Sufficient Criteria for Existence of Pullback Attractors for Stochastic Lattice Dynamical Systems with Deterministic Non-autonomous Terms [1]

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Abstract: We consider the pullback attractors for non-autonomous dynamical systems generated by stochastic lattice differential equations with non-autonomous deterministic terms. We first establish a sufficient condition for existence of pullback attractors of lattice dynamical systems with both non-autonomous deterministic and random forcing terms. As an application of the abstract theory, we prove the existence of a unique pullback attractor for the first-order lattice dynamical systems with both deterministic non-autonomous forcing terms and multiplicative white noise. Our results recover many existing ones on the existences of pullback attractors for lattice dynamical systems with autonomous terms or white noises.

Keywords: Random attractor, stochastic lattice dynamical system, multiplicative white noise.

1 Introduction

The study of non-autonomous evolution equations has attracted several interests from both mathematicians and physicists due to the effects of time-dependent linear/non-linear forces from natural phenomena are represented by non-autonomous terms in the associated models. One of the important concepts for describing the asymptotic behavior of non-autonomous evolution equations is the pullback attractor, which generalized the notation of global attractor for non-autonomous dynamical systems [2, 10, 11]. The pullback attractors are different from the uniform attractor (see e.g. [7, 8]) in that they employ techniques of non-autonomous equations more straightly.

Global attractors, uniform attractors and pullback attractors all play important roles in the fields of asymptotic behavior of autonomous and non-autonomous infinite dynamical systems [7, 8, 15, 17, 19, 20]. Sometimes, the forwards dynamics may be hard to describe, in this case, there is not even an attracting trajectory (which would in general be a moving object) that describes the dynamics. Especially, in the stochastic cases, the pullback process produces a

fixed subset of the phrase space. Pullback attractors attract all bounded set, then become appropriate alternatives to study the asymptotic behavior of dynamical systems.

Lattice dynamical systems, which are coupled systems with ODEs on infinite lattices, have drawn much attention from mathematicians and physicists recently, due to the wide range of applications in various areas (e.g. [9]). For autonomous deterministic lattice dynamical systems, we can see e.g. [3, 26, 27, 28, 31] for the existence and approximations of attractors. For non-autonomous deterministic cases, we can see e.g. [25, 30, 32, 33] for the existence and continuity of kernel section, uniform attractors and pullback exponential attractors. As in the stochastic cases, stochastic lattice dynamical systems (SLDS) arise naturally while random influences or uncertainties are taken into account in lattice dynamical systems, these noises may play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models. Since Bates et al. [4] initiated the study of SLDS, lots of work have been done regarding the existence of global random attractors for SLDS with white additive/multiplicative noises in regular or weight spaces of infinite sequences, see e.g. [5, 6, 16, 29]. For lattice dynamical systems perturbed by other "rough" noises, we can refer to e.g. [13, 14] for more details. As we can seen that all the systems above are considered with the autonomous deterministic external forcing terms (if indeed exist!). There is no results on pullback attractors for general non-autonomous SLDS (with time-dependent deterministic coefficients and external forcing terms) as far as we know.

Motivated by [22] and [30], we consider the existence of non-autonomous dynamical systems generated by lattice differential equations with both non-autonomous deterministic and stochastic forcing terms. By borrowing the main framework of [22] on two parametric space, we first set up the abstract structure for the continuous cocycle. As a typical example, we investigate the following stochastic lattice dynamical systems (SLDS) with time-dependent external forcing terms:

$$\frac{du_i(t)}{dt} = \nu_i(t)(u_{i-1} - 2u_i + u_{i+1}) - \lambda_i(t)u_i - f_i(u_i, t) + g_i(t) + u_i \circ \frac{dw(t)}{dt}, \qquad (1.1)$$

where $i \in \mathbb{Z}$, $\mathbb{Z}$ denotes the integer set; $u_i \in \mathbb{R}$, $\nu_i(t)$ and $\lambda_i(t)$ are locally integrable in t; $g_i \in C(\mathbb{R}, \mathbb{R})$ and $f_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies proper dissipative conditions; $w(t)$ is a Brownian motion (Wiener process) and $\circ$ denotes the Stratonovich sense of the stochastic term.

Stochastic systems similar to (1.1) are discrete of the *Reaction-Diffusion equation* which used to model the phenomena of stochastic resonance in biology and physics, where f is a time-dependent input signal and w is a Wiener process used to test the impact of stochastic fluctuations on f. For this topic, we can see e.g. [12, 21, 23, 24] and the references therein. The main difference between system (1.1) and the model considered in [4] is the coefficients and

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deterministic external forcing terms are time-dependent. In this case, the existing results of one parametric space cannot be applied directly. We first need to introduce two parametric space to describe the dynamics of the SLDS: one is responsible for deterministic forcing and the other is responsible for stochastic perturbations. Then we applied the skeleton to (1.1).

The outline of the paper is as follows. In the next section, we recall some results regarding pullback attractor for non-autonomous dynamical systems over two parametric spaces in [22]. In section 3, we establish the conditions on the existence of pullback attractors for cocycles over two parametric spaces. In section 4, a sufficient condition for the existence of pullback attractors for lattice differential equations with both non-autonomous deterministic and random forcing terms is given. As a example of the result in previous sections, the existence of pullback attractor for the first-order SLDS with time-dependent deterministic force and multiplicative white noise is studied in the last section.

2   Preliminaries

For the reader's convenience, we recall the theory of pullback random dynamical systems over two parametric spaces in [22].

Let Ω_1 be a nonempty set and $\{\theta_{1,t}\}_{t\in\mathbb{R}}$ be a family of mappings from Ω_1 into itself such that $\theta_{1,0}$ is the identity on Ω_1 and $\theta_{1,s+t} = \theta_{1,t}\theta_{1,s}$ for all $t, s \in \mathbb{R}$. Let $(\Omega_2, \mathcal{F}_2, P)$ be a probability space and $\theta_2 : \mathbb{R} \times \Omega_2 \to \Omega_2$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}_2, \mathcal{F}_2)$ -measurable mapping such that $\theta_2(0, \cdot)$ is the identity on Ω_2, $\theta_2(s + t, \cdot) = \theta_2(t, \cdot)\theta_2(s, \cdot)$ for all $t, s \in \mathbb{R}$ and $P\theta_2(t, \cdot) = P$ for all $t \in \mathbb{R}$. We usually write $\theta_2(t, \cdot)$ as $\theta_{2,t}$ and call both $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ a parametric dynamical system.

Let (X, d) be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. Denote by 2^X the collection of all subsets of X. A set-valued mapping $K : \Omega_1 \times \Omega_2 \to 2^X$ is called measurable with respect to $\mathcal{F}_2$ in Ω_2 if the value $K(\omega_1, \omega_2)$ is a closed nonempty subset of X for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, and the mapping $\omega_2 \in \Omega_2 \to d(x, K(\omega_1, \omega_2))$ is $(\mathcal{F}_2, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\omega_1 \in \Omega_1$. If K is measurable with respect to $\mathcal{F}_2$ in Ω_2, then we say that the family $\{K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is measurable with respect to $\mathcal{F}_2$ in Ω_2. We now define a cocycle on X over two parametric spaces.

Definition 2.1. *Let* $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ *and* $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ *be parametric dynamical systems. A mapping* $\Phi : \mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times X \to X$ *is called a continuous cocycle on* X *over* $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ *and* $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ *if for all* $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ *and* $t, \tau \in \mathbb{R}^+$, *the following conditions (i)-(iv) are satisfied:*

(i) $\Phi(\cdot, \omega_1, \cdot, \cdot) : \mathbb{R}^+ \times \Omega_2 \times X \to X$ *is* $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_2 \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \omega_1, \omega_2, \cdot)$ *is the identity on* X;

(iii) $\Phi(t + \tau, \omega_1, \omega_2, \cdot) = \Phi(t, \theta_{1,\tau}\omega_1, \theta_{2,\tau}\omega_2, \cdot)\Phi(\tau, \omega_1, \omega_2, \cdot)$;

(iv) $\Phi(t, \omega_1, \omega_2, \cdot) : X \to X$ *is continuous.*

In the sequel, we use $\mathcal{D}(X)$ to denote a collection of some families of nonempty subsets of X:

$$\mathcal{D}(X) = \{D = \{D(\omega_1, \omega_2) \subseteq X : \ D(\omega_1, \omega_2) \neq \emptyset, \ \omega_1 \in \Omega_1, \ \omega_2 \in \Omega_2\}\}.$$

Definition 2.2. *Let* $\mathcal{D}(X)$ *be a collection of some families of nonempty subsets of* X *and* $K = \{K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \ \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$. *Then* K *is called a* $\mathcal{D}(X)$-*pullback absorbing set for* Φ *if for all* $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ *and for every* $B \in \mathcal{D}(X)$, *there exists* $T = T(B, \omega_1, \omega_2) > 0$ *such that*

$$\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)) \subseteq K(\omega_1, \omega_2) \quad \text{for all } t \geq T.$$

If, in addition, for all $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$, $K(\omega_1, \omega_2)$ *is a closed nonempty subset of* X *and* K *is measurable with respect to the* P-*completion of* $\mathcal{F}_2$ *in* Ω_2, *then we say* K *is a closed measurable* $\mathcal{D}(X)$-*pullback absorbing set for* Φ.

Definition 2.3. *Let* $\mathcal{D}(X)$ *be a collection of some families of nonempty subsets of* X. *Then* Φ *is said to be* $\mathcal{D}(X)$-*pullback asymptotically compact in* X *if for all* $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$, *the sequence*

$$\{\Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X$$

whenever $t_n \to \infty$, *and* $x_n \in B(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)$ *with* $\{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \ \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$.

Definition 2.4. *Let* $\mathcal{D}(X)$ *be a collection of some families of nonempty subsets of* X *and* $\mathcal{A} = \{\mathcal{A}(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$. *Then* $\mathcal{A}$ *is called a* $\mathcal{D}(X)$-*pullback attractor for* Φ *if the following conditions (i)-(iii) are fulfilled:*

(i) $\mathcal{A}$ *is measurable with respect to the* P-*completion of* $\mathcal{F}_2$ *in* Ω_2 *and* $\mathcal{A}(\omega_1, \omega_2)$ *is compact for all* $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$.

(ii) $\mathcal{A}$ *is invariant, that is, for every* $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$,

$$\Phi(t, \omega_1, \omega_2, \mathcal{A}(\omega_1, \omega_2)) = \mathcal{A}(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2), \quad \forall \, t \geq 0.$$

(iii) $\mathcal{A}$ *attracts every member of* $\mathcal{D}(X)$, *that is, for every* $B = \{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$ *and for every* $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$,

$$\lim_{t \to \infty} d(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)), \mathcal{A}(\omega_1, \omega_2)) = 0.$$

The following result on the existence and uniqueness of $\mathcal{D}(X)$-*pullback attractors* for Φ can be found in [22].

Proposition 2.5. *Let Φ be a continuous cocycle on X over $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. Suppose that $K = \{K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$ is a closed measurable (w.r.t. the P-completion of $\mathcal{F}_2$) $\mathcal{D}(X)$-pullback absorbing set for Φ in $\mathcal{D}(X)$ and Φ is $\mathcal{D}(X)$-pullback asymptotically compact in X. Then Φ has a unique $\mathcal{D}(X)$-pullback attractor $\mathcal{A} = \{\mathcal{A}(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}(X)$ which is given by*

$$\mathcal{A}(\omega_1, \omega_2) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, K(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2))}.$$

To describe the size of subsets in a Banach space X, we introduce the concept of *Kolmogorov's ε-entropy*. Let Y be a subset of X. Given $\varepsilon > 0$, we define

$$n_\varepsilon(Y) := \min\{n \geq 1 : Y \subset \bigcup_{i=1}^n \mathcal{N}(x_i, \varepsilon) \quad \text{for some} \quad x_1, \ldots, x_n \in X\},$$

where $\mathcal{N}(x_i, \varepsilon) = \{y \in X : \|y - x_i\|_X < \varepsilon\}$. The *Kolmogorov ε-entropy* of the subset Y of X is the number

$$\mathbb{K}_\varepsilon(Y) := \ln n_\varepsilon(Y) \in [0, +\infty]. \tag{2.1}$$

3 Pullback attractors for cocycles in ℓ^2

In this section, we provide some sufficient conditions for the existence of pullback attractors for cocycles in ℓ^2.

Let D be a bounded nonempty subset of ℓ^2, denote by $\|D\| = \sup_{u\in D} \|u\|$. Suppose $D = \{D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is a family of bounded nonempty subsets of ℓ^2 satisfying, for every $\gamma > 0$,

$$\lim_{s\to+\infty} e^{-\gamma s} \|D(\theta_{1,-s}\omega_1, \theta_{2,-s}\omega_2)\|^2 = 0. \tag{3.1}$$

Denote by $\mathcal{D}(\ell^2)$ the collection of all family of bounded nonempty subsets of ℓ^2,

$$\mathcal{D}(\ell^2) = \{D = \{D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} : D \text{ satisfies } (3.1)\}.$$

Definition 3.1. *A mapping $\Phi\colon \mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times \ell^2 \to \ell^2$ is said to be asymptotically null in $\mathcal{D}(\ell^2)$ if for a.e. $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, any $B(\omega_1, \omega_2) \in \mathcal{D}(\ell^2)$, and any $\varepsilon > 0$, there exist $T(\varepsilon, \omega_1, \omega_2, B(\omega_1, \omega_2)) > 0$ and $I(\varepsilon, \omega_1, \omega_2, B(\omega_1, \omega_2)) \in \mathbb{N}$ such that*

$$\sum_{|i|>I(\varepsilon, \omega_1, \omega_2, B(\omega_1, \omega_2))} |(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, u(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)))_i|^2 \leq \varepsilon^2,$$

for all $t \geq T(\varepsilon, \omega_1, \omega_2, B(\omega_1, \omega_2))$ and $u(\omega_1, \omega_2) \in B(\omega_1, \omega_2)$.

Theorem 3.2. *Suppose that*

(**a**) *there exists a closed measurable (w.r.t. the P-completion of $\mathcal{F}_2$) $\mathcal{D}(\ell^2)$-pullback absorbing set K in $\mathcal{D}(\ell^2)$ such that for a.e. $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, any $B(\omega_1, \omega_2) \in \mathcal{D}(\ell^2)$, there exists $T_B(\omega_1, \omega_2) > 0$ yielding*
$$\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2) \subset K \text{ for all } t \geq T_B(\omega_1, \omega_2);$$

(**b**) *$\Phi: \mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times \ell^2 \to \ell^2$ is asymptotically null on K, i.e., for a.e. $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, any $B(\omega_1, \omega_2) \in \mathcal{D}(\ell^2)$, and any $\varepsilon > 0$, there exist $T(\varepsilon, \omega_1, \omega_2, K) > 0$ and $I_0(\varepsilon, \omega_1, \omega_2, K) \in \mathbb{N}$ such that*
$$\sup_{u \in K} \sum_{|i| > I_0(\varepsilon, \omega_1, \omega_2, K)} |(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, u(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)))_i|^2 \leq \varepsilon^2,$$
$$\forall t \geq T(\varepsilon, \omega_1, \omega_2, B(\omega_1, \omega_2)).$$

Then

(*i*) *Φ possesses a unique $\mathcal{D}(\ell^2)$-pullback attractor is given by, for each $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,*
$$\mathcal{A}(\omega_1, \omega_2) = \bigcap_{\tau \geq T_K(\omega_1, \omega_2)} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, K(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2))};$$

(*ii*) *the Kolmogorov ϵ-entropy of $\mathcal{A}(\omega_1, \omega_2)$ satisfies*
$$\mathbb{K}_\varepsilon \leq (2I_0(\varepsilon, \omega_1, \omega_2, K) + 1) \ln(\lfloor \frac{2r_0(\omega_1, \omega_2)\sqrt{2I_0(\varepsilon, \omega_1, \omega_2, K) + 1}}{\varepsilon} \rfloor + 1),$$

where $r_0(\omega_1, \omega_2) = \sup_{u(\omega_1, \omega_2) \in K} \|u(\omega_1, \omega_2)\|$.

Proof. The proof is based on Theorem 3.1 in [16] and Proposition 2.5 under slightly modifications.

(i) For a.e. $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$ and $t_n \to \infty$ as $n \to \infty$, let $p_n(\omega_1, \omega_2) \in K(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2) \in \mathcal{D}(\ell^2)$ $(n = 1, 2, \cdots)$ and
$$u^{(n)}(\omega_1, \omega_2) = \Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2),$$

where $u_i^{(n)}(\omega_1, \omega_2) = (\Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2))_i, i \in \mathbb{Z}$. By (**a**), there exists $N_1(\omega_1, \omega_2, K) \in \mathbb{N}$ such that $t_n \geq T_K(\omega_1, \omega_2)$ if $n \geq N_1(\omega_1, \omega_2, K)$. Hence
$$u^{(n)}(\omega_1, \omega_2) = \Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)p_n(\omega_1, \omega_2) \in K, \ \forall n \geq N_1(\omega_1, \omega_2, K).$$

Now let us prove that the set
$$\Lambda = \{u^{(n)}(\omega_1, \omega_2) = \Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)p_n(\omega_1, \omega_2)\}_{n \geq N_1(\omega_1, \omega_2, K)}$$

is pre-compact, that is, for any given $\varepsilon > 0$, Λ has a finite covering of balls of radius ε. By condition (**b**), there exists $T_1(\varepsilon, \omega_1, \omega_2, K) > 0$ and $I_0(\varepsilon, \omega_1, \omega_2, K) \in \mathbb{N}$ such that for $n \geq N_2(\varepsilon, \omega_1, \omega_2, K)$, we have that $t_n \geq T_1(\varepsilon, \omega_1, \omega_2, K)$ and

$$\sup_{n \geq N_1(\omega_1, \omega_2, K)} \left(\sum_{|i| > I_0(\varepsilon, \omega_1, \omega_2, K)} |\Phi(t_n, \theta_{1, -t_n}\omega_1, \theta_{2, -t_n}\omega_2) p_n(\omega_1, \omega_2))_i|^2 \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2}.$$

Let $N_3(\varepsilon, \omega_1, \omega_2, K) = \max\{N_1(\omega_1, \omega_2, K), N_2(\varepsilon, \omega_1, \omega_2, K)\}$. Thus for any $n \geq N_3(\varepsilon, \omega_1, \omega_2, K)$, $u^{(n)}(\omega_1, \omega_2) = (u_i^{(n)}(\omega_1, \omega_2))_{i \in \mathbb{Z}}$ can be decomposed into

$$
\begin{aligned}
u^{(n)}(\omega_1, \omega_2) &= (u_i^{(n)}(\omega_1, \omega_2))_{i \in \mathbb{Z}} \\
&= (v_i^{(n)}(\omega_1, \omega_2))_{i \in \mathbb{Z}} + (o_i^{(n)}(\omega_1, \omega_2))_{i \in \mathbb{Z}} \\
&= v^{(n)}(\omega_1, \omega_2) + o^{(n)}(\omega_1, \omega_2),
\end{aligned}
\tag{3.2}
$$

where

$$v_i^{(n)}(\omega_1, \omega_2) = \begin{cases} u_i^{(n)}(\omega_1, \omega_2), & |i| \leq I_0(\varepsilon, \omega_1, \omega_2, K), \\ 0, & |i| > I_0(\varepsilon, \omega_1, \omega_2, K), \end{cases}$$

and

$$o_i^{(n)}(\omega_1, \omega_2) = \begin{cases} 0, & |i| \leq I_0(\varepsilon, \omega_1, \omega_2, K), \\ u_i^{(n)}(\omega_1, \omega_2), & |i| > I_0(\varepsilon, \omega_1, \omega_2, K). \end{cases}$$

Then for $n \geq N_3(\varepsilon, \omega_1, \omega_2, K)$, we obtain

$$
\begin{aligned}
\|v^{(n)}(\omega_1, \omega_2)\|^2 &= \sum_{|i| \leq I_0(\varepsilon, \omega_1, \omega_2, K)} |u_i^{(n)}(\omega_1, \omega_2)|^2 \\
&\leq \|u^{(n)}(\omega_1, \omega_2)\|^2 \leq r_0^2(\omega_1, \omega_2),
\end{aligned}
$$

$$\|o^{(n)}(\omega_1, \omega_2)\|^2 = \sum_{|i| > I_0(\varepsilon, \omega_1, \omega_2, K)} |u_i^{(n)}(\omega_1, \omega_2)|^2 \leq \frac{\varepsilon^2}{4},$$

and

$$|v_i^{(n)}(\omega_1, \omega_2)| \leq r_0(\omega_1, \omega_2),$$

for all $|i| \leq I_0(\varepsilon, \omega_1, \omega_2, K)$, where $r_0(\omega_1, \omega_2)$ defined in (ii). Now let

$$
\begin{aligned}
\Gamma(\omega_1, \omega_2) = \{v = (v_i)_{|i| \leq I_0(\varepsilon, \omega_1, \omega_2, K)} &\in \mathbb{R}^{2I_0(\varepsilon, \omega_1, \omega_2, K)+1} : \\
v_i \in \mathbb{R}, |v_i| \leq r_0(\omega_1, \omega_2)\},
\end{aligned}
$$

and

$$
\begin{aligned}
&n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2)) \\
&= (\lfloor \frac{2r_0(\omega_1, \omega_2)\sqrt{2I_0(\varepsilon, \omega_1, \omega_2, K) + 1}}{\varepsilon} \rfloor + 1)^{2I_0(\varepsilon, \omega_1, \omega_2, K)+1}.
\end{aligned}
$$

Then $\Gamma(\omega_1, \omega_2) \subset \mathbb{R}^{2I_0(\varepsilon, \omega_1, \omega_2, K)+1}$ is a $(2I_0(\varepsilon, \omega_1, \omega_2, K) + 1)$-dimensional regular polyhedron which is covered by $n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$ open balls of radius $\frac{\varepsilon}{2}$ centered at $u_m^* = (u_{m,i}^*)_{|i| \leq I_0(\varepsilon, \omega_1, \omega_2, K)}$, $u_{m,i}^* \in \mathbb{R}, 1 \leq m \leq n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$, in the norm of $\mathbb{R}^{2I_0(\varepsilon, \omega_1, \omega_2, K)+1}$.

For each $1 \leq m \leq n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$, we set $v_i = (v_{m,i})_{i \in \mathbb{Z}} \in \ell^2$ such that

$$v_{m,i} = \begin{cases} u_{m,i}^*, & |i| \leq I_0(\varepsilon, \omega_1, \omega_2, K), \\ 0, & |i| > I_0(\varepsilon, \omega_1, \omega_2, K). \end{cases}$$

Then for $v^{(n)}(\omega_1, \omega_2) = (v_i^{(n)}(\omega_1, \omega_2))_{i \in \mathbb{Z}}$ $(n \geq N_3(\varepsilon, \omega_1, \omega_2, K))$ in the decomposition (3.2), there exists $m_0 \in \{1, 2, \cdots, n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))\}$ such that

$$\|v^{(n)}(\omega_1, \omega_2) - v_{m_0}\|^2 = \sum_{|i| \leq I_0(\varepsilon, \omega_1, \omega_2, K)} |u_i^{(n)}(\omega_1, \omega_2) - u_{m_0,i}|^2 \leq \frac{\varepsilon^2}{4},$$

and hence, we get

$$\|u^{(n)}(\omega_1, \omega_2) - v_{m_0}\|^2 = \|v^{(n)}(\omega_1, \omega_2) - v_{m_0} + o^{(n)}(\omega_1, \omega_2)\|^2$$
$$\leq 2\|v^{(n)}(\omega_1, \omega_2) - v_{m_0}\|^2 + 2\|o^{(n)}(\omega_1, \omega_2)\|^2 \leq \varepsilon^2.$$

Therefore, $\{u_i^{(n)}(\omega_1, \omega_2) = \Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)_{n \geq N_3(\varepsilon, \omega_1, \omega_2, K)}\} \subset \ell^2$ can be covered by $n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$ open balls of radius ε centered at $v_m = (v_{m,i})_{i \in \mathbb{Z}}, 1 \leq m \leq n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$.

(ii) By the invariant property of $\mathcal{D}(X)$-*pullback attractors*, we have

$$\mathcal{A}(\omega_1, \omega_2) = \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)\mathcal{A}(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2) \subset K$$

for $t \geq T(\omega_1, \omega_2, K)$ and a.e. $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$. For any $\varepsilon > 0$, we can see that $\mathcal{A}(\omega_1, \omega_2)$ can be covered under the norm of ℓ^2, by $n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$-balls in ℓ^2 with center $v_m = (v_{m,i})_{i \in \mathbb{Z}}, 1 \leq m \leq n_{\varepsilon, \omega_1, \omega_2}(\Gamma(\omega_1, \omega_2))$ and radius ε. Thus, by the definition of (2.1), the proof is completed. $\square$

4   Pullback attractors for lattice differential equations in ℓ^2

In this section, we discuss the proper choice of parametric spaces Ω_1 and Ω_2 to consider pullback attractors for lattice differential equations with both non-autonomous deterministic and random forcing terms by using the abstract theory presented in the previous section.

Suppose now $\Omega_1 = \mathbb{R}$. Define a family $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ of shift operators by

$$\theta_{1,t}(\tau) = \tau + t, \quad \forall t, \tau \in \mathbb{R}. \tag{4.1}$$

Let $\Phi \colon \mathbb{R}^+ \times \mathbb{R} \times \Omega_2 \times \ell^2 \to \ell^2$ be a continuous cocycle on ℓ^2 over $(\mathbb{R}, \{\theta_{1,t}\}_{t \in \mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t \in \mathbb{R}})$ where $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ is defined in (4.1). Due to Theorem 3.2, we obtain the following result:

Theorem 4.1. *Suppose that*

(**a**) *there exists a closed measurable (w.r.t. the P-completion of $\mathcal{F}_2$) $\mathcal{D}(\ell^2)$-pullback absorbing set K in $\mathcal{D}(\ell^2)$ such that for a.e. $\tau \in \mathbb{R}, \omega \in \Omega_2$, any $B(\tau,\omega) \in \mathcal{D}(\ell^2)$, there exists $T_B(\tau,\omega) > 0$ yielding $\Phi(t, \tau - t, \theta_{2,-t}\omega)B(\tau - t, \theta_{2,-t}\omega) \subset K$ for all $t \geq T_B(\tau,\omega)$;*

(**b**) *$\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega_2 \times \ell^2 \to \ell^2$ is asymptotically null on K, i.e., for a.e. $\tau \in \mathbb{R}, \omega \in \Omega_2$, any $B(\tau,\omega) \in \mathcal{D}(\ell^2)$, and any $\varepsilon > 0$, there exist $T(\varepsilon, \tau, \omega, K) > 0$ and $I_0(\varepsilon, \tau, \omega, K) \in \mathbb{N}$ such that*

$$\sup_{u \in K} \sum_{|i| > I_0(\varepsilon,\tau,\omega,K)} |(\Phi(t, \tau - t, \theta_{2,-t}\omega, u(\tau - t, \theta_{2,-t}\omega)))_i|^2 \leq \varepsilon^2,$$
$$\forall t \geq T(\varepsilon, \tau, \omega, B(\tau,\omega)). \tag{4.2}$$

Then

(i) *Φ possesses a unique $\mathcal{D}(\ell^2)$-pullback attractor is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega_2$,*

$$\mathcal{A}(\tau,\omega) = \bigcap_{s \geq T_K(\tau,\omega)} \overline{\bigcup_{t \geq s} \Phi(t, \tau - t, \theta_{2,-t}\omega, K(\tau - t, \theta_{2,-t}\omega))}; \tag{4.3}$$

(ii) *the Kolmogorov ε-entropy of $\mathcal{A}(\tau,\omega)$ satisyies*

$$\mathbb{K}_\varepsilon \leq (2I_0(\varepsilon, \tau, \omega, K) + 1) \ln(\lfloor \frac{2r_0(\tau,\omega)\sqrt{2I_0(\varepsilon, \tau, \omega, K) + 1}}{\varepsilon} \rfloor + 1),$$

where $r_0(\tau,\omega) = \sup_{u(\tau,\omega) \in K} \|u(\tau,\omega)\|$, $\forall \tau \in \mathbb{R}, \omega \in \Omega_2$.

5 Pullback attractors for SLDS in ℓ^2

In this section, we will apply Theorem 4.1 to prove the existence of a pullback attractor for non-autonomous first order stochastic lattice dynamical system.

5.1 Mathematical Settings

Denote $C_b(\mathbb{R}, \ell^2)$ be the space of all continuous bounded functions from $\mathbb{R}$ into ℓ^2. Consider the following non-autonomous first order lattice differential equations with time-dependent external forcing terms and multiplicative white noise

$$\frac{du_i(t)}{dt} = \nu_i(t)(u_{i-1} - 2u_i + u_{i+1}) - \lambda_i(t)u_i - f_i(u_i, t) + g_i(t) + u_i \circ \frac{dw(t)}{dt}, \quad i \in \mathbb{Z}, \tag{5.1}$$

with initial data

$$u_i(\tau) = u_{i,\tau}, \quad i \in \mathbb{Z}, \tau \in \mathbb{R}, \tag{5.2}$$

where $u_i \in \mathbb{R}$, $\mathbb{Z}$ denotes the integer set; $\nu_i(t)$ and $\lambda_i(t)$ are locally integrable in t; $g_i \in C(\mathbb{R}, \mathbb{R})$ and $f_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ for $i \in \mathbb{Z}$; w is an independent Brownian motion. Note that system (5.1)-(5.2) can be written as for $t \geq \tau \in \mathbb{R}$,

$$\frac{du}{dt} = -\nu(t)Au - \lambda(t)u - f(u,t) + g(t) + u \circ \frac{dw(t)}{dt}, \ u(\tau) = u_\tau = (u_{i,\tau})_{i\in\mathbb{Z}}, \tag{5.3}$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $f(u,t) = (f_i(u_i,t))_{i\in\mathbb{Z}}, g(t) = (g_i(t))_{i\in\mathbb{Z}}$, $Au = (-u_{i-1} + 2u_i - u_{i+1})_{i\in\mathbb{Z}}$ and $w(t)$ is the white noise with values in ℓ^2 defined on the probability space $(\Omega, \mathcal{F}, P)$ and

$$\Omega = \{\omega \in C(\mathbb{R}, \ell^2) : \omega(0) = 0\},$$

the Borel sigma-algebra $\mathcal{F}$ is generated by the compact open topology, and P is the corresponding Wiener measure on $\mathcal{F}$. Define a group $\{\theta_{2,t}\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$ by

$$\theta_{2,t}\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, t \in \mathbb{R}. \tag{5.4}$$

Then $(\Omega, \mathcal{F}, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ is a parametric dynamical system. We make the following assumptions:

(**A1**) $\lambda_i(t), \nu_i(t) \in L^1_{loc}(\mathbb{R})$ in t and there exist positive constants λ^0, λ_0 and ν^0, ν_0 such that for $\forall i \in \mathbb{Z}, t \in \mathbb{R}$,

$$0 < \lambda_0 \leq \lambda_i(t) \leq \lambda^0 < +\infty,$$
$$0 < \nu_0 \leq \nu_i(t) \leq \nu^0 < +\infty;$$

(**A2**) $f_i(x,t)$ is differentiable in x and continuous in t; $f_i(0,t) = 0$; $xf_i(x,t) \geq -\alpha_i^2(t)$, where $\alpha(t) = (\alpha_i(t))_{i\in\mathbb{Z}} \in C_b(\mathbb{R}, \ell^2)$, and there exists a constant $\beta \geq 0$ such that $\partial_x f_i(x,t) \geq -\beta$, $\forall x, t \in \mathbb{R}, i \in \mathbb{Z}$;

(**A3**) There exists a positive-valued continuous function $\zeta(\iota, t) \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ such that

$$\sup_{i\in\mathbb{Z}} \max_{x\in[-\iota,\iota]} |\partial_x f_i(x,t)| \leq \zeta(\iota, t), \ \forall \iota \in \mathbb{R}^+, t \in \mathbb{R};$$

(**A4**) $g(t) = (g_i(t))_{i\in\mathbb{Z}} \in C_b(\mathbb{R}, \ell^2)$.

Now, let $\{\theta_{1,t}\}_{t\in\mathbb{R}}$ be the group acting on $\mathbb{R}$ given by (4.1). We next define a continuous cocycle for system (5.3) over $(\mathbb{R}, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. This can be done by first transferring the stochastic system into a corresponding non-autonomous deterministic one. Given $\omega \in \Omega$, denote by

$$z(\omega) = -\int_{-\infty}^0 e^r \omega(r) dr. \tag{5.5}$$

Then the random variable z given in (5.5) is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$dz + zdt = dw(t).$$

In other words, we get

$$dz(\theta_{2,t}\omega) + z(\theta_{2,t}\omega)dt = dw(t). \tag{5.6}$$

By [4, 6], we know that there exists a $\theta_{2,t}$-variant set $\Omega' \subseteq \Omega$ of full P measure such that $z(\theta_{2,t}\omega)$ is continuous in t for every $\omega \in \Omega'$, and the random variable $|z(\omega)|$ is tempered. In addition, for every $\omega \in \Omega'$, we have the following limits:

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \quad \lim_{t\to\pm\infty} \frac{|z(\theta_{2,t}\omega)|}{|t|} = 0 \quad \text{and} \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_{2,s}\omega)ds = 0. \tag{5.7}$$

Hereafter, we will write Ω as Ω' and $\theta_{2,t}$ as θ_t instead.

5.2 Existence and Uniqueness of a Mild Solution

Let $u(t)$ be the solution of system (5.3), then $v(t) = u(t)e^{-z(\theta_t\omega)}$ satisfies

$$\frac{dv}{dt} = -\nu(t)Av - \lambda(t)v - e^{-z(\theta_t\omega)}f(e^{z(\theta_t\omega)}v, t) + e^{-z(\theta_t\omega)}g(t) + z(\theta_t\omega)v, \tag{5.8}$$

with initial condition $v_\tau = v(\tau, \omega) = u_\tau e^{-z(\theta_\tau\omega)}$, $t > \tau, \tau \in \mathbb{R}, \omega \in \Omega$. We recall $v : [\tau, \tau + T) \to \ell^2$ $(T > 0)$ a mild solution of the following random differential equation

$$\frac{dv(t)}{dt} = G(v, t, \theta_t\omega), \quad v = (v_i)_{i\in\mathbb{Z}}, G = (G_i)_{i\in\mathbb{Z}}, \quad t \geq \tau \in \mathbb{R},$$

where $\omega \in \Omega$, if $v \in C([\tau, \tau + T), \ell^2)$ and

$$v_i(t, \tau) = v_i(\tau) + \int_\tau^t G_i(v(s), s, \theta_s\omega)ds \quad \text{for } i \in \mathbb{Z} \text{ and } t \in [\tau, \tau + T).$$

In this subsection, we will prove the existence and uniqueness of the mild solution of system (5.8).

Proposition 5.1. *Let $T > 0$ and assumptions (**A1**-**A4**) hold. Then for $\tau \in \mathbb{R}, \omega \in \Omega$ and any initial data $v_\tau \in \ell^2$, system (5.8) has a unique $(\mathcal{F}, \mathcal{B}(\ell^2))$-measurable mild solution $v(\cdot, \tau; \omega, v_\tau, g) \in C([\tau, \tau + T), \ell^2)$ with $v(\tau, \tau; \omega, v_\tau, g) = v_\tau$, $v(t, \tau; \omega, v_\tau, g) \in \ell^2$ being continuous in $v_\tau \in \ell^2$ and $g \in C_b(\mathbb{R}, \ell^2)$. Moreover, the solution $v(t, \tau; \omega, v_\tau, g)$ exists globally on $[\tau, +\infty)$ for any $\tau \in \mathbb{R}$. Moreover, for given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in \ell^2$, the mapping*

$$\Phi(t, \tau, \omega, v_\tau, g) = v(t + \tau, \tau; \theta_{-\tau}\omega, v_\tau, g) = u(t + \tau, \tau; \theta_{-\tau}\omega, u_\tau, g)e^{-z(\theta_t\omega)},$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times \ell^2$ to ℓ^2 over $(\mathbb{R}, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, where $v_\tau = u_\tau e^{-z(\theta_\tau\omega)}$.

Proof. We first show that if $v_\tau \in \ell^2$, system (5.8) has a unique measurable mild solution $v(t, \tau; \omega, v_\tau, g) \in \ell^2$ on $[\tau, \tau + T)$ with $v(\tau, \tau; \omega, v_\tau, g) = v_\tau$ for $T > 0$ and $\omega \in \Omega$.

Given $\omega \in \Omega, v_\tau \in \ell^2$ and $g \in C_b(\mathbb{R}, \ell^2)$, let

$$F(v, t, \omega) = -\nu(t)Av - \lambda(t)v - e^{-z(\omega)}f(ve^{z(\omega)}, t) + e^{-z(\omega)}g(t) + vz(\omega).$$

Note that $F(v, t, \omega)$ is continuous in v and locally integrable in t and measurable in ω from $\ell^2 \times \mathbb{R} \times \Omega$ into ℓ^2. Denote $||| \cdot ||| = \sup_{t \in \mathbb{R}} \| \cdot (t) \|$, then by (**A1-A4**),

$$\|F(v, t, \omega)\| \leq (\lambda^0 + 4\nu^0 + \max\{\zeta(\|v\| |e^{z(\omega)}|, t), \beta\} + |z(\omega)|)\|v\|$$
$$+ |e^{-z(\omega)}| \|\|g\|\|.$$

Hence for any $v^{(1)} = (v_i^{(1)})_{i \in \mathbb{Z}}, v^{(2)} = (v_i^{(2)})_{i \in \mathbb{Z}} \in \ell^2$,

$$\|F(v^{(1)}, t, \omega) - F(v^{(2)}, t, \omega)\|$$
$$\leq (\lambda^0 + 4\nu^0 + \max\{\zeta((\|v^{(1)}\| + \|v^{(2)}\|)|e^{z(\omega)}|, t), \beta\} + |z(\omega)|)\|v^{(1)} - v^{(2)}\|.$$

For any bounded set $B \subset \ell^2$ with $\sup_{u \in B} \|v\| \leq \iota$, and define

$$\kappa_B(t, \omega) = (\lambda^0 + 4\nu^0 + \max\{\zeta(\iota|e^{z(\omega)}|, t), \beta\} + |z(\omega)|)\iota + |e^{-z(\omega)}| \|\|g\|\| \geq 0,$$

then for any $v, v^{(1)}, v^{(2)} \in B$,

$$F(v, t, \omega) \leq \kappa_B(t, \omega), \quad \|F(v^{(1)}, t, \omega) - F(v^{(2)}, t, \omega)\| \leq \kappa_B(t, \omega)\|v^{(1)} - v^{(2)}\|$$

and

$$\int_\tau^{\tau+1} \kappa_B(s, \theta_s\omega)ds < \infty, \ \forall \tau \in \mathbb{R}.$$

By [10, Proposition 2.1.1], problem (5.8) possesses a unique local mild solution $v(\cdot, \tau, \omega; v_\tau, g) \in C([\tau, \tau + T_{\max}), \ell^2)$ satisfying the integral equation

$$v(t) = v_\tau + \int_\tau^t (-\nu(s)Av - \lambda(s)v - e^{-z(\theta_s\omega)}f(ve^{z(\theta_s\omega)}, s)$$
$$+ e^{-z(\theta_s\omega)}g(s) + vz(\theta_s\omega))ds, \ t \in [\tau, \tau + T_{\max}) \ (0 < T_{\max} \leq T), \tag{5.9}$$

where $[\tau, \tau + T_{\max})$ is the maximal interval of existence of the solution of (5.8).

We next show that $T_{\max} = T$. Since $\lambda_i(t), \nu_i(t) \in L^1_{loc}(\mathbb{R})$ in t, by [18], there exist sequences of continuous functions in $t \in \mathbb{R}, \lambda_i^{(m)}(t), \nu_i^{(m)}(t), m \in \mathbb{N}$, such that

$$\lim_{m \to \infty} \int_\tau^t |\lambda_i^{(m)}(s) - \lambda_i(s)|ds = 0 \text{ and } \lambda_0 \leq \lambda_i^{(m)}(t) \leq \lambda^0, \forall \tau, t \in \mathbb{R}, \tag{5.10}$$

$$\lim_{m \to \infty} \int_\tau^t |\nu_i^{(m)}(s) - \nu_i(s)|ds = 0 \text{ and } \nu_0 \leq \nu_i^{(m)}(t) \leq \nu^0, \forall \tau, t \in \mathbb{R}. \tag{5.11}$$

Consider the following differential equations with initial data $v_\tau \in \ell^2$,

$$\frac{dv^{(m)}}{dt} = F^{(m)}(v^{(m)}, t, \omega), \tag{5.12}$$

where $F^{(m)}(v^{(m)}, t, \omega) = (F_i^{(m)}(v^{(m)}, t, \omega))_{i \in \mathbb{Z}}$ and

$$\begin{aligned} F_i^{(m)}(v^{(m)}, t, \omega) = &-\nu_i^{(m)}(t)Av_i^{(m)} - \lambda_i^{(m)}(t)v_i^{(m)} \\ &- e^{-z(\omega)}f_i(v_i^{(m)}e^{z(\omega)}, t) + e^{-z(\omega)}g_i(t) + v_i^{(m)}z(\omega). \end{aligned} \tag{5.13}$$

For $\omega \in \Omega$, by the continuity of $F_i^{(m)}(v^{(m)}, t, \omega)$ in t, (5.12) has a unique solution $v(\cdot, \tau; \omega, v_\tau, g) \in C([\tau, \tau + T_{\max}^{(m)}), \ell^2) \cap C^1((\tau, \tau + T_{\max}^{(m)}), \ell^2)$ such that

$$\frac{dv_i^{(m)}}{dt} = F_i^{(m)}(v^{(m)}, t, \omega) \tag{5.14}$$

and

$$v_i^{(m)} = v_\tau + \int_\tau^t F_i^{(m)}(v^{(m)}(s), s, \omega)ds. \tag{5.15}$$

Taking the inner product in ℓ^2 in (5.14) yields

$$\begin{aligned} \frac{d\|v^{(m)}\|^2}{dt} = &2(-\nu^{(m)}(t)Av^{(m)} - \lambda^{(m)}(t)v^{(m)} + z(\theta_t\omega)v^{(m)}, v^{(m)}) \\ &-2(e^{-z(\theta_t\omega)}f^{(m)}(v^{(m)}e^{z(\theta_t\omega)}, t), v^{(m)}) + 2(e^{-z(\theta_t\omega)}g^{(m)}(t), v^{(m)}). \end{aligned} \tag{5.16}$$

Note that

$$\begin{aligned} -\beta e^{2z(\theta_t\omega)}\|v^{(m)}\|^2 &\le (f(v^{(m)}e^{z(\theta_t\omega)}, t), v^{(m)}e^{z(\theta_t\omega)}) \\ &\le \iota(e^{z(\theta_t\omega)}\|v^{(m)}\|, s)e^{2z(\theta_t\omega)}\|v^{(m)}\|^2. \end{aligned}$$

It follows from (5.16) that

$$\frac{d\|v^{(m)}\|^2}{dt} \le (-\lambda_0 + 2\beta + 2z(\theta_s\omega))\|v^{(m)}\|^2 + (2\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0})e^{-2z(\theta_s\omega)}. \tag{5.17}$$

Applying Gronwall's inequality to (5.17), we obtain that

$$\begin{aligned} \|v^{(m)}(t)\|^2 \le &\|v_\tau\|^2 e^{(2\beta-\lambda_0)(t-\tau)+2\int_\tau^t z(\theta_r\omega)dr} \\ &+ (2\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0})e^{(2\beta-\lambda_0)t+2\int_0^t z(\theta_r\omega)dr}\int_\tau^t e^{(\lambda_0-2\beta)s-2z(\theta_s\omega)-2\int_0^s z(\theta_r\omega)dr}ds \\ &:= \eta^2(t, \tau, \omega), \quad t \in [\tau, \tau + T_{\max}^{(m)}), \end{aligned}$$

where $\eta^2(t, \tau, \omega) \in C([\tau, \tau + T), \mathbb{R}^+)$ is independent of m, which implies that

$$|v_i^{(m)}(t)| \le \eta(t, \tau, \omega), \text{ for all } m \in \mathbb{N}, t \in [\tau, \tau + T), \omega \in \Omega. \tag{5.18}$$

It then follows that for some $\tilde{\eta}(T, \tau, \omega) > 0$, which is independent on m such that $|F_i^{(m)}(v^{(m)}(t), t)| \leq \tilde{\eta}(T, \tau, \omega)$ and

$$|v_i^{(m)}(t) - v_i^{(m)}(s)| = \int_s^t |F_i^{(m)}(v^{(m)}(r), r)| dr$$
$$\leq \tilde{\eta}(T, \tau, \omega)|t - s|, \forall t, s \in [\tau, \tau + T), m \in \mathbb{N}, \omega \in \Omega.$$

By the Arzela-Ascoli Theorem, there exists a convergent subsequence $\{v_i^{(m_k)}(t), t \in [\tau, \tau + T)\}$ of $\{v_i^{(m)}(t), t \in [\tau, \tau + T)\}$ such that

$$v_i^{(m_k)}(t) \to \bar{v}_i(t) \text{ as } k \to \infty \text{ for } t \in [\tau, \tau + T), i \in \mathbb{Z}$$

and $\bar{v}_i(t)$ is continuous in $t \in [\tau, \tau + T)$. Moreover, $|\bar{v}_i(t)| \leq \eta(t, \tau, \omega)$ for $t \in [\tau, \tau + T), \omega \in \Omega$.

By (5.10), (5.11), (5.18) and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{k \to \infty} \int_\tau^t |\lambda_i^{(m_k)}(s) v_i^{(m_k)}(s) - \lambda_i(s)\bar{v}_i(s)| ds = 0, \tag{5.19}$$

$$\lim_{k \to \infty} \int_\tau^t |\nu_i^{(m_k)}(s) v_i^{(m_k)}(s) - \nu_i(s)\bar{v}_i(s)| ds = 0. \tag{5.20}$$

Thus by replacing m by m_k in (5.15) and letting $k \to \infty$, we obtain

$$\bar{v}_i(t) = v_\tau + \int_\tau^t F_i(\bar{v}(s), s, \omega) ds \text{ for all } t \in [\tau, \tau + T), \omega \in \Omega,$$

which implies that $\bar{u}(t) = (\bar{u}_i(t))_{i \in \mathbb{Z}}$ is a mild solution of (5.8). Then by the uniqueness of the mild solutions of (5.8), $T_{\max} = T$. Moreover, this means that $v(t, \tau; \omega, v_\tau, g)$ exists globally on $[\tau, +\infty)$ for any $\tau \in \mathbb{R}$. Here we remain to show for given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in \ell^2$, the mapping

$$\Phi(t, \tau, \omega, v_\tau, g) = v(t + \tau, \tau; \theta_{-\tau}\omega, v_\tau, g) = u(t + \tau, \tau; \theta_{-\tau}\omega, u_\tau, g)e^{-z(\theta_t \omega)},$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times \ell^2$ to ℓ^2 over $(\mathbb{R}, \{\theta_{1,t}\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ in the sense of Definition 2.1. In fact, the function $F(v, t, \omega)$ is continuous in v, g and measurable in t, ω, which implies that $v : (\mathbb{R}^+) \times \mathbb{R} \times \Omega \times \ell^2 \to \ell^2$, $(t, \cdot; \omega, v_\tau, g) \mapsto v(t, \cdot; \omega, v_\tau, g)$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\ell^2), \mathcal{B}(\ell^2))$-measurable (see [2]). The proof is complete.

$\square$

5.3 Existence of a Pullback Absorbing Set

In this subsection, we will get the existence of a $\mathcal{D}(\ell^2)$-pullback absorbing set for the continuous cocycle Φ.

Lemma 5.2. Let $\tilde{\lambda} = \lambda_0 - \beta - \frac{2}{\sqrt{\pi}} > 0$. Assume that (**A1**-**A4**) hold, then there exists a closed measurable $\mathcal{D}(\ell^2)$-pullback absorbing set $\mathcal{K} = \{\mathcal{K}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ for Φ in $\mathcal{D}(\ell^2)$ such that for any $B(\tau, \omega) \in \mathcal{D}(\ell^2)$, there exists $T_B = T_B(\tau, \omega) > 0$ yielding $\Phi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega) \subseteq \mathcal{K}(\tau, \omega)$ for all $t \geq T_B$ and $v_{\tau - t} \in B(\tau - t, \theta_{-t}\omega)$.

Proof. Let $\Phi^{(m)}$ be a solution of system (5.12), then $\Phi^{(m)} \in \ell^2$ for all $t \geq \tau$. From Proposition 5.1, we know that $v^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega) = \Phi^{(m)}(t, \tau - t, \theta_{-t}\omega)$. Denote $\hat{\lambda} = \lambda_0 - \beta$ and apply Gronwall's inequality over $(\tau - t, \tau)$ to (5.17), it follows that

$$\|v^{(m)}(\tau, \tau - t, \omega, v_{\tau - t})\|^2 + \beta \int_{\tau - t}^{\tau} e^{-\hat{\lambda}(\tau - s) + 2\int_s^{\tau} z(\theta_r \omega)} \|v^{(m)}(s, \tau - t, \omega, v_{\tau - t})\|^2 ds$$

$$\leq e^{-\hat{\lambda}t - 2\int_\tau^{\tau - t} z(\theta_r \omega)dr} \|v_{\tau - t}\|^2$$

$$+ (2\|\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0})e^{-\hat{\lambda}\tau + 2\int_0^{\tau} z(\theta_r \omega)dr} \int_{\tau - t}^{\tau} e^{\hat{\lambda}s - 2z(\theta_s \omega) - 2\int_0^s z(\theta_r \omega)dr} ds.$$

Since $v^{(m_k)} \to v$ for some $m_k \to \infty$, where v is the mild solution of (5.8), then the estimation above still holds with $v^{(m_k)}$ being replaced by v. Now, by replacing ω with $\theta_{-\tau}\omega$ in the expression v, we obtain

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau - t})\|^2$$

$$+ \beta \int_{\tau - t}^{\tau} e^{-\hat{\lambda}(\tau - s) + 2\int_s^{\tau} z(\theta_{r - \tau}\omega)dr} \|v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau - t})\|^2 ds$$

$$= \|v(\tau, \tau - t, \theta_{-\tau}\omega, v(\tau - t, \theta_{-\tau}\omega))\|^2$$

$$+ \beta \int_{-t}^{0} e^{-\hat{\lambda}s + 2\int_{-t}^{0} z(\theta_r \omega)dr} \|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau - t})\|^2 ds$$

$$\leq e^{-\hat{\lambda}t - 2\int_\tau^{\tau - t} z(\theta_{r - \tau}\omega)dr} \|v_{\tau - t}\|^2$$

$$+ (2\|\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0}) \int_{\tau - t}^{\tau} e^{\hat{\lambda}(s - \tau) - 2z(\theta_{s - \tau}\omega) + 2\int_s^{\tau} z(\theta_{r - \tau}\omega)dr} ds \qquad (5.21)$$

$$\leq e^{-\hat{\lambda}t - 2\int_{-t}^{0} z(\theta_r \omega)dr} \|v_{\tau - t}\|^2$$

$$+ (2\|\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0}) \int_{-t}^{0} e^{\hat{\lambda}s - 2z(\theta_s \omega) + 2\int_s^{0} z(\theta_r \omega)dr} ds$$

$$\leq e^{-\hat{\lambda}t - 2\int_{-t}^{0} z(\theta_r \omega)dr} \|v_{\tau - t}\|^2$$

$$+ (2\|\|\alpha\|\|^2 + \frac{\|\|g\|\|^2}{\lambda_0}) \int_{-\infty}^{0} e^{\hat{\lambda}s - 2z(\theta_s \omega) + 2\int_s^{0} z(\theta_r \omega)dr} ds.$$

Due to (5.7), we know that

$$\int_{-\infty}^{0} e^{\hat{\lambda}s - 2z(\theta_s \omega) + 2\int_s^{0} z(\theta_r \omega)dr} ds < +\infty,$$

and

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)| ds = \frac{1}{\sqrt{\pi}}.$$

Let $\tilde{\lambda} = \lambda_0 - \beta - \frac{2}{\sqrt{\pi}}$ and consider for any $v_{\tau-t} \in B(\tau - t, \theta_{-t}\omega)$, we have for $\tilde{\lambda} > 0$, $\tau \in \mathbb{R}$ from (3.1) that

$$
\begin{aligned}
&e^{-\hat{\lambda}t - 2\int_{-t}^0 z(\theta_r\omega)dr} \|v_{\tau-t}\|^2 \\
&\qquad \le e^{-\hat{\lambda}t - 2\int_{-t}^0 z(\theta_s\omega)ds} \|B(\tau - t, \theta_{-t}\omega)\|^2 \to 0 \text{ as } t \to +\infty.
\end{aligned}
\tag{5.22}
$$

By (5.21) and (5.22), it follows that

$$
\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \\
&\qquad \le 1 + (2\|\alpha\|^2 + \frac{\|g\|^2}{\lambda_0}) \int_{-\infty}^0 e^{\hat{\lambda}s - 2z(\theta_s\omega) + 2\int_s^0 z(\theta_r\omega)dr} ds.
\end{aligned}
$$

Now denoting

$$
R^2(\omega) = 1 + (2\|\alpha\|^2 + \frac{\|g\|^2}{\lambda_0}) \int_{-\infty}^0 e^{\hat{\lambda}s - 2z(\theta_s\omega) + 2\int_s^0 z(\theta_r\omega)dr} ds,
\tag{5.23}
$$

we conclude that

$$
\mathcal{K}(\tau, \omega) = \overline{B_{\ell^2}(0, R(\omega))}
\tag{5.24}
$$

is a closed measurable $\mathcal{D}(\ell^2)$-pullback absorbing set. In fact, for all $\gamma > 0$,

$$
\begin{aligned}
e^{-\gamma t} R^2(\theta_{-t}\omega) &= e^{-\gamma t} + (2\|\alpha\|^2 + \frac{\|g\|^2}{\lambda_0}) e^{-\gamma t} \int_{-\infty}^0 e^{\hat{\lambda}s - 2z(\theta_{s-t}\omega) + 2\int_s^0 z(\theta_{r-t}\omega)dr} ds \\
&= e^{-\gamma t} + (2\|\alpha\|^2 + \frac{\|g\|^2}{\lambda_0}) e^{-\gamma t} \int_{-\infty}^{-t} e^{\hat{\lambda}(s+t) - 2z(\theta_s\omega) + 2\int_s^0 z(\theta_r\omega)dr} ds \\
&\to 0 \text{ as } t \to +\infty.
\end{aligned}
$$

$\square$

5.4 Asymptotically Null of the Solutions

In this subsection, the property of asymptotically null for the solution Φ of system (5.8) will be established.

Lemma 5.3. *Let $\mathcal{K}(\tau, \omega)$ be the absorbing set given by (5.24). Then for every $\epsilon > 0$, there exist $\tilde{T}(\epsilon, \tau, \omega, \mathcal{K}(\tau, \omega)) > 0$ and $\tilde{N}(\epsilon, \tau, \omega, \mathcal{K}(\tau, \omega)) \ge 1$, such that the solution $\Phi(t, \tau - t, \theta_{-t}\omega) = v(\tau, \tau - t, \theta_{-\tau}\omega)$ of problem (5.8) is asymptotically null, that is, for all $t \ge \tilde{T}(\epsilon, \tau, \omega, \mathcal{K}(\tau, \omega))$, $v_{\tau-t} \in B(\tau - t, \theta_{-t}\omega)$,*

$$
\sum_{|i| > \tilde{N}(\epsilon, \tau, \omega, \mathcal{K}(\tau, \omega))} |(v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}, g)_i|^2 \le \epsilon^2.
$$

Proof. Choose a smooth cut-off function satisfying $0 \leq \rho(s) \leq 1$ for $s \in \mathbb{R}^+$ and $\rho(s) = 0$ for $0 \leq s \leq 1$, $\rho(s) = 1$ for $s \geq 2$. Suppose there exists a constant c_0 such that $|\rho'(s)| \leq c_0$ for $s \in \mathbb{R}^+$. For any $n \geq 1$, let $v_n^{(m)} = v^{(m)}(\tau, \tau - t, \omega, v_{\tau-t,n}, g_n) = (v_{n,i}^{(m)})_{i \in \mathbb{Z}}$ be a mild solution of (5.12). Let N be a fixed integer which will be specified later, and set $x_n^{(m)} = (x_{n,i}^{(m)})_{i \in \mathbb{Z}}$ where $x_{n,i}^{(m)} = \rho(\frac{|i|}{N}) v_{n,i}^{(m)}$ for any $i \in \mathbb{Z}$. Then taking the inner product of (5.12) with x in ℓ^2, we obtain

$$
\begin{aligned}
\frac{d}{dt} &\sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) |v_{n,i}^{(m)}|^2 \\
&= -2\nu^{(m)}(t)(A_m v_n^{(m)}, x_n^{(m)}) - 2(\lambda^{(m)}(t) - z(\theta_t \omega)) \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) |v_{n,i}^{(m)}|^2 \\
&\quad - 2e^{-z(\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) f(e^{z(\theta_t \omega)} v_{n,i}^{(m)}, t) v_{n,i}^{(m)} \\
&\quad + 2e^{-z(\theta_t \omega)}(g_n(t), x_n^{(m)}).
\end{aligned}
\tag{5.25}
$$

We now estimate terms in (5.25) one by one. First, we have

$$
(A_m v_n^{(m)}, x_n^{(m)}) = (\tilde{B}_m v^{(m)}, \tilde{B}_m x_n^{(m)}) \geq -\frac{2c_0}{N} \|v^{(m)}\|^2.
\tag{5.26}
$$

For the second term in (5.25), it follows from the assumption $(\mathbf{A}_2)$ that

$$
-\infty < -2e^{-z(\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) f_i(e^{z(\theta_t \omega)} v_{n,i}^{(m)}, t) v_{n,i}^{(m)} \leq 2e^{-2z(\theta_t \omega)} \sum_{|i| \geq N} \alpha_i^2(t).
$$

For the last term in (5.25),

$$
2e^{-z(\theta_t \omega)}(g_n(t), x_n^{(m)}) \leq \lambda^{(m)}(t) \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) |v_i|^2 + \frac{1}{\lambda_0} e^{-2z(\theta_t \omega)} \sum_{|i| \geq N} g_i^2(t).
\tag{5.27}
$$

Combining (5.25)-(5.27), it yields

$$
\begin{aligned}
\frac{d}{dt} &\sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) |v_{n,i}^{(m)}|^2 + (\lambda_0 - 2z(\theta_t \omega)) \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N}) |v_{n,i}^{(m)}|^2 \\
&\leq \frac{4\nu^0 c_0}{N} \|v^{(m)}\|^2 + (2 + \frac{1}{\lambda_0}) e^{-2z(\theta_t \omega)} \sum_{|i| \geq N} (\alpha_i^2(t) + g_i^2(t)).
\end{aligned}
\tag{5.28}
$$

Apply Gronwall's inequality to (5.28) over $(\tau - t, \tau)$, we obtain that

$$
\begin{aligned}
\sum_{i \in \mathbb{Z}} &\rho(\frac{|i|}{N}) |v_{n,i}^{(m)}(\tau, \tau - t, \omega, v_{\tau-t,n}^{(m)}, g_n)|^2 \\
&\leq e^{-\lambda_0 t - 2\int_\tau^{\tau-t} z(\theta_r \omega) dr} \|v_{\tau-t,n}^{(m)}\|^2 \\
&\quad + \frac{4\nu^0 c_0}{N} \int_{\tau-t}^\tau e^{-\lambda_0(\tau-s) + 2\int_s^\tau z(\theta_r \omega) dr} \|v^{(m)}(s, \tau - t, \omega, v_{\tau-t,n}^{(m)}, g_n)\|^2 ds \\
&\quad + (2 + \frac{1}{\lambda_0}) \sum_{|i| \geq N} (\alpha_i^2(t) + g_i^2(t)) \int_{\tau-t}^\tau e^{-\lambda_0(\tau-s) + 2\int_s^\tau z(\theta_r \omega) dr - 2z(\theta_s \omega)} ds.
\end{aligned}
\tag{5.29}
$$

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Now, for $\tau \in \mathbb{R}$, substitute $\theta_{-\tau}\omega$ for ω and estimate each term in (5.29)

$$
\sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N})|v_{n,i}^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)|^2
$$

$$
\leq e^{-\lambda_0 t - 2\int_\tau^{\tau-t} z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t,n}^{(m)}\|^2
$$

$$
+ \frac{4\nu^0 c_0}{N} \int_{\tau-t}^\tau e^{-\lambda_0(\tau-s)+2\int_s^\tau z(\theta_{r-\tau}\omega)dr}\|v^{(m)}(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)\|^2 ds
$$

$$
+ (2 + \frac{1}{\lambda_0})\sum_{|i| \geq N}(\alpha_i^2(t) + g_i^2(t))\int_{\tau-t}^\tau e^{-\lambda_0(\tau-s)+2\int_s^\tau z(\theta_{r-\tau}\omega)dr-2z(\theta_{s-\tau}\omega)}ds \qquad (5.30)
$$

$$
\leq e^{-\lambda_0 t - 2\int_{-t}^0 z(\theta_r\omega)dr}\|v_{\tau-t,n}^{(m)}\|^2
$$

$$
+ \frac{4\nu^0 c_0}{N} \int_{-t}^0 e^{-\lambda_0 s + 2\int_{-t}^0 z(\theta_r\omega)dr}\|v^{(m)}(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)\|^2 ds
$$

$$
+ (2 + \frac{1}{\lambda_0})\sum_{|i| \geq N}(\alpha_i^2(t) + g_i^2(t))\int_{-t}^0 e^{\lambda_0 s + 2\int_s^0 z(\theta_r\omega)dr-2z(\theta_s\omega)}ds.
$$

By Lemma 5.2, there exists $T_1(\epsilon, \tau, \omega, \mathcal{K}(\omega)) > 0$ such that for all $t \geq T_1(\epsilon, \tau, \omega, \mathcal{K}(\omega))$,

$$
\frac{4\nu^0 c_0}{N} \int_{-t}^0 e^{-\lambda_0 s + 2\int_{-t}^0 z(\theta_r\omega)dr}\|v^{(m)}(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)\|^2 ds
$$

$$
\leq \frac{4\nu^0 c_0}{\beta N}R^2(\omega), \qquad (5.31)
$$

where $R^2(\omega)$ is given by (5.23). Since $g(t), \alpha(t) \in C_b(\mathbb{R}, \ell^2)$, by using (5.7) again, we know

$$
(2 + \frac{1}{\lambda_0})\sum_{|i| \geq N}(\alpha_i^2(t) + g_i^2(t))\int_{-\infty}^0 e^{\lambda_0 s + 2\int_s^0 z(\theta_r\omega)dr-2z(\theta_s\omega)}ds < \infty,
$$

and hence

$$
\lim_{N \to \infty}(2 + \frac{1}{\lambda_0})\sum_{|i| \geq N}(\alpha_i^2(t) + g_i^2(t))\int_{-\infty}^0 e^{\lambda_0 s + 2\int_s^0 z(\theta_r\omega)dr-2z(\theta_s\omega)}ds = 0. \qquad (5.32)
$$

Now, by means of (5.22) and (5.30)-(5.32), there exist $\tilde{T}(\epsilon, \tau, \omega, \mathcal{K}(\omega)) \geq T_1(\epsilon, \tau, \omega, \mathcal{K}(\omega))$ and $\tilde{N}(\epsilon, \tau, \omega, \mathcal{K}(\omega)) \geq 1$ such that

$$
\sum_{|i| \geq \tilde{N}(\epsilon, \tau, \omega, \mathcal{K}(\omega))} |v_{n,i}^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)|^2
$$

$$
\leq \sum_{i \in \mathbb{Z}} \rho(\frac{|i|}{N})|v_{n,i}^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n)|^2 \leq \epsilon^2. \qquad (5.33)
$$

Since there is m_k such that

$$
v_{n,i}^{(m_k)}(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}^{(m)}, g_n) \to (v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}, g_n))_i
$$

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as $m_k \to \infty$, by (5.33)

$$\sum_{|i| \geq \tilde{N}(\epsilon, \tau, \omega, \mathcal{K}(\omega))} |(v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,n}, g_n))_i|^2 \leq \epsilon^2$$

for any $n \geq 1$. Now, letting $n \to \infty$ we can obtain the conclusion. $\quad\square$

5.5 Existence of Pullback Attractors

We are now in a position to give our main result in this section.

Theorem 5.4. *Suppose that* (**A1**-**A4**) *hold. The lattice dynamical system Φ with both non-autonomous deterministic and random forcing terms generated by system* (5.8) *has a unique pullback attractor.*

Proof. The result follows directly from Lemmas 5.2, 5.3 and Theorem 4.1. $\quad\square$

Acknowledgments

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