7/04/2014

Zoll and Tannery metrics from a superintegrable geodesic flow

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Abstract

We prove that for Matveev and Shevchishin superintegrable system, with a linear and a cubic integral, the metrics defined on $\mathbb{S}^2$ and on Tannery's orbifold $\mathcal{T}^2$ are either Zoll or Tannery metrics.

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1 Introduction

A family of dynamical systems on two dimensional manifolds was defined by Matveev and Shevchishin in [5] and shown to have a superintegrable geodesic flow. It was conjectured by these authors that the corresponding metrics on closed manifolds could lead to either Tannery or Zoll metrics, i. e. metrics for which all of the geodesics are closed. However a proof of this expected result was not given since the metrics were known only up to the integration of some first order non-linear ordinary differential equations. In [7] we were able on the one hand to integrate these differential equations up to the explicit form of the metrics in local coordinates and on the other hand to determine which metrics are globally defined on $\mathbb{S}^2$. It is the aim of this article to prove the conjecture of Matveev and Shevchishin and to show that it does hold also for some metrics defined on orbifolds.
Our results are the following:

1. The surface defined on $\mathbb{S}^2$ is Zoll.

2. Since in [7] only metrics defined on manifolds were of interest some of the surfaces were discarded because they exhibited a conical singularity, still being of finite measure. In fact Thurston introduced the concept of orbifold to deal with this new kind of geometrical objects. We will show that all the metrics with a conical singularity, obtained in [7], are defined on Tannery's orbifold $\mathcal{T}^2$, the simplest surface being known as Tannery's pear.

3. The conserved cubic observables which bring in superintegrability describe parametrically the geodesic trajectories.

The structure of the article is the following: in Section 2 we present some background material, coming essentially from [1], for self-containedness. In Section 3 we consider the metrics shown in [7] to be globally defined on $\mathbb{S}^2$ and prove that they are Zoll. In Section 4 we consider Tannery's pear and in Section 5 a one parameter generalization of it for which the geodesics close also after two turns. In Section 6 we consider a further generalization with two parameters which is Zoll. Remarkably enough these three metrics are defined on the same orbifold. In Section 7 we present some conclusions and prospects for open problems of interest.

2 Basic material

To any riemannian metric of the form

$$g = g_{ij}\,dx^i\,dx^j = A^2(\theta)\,d\theta^2 + \sin^2\theta\,d\phi^2 \qquad \theta \in (0,\pi) \qquad \phi \in \mathbb{S}^1 \tag{1}$$

one associates the hamiltonian

$$H = \frac{1}{2}\left(\Pi^2 + \frac{P_\phi^2}{\sin^2\theta}\right) \qquad \Pi = \frac{P_\theta}{A(\theta)} \tag{2}$$

Let us observe that we have two obvious conserved quantities: H and P_ϕ which are preserved by the geodesic flow and we will denote by E and L their values. The Hamilton equations give the following differential system for the geodesics

$$\Pi = \epsilon\sqrt{2E - \frac{L^2}{\sin^2\theta}} \qquad \epsilon = \pm 1 \qquad \sin^2\theta\,\frac{d\phi}{dt} = L > 0. \tag{3}$$

The choice $2E = 1$ is quite convenient since it ensures that the t-coordinate is nothing but the arc length s and we will take for initial conditions

$$\theta = i \in (0, \pi/2) \qquad \phi = 0 \qquad L = \sin i \qquad \Pi = 0. \tag{4}$$

The trajectory itself is determined by the integration of

$$\frac{d\phi}{d\theta} = \epsilon \, \frac{\sin i \, A(\theta)}{\sqrt{\sin^2\theta - \sin^2 i}}. \tag{5}$$

Remark: It is important to recall that the geodesic equations are made out of two pieces: in the first one θ increases from i to $\pi - i$ and then we have $\Pi \geq 0$ while in the second piece θ decreases from $\pi - i$ to i and then $\Pi \leq 0$. This should be kept in mind for all the geodesic equations given below.

The following theorem, for which a proof can be found in [1], is of paramount importance to determine the closedness of the geodesics:

Theorem 1 (Darboux) *A necessary and sufficient condition in order that all the geodesics be closed is that the rotation function*

$$R(g, i) = \int_i^{\pi - i} \frac{\sin i \, A(\theta)}{\sin\theta \sqrt{\sin^2\theta - \sin^2 i}} d\theta \tag{6}$$

be equal to $\dfrac{p}{q}\pi$ *where* p *and* q *are integers.*

An easy consequence is:

Theorem 2 *Any metric of the form*

$$g = A^2(\theta)\, d\theta^2 + \sin^2\theta \, d\phi^2 \tag{7}$$

for which A *is given by*

$$A = \frac{p}{q} + a(x) \qquad\qquad x = \cos\theta$$

where p *and* q *are integers and* $a(x)$ *is an odd function is a* $(p,\, q)$ *Tannery metric.*

Proof: The rotation function, after the change of variable $\cos\theta = (\cos i) \cdot u$, becomes

$$R(g, i) = \int_{-1}^{1} \frac{\sin i \, (p/q + a(u \, \cos i))du}{\sqrt{1 - u^2}(1 - (\cos i)^2 \, u^2)} \tag{8}$$

and since a is odd its contribution to the integral vanishes. The remaining integral is computed from the change of variable $u = \sin x$ by elementary methods and we get

$$R(g, i) = \frac{p}{q} \int_{-1}^{1} \frac{\sin i \, dx}{(1 - \cos^2 i \, \cos^2 x)} = \frac{p}{q}\pi \tag{9}$$

and we conclude using Theorem 1. $\square$

Let us add the following observations:

1. For generic values of (p,q) we have Tannery surfaces while for $p=q=1$ we have Zoll or $C_{2\pi}$-surfaces.

2. For the case of $g_0 = g(\mathbb{S}^2, \mathrm{can})$ we have $R(g_0, i) = \pi$ so that $p = q = 1$, the simplest example of a Zoll metric for which all of the geodesics have for length 2π.

3. The measure of the surface with metric (7) is

$$\mu(M,g) = 2\pi \int_{-1}^{+1} \left(\frac{p}{q} + a(x)\right) dx = \frac{p}{q}\, \mu(\mathbb{S}^2, g_0). \tag{10}$$

Let us prove now that the two dynamical systems with a superintegrable geodesic flow, globally defined on the manifold $M = \mathbb{S}^2$, which were derived in [7], give rise to Zoll metrics.

3 A Zoll metric on $\mathbb{S}^2$

In [7] two metrics, globally defined on $\mathbb{S}^2$, were given respectively in Theorems 1 and 2 of this reference. For the reader's convenience let us recall these metrics. The first one is

$$g_0 = \rho_0^2\, \frac{dv^2}{D_0} + \frac{4D_0}{P_0}\, d\phi^2 \qquad\qquad v \in (a,1) \qquad \phi \in \mathbb{S}^1 \tag{11}$$

where

$$D_0 = (v-a)(1-v^2) \quad P_0 = (v^2 - 2av + 1)^2 \quad Q_0 = -P_0 + 4(a-v)D_0 \quad \rho_0 = \frac{Q_0}{P_0} \tag{12}$$

and the parameter $a \in (-1,1)$.

The second one is

$$g = \rho^2\, \frac{dv^2}{D} + \frac{4D}{P}\, d\phi^2 \qquad\qquad x \in (-1,1) \qquad \phi \in \mathbb{S}^1 \tag{13}$$

with

$$\left\{\begin{array}{l} D = (m+x)(1-x^2) \\[2mm] P = \Big(L_+(1-x^2) + 2(m+x)\Big)\Big(L_-(1-x^2) + 2(m+x)\Big) \\[2mm] Q = 3x^4 + 4mx^3 - 6x^2 - 12mx - 4m^2 - 1 \end{array}\right. \qquad\qquad \rho = \frac{Q}{P} \tag{14}$$

with the following restrictions on the parameters:

$$l > -1 \qquad\qquad m > 1.$$

We will first prove:

Proposition 1 *The metric (11) is nothing but the limit for $l=-1$ of the metric (13).*

Proof: Let us define the change of coordinate

$$2v = (1-a)\,x + (1+a) \qquad\qquad v \in (a,1) \quad \to \quad x \in (-1,+1) \tag{15}$$

then, as a consequence of the relations

$$D_0 = \frac{(1-a)^3}{8}\,D \qquad P_0 = \frac{(1-a)^4}{16}\,P \qquad Q_0 = \frac{(1-a)^4}{16}\,Q, \tag{16}$$

we get

$$g_0 = \frac{2}{1-a}\,g(l=-1), \tag{17}$$

concluding the proof. $\square$

So, from now on we will consider only the metric (13) with the restrictions

$$l \in [-1,+\infty) \qquad\qquad m \in (1,+\infty) \tag{18}$$

and we will prove:

Proposition 2 *The metric (13) subjected to the constraints (18) is a Zoll metric on the manifold* $M = \mathbb{S}^2$.

Proof: The change of coordinate [2]

$$x = \frac{H_+ - H_- + 2c}{H_+ + H_-} \qquad\qquad H_\pm(s) = \sqrt{1 - \frac{(l \pm 1)}{(l+m)}\,s^2} \tag{19}$$

implies the following relations

$$\frac{4D}{P} = \frac{s^2}{l+m} \qquad \frac{dx}{\sqrt{D}} = -\frac{1}{\sqrt{l+m}}\,\frac{d\theta}{H_+\,H_-} \qquad -\rho = H_+\,H_- - c(H_+ - H_-) \tag{20}$$

giving for the metric the final form

$$G = (l+m)\,g = A^2\,d\theta^2 + \sin^2\theta\,d\phi^2, \qquad\qquad \theta \in (0,\pi) \qquad \phi \in \mathbb{S}^1 \tag{21}$$

where

$$A = 1 + c\left(\frac{1}{H_+} - \frac{1}{H_-}\right). \tag{22}$$

The function A has the structure required by Theorem 2, with $p = q = 1$, showing that G is indeed a Zoll metric, globally defined on $\mathbb{S}^2$. $\square$

Remarks:

1. From relation (10) the measure of this surface is $\mu(\mathbb{S}^2, G) = 4\pi$.

2. In the special case $l = 1$ the metric simplifies to

$$A = 1 + c\left(\frac{1}{H} - 1\right) \qquad H = \sqrt{1 - \varrho\,s^2} \qquad \varrho \in (0,1) \tag{23}$$

while for $l = -1$ one has

$$A = 1 + c\left(1 - \frac{1}{H}\right) \qquad H = \sqrt{1 + \varrho\,s^2} \qquad \varrho \in (0,1). \tag{24}$$

[2] From now on we will use the notations $s = \sin\theta$ and $c = \cos\theta$ to shorten the formulas.

Let us examine the cubic integrals:

Proposition 3 *The cubic integrals are given by*

$$\begin{aligned} S_1 &= +\cos\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\sin\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \\ S_2 &= -\sin\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\cos\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \end{aligned} \tag{25}$$

where

$$\begin{gathered} \alpha=\frac{1}{l+m}\left(\frac{H_+-H_-+2c}{H_++H_-}-l\right) \qquad \beta=-\frac{1}{s}\,(H_+-H_-+c) \\ \gamma=\frac{1}{s^3}\Big(H_+-H_-+c(1-H_+\,H_-)\Big) \end{gathered} \tag{26}$$

and they are constrained by

$$S_1^2+S_2^2=(2H)^3+\sigma_1\,(2H)^2\,P_\phi^2+\sigma_2\,(2H)\,P_\phi^4+\sigma_3\,P_\phi^6 \tag{27}$$

with

$$\sigma_1=-\frac{3l+m}{l+m} \qquad \sigma_2=\frac{3l^2+2lm-1}{(l+m)^2} \qquad \sigma_3=-\frac{l^2-1}{(l+m)^2}. \tag{28}$$

Proof: We have first to find the cubic integrals, taking for hamiltonian

$$H=\frac{1}{2}\left(\Pi^2+\frac{P_\phi^2}{s^2}\right) \qquad\qquad \Pi=\frac{P_\theta}{A}. \tag{29}$$

We could use the formulas given in [7] and transform them in the new $(\theta,\,\phi)$ coordinates but the computations needed are quite hairy, so we will derive them anew, writing them as in (25), where the unknown functions $(\alpha,\,\beta,\,\gamma)$ depend solely on θ. Imposing $\{H,S_1\}=0$ and $\{H,S_2\}=0$ gives one and the same differential system

$$\begin{array}{lll} (a) & s^2\,\beta'=A & \\ (b) & s^3\,\alpha'=-s\,\beta\,A-c & \\ (c) & s^2\,\gamma'=\alpha\,A & A=1+c\left(\dfrac{1}{H_+}-\dfrac{1}{H_-}\right). \\ (d) & A\left(s\gamma+\dfrac{\beta}{s}\right)=-c\left(\alpha+\dfrac{1}{s^2}\right) & \end{array} \tag{30}$$

Integrating (a) gives β which allows to integrate (b) giving α which allows in turn to integrate (c) for γ. These quadratures generate 3 unknown constants which are fixed up using (d). The results of these elementary computations are given in (26).

It follows that

$$S_1^2+S_2^2=\Pi^2\,(2H+\alpha\,P_\phi^2)^2+P_\phi^2\,(2\,\beta\,H+\gamma\,P_\phi^2)^2. \tag{31}$$

Expanding, with the help of the relation $\Pi^2=2H-\dfrac{P_\phi^2}{s^2}$, we find an homogeneous polynomial of degree 3 in $2H$ and P_ϕ^2. The computation of the various coefficients shows that they reduce to the constants given in (28). □

Using the initial conditions (4) the relation (31) becomes [3]

$$S_1^2+S_2^2=c_0^2\,H_+^2(s_0)\,H_-^2(s_0). \tag{32}$$

The interpretation of these cubic integrals will now be given:

Proposition 4 *The conserved quantities* $(S_1,\,S_2)$ *give the parametric representation of the geodesics:*

$$\left\{\begin{array}{l} c_0\,R(s_0)\,\sin\phi=\Pi\,(1+s_0^2\,\alpha(s)) \\[2mm] c_0\,R(s_0)\,\cos\phi=-s_0\,(\beta(s)+s_0^2\,\gamma(s))\end{array}\right. \qquad \Pi=\epsilon\sqrt{1-\frac{s_0^2}{s^2}} \tag{33}$$

where the functions $(\alpha,\,\beta,\,\gamma)$ *are given in (26) and* $\epsilon=+1$ *(resp.* $\epsilon=-1$*) if* θ *is increasing (resp. decreasing).*

Proof: Since S_1 and S_2 are preserved under the geodesic flow, we can compute their value at the starting point of the geodesic. With our choice (4) of initial conditions we get

$$\begin{array}{l} S_1=\cos\phi\,\Pi\,(1+s_0^2\,\alpha)+\sin\phi\,s_0\,(\beta+s_0^2\,\gamma)=0 \\[2mm] S_2=-\sin\phi\,\Pi\,(1+s_0^2\,\alpha)+\cos\phi\,s_0\,(\beta+s_0^2\,\gamma)=-c_0\,H_+(s_0)\,H_-(s_0)\end{array} \tag{34}$$

which are in agreement with (32). These relations are easily inverted and give (33), expressing the azimuthal angle ϕ parametrically in terms of the angle θ. Since ϕ is an increasing function, using these equations we can check that for $\theta=0$ we have $\phi=0$ and for $\theta=\pi-i$ we get $\phi=\pi$, hence $p=q=1$ in agreement with the rotation function. □

As to the embedding in $\mathbb{R}^3$ there is little hope to get it for generic values of the parameters $(l,\,m)$. For $l=-1$, which should be a simpler case, we may come back to the form (11). Defining the cartesian coordinates as

$$X=A(v)\,\sin\phi \qquad Y=A(v)\,\cos\phi \qquad Z=B(v) \tag{35}$$

we get for the induced metric

$$g=\Big((A')^2+(B')^2\Big)\,dv^2+A^2\,d\phi^2 \qquad v\in\,(a,1) \qquad \phi\in\mathbb{S}^1. \tag{36}$$

Identifying this metric with (11) gives for relations

$$A=2\frac{\sqrt{(v-a)(1-v^2)}}{v^2-2av+1} \qquad (B')^2=-8\frac{(v^3-3v+2a)}{(v^2-2av+1)^3}. \tag{37}$$

The positivity of $(B')^2$ is ensured for $a\in\,[0,1)$, however, since the underlying algebraic curve is hyperelliptic of genus 2, its integration will be quite technical.

Let us consider now other metrics, no longer defined on a closed manifold, but rather on an orbifold. This weakening of the concept of manifold, introduced by Thurston, allows for a finite number of conical singularities. We will begin with the simplest example.

[3] From now on we will use the shorthand notations $s_0=\sin i$ and $c_0=\cos i$.

4 Tannery's pear

In [7], Proposition 7, it is proved that the metric

$$g = \rho^2 \frac{dv^2}{D} + \frac{4D}{P} d\phi^2 \qquad v \in (-\infty, a) \qquad \phi \in \mathbb{S}^1 \tag{38}$$

where

$$D = v^2(a-v) \qquad P = v^2(v-2a)^2 \qquad \rho = \frac{v(3v-4a)}{(v-2a)^2} \qquad a > 0 \tag{39}$$

has a regular point for $v \to a-$ and a conical singularity for $v \to -\infty$. So it cannot be defined on a manifold but we will prove:

Proposition 5 *The metric (38) is the metric of Tannery's pear defined on $\mathcal{T}^2$.*

Proof: The first change of variable $w = 1 - v/a$ gives

$$a\, g = \frac{(1+3w)^2}{(1+w)^4} \frac{dw^2}{w} + \frac{4w}{(1+w)^2} d\phi^2 \qquad w \in (0, +\infty) \qquad \phi \in \mathbb{S}^1 \tag{40}$$

and it shows that we can take $a = 1$. The second change of variable

$$w = \frac{1+\cos\theta}{1-\cos\theta} \qquad w \in (0, +\infty) \to \theta \in (0, \pi) \tag{41}$$

transforms the metric into

$$g = (2+\cos\theta)^2\, d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \theta \in (0, \pi) \qquad \phi \in \mathbb{S}^1 \tag{42}$$

on which we recognize the metric on Tannery's pear [6]. This was the first example (in 1892!) of a metric, with non-constant sectional curvature, for which the geodesics close after two turns.

We will define Tannery's orbifold $\mathcal{T}^2$ by the following singularity structure of the metric:

1. At the south pole $\theta \to \pi-$ we have

$$g \sim d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \phi \in \mathbb{S}^1 \tag{43}$$

showing that this point is in fact a regular point since the apparent singularity would disappear using local cartesian coordinates.

2. However for the north pole $\theta \to 0+$ we have the conical singularity

$$\frac{g}{9} \sim d\theta^2 + \sin^2\theta \left(\frac{d\phi}{3}\right)^2 \tag{44}$$

precluding a manifold but allowed for an orbifold. □

Let us recall some known facts about this orbifold:

1. Using (10) the measure of Tannery's pear is $\mu(\mathcal{T}^2, g) = 8\pi$.

2. Its sectional curvature

$$\sigma(\mathcal{T}^2, g) = \frac{2}{(2+\cos\theta)^3} \tag{45}$$

is C^∞ and positive. This implies that G can be isometrically embedded in $\mathbb{R}^3$. If we take for explicit (global) embedding

$$X = \sin\theta\,\cos\phi \qquad Y = \sin\theta\,\sin\phi \qquad Z = 4\sqrt{2}\,\sin\frac{\theta}{2} \qquad \theta\in[0,\pi] \qquad \phi\in\mathbb{S}^1 \tag{46}$$

its cartesian equation is

$$X^2+Y^2 = \frac{Z^2}{8}\left(1-\frac{Z^2}{32}\right) \qquad\qquad Z\in[0,4\sqrt{2}]. \tag{47}$$

In this way the point $(X,Y,Z) = (0,0,4\sqrt{2})$ is regular while the point $(0,0,0)$ is the vertex of a cone with an aperture of $2\arctan\left(\frac{1}{2\sqrt{2}}\right)$ close to 39°.

Proposition 6 *The cubic integrals are given by*

$$\begin{aligned} S_1 &= +\cos\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\sin\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \\ S_2 &= -\sin\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\cos\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \end{aligned} \tag{48}$$

where

$$\alpha = -\frac{(1+c)^2}{s^2} \qquad\qquad \beta = -\frac{(1+2c)}{s} \qquad\qquad \gamma = \frac{(1+c)^2}{s^3} \tag{49}$$

and they are constrained by

$$S_1^2+S_2^2 = (2H)^3-2\,(2H)^2\,P_\phi^2+(2H)\,P_\phi^4 = c_0^4. \tag{50}$$

The geodesic equations are

$$\left\{\begin{array}{l} c_0^2\,\sin\phi = -\Pi\,(1+s_0^2\,\alpha(s)) \\ c_0^2\,\cos\phi = s_0\,(\beta(s)+s_0^2\,\gamma(s)) \end{array}\right. \qquad\qquad \Pi = \epsilon\sqrt{1-\frac{s_0^2}{s^2}}. \tag{51}$$

Proof: The function $A = 2+c$ being given, one has to integrate the differential system (30) as already explained in the proof of Proposition 3. Having fixed up S_1 and S_2 one deduces that

$$S_1^2+S_2^2 = (2H)^3-2\,(2H)^2\,P_\phi^2+(2H))\,P_\phi^4 = c_0^4 \tag{52}$$

and upon inversion of the relations $S_1 = 0$ and $S_2 = c_0^2$ one gets (51). Here too we can check that for $\theta = 0$ we have $\phi = 0$, for $\theta = \frac{\pi}{2}$ we have $\phi = \pi+i$ and for $\theta = \pi - i$ we get $\phi = 2\pi$, hence $p = 2$ and $q = 1$ in agreement with the rotation function. □

The equations (51) for the geodesics were first given by Tannery in [6].

Let us proceed with a one parametric extension of Tannery's metric.

5 A generalization of Tannery's pear

It was proved in Proposition 12 of [7] that the metric

$$g=\rho^2\frac{dv^2}{D}+\frac{4D}{P}\,d\phi^2 \qquad v\in(-\infty,v_0) \qquad \phi\in\mathbb{S}^1 \tag{53}$$

with

$$v_0>v_1 \qquad D=(v_0-v)(v-v_1)^2 \qquad P=(v-v_1)^2\,p \tag{54}$$

and

$$p=v^2-2(2v_0+3v_1)v+(2v_0+v_1)^2 \quad \Delta(p)<0 \qquad \rho=\frac{(v-v_1)(3v-4v_0+v_1)}{p} \tag{55}$$

Has a regular end-point for $v\to v_0-$ but a conical singularity for $v\to -\infty$.

Let us first clean up these formulas using the change of variable

$$w=\frac{v_0-v}{v_0-v_1} \qquad w\in(0,+\infty)$$

which gives

$$(v_0-v_1)\,g=\frac{(1+3w)^2}{(w^2+2aw+1)^2}\,\frac{dw^2}{w}+\frac{4w}{w^2+2aw+1}\,d\phi^2 \qquad a\in(-1,+1) \tag{56}$$

so we can set $v_0-v_1=1$ and we are left with a single parameter, namely a. In the limit $a\to 1$ we recover Tannery's metric (40).

Let us prove:

Proposition 7 *The metric given by (53) is a* $(2,1)$ *Tannery metric defined on* $\mathcal{T}^2$.

Proof: The change of coordinate

$$w=-a+\frac{(1+a)}{s^2}(1+c\,R) \qquad R(s)=\sqrt{1+\varrho\,s^2} \qquad \varrho=\frac{1-a}{1+a}\in(0,+\infty) \tag{57}$$

maps $w\in(0,+\infty)\ \to\ \theta\in(0,\pi)$. Using the relations

$$\begin{gathered}\frac{dw}{\sqrt{w}\,(w^2+2aw+1)}=-\sqrt{\frac{2}{1+a}}\,\frac{d\theta}{1+w}\\ \frac{2w}{w^2+2aw+1}=\frac{s^2}{1+a} \qquad \frac{1+3w}{1+w}=2+\frac{c}{R}\end{gathered} \tag{58}$$

one gets for the transformed metric

$$G\equiv\frac{(1+a)}{2}\,g=A^2\,d\theta^2+s^2\,d\phi^2 \qquad A=2+\frac{c}{R}. \tag{59}$$

Since A has the structure required by Theorem 2, with $p=2$ and $q=1$, we conclude that it is a Tannery metric. The structure of the singularities for $\theta=0$ and $\theta=\pi$ agree with Tannery's orbifold. $\square$

Let us proceed to:

Proposition 8 *The cubic integrals are given by*

$$\begin{aligned} S_1 &= +\cos\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\sin\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \\ S_2 &= -\sin\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\cos\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2) \end{aligned} \qquad (60)$$

where

$$\alpha=-\frac{1+c^2+2c\,R}{s^2} \qquad \beta=-\frac{2c+R}{s} \qquad \gamma=\frac{2c+(1+c^2)R}{s^3}, \qquad (61)$$

and they are constrained by

$$S_1^2+S_2^2=(2H)^3-(2-\varrho)(2H)^2\,P_\phi^2+(1-2\varrho)(2H)\,P_\phi^4+\varrho P_\phi^6=c_0^4\,R^2(s_0). \qquad (62)$$

The geodesics equations are

$$\left\{\begin{array}{l} c_0^2\,R(s_0)\,\sin\phi=-\Pi\,(1+s_0^2\,\alpha(s)) \\ c_0^2\,R(s_0)\,\cos\phi=s_0\,(\beta(s)+s_0^2\,\gamma(s)) \end{array}\right. \qquad \Pi=\epsilon\sqrt{1-\frac{s_0^2}{s^2}} \qquad (63)$$

Proof: We need to integrate the differential system (30) with $A=2+\dfrac{c}{R}$ following the same pattern explained in the proof of Proposition 3. The results are given in (61).

The proof of (62) is similar to the one given for (27) in Proposition 3. Using the initial conditions (4) we have $S_1=0$ and $S_2=c_0^2\,R(s_0)$. Inverting these relations for $\sin\phi$ and $\cos\phi$ leads to the geodesics equations (63). □

Let us add the following remarks:

1. We have analyzed previously Tannery's pear for its own historical interest but in fact it appears as the special case $\varrho\to 0$ of the present metric (59).

2. The measure of this surface is still $\mu(\mathcal{T}^2,G)=8\pi$.

3. Let the embedding in $\mathbb{R}^3$ be given by

$$x=A(w)\,\cos\phi \qquad y=A(w)\,\sin\phi \qquad z=B(w) \qquad (64)$$

where we come back to the initial metric g given by (56). It follows that we have

$$A=2\sqrt{\frac{w}{p}} \qquad p=w^2+2aw+1 \qquad a\in\,(-1,1)$$

and the function B is given by

$$(B')^2=\frac{(3p+2w+2a)(3wp+2aw+2)}{p^3} \qquad (65)$$

In the range allowed for a we have $p>0$ and the resultant shows that the two polynomials in the numerator have no common zero. Hence, if $3p+2w+2a$ has real zeroes the sign of the right hand side will change and this happens for $a\in\,(-1,-\dfrac{2\sqrt{2}}{3}]$, precluding any emmbedding. This corresponds to values of ϱ larger than $33.97\ldots$. For $a\in\,(-\dfrac{2\sqrt{2}}{3},1)$

then $3p+2w+2a>0$ and the discriminant of $3wp+2aw+2$ is strictly negative ensuring a single real root which can be checked, from Cardano formula, to be strictly negative and the embedding is possible. The integration of (65) will be difficult since the underlying algebraic curve is hyperelliptic of genus 3.

As a side remark let us observe that the sectional curvature is strictly positive for $a \in (-1/3, 1)$, but we have found a larger domain for a in which the embedding is still possible.

6 A Zoll metric on Tannery's orbifold

It was proved in Proposition 13 of [7] that the metric

$$g = \rho_0^2 \frac{dv^2}{D_0} + \frac{4D_0}{P_0}\, d\phi^2 \qquad v \in (-\infty, v_0) \qquad \phi \in \mathbb{S}^1 \tag{66}$$

with

$$D_0 = (v_0 - v)[(v - v_1)^2 + v_2^2] \qquad v_1 > 0 \qquad v_2 \in \mathbb{R}\backslash\{0\} \tag{67}$$

and

$$\rho_0 = -1 + \frac{4(v_0 + 2v_1 - v)D_0}{P_0} \qquad\qquad P_0 = 8v\, D_0 + (D_0')^2 \tag{68}$$

has a regular south pole for $v \to v_0$ and a conical singularity for $v \to -\infty$.

To clean up this metric let us make the change of variable

$$v = v_0 - \lambda\, w \qquad\qquad \lambda = \sqrt{(v_0 - v_1)^2 + v_2^2}$$

which yields

$$\lambda\, g = \rho^2 \frac{dw^2}{D} + \frac{4D}{P}\, d\phi^2 \qquad\qquad w \in (0, +\infty) \tag{69}$$

with

$$\left\{\begin{array}{ll} D = w(w^2 - 2aw + 1) & a = \dfrac{v_0 - v_1}{\lambda} \in (-1, 1) \\ P = \Big(w^2 + 2(m+r)w + 1\Big)\Big(w^2 + 2(m-r)w + 1\Big) & m = 2\dfrac{v_1}{\lambda} \in (0, +\infty) \\ \rho = -1 + \dfrac{4(w+m)D}{P} & r = \sqrt{m^2 + 2am + 1}. \end{array}\right. \tag{70}$$

So from now on we will take $\lambda = 1$. Let us begin with:

Proposition 9 *The metric (69), still defined on $\mathcal{T}^2$, is a Zoll metric.*

Proof: The change of coordinate

$$w = a + \frac{B + c\sqrt{2m\,B}}{s^2} \qquad w \in (0, +\infty) \;\to\; \theta \in (0, \pi) \tag{71}$$

with

$$B(s) = m\,c^2 - a\,s^2 + \mathcal{B}(s) \qquad\qquad \mathcal{B}(s) = \sqrt{(m\,c^2 - a\,s^2)^2 + (1 - a^2)s^4}$$

and the relations

$$\frac{4D}{P}=\frac{s^2}{m}\qquad\qquad \rho^2\,\frac{dw^2}{D}=\frac{1}{m}\,(1+c\,R)^2 d\theta^2\qquad R=\sqrt{m}\,\frac{(2\,B)^{3/2}}{B^2+(1-a^2)\,s^4}\tag{72}$$

give for the metric

$$G\equiv m\,g=A^2\,d\theta^2+\sin^2\theta\,d\phi^2\qquad\qquad A=1+c\,R.\tag{73}$$

Since A has the structure required by Theorem 2 with $p=q=1$, we get a Zoll metric and the singularity structure leads again to Tannery's orbifold $\mathcal{T}^2$. $\square$

Proposition 10 *The cubic integrals are given by*

$$\begin{aligned}S_1&=+\cos\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\sin\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2)\\ S_2&=-\sin\phi\,\Pi\,(2H+\alpha\,P_\phi^2)+\cos\phi\,P_\phi\,(2\beta\,H+\gamma\,P_\phi^2)\end{aligned}\tag{74}$$

where

$$\alpha=-\frac{1}{s^2}\left(\frac{\mathcal{B}}{m}+c\sqrt{\frac{2B}{m}}\right)\qquad \beta=-\frac{1}{s}\left(\sqrt{\frac{2B}{m}}+c\right)$$
$$\gamma=\frac{1}{s^3}\left(\sqrt{\frac{2B}{m}}+c\,\frac{\mathcal{B}}{m}\right)\tag{75}$$

and they are constrained by

$$S_1^2+S_2^2=(2H)^3+\sigma_1\,(2H)^2\,P_\phi^2+\sigma_2\,(2H)\,P_\phi^4+\sigma_3\,P_\phi^6=c_0^2\,\mathcal{B}^2\tag{76}$$

with

$$\sigma_1=-\frac{3m+2a}{m}\qquad \sigma_2=\frac{3m^2+4am+1}{m^2}\qquad \sigma_3=-\frac{m^2+2am+1}{m^2}.\tag{77}$$

The geodesics equations are

$$\left\{\begin{array}{l}c_0^2\,\dfrac{\mathcal{B}(s_0)}{m}\,\sin\phi=-\Pi\,(1+s_0^2\,\alpha(s))\\[2ex] c_0^2\,\dfrac{\mathcal{B}(s_0)}{m}\,\cos\phi=s_0\,(\beta(s)+s_0^2\,\gamma(s))\end{array}\right.\qquad \Pi=\epsilon\sqrt{1-\frac{s_0^2}{s^2}}\tag{78}$$

One may wonder whether one could generalize to include in the hamiltonian some potential term invariant under the Killing vector ∂_ϕ. This is not possible in contrast with the Koenigs superintegrable models [4] with quadratic integrals (special cases of Matveev and Shevchishin models) which were shown in [2] to allow for potentials.

7 Conclusion

As we have seen the conjecture of Matveev and Shevchishin is valid for their superintegrable model with cubic integrals in the momenta, be the metrics defined on manifolds or on orbifolds. It opens a very large field of research and of construction of superintegrable models with higher degrees integrals, the final target being a proof that their conjecture remains valid for any such degree higher than 3. Let us observe that Kiyohara [3] has constructed a set of *integrable* models with integrals of any degree greater than 3 for which all of their geodesics are closed. This seems to indicate that Tannery and Zoll metrics are tightly related to integrable and superintegrable models.

Acknowledgements: We are greatly indebted to Professor David Kohel, from IML at Luminy, for the determination of the genera of the algebraic curves in Section 3 and in Section 5.

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