS86].

In the following two subsections we discuss additional structures on the central extension $\mathscr{T}_{\mathcal{G}_{bas}}$, which we find by analyzing the image of the transgression functor $\mathscr{T}$.

2.2 Connections and splittings

The principal U(1)-bundles in the image of the transgression functor $\mathscr{T}$ of (2.1.7) are canonically equipped with connections [Bry93]. In other words, transgression is actually a functor

$$\mathscr{T} : \mathrm{h}_1\mathcal{G}rb^\nabla(X) \longrightarrow \mathcal{B}un^\nabla(LX) \tag{2.2.1}$$

to the category of Fréchet principal U(1)-bundles *with connection.* It satisfies

$$\mathrm{curv}(\mathscr{T}_{\mathcal{G}}) = -\tau_\Omega(\mathrm{curv}(\mathcal{G})), \tag{2.2.2}$$

where τ_Ω is the differential form counterpart of the transgression homomorphism (1.2):

$$\tau_\Omega : \Omega^n(X) \longrightarrow \Omega^{n-1}(LX) : \omega \longmapsto \int_{S^1} \mathrm{ev}^*\omega; \tag{2.2.3}$$

it integrates the pullback of a differential form along the evaluation map $\mathrm{ev} : S^1 \times LX \longrightarrow X$ over the factor S^1. If $\omega \in \Omega^2(X)$ is a 2-form, and $\mathcal{I}_\omega$ is the associated trivial bundle gerbe with connection, then the above-mentioned canonical trivialization of $\mathscr{T}_{\mathcal{I}_\omega}$ has covariant derivative $\tau_\Omega(\omega) \in \Omega^1(LX)$ [Wal11, Lemma 3.6].

We continue the analysis of the central extension $\mathscr{T}_{\mathcal{G}}$ of LG obtained by transgression of a multiplicative bundle gerbe $\mathcal{G}$ over a Lie group G with connection relative to the differential forms H and

ρ. As we have now lifted the transgression functor $\mathscr{T}$ to the category of bundles *with connection*, it follows that the central extension $\mathscr{T}_{\mathcal{G}}$ has a connection.

In the following we denote the central extension by $\widetilde{LG}$, and we denote the connection by $\nu \in \Omega^1(\widetilde{LG})$. According to Equation (2.2.2) it has curvature $\mathrm{curv}(\nu) = -\tau_\Omega(H)$. For us, the most important feature of the connection ν is that it is *not strictly compatible* with the group structure of $\widetilde{LG}$. Indeed, looking again at the transgression of the isomorphism $\mathcal{M}$, but now in the setting with connections, we obtain a connection-preserving bundle isomorphism

$$\mathrm{pr}_1^*\mathscr{T}_{\mathcal{G}} \otimes \mathrm{pr}_2^*\mathscr{T}_{\mathcal{G}} \xrightarrow{\mathscr{T}_{\mathcal{M}}} m^*\mathscr{T}_{\mathcal{G}} \otimes \mathscr{T}_{\mathcal{I}_\rho} \cong m^*\mathscr{T}_{\mathcal{G}} \otimes \mathbf{I}_{\epsilon_\nu},$$

where $\mathbf{I}_{\epsilon_\nu}$ is the trivial U(1)-bundle over LG equipped with the connection 1-form $\epsilon_\nu := \tau_\Omega(\rho) \in \Omega^1(LG \times LG)$. In terms of the connection 1-form ν and the group structure defined by the underlying bundle morphism, this can be expressed as

$$\nu_{\tilde{\tau}_1}(\tilde{X}_1) + \nu_{\tilde{\tau}_2}(\tilde{X}_2) = \nu_{\tilde{\tau}_1\tilde{\tau}_2}(\tilde{\tau}_1\tilde{X}_2 + \tilde{X}_1\tilde{\tau}_1) + \epsilon_\nu|_{\tau_1,\tau_2}(X_1, X_2) \tag{2.2.4}$$

for elements $\tilde{\tau}_1, \tilde{\tau}_2 \in \widetilde{LG}$ projecting to loops $\tau_1, \tau_2 \in LG$, and tangent vectors $\tilde{X}_1 \in T_{\tilde{\tau}_1}\widetilde{LG}$ and $\tilde{X}_2 \in T_{\tilde{\tau}_2}\widetilde{LG}$ projecting to $X_1 \in T_{\tau_1}LG$ and $X_2 \in T_{\tau_2}LG$, respectively. The 1-form ϵ_ν can be computed explicitly from the given 2-form ρ of (2.1.3),

$$\epsilon_\nu|_{\tau_1,\tau_2}(X_1, X_2) = \int_{S^1} \left\{ \left\langle \tau_1(z)^{-1}\partial_z\tau_1(z), X_2(z)\tau_2(z)^{-1} \right\rangle - \left\langle \tau_1(z)^{-1}X_1(z), \partial_z\tau_2(z)\tau_2(z)^{-1} \right\rangle \right\} \mathrm{d}z. \tag{2.2.5}$$

Here, and in the following, we regard a tangent vector $X \in T_\tau LG$ as a section of TG along τ, i.e. as a smooth map $X : S^1 \longrightarrow TG$ such that $X(z) \in T_{\tau(z)}G$, see [PS86].

In general, a connection on a central extension induces a splitting of the Lie algebra extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \widetilde{L\mathfrak{g}} \xrightarrow{p_*} L\mathfrak{g} \longrightarrow 0, \tag{2.2.6}$$

i.e. a linear map $s : L\mathfrak{g} \longrightarrow \widetilde{L\mathfrak{g}}$ such that $p_* \circ s = \mathrm{id}_{L\mathfrak{g}}$. Indeed, the connection ν determines a horizontal subspace $H_1^\nu\widetilde{LG} \subseteq T_1\widetilde{LG}$ such that $p_* \colon H_1^\nu\widetilde{LG} \longrightarrow L\mathfrak{g}$ is an isomorphism. For $X \in L\mathfrak{g}$ we let $s(X) \in H_1^\nu\widetilde{LG}$ be its preimage under p_*. An equivalent definition that uses the connection 1-form ν directly is to first choose any lift $\tilde{X} \in \widetilde{L\mathfrak{g}}$ of X and then define $s(X) := \tilde{X} - \nu(\tilde{X})$.

Given the splitting s determined by the connection ν one can define the map

$$Z : LG \times L\mathfrak{g} \longrightarrow \mathbb{R} \quad , \quad Z(\tau, X) := \mathrm{Ad}_\tau^{-1}(s(X)) - s(\mathrm{Ad}_\tau^{-1}(X)).$$

Lemma 2.2.1. $Z(\tau, X) = 2\displaystyle\int_{S^1} \langle \tau^*\bar{\theta}, X \rangle$.

Proof. We express Z in terms of the error 1-form ϵ_ν of the connection ν. Consider $\tilde{\tau} \in \widetilde{LG}$ and $\tilde{X} \in \widetilde{L\mathfrak{g}}$ projecting to $\tau \in LG$ and $X \in L\mathfrak{g}$, respectively. Then,

$$\begin{aligned} Z(\tau, X) = \mathrm{Ad}_{\tilde{\tau}}^{-1}(s(X)) - s(\mathrm{Ad}_\tau^{-1}(X)) &= \tilde{\tau}^{-1}(\tilde{X} - \nu(\tilde{X}))\tilde{\tau} - \tilde{\tau}^{-1}\tilde{X}\tilde{\tau} + \nu(\tilde{\tau}^{-1}\tilde{X}\tilde{\tau}) \\ &= -\nu(\tilde{X}) + \nu(\tilde{\tau}^{-1}\tilde{X}\tilde{\tau}) \overset{(2.2.4)}{=} -\epsilon_\nu|_{1,\tau}(X, 0) - \epsilon_\nu|_{\tau^{-1},\tau}(0, X\tau). \end{aligned}$$

With (2.2.5) we see that these two terms are equal and add up to the claimed formula. □

The formula

$$\omega(X,Y) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Z(e^{-tX},Y) = [s(X),s(Y)] - s([X,Y])$$

defines a 2-cocycle ω for the Lie algebra cohomology of $L\mathfrak{g}$ with coefficients in the trivial module $\mathbb{R}$, and classifies the Lie algebra extension. From Lemma 2.2.1 we get the following.

Lemma 2.2.2. $\omega(X,Y) = 2\int_{S^1} \langle X, \mathrm{d}Y\rangle$.

Up to the prefactor (which can always be absorbed into the normalization of the bilinear form $\langle -,-\rangle$) this is the standard cocycle on the loop algebra, see [PS86, Section 4.2]. Note that the cocycle ω is not invariant; instead it satisfies [Gom03, Lemma 5.8 (b)]

$$\omega(\mathrm{Ad}_\tau^{-1}(X), \mathrm{Ad}_\tau^{-1}(Y)) = \omega(X,Y) + Z(\tau,[X,Y]). \tag{2.2.7}$$

It is well-known that a given splitting s of a Lie algebra extension (2.2.6) induces, conversely, a connection ν_s on the central extension $\widetilde{LG}$, given by the formula

$$\nu_s = \tilde{\theta} - s\,(p^*\theta) \in \Omega^1(\widetilde{LG}),$$

where $\tilde{\theta}$ stands for the left-invariant Maurer-Cartan form on $\widetilde{LG}$. Its curvature is given by $-\frac{1}{2}\omega(\theta\wedge\theta) \in \Omega^2(LG)$, see e.g. [Gom03, Lemma 5.4]. We will later have to compare the original connection ν with the connection ν_s determined by s and hence indirectly by ν. For this purpose, we consider the 1-form $\beta \in \Omega^1(LG)$ given by the formula

$$\beta_\tau(X) := \int_0^1 \left\langle \tau(z)^{-1}\partial_z\tau(z), \tau(z)^{-1}X(z)\right\rangle \,\mathrm{d}z$$

for $\tau \in LG$ and $X \in T_\tau LG$.

Lemma 2.2.3. *The connection ν_s is obtained by shifting the connection ν by β, i.e. $\nu_s = \nu + \beta$. In particular, the curvatures obey the following relation:*

$$-\frac{1}{2}\omega(\theta\wedge\theta) = \mathrm{curv}(\nu) + \mathrm{d}\beta. \tag{2.2.8}$$

Proof. For a tangent vector $\tilde{X} \in T_{\tilde{\tau}}\widetilde{LG}$ we obtain from the definitions of the connection ν_s and the splitting s that $\nu_s(\tilde{X}) = \nu(\tilde{\tau}^{-1}\tilde{X})$. Using the multiplicativity law (2.2.4) for the connection ν, we get $\nu(\tilde{\tau}^{-1}\tilde{X}) = \nu(\tilde{X}) - \epsilon_\nu|_{\tau^{-1},\tau}(0,X)$. Looking at the explicit expression (2.2.5), we see that $-\epsilon_\nu|_{\tau^{-1},\tau}(0,X) = \beta_\tau(X)$. □

The connection ν on $\widetilde{LG}$ has an interesting property which distinguishes it from other connections on $\widetilde{LG}$, in particular from the connection ν_s. The property is that ν is *superficial*. In order to explain this, we fix the following notation: if $\tau \in LLX$ is a loop in the loop space of a smooth manifold X, then by $\tau^\vee : S^1 \times S^1 \longrightarrow X$ we denote the "adjoint" map defined by $\tau^\vee(z_1,z_2) := \tau(z_1)(z_2)$. We use the following terminology: a map $f : X \longrightarrow Y$ between smooth manifolds is said to be of rank k if its differential $\mathrm{d}f_x$ has at most rank k for all $x \in X$. The map f is called *thin*, if it is of rank $\dim(X) - 1$.

Definition 2.2.4 ([Walb, Definition 2.2.1])**.** *A connection ν on a Fréchet principal* U(1)*-bundle over the loop space LX of a smooth manifold X is called superficial, if the following two conditions are satisfied:*

(i) The holonomy of a loop $\tau \in LLX$ vanishes if $\tau^\vee$ thin.

(ii) Two loops $\tau, \tau' \in LLX$ have the same holonomy, if $\tau^\vee$ and $\tau'^\vee$ are thin homotopic.

By [Walb, Corollary 4.3.3] all connections in the image of the transgression functor (2.2.1) are superficial. This comes from the fact that the holonomy of such connections can be expressed in terms of the surface holonomy of the bundle gerbe $\mathcal{G}$ via the formula

$$\mathrm{Hol}_\nu(\tau) = \mathrm{Hol}_{\mathcal{G}}(\tau^\vee).$$

The surface holonomy of a connection on a gerbe has the two properties (i) and (ii).

2.3 Thin structures and fusion

In this article, the most important aspect of superficial connections is that they induce *thin structures*, a kind of equivariance with respect to thin homotopies. We use *diffeological spaces* as an auxiliary tool. In short, a diffeological space is a set X with specified *plots*: maps $c : U \longrightarrow X$ defined on open subsets $U \subseteq \mathbb{R}^n$, $n \geq 0$. There are full and faithful functors

$$\mathcal{M}an \hookrightarrow \mathcal{F}rech \hookrightarrow \mathcal{D}iff$$

that realize smooth manifolds and Fréchet manifolds as diffeological spaces with plots given by all smooth maps $c : U \longrightarrow X$ from all open subsets $U \subseteq \mathbb{R}^n$ for all n. In almost all aspects relevant for this article, diffeological spaces behave exactly as smooth manifolds – there are just more of them. For example, differential forms, principal bundles, and connections can be defined on diffeological spaces in a manner consistent with above inclusions, see [Wal12b].

If X is a smooth manifold, we denote by $LX^2_{thin} \subseteq LX \times LX$ the set consisting of pairs (τ_1, τ_2) of thin homotopic loops, i.e. there exists a homotopy $h : [0,1] \times S^1 \longrightarrow X$ of rank one. The set LX^2_{thin} carries a natural diffeology [Wal12c, Section 3.1].

Definition 2.3.1 ([Wal12c, Definition 3.1.1]). *A* <u>*thin homotopy equivariant structure*</u> *on a Fréchet principal* $\mathrm{U}(1)$*-bundle P over LX is a smooth bundle isomorphism*

$$d : \mathrm{pr}_1^*P \longrightarrow \mathrm{pr}_2^*P$$

over LX^2_{thin} that satisfies the cocycle condition $d_{\tau_2,\tau_3} \circ d_{\tau_1,\tau_2} = d_{\tau_1,\tau_3}$ for any triple (τ_1, τ_2, τ_3) of thin homotopic loops.

A bundle morphism $\varphi : P_1 \longrightarrow P_2$ between bundles with thin homotopy equivariant structures d_1 and d_2 is called *thin*, if the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^*P_1 & \xrightarrow{\mathrm{pr}_1^*\varphi} & \mathrm{pr}_1^*P_2 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ \mathrm{pr}_2^*P_1 & \xrightarrow[\mathrm{pr}_2^*\varphi]{} & \mathrm{pr}_2^*P_2 \end{array} \tag{2.3.1}$$

of bundle morphisms over LX^2_{thin} is commutative.

Now suppose P is equipped with a superficial connection ω. Property (i) implies that the parallel transport of ω along a path $\gamma : [0,1] \longrightarrow LX$ between two loops $(\tau_1, \tau_2) \in LX^2_{thin}$ is independent of the choice of the path (provided it is chosen so that $\gamma^\vee$ is thin). Thus, we have a well-defined map

$$d^\omega_{\tau_1,\tau_2} : P_{\tau_1} \longrightarrow P_{\tau_2}.$$

The maps $d^\omega_{\tau_1,\tau_2}$ form a thin homotopy equivariant structure [Wal12c, Lemma 3.1.5]. A thin homotopy equivariant structure d is called a *thin structure*, if there is a superficial connection ω with $d = d^\omega$.

Summarizing, a thin structure on a bundle P over a loop space LX is a consistent way of identifying its fibres over thin homotopic loops. As orientation-preserving diffeomorphisms of S^1 induce thin homotopies ($\mathcal{D}\mathit{iff}^+(S^1)$ is connected), we have the following.

Proposition 2.3.2 ([Wal12c, Proposition 3.1.2])**.** *A thin structure on a Fréchet principal* $\mathrm{U}(1)$*-bundle* P *over* LX *determines a* $\mathcal{D}\mathit{iff}^+(S^1)$*-equivariant structure on* P*.*

We continue to discuss the central extension $\widetilde{LG}$ obtained by transgression of a multiplicative bundle gerbe over G with connection relative to the forms H and ρ given by (2.1.2) and (2.1.3). Since the connection ν on $\widetilde{LG}$ is superficial, the central extension $\widetilde{LG}$ is equipped with a thin structure d^ν.

Proposition 2.3.3. *The thin structure* d^ν *is multiplicative in the sense that*

$$d^\nu_{\tau_0\gamma_0,\tau_1\gamma_1}(\tilde\tau\cdot\tilde\gamma) = d^\nu_{\tau_0,\tau_1}(\tilde\tau)\cdot d^\nu_{\gamma_0,\gamma_1}(\tilde\gamma)$$

for all $((\tau_0,\gamma_0),(\tau_1,\gamma_1)) \in L(G\times G)^2_{thin}$ *and all* $\tilde\tau,\tilde\gamma\in\widetilde{LG}$ *projecting to* τ_0 *and* γ_0*, respectively.*

Proof. We consider the connection-preserving bundle morphism

$$\mathrm{pr}_1^*\mathscr{T}_{\mathcal{G}}\otimes\mathrm{pr}_2^*\mathscr{T}_{\mathcal{G}} \xrightarrow{\mathscr{T}_{\mathcal{M}}} m^*\mathscr{T}_{\mathcal{G}}\otimes\mathscr{T}_{\mathcal{I}_\rho} = m^*\mathscr{T}_{\mathcal{G}}\otimes\mathbf{I}_{\epsilon_\nu},$$

that describes the relation between the group structure of $\widetilde{LG}$ and the superficial connection ν. Now we pass from superficial connections to thin structures, and observe that the thin structure on the trivial bundle $\mathbf{I}_{\epsilon_\nu}$ is the trivial one. This comes simply from the fact that the parallel transport of the connection $\epsilon_\nu = \tau_\Omega(\rho)$ along a path $\gamma\in P(G\times G)$ can be expressed as an integral of the 2-form ρ over $\gamma^\vee$. If $\gamma^\vee$ is of rank one, that integral vanishes; see [Wal12c, Proposition 3.1.8] for the full argument. Thus,

$$\mathrm{pr}_1^*\mathscr{T}_{\mathcal{G}}\otimes\mathrm{pr}_2^*\mathscr{T}_{\mathcal{G}} \xrightarrow{\mathscr{T}_{\mathcal{M}}} m^*\mathscr{T}_{\mathcal{G}}$$

is a thin bundle morphism over $L(G\times G)^2_{thin}$. Now, diagram (2.3.1) evaluated over the point $((\tau_0,\gamma_0),(\tau_1,\gamma_1)) \in L(G\times G)^2_{thin}$ gives the claimed identity. $\square$

We remark that $((\tau_0,\gamma_0),(\tau_1,\gamma_1)) \in L(G\times G)^2_{thin}$ means that there exists a thin path (τ,γ) in $L(G\times G)$ connecting (τ_0,γ_0) with (τ_1,γ_1). It is necessary, but not sufficient, that the paths τ, γ, and $\tau\gamma$ in LG are separately thin.

Finally, we come to another additional structure on the central extension $\widetilde{LG}$: a *fusion product*. By PX we denote the set of paths in a smooth manifold X with "sitting instants", i.e. smooth maps $\gamma\colon [0,1] \longrightarrow X$ that are locally constant near the endpoints. Due to the sitting instants, PX is not a Fréchet manifold, but still a nice diffeological space, with plots $c: U \longrightarrow PX$ those maps whose adjoints $c^\vee: U\times[0,1] \longrightarrow X$ are smooth. We denote by $PX^{[k]}$ the k-fold fibre product of PX over the evaluation map $\mathrm{ev}\colon PX \longrightarrow X\times X$, i.e. the diffeological space of k-tuples of paths with a common initial point and a common end point. Due to the sitting instants, we have a well-defined and smooth map

$$\cup\colon PX^{[2]} \longrightarrow LX\colon (\gamma_1,\gamma_2) \longmapsto \overline{\gamma_2}\star\gamma_1,$$

where $\star$ denotes the path concatenation, and $\overline{\gamma}$ denotes the reversed path; see [Wal12b, Section 2] for a more detailed discussion. For $ij \in \{12, 23, 13\}$, we denote by $\cup_{ij}$ the composition of $\cup$ with the projection $\mathrm{pr}_{ij} : PX^{[3]} \longrightarrow PX^{[2]}$.

Definition 2.3.4 ([Walb, Definition 2.1.3])**.** *A* <u>*fusion product*</u> *on a Fréchet principal* $\mathrm{U}(1)$*-bundle* P *over the loop space* LX *of a smooth manifold* X *is a smooth bundle morphism*

$$\lambda : \cup_{23}^{*} P \otimes \cup_{12}^{*} P \longrightarrow \cup_{13}^{*} P$$

over $PX^{[3]}$ *that is associative in the sense that*

$$\lambda(\lambda(p_{34} \otimes p_{23}) \otimes p_{12}) = \lambda(p_{34} \otimes \lambda(p_{23} \otimes p_{12}))$$

for all $p_{ij} \in P_{\gamma_i \cup \gamma_j}$ *and all* $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in PX^{[4]}$.

A morphism $\varphi : P_1 \longrightarrow P_2$ between principal $\mathrm{U}(1)$-bundles over LX equipped with fusion products λ_1 and λ_2, respectively, is called *fusion-preserving* if the diagram

$$\begin{array}{ccc} \cup_{23}^{*} P_1 \otimes \cup_{12}^{*} P_1 & \xrightarrow{\lambda_1} & \cup_{13}^{*} P \\ {\scriptstyle \cup_{23}^{*}\varphi \otimes \cup_{12}^{*}\varphi} \downarrow & & \downarrow {\scriptstyle \cup_{13}^{*}\varphi} \\ \cup_{23}^{*} P_2 \otimes \cup_{12}^{*} P_2 & \xrightarrow[\lambda_2]{} & \cup_{13}^{*} P_2 \end{array}$$

of bundle morphisms over $PX^{[3]}$ is commutative.

If P is equipped with a fusion product λ, then a connection ν is called *fusive*, if the following conditions are satisfied:

(i) The fusion product λ is a *connection-preserving* bundle morphism over $PX^{[3]}$.

(ii) The rotation by an angle of π is an orientation-preserving diffeomorphism of S^1 and induces a diffeomorphism $r_\pi : LX \longrightarrow LX$. The $\mathcal{D}\!\mathit{iff}^{+}(S^1)$-equivariant structure of Proposition 2.3.2 provides a lift $r_\pi^{d^\nu} : P \longrightarrow P$. We demand that the condition

$$\lambda(r_\pi^{d^\nu}(p_{12}) \otimes r_\pi^{d^\nu}(p_{23})) = r_\pi^{d^\nu}(\lambda(p_{23} \otimes p_{12}))$$

is satisfied for all $p_{12} \in P_{\gamma_1 \cup \gamma_2}$, $p_{23} \in P_{\gamma_2 \cup \gamma_3}$, and $(\gamma_1, \gamma_2) \in PX^{[2]}$.

In [Walb] a category $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX)$ is considered with objects the principal $\mathrm{U}(1)$-bundles over LX equipped with fusion products and superficial fusive connections, and morphisms the fusion-preserving, connection-preserving bundle morphisms. By a construction performed in [Walb, Section 4.2], the transgression functor (2.2.1) lifts into this category:

$$\mathscr{T} : \mathrm{h}_1\mathcal{G}rb^{\nabla}(X) \longrightarrow \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX). \tag{2.3.2}$$

Before we return to the central extension $\widetilde{LG}$, we relate fusion products to thin structures.

Definition 2.3.5. *Let* P *be a principal* $\mathrm{U}(1)$*-bundle* P *over* LX *with a fusion product* λ*. A thin structure* d *on* P *is called* <u>*fusive*</u> *with respect to* λ*, if there exists a superficial fusive connection* ν *on* P *such that* $d = d^\nu$*.*

In particular, the fusion product λ is a thin bundle morphism with respect to a fusive thin structure. In [Wal12c] a category $h\mathcal{F}us\mathcal{B}un^{th}(LX)$ is considered with objects the principal $\mathrm{U}(1)$-bundles over LX equipped with fusion products and fusive thin structures, and morphisms the homotopy classes of fusion-preserving, thin bundle morphisms.

The two categories $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX)$ and $h\mathcal{F}us\mathcal{B}un^{th}(LX)$ are loop space analogues of the categories $\mathrm{h}_1\mathcal{G}rb^{\nabla}(X)$ and $\mathrm{h}_1\mathcal{G}rb(X)$ of bundle gerbes with and without connections over X, respectively. The procedure of inducing a thin structure from a superficial connection (and projecting to the homotopy class of a morphism) defines a functor

$$\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX) \longrightarrow h\mathcal{F}us\mathcal{B}un^{th}(LX);$$

it is the loop space analogue of the 2-functor (2.1.1) that passes from gerbes with connection to gerbes without connections. These analogies are the content of the following theorem, which is the main result of the series of articles [Wal12b, Walb, Wal12c].

Theorem 2.3.6. *Let X be a connected smooth manifold. There is a strictly commutative diagram*

$$\begin{array}{ccc} \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX) & \longrightarrow & \mathrm{h}_1\mathcal{G}rb^{\nabla}(X) \\ \downarrow & & \downarrow \\ h\mathcal{F}us\mathcal{B}un^{th}(LX) & \longrightarrow & \mathrm{h}_1\mathcal{G}rb(X) \end{array}$$

of monoidal categories and functors, natural in X, whose horizontal functors are monoidal equivalences of categories.

The functor in the first row of the diagram is inverse to the transgression functor $\mathscr{T}$ of (2.3.2), i.e. the two functors form an equivalence of categories. The functor in the second row is essentially surjective, full and faithful, but has no canonical inverse functor. The two functors are called *regression* [Walb, Section 5].

Let us now return to the discussion of the central extension $\widetilde{LG}$ defined by transgression of a multiplicative bundle gerbe $\mathcal{G}$ over G. According to above discussion, $\widetilde{LG}$ is equipped with a fusion product, which we denote by $\lambda_{\mathcal{G}}$. The connection ν on $\widetilde{LG}$ and the induced thin structure d^{ν} are fusive. Since transgression is a functor, the multiplication $\mathscr{T}_{\mathcal{M}}$ is fusion-preserving. This can be rephrased as follows.

Lemma 2.3.7. *The fusion product on $\widetilde{LG}$ is multiplicative in the sense that*

$$\lambda(p_{23} \otimes p_{12}) \cdot \lambda(p'_{23} \otimes p'_{12}) = \lambda(p_{23}p'_{23} \otimes p_{12}p'_{12}) \tag{2.3.3}$$

for all elements $p_{ij}, p'_{ij} \in \widetilde{LG}$ projecting to loops $\gamma_i \cup \gamma_j$ and $\gamma'_i \cup \gamma'_j$, respectively, for all $(\gamma_1, \gamma_2, \gamma_3), (\gamma'_1, \gamma'_2, \gamma'_3) \in PG^{[3]}$.

We have now listed all additional structures and properties of the central extension $\widetilde{LG}$ that arise from our approach using the transgression multiplicative bundle gerbes, and that we need in the following. In [Walc] we show how "transgressive" central extensions can be characterized by fusion products and thin structures.

3 Thin fusion spin structures

In Section 3.1 we first recall the definition of spin structures on loop spaces following Killingback [Kil87]. Based upon this definition we develop the notion of thin fusion spin structures, which constitute our loop space analogue for string structures. In Section 3.2 we prepare one part of the proof of this analogy: we provide a lifting gerbe formulation for thin fusion spin structures.

3.1 Versions of spin structures on loop spaces

Let M be a spin manifold of dimension $n = 3$ or $n > 4$, so that $\mathrm{Spin}(n)$ is compact, simple and simply-connected. We denote by $\pi : FM \longrightarrow M$ the spin-oriented frame bundle of M, which is a $\mathrm{Spin}(n)$-principal bundle over M. Since $\mathrm{Spin}(n)$ is connected, LFM is a principal $L\mathrm{Spin}(n)$-bundle over LM.

Definition 3.1.1 ([Kil87]). *A <u>spin structure on LM</u> is a lift of the structure group of the looped frame bundle LFM from $L\mathrm{Spin}(n)$ to the universal central extension $\widetilde{L\mathrm{Spin}(n)}$.*

Thus, a spin structure on LM is a pair $(\mathcal{S}, \sigma)$ of a Fréchet principal $\widetilde{L\mathrm{Spin}(n)}$-bundle $\mathcal{S}$ over LM together with a smooth map $\sigma : \mathcal{S} \longrightarrow LFM$ such that the diagram

$$\begin{array}{ccccc} \mathcal{S} \times \widetilde{L\mathrm{Spin}(n)} & \longrightarrow & \mathcal{S} & \searrow & \\ \Big\downarrow{\scriptstyle \sigma \times p} & & \Big\downarrow{\scriptstyle \sigma} & & LM \\ LFM \times L\mathrm{Spin}(n) & \longrightarrow & LFM & \nearrow & \end{array}$$

is commutative. A morphism between spin structures $(\mathcal{S}_1, \sigma_1)$ and $(\mathcal{S}_2, \sigma_2)$ is a bundle morphism $\varphi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that $\sigma_1 = \sigma_2 \circ \varphi$. Spin structures on LM form a category that we denote by $\mathcal{S}pin(LM)$. It is a module for the monoidal category $\mathcal{B}un(LM)$ of Fréchet principal U(1)-bundles over LM, under an action functor

$$\mathcal{B}un(LM) \times \mathcal{S}pin(LM) \longrightarrow \mathcal{S}pin(LM) : (K, (\mathcal{S}, \sigma)) \longmapsto K \otimes (\mathcal{S}, \sigma). \tag{3.1.1}$$

Here, $K \otimes (\mathcal{S}, \sigma)$ is the spin structure with the $\widetilde{L\mathrm{Spin}(n)}$-bundle $K \otimes \mathcal{S} := (K \times_{LM} \mathcal{S})/\,\mathrm{U}(1)$ over LM, and the map $(k, s) \longmapsto \sigma(s)$ to LFM. By Corollary 3.2.2 proved below, the action (3.1.1) exhibits $\mathcal{S}pin(LM)$ as a *torsor* over $\mathcal{B}un(LM)$ in the sense that the associated functor

$$\mathcal{B}un(LM) \times \mathcal{S}pin(LM) \longrightarrow \mathcal{S}pin(LM) \times \mathcal{S}pin(LM) : (K, (\mathcal{S}, \sigma)) \longmapsto (K \otimes (\mathcal{S}, \sigma), (\mathcal{S}, \sigma))$$

is an equivalence of categories.

The notion of a spin structure in the sense of Definition 3.1.1 suffers from the fact that there are manifolds that are not string manifolds but whose loop space admits spin structures [PW88]. The plan we follow in this article is to add additional conditions/structure to spin structures on loop spaces, in order to better reflect string structures on the base manifold.

If $(\mathcal{S}, \sigma)$ is a spin structure, then $\sigma : \mathcal{S} \longrightarrow LFM$ is a principal U(1)-bundle under the U(1)-action obtained by restriction of the $\widetilde{L\mathrm{Spin}(n)}$-action. We use the notation $T_{\mathcal{S}}$ for explicit reference to this principal U(1)-bundle. Any morphism $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ between spin structures is a morphism $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ between the associated principal U(1)-bundles. Under the action (3.1.1) we have

$$T_{K\otimes \mathcal{S}} = L\pi^*K \otimes T_{\mathcal{S}}. \tag{3.1.2}$$

Definition 3.1.2 ([Wala, Definition 3.6]). *A fusion product on a spin structure $(\mathcal{S},\sigma)$ is a fusion product $\lambda_{\mathcal{S}}$ on $T_{\mathcal{S}}$ such that the $\widetilde{L\mathrm{Spin}(n)}$-action on $\mathcal{S}$ is fusion-preserving:*

$$\lambda_{\mathcal{S}}(t_{23}\cdot\tilde{\gamma}_{23}\otimes t_{12}\cdot\tilde{\gamma}_{12}) = \lambda_{\mathcal{S}}(t_{23}\otimes t_{12})\cdot \lambda_{\mathcal{G}_{bas}}(\tilde{\gamma}_{23}\otimes\tilde{\gamma}_{12}),$$

for all $t_{12}, t_{23} \in \mathcal{S}$ and $\tilde{\gamma}_{12}, \tilde{\gamma}_{23} \in \widetilde{L\mathrm{Spin}(n)}$ such that the fusion products are defined. A morphism $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ between spin structures with fusion products is called fusion-preserving if the associated morphism $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ is fusion-preserving.

Spin structures with fusion products form a category that we denote by $\mathcal{S}pin_{fus}(LM)$. Similar to the action functor (3.1.1), the category $\mathcal{S}pin_{fus}(LM)$ carries an action of the monoidal category $\mathcal{F}us\mathcal{B}un(LM)$ of Fréchet principal U(1)-bundles with fusion products, under which (3.1.2) holds as an equation of bundles with fusion products.

The main result of the paper [Wala] was that a spin manifold M is a string manifold if and only if its loop space LM admits a spin structure with fusion product. Next we explain how to add thin structures into the picture in order to improve that result. We recall that the central extension $\widetilde{L\mathrm{Spin}(n)}$ is equipped with a thin structure d^ν induced from the superficial connection ν.

Definition 3.1.3. *A thin structure on a spin structure $(\mathcal{S},\sigma)$ is a thin structure d on $T_{\mathcal{S}}$ such that*

$$d_{\tau_1\cdot\gamma_1,\tau_2\cdot\gamma_2}(t\cdot\tilde{\gamma}) = d_{\tau_1,\tau_2}(t)\cdot d^{\nu}_{\gamma_1,\gamma_2}(\tilde{\gamma}).$$

for all $((\tau_1,\gamma_1),(\tau_2,\gamma_2)) \in L(FM\times\mathrm{Spin}(n))^2_{thin}$, all $t \in T_{\mathcal{S}}$ projecting to τ_1, and all $\tilde{\gamma} \in \widetilde{L\mathrm{Spin}(n)}$ projecting γ_1. A thin spin structure is a spin structure together with a thin structure. A morphism between thin spin structures is a morphism φ: $\mathcal{S} \longrightarrow \mathcal{S}'$ between spin structures such that the induced morphism $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ is thin.

In this definition it is relevant to observe that $((\tau_1,\gamma_1),(\tau_2,\gamma_2)) \in L(FM\times\mathrm{Spin}(n))^2_{thin}$ implies that $(\tau_1,\tau_2)\in LFM^2_{thin}$, $(\gamma_1,\gamma_2)\in L\mathrm{Spin}(n)^2_{thin}$, and $(\tau_1\cdot\gamma_1,\tau_2\cdot\gamma_2)\in LFM^2_{thin}$.

Thin spin structures form a category that we denote by $\mathcal{S}pin^{th}(LM)$. Based on the action functor (3.1.1), it carries an action of the monoidal category $\mathcal{B}un^{th}(LM)$ of Fréchet principal U(1)-bundles with thin structures, under which (3.1.2) is an equality of bundles with thin structures.

Proposition 3.1.4. *A thin structure on a spin structure $(\mathcal{S},\sigma)$ on LM determines a $\mathcal{D}\!i\!f\!f^+(S^1)$-equivariant structure on the principal $\widetilde{L\mathrm{Spin}(n)}$-bundle $\mathcal{S}$ over LM, such that the map $\sigma : \mathcal{S} \longrightarrow LFM$ is $\mathcal{D}\!i\!f\!f^+(S^1)$-equivariant.*

Proof. We note that LFM is obviously $\mathcal{D}\!i\!f\!f(S^1)$-equivariant as a $L\mathrm{Spin}(n)$-bundle over LM, since $\mathcal{D}\!i\!f\!f^+(S^1)$ acts on LFM. By Proposition 2.3.2, the thin structure on $T_{\mathcal{S}}$ lifts this action to $\mathcal{S}$. □

Thin structures and fusion products for spin structures combine in the following way.

Definition 3.1.5. *A thin fusion spin structure on LM is a spin structure $(\mathcal{S},\sigma)$ with a fusion product λ in the sense of Definition 3.1.2 and a thin structure d in the sense of Definition 3.1.3, such that d is fusive with respect to λ in the sense of Definition 2.3.5.*

Particular care has to be taken with the correct notion of morphisms between thin fusion spin structures. If X is a smooth manifold, a *fusion map* $f : LX \longrightarrow \mathrm{U}(1)$ is a smooth map with the following properties:

(i) If $\tau, \tau' \in LX$ are thin homotopic loops, then $f(\tau) = f(\tau')$.

(ii) If $(\gamma_1, \gamma_2, \gamma_3) \in PX^{[3]}$, then $f(\gamma_1 \cup \gamma_2) \cdot f(\gamma_2 \cup \gamma_3) = f(\gamma_1 \cup \gamma_3)$.

A *fusion homotopy* is a smooth map $h : [0,1] \times LX \longrightarrow \mathrm{U}(1)$ such that $h_t : LX \longrightarrow \mathrm{U}(1)$ is a fusion map for all $t \in [0,1]$.

Definition 3.1.6. *Let $(\mathcal{S}, \sigma, \lambda, d)$ and $(\mathcal{S}', \sigma', \lambda', d')$ be thin fusion spin structures. A morphism is a smooth map $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ satisfying the following conditions:*

(i) $\sigma' \circ \varphi = \sigma$; in particular, φ covers the identity on LM.

(ii) φ is equivariant with respect to the $\mathrm{U}(1)$-actions on $\mathcal{S}$ and $\mathcal{S}'$, i.e. it induces a morphism $\varphi \colon T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ between $\mathrm{U}(1)$-bundles over LFM.

(iii) The bundle morphism $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ is fusion-preserving and thin.

(iv) φ is fusion-homotopy-equivariant with respect to the $\widetilde{L\mathrm{Spin}(n)}$-action, i.e. there exists a fusion homotopy $h : [0,1] \times LFM \times L\mathrm{Spin}(n) \longrightarrow \mathrm{U}(1)$ with $h_0 = 1$ and

$$\varphi(t \cdot \tilde{\tau}) \cdot h_1(\beta, \tau) = \varphi(t) \cdot \tilde{\tau} \tag{3.1.3}$$

for all $t \in \mathcal{S}$ and $\tilde{\tau} \in \widetilde{L\mathrm{Spin}(n)}$ over $\beta \in LFM$ and $\tau \in L\mathrm{Spin}(n)$, respectively.

Definitions 3.1.5 and 3.1.6 result in a category of thin fusion spin structures which we denote by $\mathcal{S}pin_{fus}^{th}(LM)$. It carries an action of the monoidal category $\mathcal{F}us\mathcal{B}un^{th}(LM)$.

In the end, the category that is equivalent to the category of string structures on M is the *homotopy category* $h\mathcal{S}pin_{fus}^{th}(LM)$, i.e. two morphisms $\varphi_0, \varphi_1 : \mathcal{S} \longrightarrow \mathcal{S}'$ become identified if there is a smooth map $h : [0,1] \times \mathcal{S} \longrightarrow \mathcal{S}'$ with $h_0 = \varphi_0$, $h_1 = \varphi_1$, and h_t is a morphism between thin fusion spin structures for all $t \in [0,1]$. The homotopy category $h\mathcal{S}pin_{fus}^{th}(LM)$ inherits an action of the homotopy category $h\mathcal{F}us\mathcal{B}un^{th}(LM)$. As a consequence of Theorem 7.4, this action exhibits $h\mathcal{S}pin_{fus}^{th}(LM)$ as a torsor over $h\mathcal{F}us\mathcal{B}un^{th}(LM)$.

3.2 Lifting theory for spin structures

As any lifting problem, spin structures on loop spaces can be described by a bundle gerbe, the *spin lifting gerbe* $\mathcal{S}_{LM}$ [CCM98]. We refer to [Wala, Section 4.1] for a detailed treatment. In short, the spin lifting gerbe $\mathcal{S}_{LM}$ is the following bundle gerbe over LM:

(i) it has the surjective submersion $L\pi : LFM \longrightarrow LM$.

(ii) over the 2-fold fibre product $LFM^{[2]}$ it carries the Fréchet principal $\mathrm{U}(1)$-bundle

$$P := L\delta^* \widetilde{L\mathrm{Spin}(n)},$$

where $\delta : FM^{[2]} \longrightarrow \mathrm{Spin}(n)$ is the "difference map" defined by $p' \cdot \delta(p, p') = p$.

(iii) over the 3-fold fibre product $LFM^{[3]}$ it has the bundle gerbe product

$$\mu : \mathrm{pr}_{23}^*P \otimes \mathrm{pr}_{12}^*P \longrightarrow \mathrm{pr}_{13}^*P : ((\beta_2, \beta_3, \tilde{\tau}_{23}) \otimes (\beta_1, \beta_2, \tilde{\tau}_{12})) \longmapsto (\beta_1, \beta_3, \tilde{\tau}_{23} \cdot \tilde{\tau}_{12})$$

defined from the group structure of $L\widetilde{\mathrm{Spin}(n)}$.

The purpose of the lifting bundle gerbe is to provide a reformulation of spin structures in terms of trivializations of $\mathcal{S}_{LM}$. A trivialization of $\mathcal{S}_{LM}$ is by definition a pair $\mathcal{T} = (T, \kappa)$ consisting of a principal U(1)-bundle T over LFM and a bundle isomorphism

$$\kappa\colon \mathrm{pr}_2^*T \otimes P \longrightarrow \mathrm{pr}_1^*T$$

over $LFM^{[2]}$ such that the diagram

$$\begin{array}{ccc} \mathrm{pr}_3^*T \otimes \mathrm{pr}_{23}^*P \otimes \mathrm{pr}_{12}^*P & \xrightarrow{\mathrm{pr}_{23}^*\kappa \otimes \mathrm{id}} & \mathrm{pr}_2^*T \otimes \mathrm{pr}_{12}^*P \\ {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{12}^*\kappa} \\ \mathrm{pr}_3^*T \otimes \mathrm{pr}_{13}^*P & \xrightarrow[\mathrm{pr}_{13}^*\kappa]{} & \mathrm{pr}_1^*T \end{array}$$

of bundle morphisms over $LFM^{[3]}$ is commutative. A morphism between trivializations (T, κ) and (T', κ') is a bundle isomorphism $\varphi : T \longrightarrow T'$ such that the diagram

$$\begin{array}{ccc} \mathrm{pr}_2^*T \otimes P & \xrightarrow{\kappa} & \mathrm{pr}_1^*T \\ {\scriptstyle \mathrm{pr}_2^*\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\varphi} \\ \mathrm{pr}_2^*T' \otimes P & \xrightarrow[\kappa']{} & \mathrm{pr}_1^*T' \end{array}$$

is commutative. Trivializations of $\mathcal{S}_{LM}$ form a category $\mathcal{T}riv(\mathcal{S}_{LM})$, which is a module for the monoidal category $\mathcal{B}un(LM)$ of Fréchet principal U(1)-bundles over LM under the action functor

$$\mathcal{B}un(LM) \times \mathcal{T}riv(\mathcal{S}_{LM}) \longrightarrow \mathcal{T}riv(\mathcal{S}_{LM}) : (K, \mathcal{T}) \longmapsto K \otimes \mathcal{T}. \tag{3.2.1}$$

Here, the trivialization $K \otimes \mathcal{T}$ consists of the principal U(1)-bundle $L\pi^*K \otimes T$ over LFM and of the bundle isomorphism $\mathrm{id} \otimes \kappa$. The action (3.2.1) exhibits $\mathcal{T}riv(\mathcal{S}_{LM})$ as a torsor over $\mathcal{B}un(LM)$. For a spin structure $(\mathcal{S}, \sigma)$ we have a bundle isomorphism

$$\kappa_{\mathcal{S}} : \mathrm{pr}_2^*T_{\mathcal{S}} \otimes P \longrightarrow \mathrm{pr}_1^*T_{\mathcal{S}} : (\beta_2, t) \otimes (\beta_1, \beta_2, \tilde{\tau}) \longmapsto (\beta_1, t \cdot \tilde{\tau})$$

over $LFM^{[2]}$, where $t \cdot \tilde{\tau}$ is the $L\widetilde{\mathrm{Spin}(n)}$-action on $\mathcal{S}$. It is easy to see that $(T_{\mathcal{S}}, \kappa_{\mathcal{S}})$ is a trivialization of the spin lifting gerbe $\mathcal{S}_{LM}$. A morphism $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ between spin structures induces a morphism $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ between bundles over LFM, which is in fact a morphism $(T_{\mathcal{S}}, \kappa_{\mathcal{S}}) \longrightarrow (T_{\mathcal{S}'}, \kappa_{\mathcal{S}'})$ between trivializations. As a consequence of a general theorem of Murray about lifting gerbes [Mur96] we obtain the following result; also see [Wala, Theorem 4.1.3].

Proposition 3.2.1. *The assignment $(\mathcal{S}, \sigma) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}})$ establishes an equivalence of categories:*

$$\mathcal{S}pin(LM) \cong \left\{ \begin{array}{c} \text{Trivializations of the spin} \\ \text{lifting gerbe } \mathcal{S}_{LM} \end{array} \right\}.$$

Formula (3.1.2) shows that the equivalence of Proposition 3.2.1 is equivariant under the actions (3.1.1) and (3.2.1) of $\mathcal{B}un(LM)$. In particular, we obtain the following consequence.

Corollary 3.2.2. *The category $\mathcal{S}pin(LM)$ of spin structures on LM is a torsor over the monoidal category $\mathcal{B}un(LM)$.*

The fusion product $\lambda_{\mathcal{G}_{bas}}$ of the central extension $\widetilde{L\mathrm{Spin}(n)}$ pulls back along the map $L\delta$ to a fusion product $\lambda_P := L\delta^*\lambda_{\mathcal{G}_{bas}}$ on the U(1)-bundle P of the lifting gerbe $\mathcal{S}_{LM}$. The bundle gerbe product μ of $\mathcal{S}_{LM}$ is fusion-preserving according to Lemma 2.3.7.

Suppose $\mathcal{T} = (T, \kappa)$ is a trivialization of $\mathcal{S}_{LM}$. A fusion product λ on T is called *compatible* if the bundle morphism κ is fusion-preserving (with respect to the fusion product λ_P on P). A morphism $\varphi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between two trivializations with fusion products is called *fusion-preserving*, if it is fusion-preserving as a bundle morphism $\varphi : T_1 \longrightarrow T_2$.

Proposition 3.2.3 ([Wala, Corollary 4.4.8])**.** *The assignment $(\mathcal{S}, \sigma, \lambda) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}}, \lambda)$ establishes an equivalence of categories:*

$$\mathcal{S}pin_{fus}(LM) \cong \left\{ \begin{array}{c} \textit{Trivializations of the spin} \\ \textit{lifting gerbe } \mathcal{S}_{LM} \textit{ with} \\ \textit{compatible fusion products} \end{array} \right\}.$$

Next we include thin structures into the lifting-gerbe description. The thin structure d^ν on the central extension $\widetilde{L\mathrm{Spin}(n)}$ pulls back along the map $L\delta$ to a thin structure d_P on the U(1)-bundle P of the lifting gerbe $\mathcal{S}_{LM}$. Suppose $\mathcal{T} = (T, \kappa)$ is a trivialization of $\mathcal{S}_{LM}$. A thin structure d on T is called *compatible*, if κ is a thin bundle morphism (with respect to the thin structure d_P on P). A morphism $\varphi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between trivializations with thin structures is called *thin*, if it is thin as a morphism $\varphi : T_1 \longrightarrow T_2$.

Proposition 3.2.4. *The assignment $(\mathcal{S}, \sigma, d) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}}, d)$ establishes an equivalence of categories:*

$$\mathcal{S}pin^{th}(LM) \cong \left\{ \begin{array}{c} \textit{Trivializations of } \mathcal{S}_{LM} \textit{ with} \\ \textit{compatible thin structures} \end{array} \right\}.$$

Proof. Based on the equivalence of Proposition 3.2.1, we observe that the structure on the objects on both hand sides is the same, namely a thin structure d on the U(1)-bundle $T = T_{\mathcal{S}}$. It remains to check that the conditions are the same. For the trivialization, the condition is that the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^*T \otimes \mathrm{pr}_1^*P & \xrightarrow{\mathrm{pr}_1^*\kappa} & \mathrm{pr}_1^*T \\ {\scriptstyle d\otimes d_P}\downarrow & & \downarrow{\scriptstyle d} \\ \mathrm{pr}_2^*T \otimes \mathrm{pr}_2^*P & \xrightarrow[\mathrm{pr}_2^*\kappa]{} & \mathrm{pr}_2^*T \end{array}$$

over $L(FM^{[2]})^2_{thin} = L(\widetilde{FM \times \mathrm{Spin}(n)})^2_{thin}$ is commutative. We recall the relation $\kappa(t \otimes \tilde{\tau}) = t \cdot \tilde{\tau}$ between κ and the principal $\widetilde{L\mathrm{Spin}(n)}$-action on T. Under this relation, the commutativity of the diagram is equivalent to the following equation:

$$d(t \cdot \tilde{\tau}) = d(\kappa(t \otimes \tilde{\tau}) = \kappa(d(t) \otimes d_P(\tilde{\tau})) = d(t) \cdot d_P(\tilde{\tau}).$$

This is precisely the condition of Definition 3.1.3. For the morphisms, we have on both sides the same condition, namely that $\varphi : T_1 \longrightarrow T_2$ is thin with respect to the thin structures on T_1 and T_2. □

Finally, we combine fusion products and thin structures on trivializations in the following definition.

Definition 3.2.5.

(i) A *thin fusion trivialization* *of the spin lifting gerbe $\mathcal{S}_{LM}$ is a trivialization $\mathcal{T} = (T, \kappa)$ with a fusion product λ on T compatible with λ_P and a thin structure d on T that is fusive with respect to λ and compatible with d_P.*

(ii) A *morphism between thin fusion trivializations* *(T, κ, λ, d) and $(T', \kappa', \lambda', d')$ is a fusion-preserving, thin bundle morphism φ: $T \longrightarrow T'$, such that the diagram*

$$\begin{array}{ccc} \mathrm{pr}_2^*T \otimes P & \xrightarrow{\kappa} & \mathrm{pr}_1^*T \\ {\scriptstyle \mathrm{pr}_2^*\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\varphi} \\ \mathrm{pr}_2^*T' \otimes P & \xrightarrow[\kappa']{} & \mathrm{pr}_1^*T' \end{array}$$

commutes in the homotopy category $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$.

The condition that the diagram in (ii) commutes in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$, i.e. up to homotopy through thin, fusion-preserving bundle morphisms means, explicitly, that there is a smooth map

$$H : [0,1] \times \mathrm{pr}_2^*T_{\mathcal{S}} \otimes P \longrightarrow \mathrm{pr}_1^*T'_{\mathcal{S}} \tag{3.2.2}$$

such that $H_t : \mathrm{pr}_2^*T_{\mathcal{S}} \otimes P \longrightarrow \mathrm{pr}_1^*T'_{\mathcal{S}}$ is a thin, fusion-preserving bundle morphism for all $t \in [0,1]$ and we have $H_0 = \mathrm{pr}_1^*\varphi \circ \kappa$ and $H_1 = \kappa' \circ \mathrm{pr}_2^*\varphi$.

The category of thin fusion trivializations is denoted by $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$. Based on the action functor (3.2.1), it is straightforward to see that it is a module over the monoidal category $\mathcal{F}us\mathcal{B}un^{th}(LM)$.

Proposition 3.2.6. *The assignment $(\mathcal{S}, \sigma, \lambda, d) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}}, \lambda, d)$ establishes an equivalence of categories:*

$$\mathcal{S}pin_{fus}^{th}(LM) \cong \mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}).$$

Moreover, it is equivariant with respect to the $\mathcal{F}us\mathcal{B}un^{th}(LM)$-actions on both categories.

Proof. Based on the equivalence of Proposition 3.2.1 and its extension to fusion products (Proposition 3.2.3) and thin structures (Proposition 3.2.4) it remains to notice, on the level of objects, that the compatibility condition between fusion product and thin structure is the same on both sides. Concerning the morphisms, we first observe that we have, in both categories, thin, fusion-preserving morphisms $\varphi : T_{\mathcal{S}} \longrightarrow T_{\mathcal{S}'}$ between U(1)-bundles over LFM. It remains to check that the commutativity in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$ of (ii) of Definition 3.2.5 is equivalent to (iv) of Definition 3.1.6. In order to see this, we notice that the existence of the map H in (3.2.2) is equivalent to the existence of a fusion homotopy

$$h : [0,1] \times LFM^{[2]} \longrightarrow \mathrm{U}(1)$$

with $h_0 = 1$ and

$$h_1 \cdot (\mathrm{pr}_1^*\varphi \circ \kappa) = \kappa' \circ \mathrm{pr}_2^*\varphi \tag{3.2.3}$$

Now, under the correspondence $(\kappa(t \otimes \tilde{\tau}) = t \cdot \tilde{\tau})$ between the bundle morphisms κ and κ' with the $\widetilde{L\mathrm{Spin}(n)}$-action on $\mathcal{S}$ and $\mathcal{S}'$, respectively, (3.2.3) is precisely Equation (3.1.3) in Definition 3.1.6. □

4 Superficial spin connections

In Section 4.1 we study the notion of a spin connection introduced by Coquereaux and Pilch [CP98], and circumstances under which they induce thin spin structures. We couple spin connections to fusion products and introduce the notion of a superficial geometric fusion spin structure. In Section 4.2 we develop the corresponding lifting gerbe theory.

4.1 Spin connections on loop spaces

In the following we denote by $\mathfrak{g}$ the Lie algebra of $\mathrm{Spin}(n)$. The Levi-Cevita connection on M induces a connection $A \in \Omega^1(FM, \mathfrak{g})$ on the spin-oriented frame bundle FM. One can define a 1-form $\bar{A} \in \Omega^1(LFM, L\mathfrak{g})$ by

$$\bar{A}|_\tau(X)(z) := A|_{\tau(z)}(X(z)),$$

where $\tau \in LFM$ and $X \in T_\tau LFM$. It is straightforward to check that $\bar{A}$ is a connection on LFM.

Definition 4.1.1 ([CP98])**.** *Let $(\mathcal{S}, \sigma)$ be a spin structure on LM. A* <u>*spin connection*</u> *on $(\mathcal{S}, \sigma)$ is a connection $\Omega \in \Omega^1(\mathcal{S}, \widetilde{L\mathfrak{g}})$ on $\mathcal{S}$ such that $p_*(\Omega) = \sigma^*\bar{A}$, where $p_* : \widetilde{L\mathfrak{g}} \longrightarrow L\mathfrak{g}$ is the projection in the Lie algebra extension.*

A triple $(\mathcal{S}, \sigma, \Omega)$ consisting of a spin structure and a spin connection is called a *geometric spin structure on LM*. Geometric spin structures form a category $\mathcal{S}pin^\nabla(LM)$ whose morphisms are connection-preserving morphisms between spin structures. This category is a module for the monoidal category $\mathcal{B}un^\nabla(LM)$ of Fréchet principal U(1)-bundles over LM with connection, in terms of an action functor

$$\mathcal{B}un^\nabla(LM) \times \mathcal{S}pin^\nabla(LM) \longrightarrow \mathcal{S}pin^\nabla(LM) \tag{4.1.1}$$

lifting the action (3.1.1) of $\mathcal{B}un(LM)$ on $\mathcal{S}pin(LM)$ to a setting with connections. If K is a principal U(1)-bundle over LM with connection $\eta \in \Omega^1(K)$, and $(\mathcal{S}, \sigma, \Omega)$ is a geometric spin structure, then a spin connection on the spin structure $K \otimes \mathcal{S} = (K \times_{LM} \mathcal{S})/\,\mathrm{U}(1)$ is defined by the 1-form

$$\eta \otimes \Omega := \mathrm{pr}_1^*\eta + \mathrm{pr}_2^*\Omega \in \Omega^1(K \times_{LM} \mathcal{S})$$

that descends to a connection on $K \otimes \mathcal{S}$.

We introduce a notion of *scalar curvature* of a spin connection. For this purpose, we need the splitting $s\colon L\mathfrak{g} \longrightarrow \widetilde{L\mathfrak{g}}$ described in Section 2.2 as the horizontal lift with respect to the connection ν, as well as the associated map Z of Lemma 2.2.1. Further, we need a *reduction of LFM adapted to s*, i.e. a map

$$r : LFM \times L\mathfrak{g} \longrightarrow \mathbb{R}$$

that is linear in the second argument and satisfies

$$r(\tau \cdot \gamma, \mathrm{Ad}_\gamma^{-1}(X)) = r(\tau, X) - Z(\gamma, X) \tag{4.1.2}$$

for all $\tau \in LFM$, $X \in L\mathfrak{g}$ and $\gamma \in L\mathrm{Spin}(n)$. Such a reduction can be defined using the connection A on FM, by setting [Gom03, Proposition 6.2]

$$r(\tau, X) := -2 \int_{S^1} \langle \tau^* A, X \rangle \,. \tag{4.1.3}$$

In order to define the announced scalar curvature we produce the auxiliary map

$$R : \mathcal{S} \times \widetilde{L\mathfrak{g}} \longrightarrow \mathbb{R} : (\beta, \hat{X}) \longmapsto \hat{X} - s(p_*(\hat{X})) + r(\sigma(\beta), p_*(\hat{X})).$$

Then we define a 2-form $\psi \in \Omega^2(\mathcal{S})$ by the formula $\psi_t(X,Y) := R(t, \mathrm{curv}(\Omega)_t(X,Y))$ where $t \in \mathcal{S}$ and $X, Y \in T_t\mathcal{S}$. The scalar curvature is now defined as follows.

Lemma 4.1.2. *There is a unique 2-form* $\mathrm{scurv}(\Omega) \in \Omega^2(LM)$ *such that* $L\pi^*\mathrm{scurv}(\Omega) = \psi$.

Proof. We show that ψ descends. Using (4.1.2) it is straightforward to show that

$$R(t\gamma, \mathrm{Ad}_\gamma^{-1}(\hat{X})) = R(t, \hat{X})$$

for all $t \in \mathcal{S}$, $\gamma \in \widetilde{L\mathrm{Spin}(n)}$, and $\hat{X} \in \widetilde{L\mathfrak{g}}$. On the other hand, the curvature satisfies

$$\mathrm{pr}_2^*\mathrm{curv}(\Omega) = \mathrm{Ad}_\delta^{-1}(\mathrm{pr}_1^*\mathrm{curv}(\Omega))$$

over $\mathcal{S}^{[2]}$, where $\delta : \mathcal{S}^{[2]} \longrightarrow \widetilde{L\mathrm{Spin}(n)}$ is the difference map of the principal $\widetilde{L\mathrm{Spin}(n)}$-bundle $\mathcal{S}$. This shows that $\mathrm{pr}_2^*\psi = \mathrm{pr}_1^*\psi$ over $\mathcal{S}^{[2]}$. □

We will see (Theorems 4.2.11 and 7.3) that under the correspondence between geometric string structures on M and geometric spin structures on LM, the scalar curvature is (minus) the transgression of the 3-form $K \in \Omega^3(M)$ associated to a geometric string structure (see Theorem (6.1.3) (ii)).

Let $(\mathcal{S}, \sigma, \Omega)$ be a geometric spin structure on LM, and let $T_{\mathcal{S}}$ be the associated principal U(1)-bundle over LFM. We consider the 1-form $\zeta \in \Omega^1(LFM)$ defined by

$$\zeta_\tau(X) := r(\tau, \bar{A}_\tau(X)) \tag{4.1.4}$$

for $\tau \in LFM$ and $X \in T_\tau LFM$.

Lemma 4.1.3. *For every* $x \in \mathbb{R}$*, the formula*

$$\omega_{\Omega,x} := \Omega - s(\sigma^*\bar{A}) + \frac{x}{2}\sigma^*\zeta \in \Omega^1(T_{\mathcal{S}})$$

defines a connection on $T_{\mathcal{S}}$*, of curvature*

$$\mathrm{curv}(\omega_{\Omega,x}) = L\pi^*\mathrm{scurv}(\Omega) - \frac{1}{2}\omega(\bar{A} \wedge \bar{A}) - r(\mathrm{curv}(\bar{A})) + \frac{x}{2}\mathrm{d}\zeta.$$

Proof. It suffices to prove that $\omega_{\Omega,0}$ is a connection, which is a standard calculation. The connection $\omega_{\Omega,x}$ is then obtained by shifting the connection $\omega_{\Omega,0}$ by the 1-form $\frac{x}{2}\zeta$. In order to compute the curvature of $\omega_{\Omega,0}$ we use the definition of the scalar curvature and obtain:

$$\mathrm{d}\omega_{\Omega,0} = \mathrm{d}\Omega - s(\sigma^*\mathrm{d}\bar{A}) = L\pi^*\mathrm{scurv}(\Omega) - \frac{1}{2}[\Omega \wedge \Omega] + s(p_*(\mathrm{curv}(\Omega))) - s(\sigma^*\mathrm{d}\bar{A}) - r(\sigma, p_*(\mathrm{curv}(\Omega)))$$

Then we use that $\Omega - s(\sigma^*\bar{A}) = \Omega - s(p_*\Omega) \in \mathbb{R}$, so that

$$\begin{aligned} 0 = [\Omega - s(\sigma^*\bar{A}) \wedge \Omega - s(\sigma^*\bar{A})] &= [\Omega \wedge \Omega] + [s(\sigma^*\bar{A}) \wedge s(\sigma^*\bar{A})] - 2[\Omega \wedge s(\sigma^*\bar{A})] \\ &= [\Omega \wedge \Omega] - [s(\sigma^*\bar{A}) \wedge s(\sigma^*\bar{A})]. \end{aligned}$$

This yields the claimed result. □

We recall that a morphism between geometric spin structures $(\mathcal{S},\sigma,\Omega)$ and $(\mathcal{S}',\sigma',\Omega')$ is an isomorphism $f:\mathcal{S}\longrightarrow\mathcal{S}'$ of $L\widetilde{\mathrm{Spin}(n)}$-bundles over LM such that $\Omega=f^*\Omega'$ and $\sigma'\circ f=\sigma$. It follows that $\omega_{\Omega,x}=f^*\omega_{\Omega',x}$ for all $x\in\mathbb{R}$, i.e. the induced isomorphism $f:T_{\mathcal{S}}\longrightarrow T_{\mathcal{S}'}$ of U(1)-bundles is connection-preserving for all connections $\omega_{\Omega,x}$. Further, we find under the action (4.1.1) of the monoidal category $\mathcal{B}un^{\nabla}(LM)$ on the category $\mathcal{S}pin^{\nabla}(LM)$ of geometric spin structures the formula $\omega_{\eta\otimes\Omega,x}=L\pi^*\eta+\omega_{\Omega,x}$.

The following result explains which connection of the one-parameter family $\omega_{\Omega,x}$ should be used.

Proposition 4.1.4. *Let $(\mathcal{S},\sigma,\Omega)$ be a geometric spin structure. Suppose the connection $\omega_{\Omega,x}$ is superficial for a parameter $x\in\mathbb{R}$, and let $d^{\omega_{\Omega,x}}$ be the associated thin structure on $T_{\mathcal{S}}$. Then, the pair $(\mathcal{S},\sigma,d^{\omega_x})$ is a thin spin structure if and only if $x=1$.*

We prepare the proof of this proposition with three lemmata.

Lemma 4.1.5. *Suppose $\omega_{\Omega,x}$ is superficial. Then, $(\mathcal{S},\sigma,d^{\omega_{\Omega,x}})$ is a thin spin structure if and only if the following holds: for $(\gamma_1,\gamma_2):[0,1]\longrightarrow LFM^{[2]}$ a thin path, $\tilde\gamma_2$ a $\omega_{\Omega,x}$-horizontal lift of γ_2, $\tilde\delta$ a ν-horizontal lift of the path δ defined by $\delta(t):=\delta(\gamma_1(t),\gamma_2(t))$, we have for $\tilde\gamma_1:=\tilde\gamma_2\cdot\tilde\delta$*

$$\int_{\tilde\gamma_1}\omega_{\Omega,x}\in\mathbb{Z}.$$

Proof. $(\mathcal{S},\sigma,d^{\omega_{\Omega,x}})$ is a thin spin structure if and only condition of Definition 3.1.3 is satisfied:

$$d^{\omega_{\Omega,x}}_{\tau_1\cdot\beta_1,\tau_2\cdot\beta_2}(t\cdot\tilde\tau)=d^{\omega_{\Omega_x}}_{\tau_1,\tau_2}(t)\cdot d^{\nu}_{\beta_1,\beta_2}(\tilde\tau)$$

for all $((\tau_1,\beta_1),(\tau_2,\beta_2))\in L(FM\times\mathrm{Spin}(n))^2_{thin}$, all $t\in T_{\mathcal{S}}$ projecting to τ_1, and all $\tilde\tau\in L\widetilde{\mathrm{Spin}(n)}$ projecting β_1. We denote by pt^{ν}_{δ} and $pt^{\omega_{\Omega,x}}_{\gamma_k}$ the parallel transport maps associated to the connections ν on $L\widetilde{\mathrm{Spin}(n)}$ and $\omega_{\Omega,x}$ on $T_{\mathcal{S}}$, along the paths δ, γ_1, and γ_2, respectively. Then, above condition is equivalent to the assertion that

$$pt^{\omega_{\Omega,x}}_{\gamma_1}(t\cdot\tilde\tau)=pt^{\omega_{\Omega,x}}_{\gamma_2}(t)\cdot pt^{\nu}_{\delta}(\tilde\tau) \tag{4.1.5}$$

holds for all thin paths $(\gamma_1,\gamma_2):[0,1]\longrightarrow LFM^{[2]}$ with difference path $\delta:=L\delta(\gamma_1,\gamma_2)$, elements $t\in\mathcal{S}$ projecting to $\gamma_1(0)$ and $\tilde\tau\in L\widetilde{\mathrm{Spin}(n)}$ projecting to $\delta(0)$. We note that

$$pt^{\omega_{\Omega,x}}_{\gamma_1}(t\cdot\tilde\tau)=\tilde\gamma_1(1)\cdot\exp\left(2\pi\mathrm{i}\int_{\tilde\gamma_1}\omega_{\Omega,x}\right)$$

for *all* lifts $\tilde\gamma_1$ of γ_1 with $\tilde\gamma_1(0)=t\cdot\tilde\tau$. For horizontal lifts $\tilde\gamma_2$ with $\tilde\gamma_2(0)=t$ and $\tilde\delta$ with $\tilde\delta(0)=\tilde\tau$ we have $pt^{\omega_{\Omega,x}}_{\gamma_2}(t)=\tilde\gamma_2(1)$ and $pt^{\nu}_{\delta}(\tilde\tau)=\tilde\delta(1)$. With these formulas, (4.1.5) is equivalent to the assertion that

$$\exp\left(2\pi\mathrm{i}\int_{\tilde\gamma_1}\omega_{\Omega,x}\right)=1$$

for every thin path (γ_1,γ_2), difference δ, horizontal lifts $\tilde\gamma_2$ and $\tilde\delta$, and $\tilde\gamma_1:=\tilde\gamma_2\cdot\tilde\delta$. □

The following straightforward calculation only uses property (4.1.2) of the reduction r and the defining property of the connection A.

Lemma 4.1.6. *The 1-form ζ satisfies the identity*

$$(\Delta\zeta)_{\tau_1,\tau_2}(X_1,X_2) = -r(\tau_2, Y\delta^{-1}) + Z(\delta, \bar{A}_{\tau_2}(X_2)) + Z(\delta, Y\delta^{-1}),$$

where $(\tau_1,\tau_2) \in LFM^{[2]}$, $X_1 \in T_{\tau_1}LFM$, $X_2 \in T_{\tau_2}LFM$, $\delta \in L\mathrm{Spin}(n)$ is defined by the formula $\delta(z) := \delta(\tau_1(z),\tau_2(z))$, and $Y \in T_\delta L\mathrm{Spin}(n)$ is defined by $Y(z) := \mathrm{d}\delta(X_1(z), X_2(z))$.

Now we are prepared for the following key calculation.

Lemma 4.1.7. *Let $(\gamma_1,\gamma_2) : [0,1] \longrightarrow LFM^{[2]}$ be a path with $\delta : [0,1] \longrightarrow L\mathrm{Spin}(n)$ defined by $\delta(t) := \delta(\gamma_1(t),\gamma_2(t))$. Assume that $\tilde{\gamma}_2$ is a $\omega_{\Omega,x}$-horizontal lift of γ_2, and that $\tilde{\delta}$ is a ν-horizontal lift of δ. Then, $\tilde{\gamma}_1 := \tilde{\gamma}_2 \cdot \tilde{\delta}$ is a lift of γ_1, and*

$$\omega_{\Omega,x}(\partial_t\tilde{\gamma}_1(t)) \;=\; \frac{1-x}{2}Z(\delta(t), \partial_t\delta(t)\delta(t)^{-1}) + \frac{2-x}{2}Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + \frac{x}{2}r(\gamma_2(t), \partial_t\delta(t)\delta(t)^{-1}).$$

Proof. The assumptions of horizontality mean that

$$\omega_{\Omega,t}(\partial_t\tilde{\gamma}_2(t)) = 0 \quad\text{and}\quad \nu(\partial_t\tilde{\delta}(t)) = 0 \tag{4.1.6}$$

for all $t \in [0,1]$. The relation $\omega_{\Omega,x} = \Omega - s(\sigma^*\bar{A}) + \frac{x}{2}\sigma^*\zeta$ from the definition of $\omega_{\Omega,x}$ shows:

$$\begin{aligned}
\mathrm{Ad}^{-1}_{\tilde{\delta}(t)}(\Omega(\partial_t\tilde{\gamma}_2(t)))) &= \mathrm{Ad}^{-1}_{\tilde{\delta}(t)}(s(\sigma^*\bar{A}(\partial_t\tilde{\gamma}_2(t))) + \omega_{\Omega,x}(\partial_t\tilde{\gamma}_2(t))) - \frac{x}{2}\zeta(\partial_t\gamma_2(t)) \qquad (4.1.7)\\
&\overset{(4.1.6)}{=} \mathrm{Ad}^{-1}_{\tilde{\delta}(t)}(s(\bar{A}(\partial_t\gamma_2(t))) - \frac{x}{2}\zeta(\partial_t\gamma_2(t))\\
&= Z(\tilde{\delta}(t), \bar{A}(\partial_t\gamma_2(t))) + s(\mathrm{Ad}^{-1}_{\delta(t)}(\bar{A}(\partial_t\gamma_2(t)))) - \frac{x}{2}\zeta(\partial_t\gamma_2(t)).
\end{aligned}$$

That Ω is a $\widetilde{L\mathrm{Spin}(n)}$-connection implies then the following:

$$\begin{aligned}
\Omega(\partial_t\tilde{\gamma}_1(t)) &= \mathrm{Ad}^{-1}_{\tilde{\delta}(t)}(\Omega(\partial_t\tilde{\gamma}_2(t))) + \tilde{\delta}(t)^{-1}\partial_t\tilde{\delta}(t) \qquad (4.1.8)\\
&\overset{(4.1.7)}{=} Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + s(\mathrm{Ad}^{-1}_{\delta(t)}(\bar{A}(\partial_t\gamma_2(t)))) - \frac{x}{2}\zeta(\partial_t\gamma_2(t)) + \tilde{\delta}(t)^{-1}\partial_t\tilde{\delta}(t).
\end{aligned}$$

The definition of the splitting s from the connection ν implies:

$$X - s(p_*(X)) = \nu_1(g^{-1}X) \overset{(2.2.4)}{=} \nu_g(X) - \epsilon_\nu|_{g^{-1},g}(0,X). \tag{4.1.9}$$

Now we put the pieces together and obtain:

$$\begin{aligned}
&\omega_{\Omega,x}(\partial_t\tilde{\gamma}_1(t))\\
&= \Omega(\partial_t\tilde{\gamma}_1(t)) - s(\sigma^*\bar{A}(\partial_t\tilde{\gamma}_1(t))) + \frac{x}{2}\sigma^*\zeta(\partial_t\tilde{\gamma}_1(t))\\
&\overset{(4.1.8)}{=} Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + s(\mathrm{Ad}^{-1}_{\delta(t)}(\bar{A}(\partial_t\gamma_2(t))))\\
&\qquad - \frac{x}{2}\zeta(\partial_t\gamma_2(t)) + \tilde{\delta}(t)^{-1}\partial_t\tilde{\delta}(t) - s(\bar{A}(\partial_t\gamma_1(t))) + \frac{x}{2}\zeta(\partial_t\gamma_1(t))\\
&= -s(\delta(t)^{-1}\partial_t\delta(t)) + \tilde{\delta}(t)^{-1}\partial_t\tilde{\delta}(t) + Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + \frac{x}{2}\Delta\zeta(\partial_t\gamma_1(t), \partial_t\gamma_2(t))\\
&\overset{(4.1.9)}{=} \nu_{\tilde{\delta}(t)}(\partial_t\tilde{\delta}(t)) - \epsilon_\nu|_{\tilde{\delta}(t)^{-1},\tilde{\delta}(t)}(0, \partial_t\tilde{\delta}(t)) + Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + \frac{x}{2}\Delta\zeta(\partial_t\gamma_1(t), \partial_t\gamma_2(t))\\
&\overset{(4.1.6)}{=} -\epsilon_\nu|_{\delta(t)^{-1},\delta(t)}(0, \partial_t\delta(t)) + Z(\delta(t), \bar{A}(\partial_t\gamma_2(t))) + \frac{x}{2}\Delta\zeta(\partial_t\gamma_1(t), \partial_t\gamma_2(t)).
\end{aligned}$$

With Lemma 4.1.6 this simplifies to

$$\omega_{\Omega,x}(\partial_t\tilde{\gamma}_1(t)) = -\epsilon_\nu|_{\delta(t)^{-1},\delta(t)}(0,\partial_t\delta(t)) + \frac{2-x}{2}Z(\delta(t),\bar{A}(\partial_t\gamma_2(t))) \\ - \frac{x}{2}Z(\delta(t),\partial_t\delta(t)\delta(t)^{-1}) + \frac{x}{2}r(\gamma_2(t),\partial_t\delta(t)\delta(t)^{-1}).$$

Substituting explicit expressions, we get $\frac{1}{2}Z(\delta(t),\partial_t\delta(t)\delta(t)^{-1}) = -\epsilon_\nu|_{\delta(t)^{-1},\delta(t)}(0,\partial_t\delta(t))$; this yields the claimed formula. □

Now we are in position to prove Proposition 4.1.4, and start with the "if"-part. Suppose $\omega_{\Omega,1}$ is superficial. According to Lemma 4.1.5, it suffices to prove that for all thin paths $(\gamma_1,\gamma_2):[0,1]\longrightarrow LFM^{[2]}$, all horizontal lifts $\tilde{\gamma}_2$ of γ_2 and $\tilde{\delta}$ of δ we get $\omega_{\Omega,1}(\partial_t\tilde{\gamma}_1(t)) = 0$ for $\tilde{\gamma}_1 := \tilde{\gamma}_2\cdot\tilde{\delta}$ and all $t\in[0,1]$. By Lemma 4.1.7 this is given by

$$\omega_{\Omega,1}(\partial_t\tilde{\gamma}_1(t)) = \frac{1}{2}Z(\delta(t),\bar{A}(\partial_t\gamma_2(t))) + \frac{1}{2}r(\gamma_2(t),\partial_t\delta(t)\delta(t)^{-1}).$$

Explicitly, this is

$$\omega_{\Omega,1}(\partial_t\tilde{\gamma}_1(t)) = \int_0^1 \Big\{ \big\langle \partial_z\delta^\vee(t,z)\delta^\vee(t,z)^{-1}, A(\partial_t\gamma_2^\vee(t,z))\big\rangle - \big\langle A(\partial_z\gamma_2^\vee(t,z)), \partial_t\delta^\vee(t,z)\delta^\vee(t,z)^{-1}\big\rangle \Big\}\,\mathrm{d}z.$$

The assumption that $(\gamma_1,\gamma_2)^\vee:[0,1]\times S^1\longrightarrow FM^{[2]}$ is a rank one map implies that for every (t,z) there exist $\alpha,\beta\in\mathbb{R}$, not both equal to zero, such that

$$\alpha\partial_t(\gamma_1,\gamma_2)^\vee(t,z) = \beta\partial_z(\gamma_1,\gamma_2)^\vee(t,z)$$

in $TFM\oplus TFM$, i.e. $\alpha\partial_t\gamma_1^\vee(t,z) = \beta\partial_z\gamma_1^\vee(t,z)$ and $\alpha\partial_t\gamma_2^\vee(t,z) = \beta\partial_z\gamma_2^\vee(t,z)$. These imply $\alpha\partial_t\delta^\vee(t,z) = \beta\partial_z\delta^\vee(t,z)$. Assuming either $\alpha\neq 0$ or $\beta\neq 0$, one can see by inspection that the integrand in above formula vanishes identically for every (t,z). □

We are left with the proof of the "only if"-part of Proposition 4.1.4. We assume $x\neq 1$ and produce a counterexample, i.e. appropriate paths for which the integral in Lemma 4.1.5 does not vanish. Let γ_2 be the constant path at a constant loop at a point $p\in FM$, i.e. $\gamma_2(t)(z) := p$. Let δ be a thin path in $L\mathrm{Spin}(n)$, to be specified later. Then, with $\gamma_1 := \gamma_2\cdot\delta$ we have a thin path (γ_1,γ_2) in $LFM^{[2]}$. We compute the quantity $\omega_{\Omega,x}(\partial_t\tilde{\gamma}_1(t))$ of Lemma 4.1.7. The second term in the formula of Lemma 4.1.7, $Z(\delta(t),\bar{A}(\partial_t\gamma_2(t)))$, vanishes since γ_2 is constant and Z is linear in the second argument. Likewise, the third term vanishes, using the definition (4.1.3) of the reduction r and again that γ_2 is constant: For the first term, however, we have

$$Z(\delta(t),\partial_t\delta(t)\delta(t)^{-1}) = 2\int_0^1 \big\langle \partial_z\delta^\vee(t,z)\delta^\vee(t,z)^{-1}, \partial_t\delta^\vee(t,z)\delta^\vee(t,z)^{-1}\big\rangle\,\mathrm{d}z. \tag{4.1.10}$$

Now we construct a specific thin path δ. Let $\tau\in L\mathrm{Spin}(n)$ be a non-constant loop, and let $\delta(t)(z) := \tau(ze^{2\pi\mathrm{i}t})$, the full rotation of the loop τ. Note that

$$\partial_z\delta^\vee(t,z) = \partial_z\tau(ze^{2\pi\mathrm{i}t}) = \partial_t\delta^\vee(t,z),$$

i.e. $\delta^\vee$ is thin and the linear dependence is expressed by constant coefficients. Thus, the integrand in (4.1.10) is quadratic, hence non-negative, and even positive at at least one $z\in S^1$ point as τ is non-constant. Thus therefore,

$$y_t := Z(\delta(t),\partial_t\delta(t)\delta(t)^{-1}) > 0$$

for all $t \in [0,1]$. It follows that

$$\int_{\tilde{\gamma}_1} \omega_{\Omega,x} = \int_0^1 \omega_{\Omega,x}(\partial_t \tilde{\gamma}_1(t))\mathrm{d}t = \frac{1-x}{2}\int_0^1 y_t \,\mathrm{d}t,$$

which is non-zero as $x \neq 1$. Note that one can scale this quantity continuously down to zero with a parameter $0 \leq \epsilon \leq 1$, by simply letting all paths end at ϵ instead at 1. In particular, it can be arranged to be not an integer, hence

$$\exp\left(2\pi\mathrm{i}\int_{\tilde{\gamma}_1} \omega_{\Omega,x}\right) \neq 1.$$

□

Due to Proposition 4.1.4 we promptly set $\omega_\Omega := \omega_{\Omega,1}$ as the connection of our choice on $T_{\mathcal{S}}$. Note that this is a non-standard choice, other treatments of geometric lifting problems choose $\omega_{\Omega,0}$ – e.g. [Gom03, Wal11].

Definition 4.1.8. *A spin connection Ω on $\mathcal{S}$ is called* <u>*superficial*</u>*, if the connection ω_Ω on $T_{\mathcal{S}}$ is superficial. A geometric spin structure with superficial spin connection is called a* <u>*superficial geometric spin structure*</u>*.*

Together with the connection-preserving isomorphisms between spin structures, superficial geometric spin structures form a category that we denote by $\mathcal{S}pin^{\nabla_{sf}}(LM)$. Due to Proposition 4.1.4, we obtain a functor

$$\mathcal{S}pin^{\nabla_{sf}}(LM) \longrightarrow \mathcal{S}pin^{th}(LM) : (\mathcal{S},\sigma,\Omega) \longmapsto (\mathcal{S},\sigma,d^{\omega_\Omega}).$$

This functor guarantees the consistency of the various versions of spin structures upon passing from the setting with connections to the setting without connections.

Definition 4.1.9. *A* <u>*superficial geometric fusion spin structure*</u> *on LM is a spin structure $(\mathcal{S},\sigma)$ together with a fusion product λ and a superficial spin connection Ω, such that ω_Ω is fusive with respect to λ.*

Morphisms between superficial geometric fusion spin structures are connection-preserving, fusion-preserving morphisms of spin structures. Superficial geometric fusion spin structures form a category that we denote by $\mathcal{S}pin_{fus}^{\nabla_{sf}}(LM)$. This category is our loop space formulation of the category of geometric string structures. The action (4.1.1) of the monoidal category $\mathcal{B}un^{\nabla}(LM)$ on geometric spin structures extends to an action

$$\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM) \times \mathcal{S}pin_{fus}^{\nabla_{sf}}(LM) \longrightarrow \mathcal{S}pin_{fus}^{\nabla_{sf}}(LM).$$

We will see (Corollary 4.2.12 and Theorem 7.3) that this action exhibits $\mathcal{S}pin_{fus}^{\nabla_{sf}}(LM)$ as a torsor over $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM)$.

It is clear from the construction that the passage from the setting with connections to the setting without connections also works in the presence of fusion products, i.e. we have a functor

$$\mathcal{S}pin_{fus}^{\nabla_{sf}}(LM) \longrightarrow \mathcal{S}pin_{fus}^{th}(LM) : (\mathcal{S},\sigma,\lambda,\Omega) \longmapsto (\mathcal{S},\sigma,\lambda,d^{\omega_\Omega}). \tag{4.1.11}$$

On the level of morphisms, this functor produces honest morphisms $f : \mathcal{S} \longrightarrow \mathcal{S}'$ between $\widetilde{L\mathrm{Spin}(n)}$-bundles; these form a subset of the morphisms of $\mathcal{S}pin_{fus}^{th}(LM)$ defined in Definition 3.1.6 that is characterized by the condition that the fusion homotopy in (iv) of that definition is constant. In particular, the functor (4.1.11) is not full – just as one would expect it from the passage from a setting with connections to a setting without connections.

4.2 Lifting theory for spin connections

We equip the spin lifting gerbe $\mathcal{S}_{LM}$ with a connection. We recall that $P := L\delta^* L\widetilde{\mathrm{Spin}(n)}$ is the principal U(1)-bundle of $\mathcal{S}_{LM}$ over $LFM^{[2]}$. It is equipped with the pullback connection $L\delta^*\nu$, but the bundle gerbe product μ is not connection-preserving for the connection ν. Indeed, we have seen in Section 2.2 that

$$\mathrm{pr}_{23}^* L\delta^*\nu + \mathrm{pr}_{12}^* L\delta^*\nu = \mathrm{pr}_{13}^* L\delta^*\nu + L\delta_2^*\epsilon_\nu. \tag{4.2.1}$$

We now modify the connection $L\delta^*\nu$ such that μ becomes connection-preserving. For this purpose, we consider the 1-form $Z(L\delta, \mathrm{pr}_2^*\bar{A}) \in \Omega^1(LFM^{[2]})$ and the sum

$$\xi := L\delta^*\beta + Z(L\delta, \mathrm{pr}_2^*\bar{A}) \in \Omega^1(LFM^{[2]}), \tag{4.2.2}$$

where β is the 1-form defined in Lemma 2.2.3. Under the simplicial operator

$$\Delta := \mathrm{pr}_{23}^* + \mathrm{pr}_{12}^* - \mathrm{pr}_{13}^* : \Omega^k(FM^{[2]}) \longrightarrow \Omega^k(FM^{[3]})$$

we obtain the following result.

Lemma 4.2.1. $\Delta\xi = -L\delta_2^*\epsilon_\nu$.

Proof. We make two calculations. First we calculate $\Delta\beta := \mathrm{pr}_1^*\beta + \mathrm{pr}_2^*\beta - m^*\beta \in \Omega^1(LG^2)$, using that $\mathrm{d}m_{\tau_1,\tau_2}(X_1, X_2) = X_1\tau_2 + \tau_1 X_2$. The result is

$$(\Delta\beta)_{\tau_1,\tau_2}(X_1, X_2) = -\int_0^1 \left\{\left\langle \tau_1(z)^{-1}\partial_z\tau_1(z), X_2(z)\tau_2^{-1}(z)\right\rangle - \left\langle \partial_z\tau_2(z)\tau_2(z)^{-1}, \tau_1(z)^{-1}X_1(z)\right\rangle\right\} \mathrm{d}z.$$

For the second calculation we use the notation $\delta_{ij} := L\delta(\tau_i, \tau_j)$, in which $\delta_{13} = \delta_{23}\delta_{12}$ holds, and obtain

$$(\Delta Z(L\delta, \mathrm{pr}_2^*\bar{A}))_{\tau_1,\tau_2,\tau_3}(X_1, X_2, X_3) = 2\int_0^1 \left\langle \partial_z\delta_{12}(z)\delta_{12}(z)^{-1}, \delta_{23}(z)^{-1}Y_{23}(z)\right\rangle \mathrm{d}z,$$

where $Y_{23} := \mathrm{d}L\delta(X_2, X_3)$. We have the relation $\Delta \circ L\delta^* = L\delta_2^* \circ \Delta$ which implements the fact that δ is a chain map between simplicial manifolds. Putting the two calculations together and identifying the result under (2.2.4) we obtain the claimed result. □

We also need to calculate the derivative of the 1-form ξ.

Lemma 4.2.2. $\mathrm{d}\xi = -\frac{1}{2}L\delta^*\omega(\theta\wedge\theta) - L\delta^*\mathrm{curv}(\nu) + Z(L\delta, \mathrm{pr}_2^*\mathrm{d}\bar{A}) + \omega(L\delta^*\theta \wedge \mathrm{Ad}_{L\delta}^{-1}(\mathrm{pr}_2^*\bar{A}))$.

Proof. We have [Gom03, Lemma 5.8 (a)]

$$\mathrm{d}Z(L\delta, \mathrm{pr}_2^*\bar{A}) = Z(L\delta, \mathrm{pr}_2^*\mathrm{d}\bar{A}) - \omega(L\delta^*\theta \wedge \mathrm{Ad}_{L\delta}^{-1}(\mathrm{pr}_2^*\bar{A})).$$

With Lemma 2.2.3 we obtain the claimed formula. □

In the following we consider the 1-form $\xi - \frac{1}{2}\Delta\zeta \in \Omega^1(FM^{[2]})$, with ζ the 1-form defined at the beginning of Section 4.1.

Lemma 4.2.3. *The 1-form $\xi - \frac{1}{2}\Delta\zeta \in \Omega^1(LFM^{[2]})$ is a superficial fusion form, i.e. a superficial fusive connection on the trivial bundle with respect to the trivial fusion product.*

Proof. This can be verified directly; however, we show in Lemma 7.2 that $\xi - \frac{1}{2}\Delta\zeta$ is in the image of the transgression homomorphism 2.2.3; such forms are automatically superficial [Wal12c, Lemma 3.1.7] and fusion [Wal12c, Proposition 3.2.3]. □

We consider on $P = L\delta^* L\widetilde{\mathrm{Spin}(n)}$ the connection

$$\chi_{spin} := L\delta^*\nu + (\xi - \frac{1}{2}\Delta\zeta). \tag{4.2.3}$$

Proposition 4.2.4. *The connection χ_{spin} has the following properties:*

(i) It makes the bundle gerbe product μ connection-preserving.

(ii) It is superficial, and the induced thin structure on P coincides with the one induced by the original connection: $d^{\chi_{spin}} = d_P = d^{L\delta^*\nu}$.

(iii) It is a fusive connection with respect to the fusion product $\lambda_P = L\delta^*\lambda$ on P.

Proof. (i) holds because the correction term satisfies

$$\Delta(\xi - \frac{1}{2}\Delta\zeta) = \Delta\xi = -L\delta_2^*\epsilon_\nu,$$

by Lemma 4.2.1 and so cancels the error in the multiplicativity of ν. (ii) and (iii) hold because of Lemma 4.2.3. We regard P (equipped with the fusion product λ_P and connection χ_{spin}) as the tensor product of P (equipped with λ_P and the superficial fusive connection $L\delta^*\nu$) and the trivial bundle (equipped with the trivial fusion product and the superficial fusive connection $\xi - \frac{1}{2}\Delta\zeta$). Since the conditions of being superficial and fusion are preserved under the tensor product, χ_{spin} is superficial and fusion. The same argument works for thin structures instead of connections. Here, the thin structure $d^{\,\xi-\frac{1}{2}\zeta}$ is the trivial one [Wal12c, Proposition 3.1.8], so that $d^{\chi_{spin}} = d^{L\delta^*\nu}$. □

It remains to find a *curving* adapted to the connection χ_{spin}, i.e. a 2-form B_{spin} on LFM such that $\Delta B_{spin} = \mathrm{curv}(\chi_{spin})$.

Proposition 4.2.5. *The 2-form*

$$B_{spin} := \frac{1}{2}\omega(\bar{A}\wedge\bar{A}) + r(\mathrm{curv}(\bar{A})) - \frac{1}{2}\mathrm{d}\zeta \in \Omega^2(LFM^{[2]})$$

is a curving for the connection χ_{spin}.

Proof. With Lemmata 2.2.3 and 4.2.2 the curvature of χ_{spin} is:

$$\mathrm{curv}(\chi_{spin}) = -\frac{1}{2}L\delta^*\omega(\theta\wedge\theta) + Z(L\delta, \mathrm{pr}_2^*\mathrm{d}\bar{A}) + \omega(L\delta^*\theta \wedge \mathrm{Ad}_{L\delta}^{-1}(\mathrm{pr}_2^*\bar{A})) - \frac{1}{2}\Delta\mathrm{d}\zeta.$$

In order to calculate ΔB_{spin}, we compute with (2.2.7) and (4.1.2) the formulas

$$\begin{aligned} \Delta\omega(\bar{A}\wedge\bar{A}) &= -Z(L\delta, [\mathrm{pr}_2^*\bar{A}\wedge\mathrm{pr}_2^*\bar{A}]) - 2\omega(\mathrm{Ad}_{L\delta}^{-1}(\mathrm{pr}_2^*\bar{A})\wedge L\delta^*\theta) - \omega(L\delta^*\theta\wedge L\delta^*\theta) \\ \Delta r(\mathrm{curv}(\bar{A})) &= Z(L\delta, \mathrm{curv}(\mathrm{pr}_2^*\bar{A})) \end{aligned}$$

These show the required identity $\Delta B_{spin} = \mathrm{curv}(\chi_{spin})$. □

It is worthwhile to compare the connection (χ_{spin}, B_{spin}) on $\mathcal{S}_{LM}$ with another connection developed by Gomi [Gom03] for general lifting gerbes (not only for *loop* group extensions). That connection takes as input data just the splitting s of the Lie algebra extension and the reduction r adapted to s. It is defined by

$$\chi_{Go} = L\delta^*\nu_s + Z(L\delta, \mathrm{pr}_2^*\bar{A}) \in \Omega^1(P),$$

where ν_s is the connection on $L\widetilde{\mathrm{Spin}(n)}$ determined by s. The corresponding curving is given by

$$B_{Go} := \frac{1}{2}\omega(\bar{A} \wedge \bar{A}) + r(\mathrm{curv}(\bar{A})) \in \Omega^2(LFM).$$

Since connections on bundle gerbes form an affine space [Mur96], we obtain the following.

Lemma 4.2.6. *The assignment*

$$x \longmapsto (\chi_x, B_x) := (\chi_{Go} - \frac{x}{2}\Delta\zeta, B_{Go} - \frac{x}{2}\mathrm{d}\zeta)$$

is a one-parameter family of connections on the spin lifting gerbe $\mathcal{S}_{LM}$, which contains the connection of Gomi at $x = 0$ and the connection (χ_{spin}, B_{spin}) at $x = 1$.

We recall from Section 3.2 that spin structures on LM correspond to trivializations of the spin lifting gerbe $\mathcal{S}_{LM}$, under the assignment of sending a spin structure $(\mathbb{S}, \sigma)$ to the trivialization $(T_{\mathbb{S}}, \kappa_{\mathbb{S}})$ consisting of the principal U(1)-bundle $T_{\mathbb{S}}$ over LFM, and of the bundle isomorphism $\kappa_{\mathbb{S}} : \mathrm{pr}_2^*T_{\mathbb{S}} \otimes P \longrightarrow \mathrm{pr}_1^*T_{\mathbb{S}}$ defined by $\kappa_{\mathbb{S}}(t \otimes q) := t \cdot q$. To Gomi's connection on the spin lifting gerbe, and to the connection $\omega_{\Omega,0}$ on $T_{\mathbb{S}}$ applies a general lifting theorem, see [Gom03] and [Wal11, Theorem 2.2], which in the present situation has the following form.

Proposition 4.2.7. *The assignment $(\mathbb{S}, \sigma, \Omega) \longmapsto (T_{\mathbb{S}}, \kappa_{\mathbb{S}}, \omega_{\Omega,0})$ induces an equivalence of categories:*

$$\mathcal{S}pin^{\nabla}(LM) \cong \left\{ \begin{array}{c} \textit{Trivializations of } \mathcal{S}_{LM} \textit{ with} \\ \textit{connection compatible with} \\ (\chi_{Go}, B_{Go}) \end{array} \right\}.$$

We recall that a connection on a trivialization $\mathcal{T} = (T, \kappa)$ is a connection ω on T, and it is called *compatible* with a connection (χ, B) on the lifting gerbe if κ is connection-preserving. The curving B is used in order to associate to each compatible connection on $\mathcal{T}$ a *covariant derivative*: a 2-form $\rho_{\mathcal{T}} \in \Omega^2(LM)$ uniquely determined by the condition that $L\pi^*\rho_{\mathcal{T}} = \mathrm{curv}(\omega) + B$.

Together with Lemma 4.1.3 we deduce the following result.

Corollary 4.2.8. *Under the equivalence of Proposition 4.2.7, the scalar curvature of a geometric spin connection corresponds to the covariant derivative of a trivialization, i.e.*

$$L\pi^*\mathrm{scurv}(\Omega) = \mathrm{curv}(\omega_{\Omega,0}) + B_{Go}.$$

Trivializations with compatible connections together with the connection-preserving isomorphisms between trivializations form a category, which is, analogously to (3.2.1) a torsor over the monoidal category $\mathcal{B}un^{\nabla}(LM)$ of principal U(1)-bundles with connection over LM. The equivalence of Proposition 4.2.7 is equivariant with respect to the $\mathcal{B}un^{\nabla}(LM)$-actions on both categories; in particular, we have the following consequence.

Corollary 4.2.9. *The category $\mathcal{S}pin^{\nabla}(LM)$ is a torsor over the monoidal category $\mathcal{B}un^{\nabla}(LM)$ of principal* U(1)*-bundles with connection over LM.*

We want to generalize the equivalence of Proposition 4.2.7 to a version for the connection $\omega_{\Omega,x}$ for all $x \in \mathbb{R}$, and so in particular to the case $x = 1$. In order to do so, we have the following result.

Lemma 4.2.10. *The assignment* $(T, \kappa, \omega) \longmapsto (T, \kappa, \omega + \frac{x}{2}\zeta)$ *induces an equivalence of categories:*

$$\left\{\begin{array}{c}\textit{Trivializations of } \mathcal{S}_{LM} \textit{ with}\\ \textit{connection compatible with}\\ (\chi_0, B_0)\end{array}\right\} \cong \left\{\begin{array}{c}\textit{Trivializations of } \mathcal{S}_{LM}\\ \textit{with connection}\\ \textit{compatible with } (\chi_x, B_x)\end{array}\right\}.$$

Moreover, the equivalence is equivariant with respect to the $\mathcal{B}un^{\nabla}(LM)$*-actions, and it preserves the covariant derivative of trivializations.*

Proof. It is enough to show that the given functor is well-defined for all $x \in \mathbb{R}$; it is then invertible by the functor associated to $-x$. For well-definedness it suffices to show that the given isomorphism κ is connection-preserving for the shifted connections. Indeed, in

$$\kappa : \mathrm{pr}_2^*T \otimes P \longrightarrow \mathrm{pr}_1^*T$$

we shift on the right hand side by $\frac{x}{2}\mathrm{pr}_1^*\zeta$ and on the left hand side by $\frac{x}{2}\mathrm{pr}_2^*\zeta - \frac{x}{2}\Delta\zeta = \frac{x}{2}\mathrm{pr}_1^*\zeta$; thus, κ is connection-preserving.

The equivariance under the $\mathcal{B}un^{\nabla}(LM)$-actions follows directly from the definitions. If ρ is the covariant derivative of (T, κ, ω) with respect to B_0, i.e. $L\pi^*\rho = \mathrm{curv}(\omega) + B_0$, then the covariant derivative of $(T, \kappa, \omega + \frac{x}{2}\zeta)$ with respect to $B_t = B_0 - \frac{x}{2}\mathrm{d}\zeta$ is the same ρ, since

$$L\pi^*\rho = \mathrm{curv}(\omega) + B_0 = \mathrm{curv}(\omega + \frac{x}{2}\zeta) - \frac{x}{2}\mathrm{d}\zeta + B_0 = \mathrm{curv}(\omega + \frac{x}{2}\zeta) + B_t.$$

□

From Proposition 4.2.7 and Lemma 4.2.10 we obtain for each $x \in \mathbb{R}$ an equivalence between geometric spin structures on LM and trivializations of the spin lifting gerbe equipped with the connection (χ_x, B_x). In particular, we have for $x = 1$:

Theorem 4.2.11. *The assignment* $(\mathcal{S}, \sigma, \Omega) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}}, \omega_{\Omega})$ *induces an equivalence of categories*

$$\mathcal{S}pin^{\nabla}(LM) \cong \left\{\begin{array}{c}\textit{Trivializations of } \mathcal{S}_{LM} \textit{ with}\\ \textit{connection compatible with}\\ (\chi_{spin}, B_{spin})\end{array}\right\}.$$

This equivalence is equivariant for the $\mathcal{B}un^{\nabla}(LM)$*-actions, and the scalar curvature of a geometric spin structure corresponds to the covariant derivative of the trivialization.*

Definition 4.1.8 of superficial spin connections shows that the equivalence of Theorem 4.2.11 exchanges superficial spin connections Ω with trivializations of $\mathcal{S}_{LM}$ whose connection ω_Ω is superficial. Likewise, Definition 4.1.9 of geometric fusion spin structures shows that the equivalence persists in the setting with fusion products, where it becomes the following result.

Corollary 4.2.12. *The assignment* $(\mathcal{S}, \sigma, \lambda, \Omega) \longmapsto (T_{\mathcal{S}}, \kappa_{\mathcal{S}}, \lambda, \omega_{\Omega})$ *induces an equivalence of categories,*

$$\mathcal{S}pin_{fus}^{\nabla_{sf}}(LM) \cong \left\{\begin{array}{c}\textit{Fusion trivializations of } \mathcal{S}_{LM} \textit{ with}\\ \textit{superficial fusive connection}\\ \textit{compatible with } (\chi_{spin}, B_{spin})\end{array}\right\}.$$

This equivalence is equivariant under the action of the monoidal category $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM)$.

To close, we observe by inspection that the passage from a setting with connections to a setting without connections is consistent with the lifting theory, i.e. with Proposition 3.2.6 and Corollary 4.2.12): there is a commutative diagram of categories and functors

$$\begin{array}{ccc} \mathcal{S}pin^{\nabla_{sf}}_{fus}(LM) & \longrightarrow & \left\{ \begin{array}{c} \text{Fusion trivializations of } \mathcal{S}_{LM} \text{ with} \\ \text{superficial fusive connection} \\ \text{compatible with } (\chi_{spin}, B_{spin}) \end{array} \right\} \\ \downarrow & & \downarrow \\ \mathcal{S}pin^{th}_{fus}(LM) & \longrightarrow & \mathcal{T}riv^{th}_{fus}(\mathcal{S}_{LM}), \end{array} \tag{4.2.4}$$

with the horizontal functors equivalences of categories.

5 String structures and decategorification

String structures as defined in [Wal13] form a bicategory. The main point of this section is to introduce several decategorified versions of this bicategory of string structures, which are tailored into a form that allows a direct application of the duality between gerbes and U(1)-bundles over loop spaces, see Section 2.

5.1 String structures as trivializations

The idea behind string structures as defined in [Wal13] is to realize the class $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$ using bundle 2-gerbes.

Definition 5.1.1 ([Ste04, Definition 5.3]). *A bundle 2-gerbe over a smooth manifold M is a surjective submersion $\pi : Y \longrightarrow M$ together with a bundle gerbe $\mathcal{P}$ over $Y^{[2]}$, an isomorphism*

$$\mathcal{M} : \mathrm{pr}_{23}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} \longrightarrow \mathrm{pr}_{13}^*\mathcal{P}$$

of bundle gerbes over $Y^{[3]}$, and a transformation

$$\begin{array}{ccc} \mathrm{pr}_{34}^*\mathcal{P} \otimes \mathrm{pr}_{23}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} & \xrightarrow{\mathrm{pr}_{234}^*\mathcal{M} \otimes \mathrm{id}} & \mathrm{pr}_{24}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} \\ {\scriptstyle \mathrm{id} \otimes \mathrm{pr}_{123}^*\mathcal{M}} \downarrow & \swarrow\!\!\!\!\swarrow \, \mu & \downarrow {\scriptstyle \mathrm{pr}_{124}^*\mathcal{M}} \\ \mathrm{pr}_{34}^*\mathcal{P} \otimes \mathrm{pr}_{13}^*\mathcal{P} & \xrightarrow[\mathrm{pr}_{134}^*\mathcal{M}]{} & \mathrm{pr}_{14}^*\mathcal{P} \end{array}$$

over $Y^{[4]}$ that satisfies a pentagon axiom.

The isomorphism $\mathcal{M}$ is called the *bundle 2-gerbe product* and the transformation μ is called the *associator*. The pentagon axiom implies the cocycle condition for a certain degree three Čech cocycle on M with values in U(1), which defines – via the exponential sequence – a class

$$\mathrm{CC}(\mathbb{G}) \in \mathrm{H}^4(M, \mathbb{Z});$$

see [Ste04, Proposition 7.2] for the details.

We recall from [CJM$^+$05] the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_M$, whose characteristic class is $\mathrm{CC}(\mathbb{CS}_M) = \frac{1}{2}p_1(M)$. It uses the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(n)$, together with its multiplicative structure $(\mathcal{M}, \alpha)$ described in Section 2.1. Here we first ignore the connections – they become relevant in Section 6.1. The Chern-Simons 2-gerbe $\mathbb{CS}_M$ consists of the following structure:

- Its surjective submersion is the frame bundle $\pi : FM \longrightarrow M$.
- Its bundle gerbe $\mathcal{P}$ over $FM^{[2]}$ is $\mathcal{P} := \delta^*\mathcal{G}_{bas}$, where $\delta : FM^{[2]} \longrightarrow \mathrm{Spin}(n)$ is the difference map (i.e. $p' \cdot \delta(p, p') = p$).
- Its bundle 2-gerbe product is

  $$\mathcal{M}' := \delta_2^*\mathcal{M} : \mathrm{pr}_{23}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} \longrightarrow \mathrm{pr}_{13}^*\mathcal{P},$$

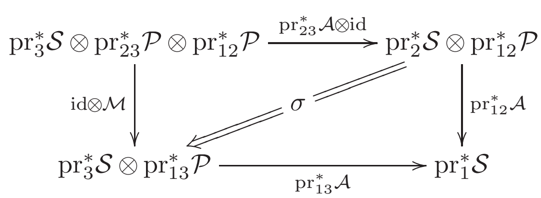
  where $\delta_2 : FM^{[3]} \longrightarrow \mathrm{Spin}(n)^2$ is defined by $(p'', p') \cdot \delta_2(p, p', p'') = (p', p)$.
- Its associator is $\mu := \delta_3^*\alpha$, where $\delta_3 : FM^{[4]} \longrightarrow \mathrm{Spin}(n)$ is defined analogously. The pentagon axiom for α implies the pentagon axiom for μ.

More detailed discussions of the Chern-Simons 2-gerbe are given in [CJM$^+$05, Wal10, Wal13, NW13].

Definition 5.1.2 ([Ste04, Definition 11.1])**.** *A trivialization of a bundle 2-gerbe $\mathbb{G}$ as in Definition 5.1.1 is a bundle gerbe $\mathcal{S}$ over Y, together with an isomorphism*

$$\mathcal{A} : \mathrm{pr}_2^*\mathcal{S} \otimes \mathcal{P} \longrightarrow \mathrm{pr}_1^*\mathcal{S}$$

of bundle gerbes over $Y^{[2]}$ and a connection-preserving transformation



(5.1.1)

over $Y^{[3]}$ that is compatible with the associator μ in the sense of Figure 1.

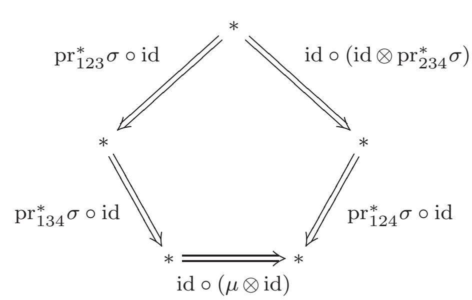


Figure 1: The compatibility condition between the associator μ of a bundle 2-gerbe and the transformation σ of a trivialization. It is an equation of transformations over $Y^{[4]}$.

The characteristic class $\mathrm{CC}(\mathbb{G}) \in \mathrm{H}^4(M,\mathbb{Z})$ of $\mathbb{G}$ vanishes if and only if $\mathbb{G}$ admits a trivialization [Ste04, Proposition 11.2]. In particular, the Chern-Simons 2-gerbe $\mathbb{CS}_M$ has trivializations if and only if M is a string manifold. This is the motivation for the following definition.

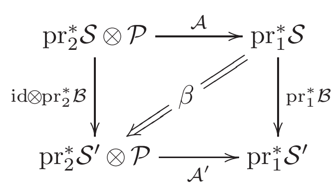
Definition 5.1.3 ([Wal13, Definition 1.1.5])**.** *A string structure on M is a trivialization $\mathbb{T}$ of $\mathbb{CS}_M$.*

The main problem with establishing a relation between string structures and loop space geometry via the transgression and regression functors of Section 2 is that these functors are defined on the truncated *categories* $\mathrm{h}_1\mathcal{G}rb^{\nabla}(X)$ and $\mathrm{h}_1\mathcal{G}rb(X)$ of bundle gerbes and not on the full *bicategories*. This problem is solved in the next subsections by reformulating the notion of trivializations of bundle 2-gerbes internal to these truncated categories.

5.2 Decategorification of trivializations

In this section $\mathbb{G}$ is a general bundle 2-gerbe over M, composed of the same structure as in Definition 5.1.1. According to [Wal13, Lemma 2.2.4], trivializations of $\mathbb{G}$ form a bicategory, which we denote by $\mathcal{T}riv(\mathbb{G})$. We recall how the 1-morphisms and 2-morphisms are defined.

Given trivializations $\mathbb{T} = (\mathcal{S},\mathcal{A},\sigma)$ and $\mathbb{T}' = (\mathcal{S}',\mathcal{A}',\sigma')$ of $\mathbb{G}$, a *1-morphism* $\mathbb{B} : \mathbb{T} \longrightarrow \mathbb{T}'$ in $\mathcal{T}riv(\mathbb{G})$ is an isomorphism $\mathcal{B} : \mathcal{S} \longrightarrow \mathcal{S}'$ between bundle gerbes over Y together with a transformation



(5.2.1)

over $Y^{[2]}$ that is compatible with the transformations σ and σ' in the sense of the pentagon diagram shown in Figure 2. If $\mathbb{B}_1 = (\mathcal{B}_1, \beta_1)$ and $\mathbb{B}_2 = (\mathcal{B}_2, \beta_2)$ are 1-morphisms between $\mathbb{T}$ and $\mathbb{T}'$, a *2-*

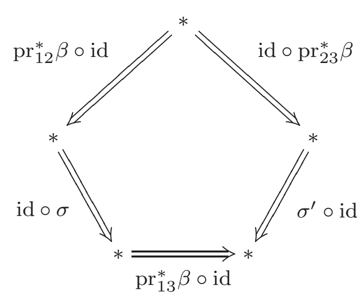


Figure 2: The compatibility between the transformations σ and σ' of two trivializations $\mathbb{T}$ and $\mathbb{T}'$ and the transformation β of a 1-morphism $\mathbb{B} = (\mathcal{B}, \beta)$ between $\mathbb{T}$ and $\mathbb{T}'$. It is an equation of transformations over $Y^{[3]}$.

morphism is a transformation $\varphi : \mathcal{B}_1 \Longrightarrow \mathcal{B}_2$ that is compatible with the transformations β_1 and β_2 in such a way that the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^*\mathcal{B}_1 \circ \mathcal{A} & \overset{\beta_1}{\Longrightarrow} & \mathcal{A}' \circ (\mathrm{pr}_2^*\mathcal{B}_1 \otimes \mathrm{id}) \\ {\scriptstyle \mathrm{pr}_1^*\varphi\circ\mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{id}\circ(\mathrm{pr}_2^*\varphi\otimes\mathrm{id})} \\ \mathrm{pr}_1^*\mathcal{B}_2 \circ \mathcal{A} & \underset{\beta_2}{\Longrightarrow} & \mathcal{A}' \circ (\mathrm{pr}_2^*\mathcal{B}_2 \otimes \mathrm{id}) \end{array} \tag{5.2.2}$$

of transformations over $Y^{[2]}$ is commutative.

The bicategory $\mathcal{T}riv(\mathbb{G})$ is a module over the monoidal bicategory $\mathcal{G}rb(M)$ in terms of an action 2-functor

$$\mathcal{G}rb(M) \otimes \mathcal{T}riv(\mathbb{G}) \longrightarrow \mathcal{T}riv(\mathbb{G}) : (\mathcal{K}, \mathbb{T}) \longmapsto \mathcal{K} \otimes \mathbb{T}, \tag{5.2.3}$$

For a trivialization $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ and a bundle gerbe $\mathcal{K}$ over M, the trivialization $\mathcal{K}\otimes\mathbb{T}$ is given by the bundle gerbe $\pi^*\mathcal{K} \otimes \mathcal{S}$, the isomorphism $\mathrm{id} \otimes \mathcal{A}$ and the transformation $\mathrm{id} \otimes \sigma$. We recall the following result.

Lemma 5.2.1 ([Wal13, Lemma 2.2.5])**.** *The action (5.2.3) exhibits the bicategory $\mathcal{T}riv(\mathbb{G})$ as a torsor over the monoidal bicategory $\mathcal{G}rb(M)$.*

There are two methods to produce a category from the bicategory $\mathcal{T}riv(\mathbb{G})$ of trivializations of a bundle 2-gerbe. The first method is to take the truncation $\mathrm{h}_1\mathcal{T}riv(\mathbb{G})$, whose objects are those of $\mathcal{T}riv(\mathbb{G})$, and whose morphisms are 2-isomorphism classes of 1-morphisms in $\mathcal{T}riv(\mathbb{G})$.

The second method is to consider the truncated presheaf of categories $\mathrm{h}_1\mathcal{G}rb$ and then formally repeat the definition of trivializations in that ambient category. This gives a category which we denote by $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$. An object in $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$ is a pair $(\mathcal{S}, [\mathcal{A}])$ of a bundle gerbe $\mathcal{S}$ over Y and an equivalence class of isomorphisms

$$\mathcal{A} : \mathrm{pr}_2^*\mathcal{S} \otimes \mathcal{P} \longrightarrow \mathrm{pr}_1^*\mathcal{S},$$

such that there exists a transformation σ as in (5.1.1). Note that it is *not* required that σ makes the diagram of Figure 1 commutative. A morphism $(\mathcal{S}_1, [\mathcal{A}_1]) \longrightarrow (\mathcal{S}_2, [\mathcal{A}_2])$ in $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$ is an equivalence class $[\mathcal{B}]$ of isomorphisms $\mathcal{B} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that there exists a transformation β as in (5.2.1). Note that it is *not* required that β makes the diagram of Figure 2 commutative.

The two categories of trivializations are related by a functor

$$\mathrm{t}_1 : \mathrm{h}_1\mathcal{T}riv(\mathbb{G}) \longrightarrow \mathrm{t}_1\mathcal{T}riv(\mathbb{G}).$$

Indeed, an object in $\mathrm{h}_1\mathcal{T}riv(\mathbb{G})$ is just an object $(\mathcal{S}, \mathcal{A}, \sigma)$ in $\mathcal{T}riv(\mathbb{G})$, and the functor sets $\mathrm{t}_1(\mathcal{S}, \mathcal{A}, \sigma) := (\mathcal{S}, [\mathcal{A}])$. A morphism in $\mathrm{h}_1\mathcal{T}riv(\mathbb{G})$ is a 2-isomorphism class $[(\mathcal{B}, \beta)]$ of 1-morphisms, and the constructions above show that $\mathrm{t}_1([(\mathcal{B}, \beta)]) := [\mathcal{B}]$ is well-defined.

We recall that $\mathcal{T}riv(\mathbb{G})$ is a torsor over $\mathcal{G}rb(M)$, see Lemma 5.2.1. By purely formal reasons, it follows that $\mathrm{h}_1\mathcal{T}riv(\mathbb{G})$ is a torsor over $\mathrm{h}_1\mathcal{G}rb(M)$. There is a similar action of $\mathrm{h}_1\mathcal{G}rb(M)$ on $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$, i.e. a functor

$$\mathrm{h}_1\mathcal{G}rb(M) \times \mathrm{t}_1\mathcal{T}riv(\mathbb{G}) \longrightarrow \mathrm{t}_1\mathcal{T}riv(\mathbb{G}) \tag{5.2.4}$$

that exhibits $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$ as a module category over the monoidal category $\mathrm{h}_1\mathcal{G}rb(M)$. On objects, it is given by $(\mathcal{K}, (\mathcal{S}, [\mathcal{A}])) \longmapsto (\pi^*\mathcal{K}\otimes\mathcal{S}, [\mathcal{A}\otimes\mathrm{id}])$, and on morphisms it is given by $([\mathcal{J}], [\mathcal{B}]) \longmapsto [\pi^*\mathcal{J}\otimes\mathcal{B}]$. The functor t_1 is obviously $\mathrm{h}_1\mathcal{G}rb(M)$-equivariant.

Next we produce three sets from the bicategory $\mathcal{T}riv(\mathbb{G})$ of trivializations. The first is the set $\mathrm{h}_0\mathcal{T}riv(\mathbb{G})$ of isomorphism classes of trivializations of $\mathbb{G}$. The second is the set $\mathrm{h}_0(\mathrm{t}_1\mathcal{T}riv(\mathbb{G}))$ of isomorphism classes of objects in $\mathrm{t}_1\mathcal{T}riv(\mathbb{G})$. The third is the set $\mathrm{t}_2\mathcal{T}riv(\mathbb{G})$ obtained by formally repeating the definition of a trivialization ambient to the presheaf $\mathrm{h}_0\mathcal{G}rb$. In detail, an element of $\mathrm{t}_2\mathcal{T}riv(\mathbb{G})$ is an isomorphism class $[\mathcal{S}]$ of bundle gerbes $\mathcal{S}$ over Y, such that there exists an isomorphism $\mathrm{pr}_2^*\mathcal{S}\otimes\mathcal{P} \cong \mathrm{pr}_1^*\mathcal{S}$ over $Y^{[2]}$.

The three sets of trivializations are related by maps

$$\mathrm{h}_0\mathcal{T}riv(\mathbb{G}) \xrightarrow{\mathrm{h}_0\mathrm{t}_1} \mathrm{h}_0(\mathrm{t}_1\mathcal{T}riv(\mathbb{G})) \xrightarrow{\mathrm{t}_2} \mathrm{t}_2\mathcal{T}riv(\mathbb{G}),$$

where $\mathrm{h}_0\mathrm{t}_1$ is the map induced by the functor t_1 on isomorphism classes, and t_2 sends an element $[(\mathcal{S},[\mathcal{A}])]$ to $[\mathcal{S}]$. Again by purely formal reasons, $\mathrm{h}_0\mathcal{T}riv(\mathbb{G})$ is a torsor over the group $\mathrm{h}_0\mathcal{G}rb(M)$. Further, the group $\mathrm{h}_0\mathcal{G}rb(M)$ acts on $\mathrm{h}_0(\mathrm{t}_1\mathcal{T}riv(\mathbb{G}))$, and $\mathrm{h}_0\mathrm{t}_1$ is equivariant. Finally, we have an action of $\mathrm{h}_0\mathcal{G}rb(M)$ on $\mathrm{t}_2\mathcal{T}riv(\mathbb{G})$, defined by $([\mathcal{K}],[\mathcal{S}]) \longmapsto [\pi^*\mathcal{K}\otimes\mathcal{S}]$, for which the map t_2 is equivariant.

5.3 From string structures to string classes

We now have the following versions of string structures:

(a) a *bicategory of string structures*, $\mathcal{S}tring(M) := \mathcal{T}riv(\mathbb{CS}_M)$,

(b) two *categories of string structures*, $\mathrm{h}_1\mathcal{S}tring(M)$ and $\mathcal{S}tring_1(M) := \mathrm{t}_1\mathcal{T}riv(\mathbb{CS}_M)$, related by a functor

$$\mathrm{t}_1 : \mathrm{h}_1\mathcal{S}tring(M) \longrightarrow \mathcal{S}tring_1(M),$$

(c) three *sets of string structures*, namely $\mathrm{h}_0\mathcal{S}tring(M)$, $\mathrm{h}_0\mathcal{S}tring_1(M)$ and $\mathcal{S}tring_0(M) := \mathrm{t}_2\mathcal{T}riv(\mathbb{CS}_M)$, related by maps

$$\mathrm{h}_0\mathcal{S}tring(M) \xrightarrow{\mathrm{h}_0\mathrm{t}_1} \mathrm{h}_0\mathcal{S}tring_1(M) \xrightarrow{\mathrm{t}_2} \mathcal{S}tring_0(M).$$

Additionally, there is a fourth set consisting of so-called string classes. A *string class* on M is a class $\xi \in \mathrm{H}^3(FM,\mathbb{Z})$ that restricts on each fibre to the generator $\gamma \in \mathrm{H}^3(\mathrm{Spin}(n),\mathbb{Z})$. We denote the set of string classes on M by $\mathcal{S}tr\mathcal{C}l(M)$. We have a map

$$\mathrm{t}_3 : \mathcal{S}tring_0(M) \longrightarrow \mathcal{S}tr\mathcal{C}l(M) : [\mathcal{S}] \longmapsto \mathrm{DD}(\mathcal{S}).$$

This map is well-defined: indeed, for a point $p \in FM$ we have the map $\iota_p\colon \mathrm{Spin}(n) \longrightarrow FM$ defined by $\iota_p(g) := pg$, implementing the "restriction to the fibre of p". We have another map $j_p\colon \mathrm{Spin}(n) \longrightarrow FM^{[2]}$ defined by $j_p(g) := (pg,p)$. Recall that an element $[\mathcal{S}]$ in $\mathcal{S}tring_0(M)$ is represented by a bundle gerbe $\mathcal{S}$ over FM that admits an isomorphism $\mathrm{pr}_2^*\mathcal{S}\otimes\delta^*\mathcal{G}_{bas} \cong \mathrm{pr}_1^*\mathcal{S}$. Pullback along j_p followed by taking the Dixmier-Douady class yields $\gamma = \mathrm{DD}(\mathcal{G}_{bas}) = \iota_p^*\mathrm{DD}(\mathcal{S})$. Thus, $\mathrm{DD}(\mathcal{S})$ is a string class.

The set $\mathcal{S}tr\mathcal{C}l(M)$ of string classes carries an action of $\mathrm{H}^3(M,\mathbb{Z})$ via pullback to FM and addition, and under the identification $\mathrm{h}_0\mathcal{G}rb(M) \cong \mathrm{H}^3(M,\mathbb{Z})$ the map t_3 is equivariant.

Theorem 5.3.1. *The functor*

$$\mathrm{t}_1 : \mathrm{h}_1\mathcal{S}tring(M) \longrightarrow \mathcal{S}tring_1(M)$$

is an equivalence of categories; in particular, it is an equivariant functor between $\mathrm{h}_1\mathcal{G}rb(M)$*-torsors. The maps*

$$\mathrm{h}_0\mathcal{S}tring(M) \xrightarrow{\mathrm{h}_0\mathrm{t}_1} \mathrm{h}_0\mathcal{S}tring_1(M) \xrightarrow{\mathrm{t}_2} \mathcal{S}tring_0(M) \xrightarrow{\mathrm{t}_3} \mathcal{S}tr\mathcal{C}l(M)$$

are all bijections; in particular, they are equivariant maps between $\mathrm{H}^3(M,\mathbb{Z})$*-torsors.*

The proof is split into a couple of lemmata. We start with the following Serre spectral sequence calculation.

Lemma 5.3.2. *Let* $\pi : P \longrightarrow M$ *be a principal* G*-bundle over* M*, for* G *a compact, simple, simply-connected Lie group. Let* A *be an abelian group. For* $k = 0, 1, 2$ *the pullback map*

$$\pi^* : \mathrm{H}^k(M, A) \longrightarrow \mathrm{H}^k(P, A)$$

is an isomorphism. Let $p \in P$ *be a point and* $i_p : G \longrightarrow P : g \longmapsto pg$*. Then, the sequence*

$$0 \longrightarrow \mathrm{H}^3(M, A) \xrightarrow{\pi^*} \mathrm{H}^3(P, A) \xrightarrow{i_p^*} \mathrm{H}^3(G, A) \xrightarrow{tr} \mathrm{H}^4(M, A)$$

is exact, where tr *is the "transgression" homomorphism of the Serre spectral sequence.*

In case of $G = \mathrm{Spin}(n)$ and $A = \mathbb{Z}$, the transgression homomorphism tr of the Serre spectral sequence sends the generator $\gamma \in \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$ to the class $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$. Thus, we obtain the following result about string classes.

Corollary 5.3.3 ([Red06, Proposition 6.1.5])**.** *Let* M *be a spin manifold.*

(i) String classes exist if and only if M *is a string manifold.*

(ii) The set of string classes $\mathcal{S}tr\mathcal{C}l(M)$ *is a torsor over* $\mathrm{H}^3(M, \mathbb{Z})$*.*

Now we are in position to contribute first partial results to the proof of Theorem 5.3.1.

Lemma 5.3.4. *The maps* t_3 *and* $\mathrm{t}_2 \circ \mathrm{h}_0\mathrm{t}_1$ *are bijections.*

Proof. If M is not string, then $\mathrm{h}_0\mathcal{S}tring(M)$ and $\mathcal{S}tr\mathcal{C}l(M)$ are empty. Hence $\mathcal{S}tring_0(M)$ is also empty, and both maps in the claim are maps between empty sets, hence bijections. If M is string, $\mathrm{h}_0\mathcal{S}tring(M)$ and $\mathcal{S}tr\mathcal{C}l(M)$ are non-empty. Then $\mathrm{h}_0\mathcal{S}tring(M)$ and $\mathcal{S}tr\mathcal{C}l(M)$ are both torsors over $\mathrm{H}^3(M, \mathbb{Z})$ by Lemma 5.2.1 and Corollary 5.3.3, and the over all composition $\mathrm{t}_3 \circ \mathrm{t}_2 \circ \mathrm{h}_0\mathrm{t}_1$ is an equivariant map between torsors over the same group, hence a bijection. In particular, t_3 is surjective. The definition of t_3 shows immediately that it is also injective. Hence, t_3 and $t_2 \circ \mathrm{h}_0\mathrm{t}_1$ are bijections. □

Lemma 5.3.5. *Let* $\pi : P \longrightarrow M$ *be a principal* G*-bundle over* M*, for* G *a compact, simple, simply-connected Lie group. We denote by* $P^{[k]}$ *the* k*-fold fibre product of* P *with itself over* M*, and by* $\pi_k : P^{[k]} \longrightarrow M$ *the projection. Then,*

$$\pi_k^* : \mathrm{H}^p(M, A) \longrightarrow \mathrm{H}^p(P^{[k]}, A)$$

is an isomorphism for all $k \in \mathbb{N}$*,* $p = 0, 1, 2$*, and* $A = \mathbb{R}, \mathbb{Z}, \mathrm{U}(1)$*.*

Proof. Since P is a principal G-bundle, we have diffeomorphisms

$$\varphi_k : P^{[k]} \longrightarrow P \times G^{k-1} : (p_1, ..., p_k) \longmapsto (p_1, \delta(p_1,p_2), \delta(p_1,p_3), ..., \delta(p_1,p_k))$$

with $\pi \circ \mathrm{pr}_1 \circ \varphi_k = \pi_k$. The projection pr_1: $P \times G^{k-1} \longrightarrow P$ induces an isomorphism in the cohomology with coefficients in $A = \mathbb{R}, \mathbb{Z}$ and in degrees $p = 0,1,2$ via the Künneth formula (using that $\mathbb{R}$ and $\mathbb{Z}$ have no torsion) and the 2-connectedness of G. For $A = \mathrm{U}(1)$ the same statement holds due to the exactness of the exponential sequence and the five lemma. The bundle projection $\pi : Y \longrightarrow M$ induces an isomorphism in cohomology in degrees $p = 0,1,2$ according to Lemma 5.3.2. □

Lemma 5.3.6. *Suppose that $\mathcal{F}$ is a presheaf of abelian groups over smooth manifolds, and $\pi\colon Y \longrightarrow M$ is a surjective submersion. We denote by $\pi_k : Y^{[k]} \longrightarrow M$ the projection, and by $\Delta : \mathcal{F}(Y^{[k]}) \longrightarrow \mathcal{F}(Y^{[k+1]})$ the Čech coboundary operator. Then,*

$$\Delta \circ \pi_k^* = \begin{cases} \pi_{k+1}^* & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

Proof. For $\alpha \in \mathcal{F}(M)$ and $\beta = \pi_k^*\alpha$ we have $\partial_i^*\beta = \partial_i^*\pi_k^*\alpha = \pi_{k+1}^*\alpha$. Now the claim is proved by counting the number of terms in the alternating sum. □

Applying Lemma 5.3.6 to the presheaf $\mathcal{F} = \mathrm{H}^p(-,\mathbb{Z})$, we obtain the following.

Corollary 5.3.7. *Let $\pi : P \longrightarrow M$ be a principal G-bundle over M, for G a compact, simple, simply-connected Lie group. Then, for $p = 0,1,2$,*

$$\Delta : \mathrm{H}^p(P^{[k]},\mathbb{Z}) \longrightarrow \mathrm{H}^p(P^{[k+1]},\mathbb{Z})$$

is an isomorphism if k is even, and the zero map if k is odd.

Now we finish the proof of Theorem 5.3.1 with the following two lemmata.

Lemma 5.3.8. *The map $\mathrm{t}_2 : \mathrm{h}_0\mathcal{S}tring_1(M) \longrightarrow \mathcal{S}tring_0(M)$ is a bijection.*

Proof. By Lemma 5.3.4 it is surjective; hence it remains to prove injectivity. Suppose $(\mathcal{S}_1, [\mathcal{A}_1])$ and $(\mathcal{S}_2, [\mathcal{A}_2])$ are objects in $\mathcal{S}tring_1(M)$ such that there images under t_2 are equal, i.e. there exists an isomorphism $\mathcal{B} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$. We have to construct a transformation

$$\beta : \mathrm{pr}_1^*\mathcal{B} \circ \mathcal{A}_1 \Longrightarrow \mathcal{A}_2 \circ (\mathrm{pr}_2^*\mathcal{B} \otimes \mathrm{id})$$

over $Y^{[2]}$, see (5.2.1), so that $\mathbb{B} = (\mathcal{B}, \beta)$ is an isomorphism in $\mathcal{S}tring_1(M)$. A priori, the two isomorphisms $\mathrm{pr}_1^*\mathcal{B} \circ \mathcal{A}_1$ and $\mathcal{A}_2 \circ (\mathrm{pr}_2^*\mathcal{B} \otimes \mathrm{id})$ are not 2-isomorphic. We recall that the category $\mathcal{H}om(\mathcal{G},\mathcal{H})$ of isomorphisms between two fixed bundle gerbes $\mathcal{G}$ and $\mathcal{H}$ over an arbitrary smooth manifold X is a torsor over the monoidal category $\mathcal{B}un(X)$ of principal $\mathrm{U}(1)$-bundles over X, under an action functor

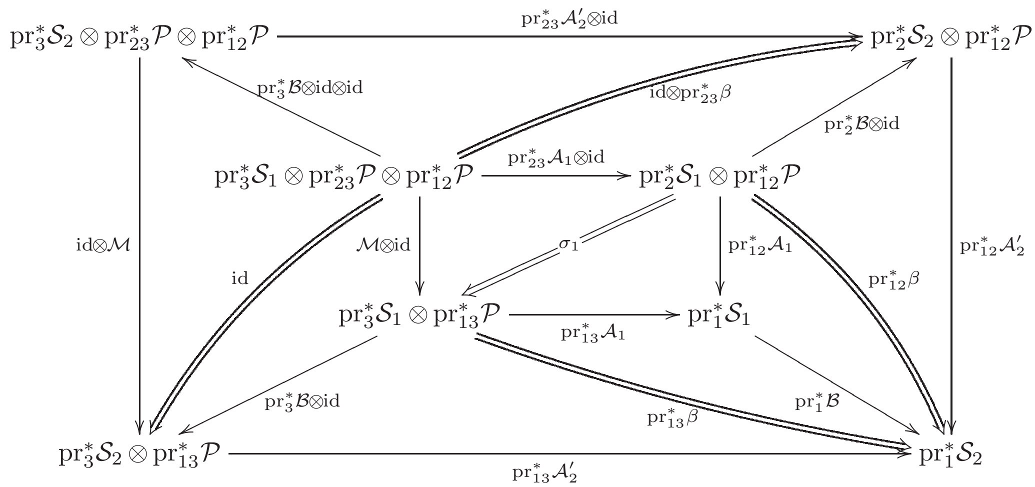
$$\mathcal{B}un(X) \times \mathcal{H}om(\mathcal{G},\mathcal{H}) \longrightarrow \mathcal{H}om(\mathcal{G},\mathcal{H}) : (B, \mathcal{A}) \longmapsto B \otimes \mathcal{A}, \tag{5.3.1}$$

see [CJM02]. In our situation, this means that there exists a principal $\mathrm{U}(1)$-bundle B over $Y^{[2]}$ such that with $\mathcal{A}_2' := B \otimes \mathcal{A}_2$ we do have a transformation

$$\beta : \mathrm{pr}_1^*\mathcal{B} \circ \mathcal{A}_1 \Longrightarrow \mathcal{A}_2' \circ (\mathrm{pr}_2^*\mathcal{B} \otimes \mathrm{id}).$$

The bundle B has a first Chern class $c_1(B) \in \mathrm{H}^2(Y^{[2]}, \mathbb{Z})$. We claim that $c_1(B) = 0$, meaning that $\mathcal{A}_2' \cong \mathcal{A}_2$ and β is the claimed transformation; thus, t_2 is injective.

Indeed, since $(\mathcal{S}_1, [\mathcal{A}_1])$ and $(\mathcal{S}_2, [\mathcal{A}_2])$ are objects in $\mathcal{S}tring_1(M) = \mathrm{t}_1\mathcal{T}riv(\mathbb{G})$, there exist transformations σ_1, σ_2 making diagram (5.1.1) commutative. One can paste together σ_1 and β and produce the following transformation:



Using the relation $\mathcal{A}_2' = B \otimes \mathcal{A}_2$ and using that the action (5.3.1) commutes with composition, this transformation induces another transformation

$$(\mathrm{pr}_{12}^*\mathcal{A}_2 \circ (\mathrm{pr}_{23}^*\mathcal{A}_2 \otimes \mathrm{id})) \otimes (\mathrm{pr}_{12}^*B \otimes \mathrm{pr}_{23}^*B \otimes \mathrm{pr}_{13}^*B^\vee) \Longrightarrow \mathrm{pr}_{13}^*\mathcal{A}_2 \circ (\mathrm{id} \otimes \mathcal{M})$$

with $B^\vee$ the dual bundle. On the other hand, σ_2 is a transformation

$$\sigma_2 : \mathrm{pr}_{12}^*\mathcal{A}_2 \circ (\mathrm{pr}_{23}^*\mathcal{A}_2 \otimes \mathrm{id}) \Longrightarrow \mathrm{pr}_{13}^*\mathcal{A}_2 \circ (\mathrm{id} \otimes \mathcal{M})$$

It follows that $\mathrm{pr}_{12}^*B \otimes \mathrm{pr}_{23}^*B \otimes \mathrm{pr}_{13}^*B^\vee$ must be trivializable, i.e. $\Delta c_1(B) = 0$. By Corollary 5.3.7 this means $c_1(B) = 0$. $\square$

Lemma 5.3.9. *The functor* t_1 *is an equivalence of categories.*

Proof. We know already that $\mathrm{h}_0\mathrm{t}_1$ is a bijection. This implies that t_1 is essentially surjective, and it implies that for two objects $\mathbb{T}_1 = (\mathcal{S}_1, \mathcal{A}_1, \sigma_1)$ and $\mathbb{T}_2 = (\mathcal{S}_2, \mathcal{A}_2, \sigma_2)$ the Hom-sets $\mathcal{H}om_{\mathrm{h}_1\mathcal{T}riv(\mathbb{CS}_M)}(\mathbb{T}_1, \mathbb{T}_2)$ and $\mathcal{H}om_{\mathrm{t}_1\mathcal{T}riv(\mathbb{CS}_M)}(\mathrm{t}_1(\mathbb{T}_1), \mathrm{t}_1(\mathbb{T}_2))$ are either both empty or both non-empty. It remains to show that t_1 induces in the non-empty case a bijection between these sets. We have already seen that t_1 is equivariant with respect to the $\mathrm{h}_1\mathcal{G}rb(M)$-action, namely the one induced from (5.2.3), and the action (5.2.4). Over the objects $(\mathcal{I}, \mathcal{I})$ and $(\mathbb{T}_1, \mathbb{T}_2)$ these induce actions of the group

$$\mathcal{H}om_{\mathrm{h}_1\mathcal{G}rb(M)}(\mathcal{I}, \mathcal{I}) = \mathrm{h}_0\mathcal{H}om_{\mathcal{G}rb(M)}(\mathcal{I}, \mathcal{I}) \overset{(5.3.1)}{=} \mathrm{h}_0\mathcal{B}un(M)$$

on the sets $\mathcal{H}om_{\mathrm{h}_1\mathcal{T}riv(\mathbb{CS}_M)}(\mathbb{T}_1, \mathbb{T}_2)$ and $\mathcal{H}om_{\mathrm{t}_1\mathcal{T}riv(\mathbb{CS}_M)}(\mathrm{t}_1(\mathbb{T}_1), \mathrm{t}_1(\mathbb{T}_2))$, respectively. By Lemma 5.2.1, the first set is even a torsor under this action. We prove that the second is also a torsor, so that t_1 is an equivariant map between torsors, hence a bijection.

We recall that the elements of $\mathcal{H}om_{\mathrm{t}_1\mathcal{T}riv(\mathbb{CS}_M)}(\mathrm{t}_1(\mathbb{T}_1), \mathrm{t}_1(\mathbb{T}_2))$ are equivalence classes $[\mathcal{B}]$ of isomorphisms $\mathcal{B} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that there exists a transformation β as in (5.2.1). The action of

$[K] \in \mathrm{h}_0\mathcal{B}un(M)$ sends $[\mathcal{B}]$ to $[\pi^*K \otimes \mathcal{B}]$. This action is free because $\pi^* : \mathrm{H}^2(M,\mathbb{Z}) \longrightarrow \mathrm{H}^2(FM,\mathbb{Z})$ is an isomorphism by Lemma 5.3.5. If $\mathcal{B}$ and $\mathcal{B}'$ are both isomorphisms from $\mathcal{S}_1$ to $\mathcal{S}_2$, then by (5.3.1) there exists a principal U(1)-bundle P on FM such that $\mathcal{B}' \cong \mathcal{B} \otimes P$. But again, since π^* is an isomorphism, $P \cong \pi^*K$ for some K in $\mathcal{B}un(M)$. Hence the action is transitive. □

6 String connections and decategorification

This section is the analogue of Section 5 in the setting with connections. We first recall the definition of string connections and geometric string structures on the basis of [Wal13]. Geometric string structures form a bicategory, of which we discuss various decategorified versions. At the end of a sequence of decategorification we naturally find the notion of a differential string class.

6.1 String connections as connections on trivializations

The Levi-Cevita connection on M lifts to a spin connection A on FM; in turn it defines a connection on the Chern-Simons 2-gerbe $\mathbb{CS}_M$ [Wal13, Theorem 1.2.1]. In the following we recall this construction. We suppose first that $\mathbb{G}$ is a bundle 2-gerbe over a smooth manifold M as in Definition 5.1.1.

Definition 6.1.1. *A* <u>*connection*</u> *on $\mathbb{G}$ is a 3-form $C \in \Omega^3(Y)$ and a connection on the bundle gerbe $\mathcal{P}$ such that*

$$\mathrm{pr}_2^*C - \mathrm{pr}_1^*C = \mathrm{curv}(\mathcal{P}), \tag{6.1.1}$$

and the bundle 2-gerbe product $\mathcal{M}$ as well as the associator μ are connection-preserving.

The 3-form C is called the *curving* of the connection. In case of the Chern-Simons 2-gerbe, the announced connection is constructed using the connection A on FM and the connection on the basic bundle gerbe $\mathcal{G}_{bas}$ described in Section 2.1:

- The curving is the Chern-Simons 3-form $CS(A) \in \Omega^3(FM)$ associated to A,

  $$CS(A) = \langle A \wedge \mathrm{d}A \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \,, \tag{6.1.2}$$

  where $\langle -, - \rangle$ is the same symmetric invariant bilinear form on the Lie algebra $\mathfrak{g}$ of Spin(n) that was used to fix the curvature of the basic gerbe $\mathcal{G}_{bas}$ in Section 2.1.

- The connection on the bundle gerbe $\mathcal{P} = \delta^*\mathcal{G}_{bas}$ over $FM^{[2]}$ is given by the connection on $\delta^*\mathcal{G}_{bas}$ shifted by the 2-form

  $$\omega := \langle \delta^*\theta \wedge \mathrm{pr}_1^*A \rangle \in \Omega^2(FM^{[2]}), \tag{6.1.3}$$

  i.e. we have $\mathcal{P} = \delta^*\mathcal{G}_{bas} \otimes \mathcal{I}_\omega$ as bundle gerbes with connection. The well-known identity

  $$CS(\mathrm{pr}_2^*A) = CS(\mathrm{pr}_1^*A) + \delta^*H + \mathrm{d}\omega \tag{6.1.4}$$

  for the Chern-Simons 3-form implies the condition (6.1.1) for the curving.

- We recall that the isomorphism

  $$\mathcal{M} : \mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G} \longrightarrow m^*\mathcal{G} \otimes \mathcal{I}_\rho$$

from the multiplicative structure on $\mathcal{G}_{bas}$ is connection-preserving. Under pullback with $\delta_2 : FM^{[3]} \longrightarrow \mathrm{Spin}(n)^2$, the 2-form ρ satisfies

$$\mathrm{pr}_{13}^*\omega = \delta_2^*\rho + \mathrm{pr}_{12}^*\omega + \mathrm{pr}_{23}^*\omega. \tag{6.1.5}$$

This permits to define a connection-preserving bundle 2-gerbe product $\mathcal{M}'$ by

$$\mathrm{pr}_{23}^*\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathcal{P} = \delta_2^*(\mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G}) \otimes \mathcal{I}_{\mathrm{pr}_{13}^*\omega - \delta_2^*\rho} \xrightarrow{\delta_2^*\mathcal{M}\otimes\mathrm{id}} \delta_2^* m^*\mathcal{G} \otimes \mathcal{I}_{\mathrm{pr}_{13}^*\omega} = \mathrm{pr}_{13}^*\mathcal{P}.$$

- The connection-preserving transformation α from the multiplicative structure on $\mathcal{G}_{bas}$ gives a connection-preserving associator.

If a bundle 2-gerbe $\mathbb{G}$ is equipped with a connection, and $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ is a trivialization of $\mathbb{G}$, then a *compatible connection* on $\mathbb{T}$ is a connection on the bundle gerbe $\mathcal{S}$ such that the isomorphism $\mathcal{A}$ and the transformation σ are connection-preserving.

Definition 6.1.2 ([Wal13, Definition 1.2.2])**.** *Let $\mathbb{T}$ be a string structure on M. A string connection on $\mathbb{T}$ is a compatible connection on $\mathbb{T}$. A geometric string structure on M is a pair of a string structure on M and a string connection.*

Geometric string structures form a bicategory $\mathcal{S}tring^\nabla(M) := \mathcal{T}riv^\nabla(\mathbb{CS}_M)$, with 1-morphisms and 2-morphisms defined exactly as in the setting without connections, just that all occurring isomorphisms and transformations are connection-preserving [Wal13, Remark 6.1.1]. We recall the following results about string connections for later reference.

Theorem 6.1.3 ([Wal13, Theorems 1.3.4 and 1.3.3])**.**

(i) Every string structure admits a string connection.

(ii) For every geometric string structure $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$, there exists a unique 3-form $K \in \Omega^3(M)$ such that $\pi^*K = CS(A) + \mathrm{curv}(\mathcal{S})$.

6.2 Decategorified string connections

Just like in the setting without connections, we may consider various truncations of the bicategory $\mathcal{T}riv^\nabla(\mathbb{G})$ for a general bundle 2-gerbe $\mathbb{G}$ with connection. So we have categories $\mathrm{h}_1\mathcal{T}riv^\nabla(\mathbb{G})$ and $\mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{G})$ and a functor

$$\mathrm{t}_1^\nabla : \mathrm{h}_1\mathcal{T}riv^\nabla(\mathbb{G}) \longrightarrow \mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{G}).$$

Further we have three sets $\mathrm{h}_0\mathcal{T}riv^\nabla(\mathbb{G})$, $\mathrm{h}_0\mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{G})$ and $\mathrm{t}_2\mathcal{T}riv^\nabla(\mathbb{G})$, and maps

$$\mathrm{h}_0\mathcal{T}riv^\nabla(\mathbb{G}) \xrightarrow{\mathrm{h}_0\mathrm{t}_1^\nabla} \mathrm{h}_0(\mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{G})) \xrightarrow{\mathrm{t}_2^\nabla} \mathrm{t}_2\mathcal{T}riv^\nabla(\mathbb{G}).$$

The passage from the setting with connections to the one without connections is attended by a 2-functor $F_2 : \mathcal{T}riv^\nabla(\mathbb{G}) \longrightarrow \mathcal{T}riv(\mathbb{G})$, a functor $F_1 : \mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{G}) \longrightarrow \mathrm{t}_1\mathcal{T}riv(\mathbb{G})$, and a map $F_0 \colon \mathrm{t}_2\mathcal{T}riv^\nabla(\mathbb{G}) \longrightarrow \mathrm{t}_2\mathcal{T}riv(\mathbb{G})$. It is fully consistent with the various functors and maps introduced above, in the sense that the diagrams

$$\begin{array}{ccc} \mathrm{h}_1\mathcal{T}riv^{\nabla}(\mathbb{G}) & \longrightarrow & \mathrm{t}_1\mathcal{T}riv^{\nabla}(\mathbb{G}) \\ {\scriptstyle \mathrm{h}_1 F_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathrm{h}_1\mathcal{T}riv(\mathbb{G}) & \longrightarrow & \mathrm{t}_1\mathcal{T}riv(\mathbb{G}) \end{array} \qquad \begin{array}{ccccc} \mathrm{h}_0\mathcal{T}riv^{\nabla}(\mathbb{G}) & \longrightarrow & \mathrm{h}_0\mathrm{t}_1\mathcal{T}riv^{\nabla}(\mathbb{G}) & \longrightarrow & \mathrm{t}_2\mathcal{T}riv^{\nabla}(\mathbb{G}) \\ {\scriptstyle \mathrm{h}_0 F_2}\downarrow & & \downarrow{\scriptstyle \mathrm{h}_0 F_1} & & \downarrow{\scriptstyle F_0} \\ \mathrm{h}_0\mathcal{T}riv(\mathbb{G}) & \longrightarrow & \mathrm{h}_0\mathrm{t}_1\mathcal{T}riv(\mathbb{G}) & \longrightarrow & \mathrm{t}_2\mathcal{T}riv(\mathbb{G}) \end{array} \tag{6.2.1}$$

of functors and maps, respectively, are commutative.

We have again various actions. The bicategory $\mathcal{T}riv^{\nabla}(\mathbb{G})$ is a torsor for the monoidal bicategory $\mathcal{G}rb^{\nabla}(M)$ of bundle gerbes with connection over M [Wal13, Lemma 2.2.5]. As before, a bundle gerbe $\mathcal{K}$ with connection over M acts on a trivialization $\mathbb{T} = (\mathcal{S}, \mathcal{A}, \sigma)$ with compatible connection by sending it to $(\pi^*\mathcal{K} \otimes \mathcal{S}, \mathrm{id} \otimes \mathcal{A}, \mathrm{id} \otimes \sigma)$. Correspondingly, the category $\mathrm{h}_1\mathcal{T}riv^{\nabla}(\mathbb{G})$ is a torsor category over the monoidal category $\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)$, and the set $\mathrm{h}_0\mathcal{T}riv^{\nabla}(\mathbb{G})$ is a torsor over the group $\mathrm{h}_0\mathcal{G}rb^{\nabla}(M)$.

In the same natural way, the monoidal category $\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)$ acts on the category $\mathrm{t}_1\mathcal{T}riv^{\nabla}(\mathbb{G})$ such that the functor t_1^{∇} is equivariant, and the group $\mathrm{h}_0\mathcal{G}rb^{\nabla}(M)$ acts on the set $\mathrm{t}_2\mathcal{T}riv^{\nabla}(\mathbb{G})$ such that the map t_2^{∇} is equivariant.

Now we specialize to the case of the Chern-Simons 2-gerbe $\mathbb{G} = \mathbb{CS}_M$, and discuss the bicategory $\mathcal{S}tring^{\nabla}(M) = \mathcal{T}riv^{\nabla}(\mathbb{CS}_M)$ of geometric string structures, the two categories $\mathrm{h}_1\mathcal{S}tring^{\nabla}(M)$ and $\mathcal{S}tring_1^{\nabla}(M) := \mathrm{t}_1\mathcal{T}riv^{\nabla}(\mathbb{CS}_M)$ of geometric string structures, and the three sets $\mathrm{h}_0\mathcal{S}tring^{\nabla}(M)$, $\mathrm{h}_0\mathcal{S}tring_1^{\nabla}(M)$, and $\mathcal{S}tring_0^{\nabla}(M) := \mathrm{t}_2\mathcal{T}riv^{\nabla}(\mathbb{CS}_M)$ of geometric string structures. We first note the following result about the passage from the setting with connections to the setting without connections.

Proposition 6.2.1.

(i) The 2-functor $F_2 : \mathcal{S}tring^{\nabla}(M) \longrightarrow \mathcal{S}tring(M)$ is essentially surjective.

(ii) The functor $F_1 \colon \mathcal{S}tring_1^{\nabla}(M) \longrightarrow \mathcal{S}tring_1(M)$ is essentially surjective.

(iii) The map $F_0 \colon \mathcal{S}tring_0^{\nabla}(M) \longrightarrow \mathcal{S}tring_0(M)$ is surjective.

Proof. (i) is Theorem 6.1.3 (i). It follows that the functor $\mathrm{h}_1 F_2$ is essentially surjective, and that the map $\mathrm{h}_0 F_2$ is surjective. Then, (ii) and (iii) follow from the commutativity of the diagrams (6.2.1) and Theorem 5.3.1. □

Next we present the main theorem of this section, which is the analogue of Theorem 5.3.1 in the setting with connections.

Theorem 6.2.2. *The functor*

$$\mathrm{t}_1^{\nabla} : \mathrm{h}_1\mathcal{S}tring^{\nabla}(M) \longrightarrow \mathcal{S}tring_1^{\nabla}(M)$$

is an equivalence of categories; in particular, it is an equivariant functor between $\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)$-torsors. The map

$$\mathrm{t}_2^{\nabla} : \mathrm{h}_0\mathcal{S}tring_1^{\nabla}(M) \longrightarrow \mathcal{S}tring_0^{\nabla}(M)$$

is a bijection; in particular, it is an equivariant map between $\hat{\mathrm{H}}^3(M)$-torsors.

Here, $\hat{\mathrm{H}}^n(M)$ stands for the *differential cohomology* of M. We recall from [Bry93] that the degree n differential cohomology of a smooth manifold X is a group $\hat{\mathrm{H}}^n(X)$ that fits into the exact sequences

$$0 \longrightarrow \Omega^{n-1}_{cl,\mathbb{Z}}(X) \longrightarrow \Omega^{n-1}(X) \xrightarrow{a} \hat{\mathrm{H}}^n(X) \xrightarrow{c} \mathrm{H}^n(X, \mathbb{Z}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{H}^{n-1}(X, \mathrm{U}(1)) \longrightarrow \hat{\mathrm{H}}^n(X) \xrightarrow{R} \Omega^n_{cl,\mathbb{Z}}(X) \longrightarrow 0,$$

in which $\Omega^n_{cl,\mathbb{Z}}(X)$ denotes the closed n-forms on X with integral periods. Bundle gerbes with connection are classified by degree three differential cohomology in terms of a differential Dixmier-Douady class $\widehat{\mathrm{DD}} : \mathrm{h}_0\mathcal{G}rb^\nabla(X) \longrightarrow \hat{\mathrm{H}}^3(X)$; the map $c : \hat{\mathrm{H}}^3(X) \longrightarrow \mathrm{H}^3(X,\mathbb{Z})$ corresponds to projecting to the underlying (non-differential) Dixmier-Douady class, the map $R : \hat{\mathrm{H}}^3(X) \longrightarrow \Omega^3_{cl,\mathbb{Z}}(X)$ corresponds to taking the curvature, and the map a: $\Omega^2(X) \longrightarrow \hat{\mathrm{H}}^3(X)$ corresponds to taking the trivial bundle gerbe $\mathcal{I}_\rho$ associated to a 2-form ρ.

In the remainder of this section we prove Theorem 6.2.2, see Lemmata 6.2.5 and 6.2.6 below. First we generalize one aspect of the Serre spectral sequence calculation of Lemma 5.3.2 from ordinary cohomology to differential cohomology.

Lemma 6.2.3. *The pullback $\pi^* : \hat{\mathrm{H}}^3(M) \longrightarrow \hat{\mathrm{H}}^3(FM)$ is injective.*

Proof. Let $\hat{\eta} \in \hat{\mathrm{H}}^3(M)$. We show that $\hat{\eta} \neq 0$ implies $\pi^*\hat{\eta} \neq 0$. Indeed, if the underlying class $\eta := c(\hat{\eta}) \in \mathrm{H}^3(M,\mathbb{Z})$ is non-zero, than $\pi^*\eta \neq 0$ because of Lemma 5.3.2, and so is $\pi^*\hat{\eta} \neq 0$. If $\eta = 0$, then $\hat{\eta} = a(\mu)$ for a 2-form $\mu \in \Omega^2(M)$. Since π is a surjective submersion, π^* is injective. Thus, if μ is not closed, then $\pi^*\mu$ is also not closed and $\pi^*\hat{\eta} = a(\pi^*\mu)$ must be non-trivial. It remains to discuss the case that μ is closed but its class is not integral. Then, $\pi^*\mu$ is also closed. By Lemma 5.3.2, $\pi^* : \mathrm{H}^2(M,\mathbb{Z}) \longrightarrow \mathrm{H}^2(FM,\mathbb{Z})$ is an isomorphism, so since μ is not integral, $\pi^*\mu$ is not integral. Hence $\pi^*\hat{\eta}$ is non-trivial. □

Corollary 6.2.4. *The actions of $\hat{\mathrm{H}}^3(M) \cong \mathrm{h}_0\mathcal{G}rb^\nabla(M)$ on $\mathrm{h}_0\mathcal{S}tring^\nabla_1(M)$ and $\mathcal{S}tring^\nabla_0(M)$ are free.*

Proof. These actions are defined by pullback (injective by Lemma 6.2.3) and then addition in the group $\hat{\mathrm{H}}^3(FM)$. □

Now we are in position to prove the first part of Theorem 6.2.2.

Lemma 6.2.5. *The functor $\mathrm{t}^\nabla_1 : \mathrm{h}_1\mathcal{S}tring^\nabla(M) \longrightarrow \mathcal{S}tring^\nabla_1(M)$ is an equivalence.*

Proof. Part I of the proof is to show that t^∇_1 is essentially surjective. We consider an object $(\mathcal{S}, [\mathcal{A}])$ in $\mathcal{S}tring^\nabla_1(M) = \mathrm{t}_1\mathcal{T}riv^\nabla(\mathbb{CS}_M)$, i.e. $\mathcal{S}$ is a bundle gerbe with connection over FM, and $\mathcal{A} : \mathrm{pr}_2^*\mathcal{S} \otimes \mathcal{P} \longrightarrow \mathrm{pr}_1^*\mathcal{S}$ is a connection-preserving 1-isomorphism such that there exists a connection-preserving transformation σ as in (5.1.1). A priori, σ does not satisfy the compatibility condition with the associator μ of $\mathbb{CS}_M$, see Figure 1. We show that it yet does, using the assumption that it is connection-preserving. The error in the commutativity of the diagram of Figure 1 is a smooth map $\epsilon : FM^{[4]} \longrightarrow \mathrm{U}(1)$. Since both μ and σ are connection-preserving, ϵ is locally constant. Since M is connected, FM and all fibre products $FM^{[k]}$ are connected, in particular $FM^{[4]}$. Thus, ϵ is constant. From the pentagon axiom for μ over $FM^{[5]}$ it follows that $\Delta\epsilon = 1$. Since $\Delta\epsilon$ has five terms, which are all equal as ϵ is constant, this implies $\epsilon = 1$. Thus, σ automatically satisfies the compatibility condition, $(\mathcal{S}, \mathcal{A}, \sigma)$ is a geometric string structure, and a preimage of $(\mathcal{S}, [\mathcal{A}])$ under t^∇_1. Hence, t^∇_1 is essentially surjective.

Part II of the proof is to show that t^∇_1 is full and faithful. First of all, we note that the map $\mathrm{h}_0\mathrm{t}^\nabla_1$ is equivariant under free actions and defined on a torsor, and thus injective. We have just proved that it also is surjective; hence, $\mathrm{h}_0\mathrm{t}^\nabla_1$ is a bijection. Now we proceed similar to the proof of Lemma 5.3.9. That $\mathrm{h}_0\mathrm{t}^\nabla_1$ is a bijection implies that for two objects $\mathbb{T}_1 = (\mathcal{S}_1, \mathcal{A}_1, \sigma_1)$ and $\mathbb{T}_2 = (\mathcal{S}_2, \mathcal{A}_2, \sigma_2)$

of $\mathrm{h}_1\mathcal{S}tring^{\nabla}(M)$ the Hom-sets $\mathcal{H}om_{\mathrm{h}_1\mathcal{S}tring^{\nabla}(M)}(\mathbb{T}_1,\mathbb{T}_2)$ and $\mathcal{H}om_{\mathcal{S}tring_1^{\nabla}(M)}(\mathrm{t}_1^{\nabla}(\mathbb{T}_1),\mathrm{t}_1^{\nabla}(\mathbb{T}_2))$ are either both empty or both non-empty. It remains to show that t_1^{∇} induces in the non-empty case a bijection between these sets.

The action of the monoidal bicategory $\mathcal{B}un(X)$ on the category of homomorphisms between two fixed bundle gerbes over X, see 5.3.1, has a counterpart in the setting with connections, namely an action

$$\mathcal{B}un^{\nabla_0}(X)\times\mathcal{H}om^{\nabla}(\mathcal{G},\mathcal{H})\longrightarrow\mathcal{H}om^{\nabla}(\mathcal{G},\mathcal{H}) \tag{6.2.2}$$

of the monoidal category of principal U(1)-bundles with *flat* connections on the category of connection preserving isomorphisms between $\mathcal{G}$ and $\mathcal{H}$, and connection-preserving transformations. This action exhibits again $\mathcal{H}om^{\nabla}(\mathcal{G},\mathcal{H})$ as a torsor over $\mathcal{B}un^{\nabla_0}(X)$.

The functor t_1^{∇} is equivariant with respect to the $\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)$-actions. Over the objects $(\mathcal{I}_0,\mathcal{I}_0)$ and $(\mathbb{T}_1,\mathbb{T}_2)$ these induce actions of the group

$$\mathcal{H}om_{\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)}(\mathcal{I}_0,\mathcal{I}_0)=\mathrm{h}_0\mathcal{H}om_{\mathcal{G}rb^{\nabla}(M)}(\mathcal{I}_0,\mathcal{I}_0)\overset{(6.2.2)}{=}\mathrm{h}_0\mathcal{B}un^{\nabla_0}(M)$$

on the sets $\mathcal{H}om_{\mathrm{h}_1\mathcal{S}tring^{\nabla}(M)}(\mathbb{T}_1,\mathbb{T}_2)$ and $\mathcal{H}om_{\mathcal{S}tring_1^{\nabla}(M)}(\mathrm{t}_1(\mathbb{T}_1),\mathrm{t}_1(\mathbb{T}_2))$, respectively. The first set is a torsor under this action. We prove that the second is also a torsor, so that t_1^{∇} is an equivariant map between torsors, hence a bijection.

We recall that the elements of $\mathcal{H}om_{\mathcal{S}tring_1^{\nabla}(M)}(\mathrm{t}_1(\mathbb{T}_1),\mathrm{t}_1(\mathbb{T}_2))$ are equivalence classes $[\mathcal{B}]$ of connection-preserving isomorphisms $\mathcal{B}:\mathcal{S}_1\longrightarrow\mathcal{S}_2$ such that there exists a connection-preserving transformation β as in (5.2.1). The action of $[K]\in\mathrm{h}_0\mathcal{B}un^{\nabla_0}(M)\cong\mathrm{H}^1(M,\mathrm{U}(1))$ sends $[\mathcal{B}]$ to $[\pi^*K\otimes\mathcal{B}]$. This action is free because $\pi^*:\mathrm{H}^1(M,\mathrm{U}(1))\longrightarrow\mathrm{H}^1(FM,\mathrm{U}(1))$ is an isomorphism by Lemma 5.3.2. If $\mathcal{B}$ and $\mathcal{B}'$ are both connection-preserving isomorphisms from $\mathcal{S}_1$ to $\mathcal{S}_2$, then by (6.2.2) there exists a flat principal U(1)-bundle P on FM such that $\mathcal{B}'\cong\mathcal{B}\otimes P$. But again, since π^* is an isomorphism, $P\cong\pi^*K$ for some K in $\mathcal{B}un^{\nabla_0}(M)$. Hence the action is transitive. □

The second part of Theorem 6.2.2 is proved by the following lemma.

Lemma 6.2.6. *The map* $\mathrm{t}_2^{\nabla}:\mathrm{h}_0\mathcal{S}tring_1^{\nabla}(M)\longrightarrow\mathcal{S}tring_0^{\nabla}(M)$ *is a bijection.*

Proof. t_2^{∇} is equivariant with respect to free actions and is defined on a torsor. Hence it is injective. Now we prove that it is surjective. Consider an element in $\mathcal{S}tring_0^{\nabla}(M)$, represented by a bundle gerbe $\mathcal{S}$ with connection over FM that admits a connection-preserving isomorphism $\mathcal{A}:\mathrm{pr}_2^*\mathcal{S}\otimes\mathcal{P}\longrightarrow\mathrm{pr}_1^*\mathcal{S}$. The existence of a connection-preserving transformation σ as in (5.1.1) is obstructed by a flat principal U(1)-bundle A over $Y^{[3]}$, i.e. there exists a connection-preserving transformation

$$\sigma:(\mathrm{pr}_{12}^*\mathcal{A}\circ(\mathrm{pr}_{23}^*\mathcal{A}\otimes\mathrm{id}))\otimes A\Longrightarrow\mathrm{pr}_{13}^*\mathcal{A}\circ(\mathrm{id}\otimes\mathcal{M}).$$

Since flat principal U(1)-bundles are classified by $\mathrm{H}^1(X,\mathrm{U}(1))$, we infer from Lemma 5.3.5 that this bundle is the pullback of a flat bundle A' over M along π_3: $Y^{[3]}\longrightarrow M$. Now we consider $\mathcal{A}':=\mathcal{A}\otimes\pi_2^*A'$, which is another connection-preserving isomorphism $\mathcal{A}'$: $\mathrm{pr}_2^*\mathcal{S}\otimes\mathcal{P}\longrightarrow\mathrm{pr}_1^*\mathcal{S}$. Then, σ induces the required transformation for $\mathcal{A}'$. This means that $(\mathcal{S},[\mathcal{A}'])$ is an object in $\mathcal{S}tring_1^{\nabla}(M)=\mathrm{t}_1\mathcal{T}riv^{\nabla}(\mathbb{CS}_M)$ with $\mathrm{t}_2^{\nabla}(\mathcal{S},[\mathcal{A}])=[\mathcal{S}]$. □

6.3 Differential string classes

With Theorem 6.2.2 proved in the previous subsection we are well prepared to introduce an analog of string classes in the setting with connections – we call it *differential string classes*. Like string classes, differential string classes have the advantage to be based solely on differential cohomology theory, and no bundle gerbe theory is needed.

We let $\hat{\gamma} := \widehat{\mathrm{DD}}(\mathcal{G}_{bas}) \in \hat{\mathrm{H}}^3(\mathrm{Spin}(n))$ denote the differential cohomology class of the basic gerbe, with underlying class $\gamma \in \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$. As explained in Section 2.1 this class is uniquely determined by just the 3-form H. We let $\hat{\omega} := a(\omega) \in \hat{\mathrm{H}}^3(FM^{[2]})$ denote the differential cohomology class associated to the 2-form ω of (6.1.3).

Definition 6.3.1. *Let M be a spin manifold with spin-oriented frame bundle FM. A* *differential string class* *is a class $\hat{\xi} \in \hat{\mathrm{H}}^3(FM)$ such that the condition*

$$\mathrm{pr}_2^*\hat{\xi} + \delta^*\hat{\gamma} + \hat{\omega} = \mathrm{pr}_1^*\hat{\xi} \tag{6.3.1}$$

over $FM^{[2]}$ is satisfied, where $\mathrm{pr}_1, \mathrm{pr}_2 : FM^{[2]} \longrightarrow FM$ are the two projections, and $\delta\colon FM^{[2]} \longrightarrow \mathrm{Spin}(n)$ is the difference map (i.e. $p' \cdot \delta(p,p') = p$).

We denote by $\mathcal{S}tr\mathcal{C}l^{\nabla}(M)$ the set of differential string classes. Condition (6.3.1) implies

$$\hat{\gamma} = i_p^*\hat{\xi} \in \hat{\mathrm{H}}^3(\mathrm{Spin}(n)) \tag{6.3.2}$$

for all $p \in FM$ and $i_p : \mathrm{Spin}(n) \longrightarrow FM : g \longmapsto pg$ the inclusion of the fibre of p. Indeed, for $j_p : \mathrm{Spin}(n) \longrightarrow FM^{[2]} : g \longmapsto (pg, p)$ we have

$$j_p^*\omega = j_p^*\left\langle \delta^*\theta \wedge \mathrm{pr}_1^*A \right\rangle = \left\langle \theta \wedge \iota_p^*A \right\rangle = \left\langle \theta \wedge \theta \right\rangle = 0.$$

Further we have $j_p^*\delta^*\hat{\gamma} = \hat{\gamma}$ and $j_p^*\mathrm{pr}_2^*\hat{\xi} = 0$ and $j_p^*\mathrm{pr}_1^*\hat{\xi} = \iota_p^*\hat{\xi}$ and so the pullback of (6.3.1) is (6.3.2).

Under the projection $c : \hat{\mathrm{H}}^3(X) \longrightarrow \mathrm{H}^3(X, \mathbb{Z})$ from differential to ordinary cohomology, condition (6.3.2) becomes the condition for string classes. In other words, c induces a well-defined map

$$c : \mathcal{S}tr\mathcal{C}l^{\nabla}(M) \longrightarrow \mathcal{S}tr\mathcal{C}l(M)$$

from differential string classes to ordinary string classes.

Remark 6.3.2. Above considerations raise the question of whether we could replace the defining condition (6.3.1) by condition (6.3.2). However, this is not the case. In order to see this, let $\hat{\xi}$ be a differential string class. Let $\mu \in \Omega^2(M)$ and $f \in C^\infty(FM, \mathbb{R})$ satisfy the following assumptions:

- μ is closed and non-zero at at least one point $x \in M$.
- $\mathrm{d}f$ is non-zero at a point $p \in FM$ in the fibre over x, and $\mathrm{d}f = 0$ at another point $p' \in FM$ in the same fibre.

Such μ, f clearly exist. We consider the class $\hat{\epsilon} = a(\epsilon) \in \hat{\mathrm{H}}^3(FM)$ associated to the 2-form $\epsilon := f \cdot \pi^*\mu \in \Omega^2(FM)$. We have $\mathrm{d}\epsilon = \mathrm{d}f \wedge \pi^*\mu + f \cdot \pi^*\mathrm{d}\mu = \mathrm{d}f \wedge \pi^*\mu$. This is non-zero at the point p, in particular, ϵ is not closed and $\hat{\epsilon} \neq 0 \in \hat{\mathrm{H}}^3(FM)$. We show that $\hat{\xi}' := \hat{\xi} + \hat{\epsilon}$ satisfies (6.3.2) but is not a differential string class. Firstly, we have $i_p^*\epsilon = i_p^*f \cdot i_p^*\pi^*\mu = 0$, since $\pi \circ i_p$ is constant. Thus, $i_p^*\hat{\xi}' = i_p^*\hat{\xi} = \hat{\gamma}$. Secondly, we have $\mathrm{d}(\mathrm{pr}_1^*\epsilon - \mathrm{pr}_2^*\epsilon) = \mathrm{d}(\mathrm{pr}_1^*f - \mathrm{pr}_2^*f) \cdot \mathrm{pr}^*\mu$, which is non-zero at the point $(p,p') \in FM^{[2]}$. In particular, $\mathrm{pr}_1^*\hat{\epsilon} \neq \mathrm{pr}_2^*\hat{\epsilon} \in \hat{\mathrm{H}}^3(FM^{[2]})$. Thus, condition (6.3.1) is not satisfied.

We have an action of $\hat{\mathrm{H}}^3(M)$ on the set $\mathcal{S}tr\mathcal{C}l^{\nabla}(M)$ of differential string classes, defined by $(\hat{\eta}, \hat{\xi}) \longmapsto \pi^*\hat{\eta} + \hat{\xi}$. Under the projection to ordinary string classes, it covers the action of $\mathrm{H}^3(M, \mathbb{Z})$ on $\mathcal{S}tr\mathcal{C}l(M)$. The differential Dixmier-Douady class gives a $\hat{\mathrm{H}}^3(M)$-equivariant map

$$\mathcal{S}tring_0^{\nabla}(M) \longrightarrow \mathcal{S}tr\mathcal{C}l^{\nabla}(M) : [\mathcal{S}] \longmapsto \widehat{\mathrm{DD}}(\mathcal{S}).$$

This map is a bijection, because $\widehat{\mathrm{DD}}$ is a bijection and the conditions on both sides are the same. Thus, we may identify the set of differential string classes with the set $\mathcal{S}tring_0^{\nabla}(M)$ introduced in the previous section. With this identification, Theorem 6.2.2 applies to differential string classes, and we obtain the following result.

Theorem 6.3.3.

(i) *The set $\mathcal{S}tr\mathcal{C}l^{\nabla}(M)$ of differential string classes is non-empty if and only if M is a string manifold; in this case it is a torsor over $\hat{\mathrm{H}}^3(M)$.*

(ii) *The map*

$$\mathrm{h}_0\mathcal{S}tring^{\nabla}(M) \longrightarrow \mathcal{S}tr\mathcal{C}l^{\nabla}(M) : (\mathcal{S}, \mathcal{A}, \sigma) \longmapsto \widehat{\mathrm{DD}}(S)$$

is a $\hat{\mathrm{H}}^3(M)$-equivariant bijection between isomorphism classes of geometric string structures and differential string classes.

(iii) *The projection from differential string classes to ordinary string classes,*

$$\mathcal{S}tr\mathcal{C}l^{\nabla}(M) \longrightarrow \mathcal{S}tr\mathcal{C}l(M), \tag{6.3.3}$$

is surjective and its fibres are torsors over $\Omega^2(M)/\Omega^2_{cl,\mathbb{Z}}(M)$.

(iv) *For every differential string class $\hat{\xi}$ there exists a unique 3-form $K \in \Omega^3(M)$ such that $\pi^*K = CS(A) + R(\hat{\xi})$ as 3-forms over FM.*

Proof. By Theorem 6.1.3 (i) and Corollary 5.3.3 (i), M is a string manifold if and only if it admits geometric string structures. By Theorem 6.2.2 we have $\hat{\mathrm{H}}^3(M)$-equivariant bijections $\mathrm{h}_0\mathcal{S}tring^{\nabla}(M) \cong \mathcal{S}tring_0^{\nabla}(M) \cong \mathcal{S}tr\mathcal{C}l^{\nabla}(M)$; this shows (i) and (ii). The projection $\mathcal{S}tr\mathcal{C}l^{\nabla}(M) \longrightarrow \mathcal{S}tr\mathcal{C}l(M)$ is a map from a $\hat{\mathrm{H}}^3(M)$-torsor to a $\mathrm{H}^3(M,\mathbb{Z})$-torsor, and equivariant along the projection c: $\mathrm{H}^3(M) \longrightarrow \mathrm{H}^3(M,\mathbb{Z})$. This projection is surjective and has kernel $\Omega^2(M)/\Omega^2_{cl,\mathbb{Z}}(M)$; this shows (iii). Assertion (iv) is Theorem 6.1.3 (ii). □

7 Transgression of string geometry

In [Wala, Section 5.2] and [NW13] we have described transgression for bundle 2-gerbes $\mathbb{G}$ with connections and with loopable surjective submersions, i.e. surjective submersions $\pi : Y \longrightarrow M$ for which $L\pi\colon LP \longrightarrow LM$ is again a surjective submersion. Suppose $\mathbb{G}$ is such a bundle 2-gerbe over M, with loopable surjective submersion $\pi\colon Y \longrightarrow M$, a curving 3-form $C \in \Omega^3(Y)$, over $Y^{[2]}$ a bundle gerbe $\mathcal{P}$ with connection, and over $Y^{[3]}$ a connection-preserving bundle 2-gerbe product $\mathcal{M}$ with associator μ. Then, the bundle gerbe $\mathscr{T}_{\mathcal{G}}$ over LM with connection and internal fusion product is given as follows: the surjective submersion is $L\pi : LY \longrightarrow LM$, the curving is $-\tau_{\Omega}(C) \in \Omega^2(LY)$, the principal U(1)-bundle with connection and fusion product over $LY^{[2]}$ is $\mathscr{T}_{\mathcal{P}}$, and the connection-preserving, fusion-preserving bundle gerbe product over $LY^{[3]}$ is $\mathscr{T}_{\mathcal{M}}$.

Theorem 7.1. *The transgression of the Chern-Simons 2-gerbe $\mathbb{CS}_M$ is canonically isomorphic to the spin lifting gerbe $\mathcal{S}_{LM}$ as bundle gerbes with connections and internal fusion products, where $\mathcal{S}_{LM}$ is equipped with the connection (χ_{spin}, B_{spin}) constructed in Section 4.2.*

Proof. The bare isomorphism has been constructed in [NW13, Proposition 6.2.1], and in [Wala, Proposition 5.2.3] it is proved that it is fusion-preserving. We only have to prove that it is connection-preserving, and for this purpose we have to recall the construction.

We start by noticing that both bundle gerbes, $\mathscr{T}_{\mathbb{CS}_M}$ and $\mathcal{S}_{LM}$, have the same surjective submersion $L\pi : LFM \longrightarrow LM$. The curving of $\mathscr{T}_{\mathbb{CS}_M}$ is $-\tau_\Omega(CS(A))$, and the curving of $\mathcal{S}_{LM}$ is B_{spin} from Proposition 4.2.5. These two 2-forms on LFM coincide [CP98, Eq. 24]. The bundle gerbe $\mathscr{T}_{\mathbb{CS}_M}$ has over $LFM^{[2]}$ the principal U(1)-bundle $\mathscr{T}_{\mathcal{P}}$, where $\mathcal{P} := \delta^*\mathcal{G}_{bas} \otimes \mathcal{I}_\omega$, equipped with a connection $\nu_{\mathcal{P}}$ and an internal fusion product $\lambda_{\mathcal{P}}$ induced from transgression. The bundle gerbe $\mathcal{S}_{LM}$ has over $LFM^{[2]}$ the principal U(1)-bundle $P = L\delta^*\widetilde{L\mathrm{Spin}(n)} = L\delta^*\mathscr{T}_{\mathcal{G}_{bas}}$, equipped with the fusion product $L\delta^*\lambda_{\mathcal{G}_{bas}}$ and the connection χ_{spin} defined in (4.2.3). Naturality of transgression and the canonical connection-preserving, fusion-preserving bundle isomorphism $\mathscr{T}_{\mathcal{I}_\omega} \cong \mathbf{I}_{\tau_\Omega(\omega)}$ provide a connection-preserving, fusion-preserving isomorphism

$$\mathscr{T}_{\mathcal{P}} \cong L\delta^*\mathscr{T}_{\mathcal{G}_{bas}} \otimes \mathscr{T}_{\mathcal{I}_\omega} \cong L\delta^*\mathscr{T}_{\mathcal{G}_{bas}} \otimes \mathbf{I}_{\tau_\Omega(\omega)}.$$

In the first place, this is an isomorphism

$$\mathscr{T}_{\mathcal{P}} \cong P \tag{7.1}$$

between the principal U(1)-bundles of the two bundle gerbes. It commutes with the bundle gerbe products ([NW13, Proposition 6.2.1]) and so we have completed the construction of an isomorphism $\mathscr{T}_{\mathbb{CS}_M} \cong \mathcal{S}_{LM}$. Moreover, the isomorphism (7.1) is fusion-preserving for the fusion product $\lambda_{\mathcal{P}}$ on the left and $L\delta^*\lambda_{\mathcal{G}_{bas}}$ on the right, as $\mathbf{I}_{\tau_\Omega(\omega)}$ is equipped with the trivial fusion product. Finally, it is connection-preserving for the connection $\nu_{\mathcal{P}}$ on the left and $L\delta^*\nu + \tau_\Omega(\omega)$ on the right: we show below in Lemma 7.2 the equality

$$\tau_\Omega(\omega) = \xi - \frac{1}{2}\Delta\zeta$$

of 2-forms on $LFM^{[2]}$, where ξ and ζ are defined in (4.1.4) and (4.2.2), respectively. This shows that $L\delta^*\nu + \tau_\Omega(\omega) = \chi_{spin}$. Thus, (7.1) is connection-preserving. □

Lemma 7.2. *The transgression of the 2-form $\omega \in \Omega^2(P^{[2]})$ of (6.1.3) is*

$$\tau_\Omega(\omega) = \xi - \frac{1}{2}\Delta\zeta,$$

with ξ and ζ the differential forms defined in (4.1.4) and (4.2.2), respectively.

Proof. We use the reformulation

$$\omega = \langle \delta^*\theta \wedge \mathrm{pr}_1^*A\rangle = \left\langle \delta^*\bar\theta \wedge \mathrm{pr}_2^*A\right\rangle.$$

Then, we calculate for tangent vectors $X_1 \in T_{\tau_1}LP$, $X_2 \in T_{\tau_2}LP$, and their differences $\delta := L\delta(\tau_1,\tau_2) \in LG$ and $Y := \mathrm{d}L\delta(X_1,X_2) \in T_\delta LG$:

$$\begin{aligned}
&\tau_\Omega(\omega)|_{(\tau_1,\tau_2)}(X_1,X_2)\\
&= \int_0^1 \omega_{(\tau_1(z),\tau_2(z))}((\partial_z\tau_1(z),\partial_z\tau_2(z)),(X_1(z),X_2(z)))\mathrm{d}z\\
&= \int_0^1 \left\langle \partial_z\delta(z)\delta(z)^{-1}, A_{\tau_2(z)}(X_2(z))\right\rangle \mathrm{d}z - \int_0^1 \left\langle Y(z)\delta(z)^{-1}, A_{\tau_2(z)}(\partial_z\tau_2(z))\right\rangle \mathrm{d}z\\
&= (\xi - \frac{1}{2}\Delta\zeta)|_{\tau_1,\tau_2}(X_1,X_2),
\end{aligned}$$

where the last step is obtained using Lemma 4.1.6. □

We are now in position to provide the second half of the main result of this article, an equivalence between string structures in M and trivializations of the spin lifting gerbe.

Theorem 7.3. *Let M be a connected spin manifold. Then, transgression and regression functors induce an equivalence of categories,*

$$\mathcal{S}tring_1^{\nabla}(M) \cong \left\{ \begin{array}{c} \text{Fusion trivializations of } \mathcal{S}_{LM} \text{ with} \\ \text{superficial fusive connection} \\ \text{compatible with } (\chi_{spin}, B_{spin}) \end{array} \right\}.$$

This equivalence is equivariant with respect to the action of $\mathrm{h}_1\mathcal{G}rb^{\nabla}(M)$ on the left hand side and the action of $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM)$ on the right hand side, under the equivalence between these monoidal categories. Moreover, if $K \in \Omega^3(M)$ is the 3-form associated to a geometric string structure by Theorem 6.1.3 (ii), and $\rho \in \Omega^2(LM)$ is the covariant derivative of the corresponding trivialization of $\mathcal{S}_{LM}$, then $\tau_\Omega(K) = -\rho$.

Proof. The purpose of the category $\mathcal{S}tring_1^{\nabla}(M)$ introduced in Section 6.2 was that its definition is purely in terms of the presheaf $\mathrm{h}_1\mathcal{G}rb^{\nabla}$: its objects are pairs $(\mathcal{S}, [\mathcal{A}])$ consisting of an object $\mathcal{S}$ in $\mathrm{h}_1\mathcal{G}rb^{\nabla}(FM)$ and a morphism

$$[\mathcal{A}] : \mathrm{pr}_2^*\mathcal{S} \otimes \mathcal{P} \longrightarrow \mathrm{pr}_1^*\mathcal{S}$$

of $\mathrm{h}_1\mathcal{G}rb^{\nabla}(FM^{[2]})$ such that an equality of morphisms of $\mathrm{h}_1\mathcal{G}rb^{\nabla}(FM^{[3]})$ holds, namely (5.1.1). Likewise, the morphisms of $\mathcal{S}tring_1^{\nabla}(M)$ are morphisms $[\mathcal{B}] : \mathcal{S} \longrightarrow \mathcal{S}'$ of $\mathrm{h}_1\mathcal{G}rb^{\nabla}(FM^{[2]})$ such that an equality of morphisms of $\mathrm{h}_1\mathcal{G}rb^{\nabla}(FM^{[3]})$ holds, namely (5.2.1). Now, we recall from Theorem 2.3.6 that transgression and regression form an equivalence

$$\mathrm{h}_1\mathcal{G}rb^{\nabla}(X) \cong \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX)$$

that is natural in X and monoidal. Hence, $\mathcal{S}tring_1^{\nabla}(M)$ is equivalent to the following category:

- An object is a pair (T, κ) of an object T in $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LFM)$ and a morphism

  $$\kappa : \mathrm{pr}_2^*T \otimes \mathscr{T}_{\mathcal{P}} \longrightarrow \mathrm{pr}_1^*T$$

  in $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LFM^{[2]})$ such that the equality

  $$\begin{array}{ccc} \mathrm{pr}_3^*T \otimes \mathrm{pr}_{23}^*\mathscr{T}_{\mathcal{P}} \otimes \mathrm{pr}_{12}^*\mathscr{T}_{\mathcal{P}} & \xrightarrow{\mathrm{pr}_{23}^*\kappa \otimes \mathrm{id}} & \mathrm{pr}_2^*T \otimes \mathrm{pr}_{12}^*\mathscr{T}_{\mathcal{P}} \\ \downarrow{\scriptstyle \mathrm{id}\otimes\mathscr{T}_{\mathcal{M}'}} & & \downarrow{\scriptstyle \mathrm{pr}_{12}^*\kappa} \\ \mathrm{pr}_3^*T \otimes \mathrm{pr}_{13}^*\mathscr{T}_{\mathcal{P}} & \xrightarrow[\mathrm{pr}_{13}^*\kappa]{} & \mathrm{pr}_1^*T \end{array}$$

  of morphisms of $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LFM^{[3]})$ holds.

- A morphism is a morphism $\varphi : T \longrightarrow T'$ in $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LFM^{[2]})$, such that the equality

  $$\begin{array}{ccc} \mathrm{pr}_2^*T \otimes \mathscr{T}_{\mathcal{P}} & \xrightarrow{\kappa} & \mathrm{pr}_1^*T \\ \downarrow{\scriptstyle \mathrm{pr}_2^*\varphi} & & \downarrow{\scriptstyle \mathrm{pr}_1^*\varphi} \\ \mathrm{pr}_2^*T' \otimes \mathscr{T}_{\mathcal{P}} & \xrightarrow[\kappa']{} & \mathrm{pr}_1^*T' \end{array}$$

  of morphisms in $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LFM^{[3]})$ holds.

This is precisely the category of fusion trivializations of $\mathscr{T}_{\mathbb{CS}_M}$ with superficial fusive connection compatible with $(\nu_P, \tau_\Omega(CS(A)))$, the connection on the transgression of the Chern-Simons 2-gerbe. The connection-preserving, fusion-preserving isomorphism of Theorem 7.1 identifies this category with the claimed one.

The equivariance under the $\mathrm{h}_1\mathcal{G}rb^\nabla(M)$-actions follows immediately from the definitions. Suppose $K \in \Omega^3(M)$ is the 3-form associated to a string structure $(\mathcal{S}, [\mathcal{A}])$, i.e. $\pi^*K = CS(A) + \mathrm{curv}(\mathcal{S})$, then $\tau_\Omega(K)$ satisfies

$$L\pi^*\tau_\Omega(K) = \tau_\Omega(CS(A)) + \tau_\Omega(\mathrm{curv}(\mathcal{S})) = -B_{spin} - \mathrm{curv}(\mathscr{T}_\mathcal{S}),$$

see [CP98, Eq. 24] and (2.2.2). Thus, $-\tau_\Omega(K)$ is the covariant derivative of $(\mathscr{T}_\mathcal{S}, \mathscr{T}_\mathcal{A})$. □

Now we come to the correspondence between string structures and trivializations of the spin lifting gerbe in the setting without connections.

Theorem 7.4. *Let M be a connected spin manifold. Then, regression induces an equivalence*

$$h\mathcal{T}riv^{th}_{fus}(\mathcal{S}_{LM}) \cong \mathcal{S}tring_1(M)$$

between the homotopy category of thin fusion trivializations of the spin lifting gerbe $\mathcal{S}_{LM}$ and the category of string structures on LM. This equivalence is equivariant with respect to the action of $h\mathcal{F}us\mathcal{B}un^{th}(LM)$ on the left hand side and the action of $\mathrm{h}_1\mathcal{G}rb(M)$ on the right hand side, along the equivalence between the two monoidal categories.

Proof. We proceed as in the proof of Theorem 7.3, now using the equivalence

$$h\mathcal{F}us\mathcal{B}un^{th}(LX) \cong \mathrm{h}_1\mathcal{G}rb(X)$$

established by the regression functor $\mathscr{R}_x$ which is natural in X, monoidal, and depends on the choice of a point $x \in X$. Choosing a point $p \in FM$ (and then using the point $(p, ..., p) \in FM^{[k]}$ in all higher fibre products) we find an equivalence $K : \mathcal{C} \longrightarrow \mathcal{S}tring_1(M)$, where $\mathcal{C}$ stands for the following category:

- An object is a pair (T, κ) of an object T in $h\mathcal{F}us\mathcal{B}un^{th}(LFM)$ and a morphism

  $$\kappa : \mathrm{pr}_2^*T \otimes \mathscr{T}_\mathcal{P} \longrightarrow \mathrm{pr}_1^*T$$

  in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$ such that the equality

  $$\begin{array}{ccc} \mathrm{pr}_3^*T \otimes \mathrm{pr}_{23}^*\mathscr{T}_\mathcal{P} \otimes \mathrm{pr}_{12}^*\mathscr{T}_\mathcal{P} & \xrightarrow{\mathrm{pr}_{23}^*\kappa \otimes \mathrm{id}} & \mathrm{pr}_2^*T \otimes \mathrm{pr}_{12}^*\mathscr{T}_\mathcal{P} \\ {\scriptstyle \mathrm{id}\otimes\mathscr{T}_{\mathcal{M}'}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{12}^*\kappa} \\ \mathrm{pr}_3^*T \otimes \mathrm{pr}_{13}^*\mathscr{T}_\mathcal{P} & \xrightarrow[\mathrm{pr}_{13}^*\kappa]{} & \mathrm{pr}_1^*T \end{array} \tag{7.2}$$

  of morphisms of $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[3]})$ holds.

- A morphism is a morphism $\varphi : T \longrightarrow T'$ in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$, such that the equality

  $$\begin{array}{ccc} \mathrm{pr}_2^*T \otimes \mathscr{T}_\mathcal{P} & \xrightarrow{\kappa} & \mathrm{pr}_1^*T \\ {\scriptstyle \mathrm{pr}_2^*\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*\varphi} \\ \mathrm{pr}_2^*T' \otimes \mathscr{T}_\mathcal{P} & \xrightarrow[\kappa']{} & \mathrm{pr}_1^*T' \end{array} \tag{7.3}$$

  of morphisms in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$ holds.

The connection-preserving, fusion-preserving isomorphism between $\mathscr{T}_{\mathbb{CS}_M}$ and $\mathcal{S}_{LM}$ induces an equivalence between $\mathcal{C}$ and a category $\mathcal{C}'$ obtained from $\mathcal{C}$ by replacing $\mathscr{T}_{\mathcal{P}}$ by the principal U(1)-bundle P of $\mathcal{S}_{LM}$ and $\mathscr{T}_{\mathcal{M}'}$ by the bundle gerbe product μ of $\mathcal{S}_{LM}$.

Now we go into the details of the category $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$ of thin fusion trivializations of the spin lifting gerbe, as introduced in Definition 3.2.5. The category $\mathcal{C}'$ receives a functor

$$K' : \mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}) \longrightarrow \mathcal{C}'$$

defined in the following natural way:

- It takes a thin fusion trivialization (T, κ, λ, d) and sends it to the pair composed of the object (T, λ, d) in $\mathcal{F}us\mathcal{B}un^{th}(FM)$ and of the homotopy class of κ, which is a morphism in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$ making diagram (7.2) commutative.
- A morphism φ between thin fusion trivializations (T, κ, λ, d) and $(T', \kappa', \lambda', d')$ is sent to its homotopy class, which is a morphism in $h\mathcal{F}us\mathcal{B}un^{th}(LFM)$. The condition, i.e. the commutativity of diagram (7.3), is exactly the same as in Definition 3.2.5.

We obtain a commutative diagram of categories and functors,

$$\begin{array}{ccccc} \left\{\begin{array}{c}\text{Fusion trivializations of } \mathcal{S}_{LM} \\ \text{with compatible superficial} \\ \text{fusive connection}\end{array}\right\} & & \xrightarrow{\text{Theorem 7.3}} & & \mathcal{S}tring_1^{\nabla}(M) \\ \downarrow & & & & \downarrow{\scriptstyle F_1} \\ \mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}) & \xrightarrow[K']{} & \mathcal{C}' \cong \mathcal{C} & \xrightarrow[K]{} & \mathcal{S}tring_1(M). \end{array} \tag{7.4}$$

Since the functor F_1 is essentially surjective (Proposition 6.2.1 (ii)), and the functor on the top is an equivalence (Theorem 7.3), it follows that K' is essentially surjective. It is also full: suppose φ is a morphism in $\mathcal{C}'$. It is represented by a fusion-preserving, thin bundle morphism $\varphi : T \longrightarrow T'$ such that diagram (7.3) is commutative in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$. This means that the representative φ is a morphism in $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$.

However, the functor K' is not faithful. This problem is solved by passing to the *homotopy category* of thin fusion trivializations, $h\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$. We note that the functor K' is well-defined on the homotopy category, i.e. it induces a functor hK' making the diagram

$$\begin{array}{ccc} \mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}) & \xrightarrow{K'} & \mathcal{C}' \\ \downarrow & \nearrow{\scriptstyle hK'} & \\ h\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}) & & \end{array}$$

strictly commutative. Indeed, a morphism on the left is a class $[\varphi]$ of morphisms between thin fusion trivializations, in which two morphisms φ_0 and φ_1 are identified, if there exists a homotopy through morphisms of $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$. Such a homotopy is, in particular, a homotopy through fusion-preserving, thin bundle morphisms. Thus, $hK'([\varphi]) := [\varphi]$ is well-defined.

As the projection $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM}) \longrightarrow h\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$ is surjective, we deduce from the fact that K' is essentially surjective and full, that hK' is essentially surjective and full, too. It remains to show that hK' is faithful. Suppose φ_0 and φ_1 are morphisms in $\mathcal{T}riv_{fus}^{th}(\mathcal{S}_{LM})$, such that they are equal in

$\mathcal{C}'$. That is, there exists a homotopy h between $\varphi_0 = h_0$ and $\varphi_1 = h_1$ through fusion-preserving, thin bundle morphisms $h_t : T \longrightarrow T'$, identifying φ_0 and φ_1 in $h\mathcal{F}us\mathcal{B}un^{th}(LFM)$. We show that the same homotopy h is a homotopy through morphisms in $\mathcal{T}riv^{th}_{fus}(\mathcal{S}_{LM})$, i.e. the diagram

$$\begin{array}{ccc} \mathrm{pr}_2^*T \otimes P & \xrightarrow{\kappa} & \mathrm{pr}_1^*T \\ {\scriptstyle \mathrm{pr}_2^*h_t \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*h_t} \\ \mathrm{pr}_2^*T' \otimes P & \xrightarrow[\kappa']{} & \mathrm{pr}_1^*T' \end{array} \tag{7.5}$$

commutes in $h\mathcal{F}us\mathcal{B}un^{th}(LFM^{[2]})$ for all $t \in [0,1]$. Indeed, as φ_0 and φ_1 are morphisms in $\mathcal{T}riv^{th}_{fus}(\mathcal{S}_{LM})$ the diagram commutes for $t = 0$ (and $t = 1$). That is, there is a homotopy H between $\mathrm{pr}_1^*\varphi_0 \circ \kappa = H_0$ and $\kappa' \circ (\mathrm{pr}_2^*\varphi_0 \otimes \mathrm{id}) = H_1$ through fusion-preserving, thin bundle morphisms $H_s : \mathrm{pr}_2^*T \otimes P \longrightarrow \mathrm{pr}_1^*T$. Now we have the following homotopies:

1. from $\mathrm{pr}_1^*h_t \circ \kappa$ to $\mathrm{pr}_1^*\varphi_0 \circ \kappa$, namely $s \longmapsto \mathrm{pr}_1^*h_{t(1-s)} \circ \kappa$
2. from $\mathrm{pr}_1^*\varphi_0 \circ \kappa$ to $\kappa' \circ (\mathrm{pr}_2^*\varphi_0 \otimes \mathrm{id})$, namely $s \longmapsto H_s$
3. from $\kappa' \circ (\mathrm{pr}_2^*\varphi_0 \otimes \mathrm{id})$ to $\kappa' \circ (\mathrm{pr}_2^*h_t \otimes \mathrm{id})$, namely $s \longmapsto \kappa' \circ (\mathrm{pr}_2^*h_{st} \otimes \mathrm{id})$

These can be concatenated to a smooth homotopy showing that diagram (7.5) is commutative in $h\mathcal{F}us\mathcal{B}un^{th}(LFM)$. □

8 Proof of Theorem A

The various functors we have introduced form the commutative diagram of Theorem A:

$$\begin{array}{ccccccc} \mathcal{S}pin^{\nabla_{sf}}_{fus}(LM) & \xrightarrow{\text{Cor. 4.2.12}} & \left\{\begin{array}{c}\text{Fusion trivializations}\\ \text{of the spin lifting}\\ \text{gerbe } \mathcal{S}_{LM} \text{ with}\\ \text{superficial connection}\end{array}\right\} & \xrightarrow{\text{Th. 7.3}} & \mathcal{S}tring^{\nabla}_1(M) & \xleftarrow{\text{Th. 6.2.2}} & \mathrm{h}_1\mathcal{S}tring^{\nabla}(M) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle \mathrm{h}_1F_2} \\ h\mathcal{S}pin^{th}_{fus}(LM) & \xrightarrow[\text{Prop. 3.2.6}]{} & \left\{\begin{array}{c}\text{Homotopy category of}\\ \text{thin fusion}\\ \text{trivializations of the}\\ \text{spin lifting gerbe } \mathcal{S}_{LM}\end{array}\right\} & \xrightarrow[\text{Th. 7.4}]{} & \mathcal{S}tring_1(M) & \xleftarrow[\text{Th. 5.3.1}]{} & \mathrm{h}_1\mathcal{S}tring(M). \end{array}$$

The separate diagrams have been discussed in Section 4.2, see (4.2.4), in Section 7, see (7.4), and in Section 6.2, see (6.2.1). The arrows are labelled with references to those statements where we have proved that they are equivalences of categories and have the claimed equivariance properties.

If M is not a string manifold, $\mathcal{S}tring(M)$ is empty. Thus, all categories in above diagram must be empty since they have functors to $\mathrm{h}_1\mathcal{S}tring(M)$. Now suppose M is a string manifold. Then, $\mathcal{S}tring(M)$ is non-empty. Since the functor F_2 is essentially surjective by Theorem 6.1.3 (i), $\mathcal{S}tring^{\nabla}(M)$ is non-empty, too, and so are all other categories in the diagram. It also follows that all vertical functors in the diagram are essentially surjective.

Table of notation

$\mathcal{B}un(X)$	The category of Fréchet principal U(1)-bundles over X
$\mathcal{B}un^{\nabla}(X)$	The category of Fréchet principal U(1)-bundles with connection
$\mathcal{G}rb(X)$	The bicategory of bundle gerbes over X
$\mathcal{G}rb^{\nabla}(X)$	The bicategory of bundle gerbes with connection over X
$h\mathcal{C}$	The homotopy category of a (topological) category, defined by identifying homotopic morphisms.
$\mathrm{h}_k\mathcal{C}$	The k-truncation of a (higher) category, defined using $(k+1)$-isomorphism classes of k-morphisms.
$\mathcal{S}pin^{th}_{fus}(LM)$	The category of thin fusion spin structures on LM(Section 3.1)
$\mathcal{S}pin^{\nabla_{sf}}_{fus}(LM)$	The category of superficial geometric fusion spin structures (Section 4.1)
$\mathcal{S}tr\mathcal{C}l(M)$	The set of string classes on M(Section 5.3)
$\mathcal{S}tr\mathcal{C}l^{\nabla}(M)$	The set of differential string classes on M(Section 6.3)
$\mathcal{S}tring(M)$	The bicategory of string structures on M(Section 5.3)
$\mathcal{S}tring_k(M)$	A decategorification of the bicategory of string structures (Section 5.3)
$\mathcal{S}tring^{\nabla}(M)$	The bicategory of geometric string structures on M(Section 6.2)
$\mathcal{S}tring^{\nabla}_k(M)$	A decategorification of the bicategory of geometric string structures on M(Section 6.2)
$\mathcal{F}us\mathcal{B}un^{th}(LX)$	The category of Fréchet principal U(1)-bundles over LX with fusion product and thin structure
$\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LX)$	The category of Fréchet principal U(1)-bundles over LX with fusion product and superficial connection

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