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Integrability of classical strings dual for noncommutative gauge theories

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Abstract

We derive the gravity duals of noncommutative gauge theories from the Yang-Baxter sigma model description of the $AdS_5 \times S^5$ superstring with classical r-matrices. The corresponding classical r-matrices are 1) solutions of the classical Yang-Baxter equation (CYBE), 2) skew-symmetric, 3) nilpotent and 4) abelian. Hence these should be called *abelian Jordanian deformations*. As a result, the gravity duals are shown to be integrable deformations of $AdS_5 \times S^5$. Then, abelian twists of AdS_5 are also investigated. These results provide a support for the gravity/CYBE correspondence proposed in arXiv:1404.1838.

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1 Introduction

A particular class of gauge/gravity dualities can be seen as deformations of AdS/CFT [1]. With great progress, an integrable structure inhabiting AdS/CFT is well recognized now [2]. The Green-Schwarz string action on $AdS_5 \times S^5$ is constructed from a supercoset [3]

$$PSU(2,2|4)/[SO(1,4) \times SO(5)]$$

and the classical integrability follows from the $\mathbb{Z}_4$-grading [4][1]. Some deformations of the AdS/CFT correspondence may preserve the integrability and hence it would be interesting to consider a method to classify the integrable deformations.

A possible way is to employ the Yang-Baxter sigma model description [10, 11] (For q-deformed su(2) and its affine extension, see [12] and [13], respectively). It has been applied to the $AdS_5 \times S^5$ superstring in [14]. According to this approach, integrable deformations of $AdS_5 \times S^5$ are given in terms of classical r-matrices satisfying modified classical Yang-Baxter equation (mCYBE). The case of [14] corresponds to the classical r-matrix of Drinfeld-Jimbo type [15–17]. The metric in the string frame and NS-NS two-form are obtained [18] and some generalizations to other cases are discussed in [19]. It is an intriguing issue to look for the complete gravitational solution. A mirror TBA is also proposed [20].

As a generalization of the Yang-Baxter sigma model description, one may consider classical Yang-Baxter equation (CYBE) rather than mCYBE. The classical action of the $AdS_5 \times S^5$ superstring has been constructed in [21]. The integrable deformations are basically regarded as Drinfeld-Reshetikhin twists [15, 16, 22] including Jordanian twists [23, 24] and abelian twists. Hence one can classify integrable deformations of this kind in terms of classical r-matrices. We will refer this picture as to the gravity/CYBE correspondence. The first example is presented in [25][2]. As another example, Lunin-Maldacena backgrounds [27, 28] have also been derived [29].

In this note, we derive the gravity duals of noncommutative (NC) gauge theories [33, 34] from the Yang-Baxter sigma model description of the $AdS_5 \times S^5$ superstring with classical r-matrices. The corresponding classical r-matrices are 1) solutions of CYBE, 2) skew-symmetric, 3) nilpotent and 4) abelian. Hence these should be called *abelian Jordanian deformations*. As a result, the gravity duals of NC gauge theories are shown to be integrable deformations of $AdS_5 \times S^5$. Then, abelian twists of AdS_5 are also investigated. These results provide a support for the gravity/CYBE correspondence proposed in [29].

[1] For another formulation [5] of the $AdS_5 \times S^5$ superstring, the classical integrability is argued in [6]. For a classification of integrable supercosets, see [7, 8]. For an argument on non-symmetric cosets, see [9].

[2] The solution is closely related to the one in Appendix C of [26].

This note is organized as follows. Section 2 gives a short summary of the Yang-Baxter sigma model description of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring with classical r-matrices satisfying CYBE. Then we introduce three classes of skew-symmetric solutions of CYBE. A new class of r-matrices induces abelian Jordanian deformations. Section 3 presents examples of abelian Jordanian type, which lead to the gravity duals of NC gauge theories. In section 4, we consider a deformation of AdS_5 with an abelian r-matrix concerned with a TsT transformation of AdS_5. Section 5 is devoted to conclusion and discussion. We argue some implications of this result and future directions in studies of the gravity/CYBE correspondence. In Appendix A our notation and convention are summarized. Appendix B presents the gravity duals of NC gauge theories with six deformation parameters. Appendix C describes the detailed computation of three-parameter abelian twists of AdS_5.

2 Integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring

We introduce here integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring based on the Yang-Baxter sigma model description with CYBE [21]. After giving a short review on the general form of deformed actions, we present three classes of classical r-matrices.

2.1 Deforming the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action with CYBE

A class of integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring can be described with classical r-matrices satisfying CYBE [21]. The deformed action is given by

$$S = -\frac{1}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \; \mathrm{Str}\Big(A_\alpha d \circ \frac{1}{1-\eta R_g \circ d} A_\beta\Big)\,, \tag{2.1}$$

where the left-invariant one-form A_α is defined as

$$A_\alpha \equiv g^{-1}\partial_\alpha g\,, \qquad g \in SU(2,2|4)\,. \tag{2.2}$$

Here $\gamma^{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are the flat metric and the anti-symmetric tensor on the string world-sheet. The operator R_g is defined as

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g\,, \tag{2.3}$$

where a linear operator R satisfies CYBE rather than mCYBE [14]. The R-operator is related to the tensorial representation of classical r-matrix through

$$R(X) = \mathrm{Tr}_2[r(1\otimes X)] = \sum_i \big(a_i \mathrm{Tr}(b_i X) - b_i \mathrm{Tr}(a_i X)\big) \tag{2.4}$$
$$\text{with} \quad r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i)\,.$$

The operator d is given by the following,

$$d = P_1 + 2P_2 - P_3\,, \tag{2.5}$$

where P_i $(i = 0, 1, 2, 3)$ are the projections to the $\mathbb{Z}_4$-graded components of $\mathfrak{su}(2,2|4)$. $P_0\,, P_2$ and $P_1\,, P_3$ are the projectors to the bosonic and fermionic generators, respectively. In particular, $P_0(\mathfrak{su}(2,2|4))$ is nothing but $\mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$.

For the action (2.1) with an R-operator satisfying CYBE, the Lax pair has been constructed [21] and the classical integrability is ensured in this sense. The κ-invariance has been proven as well [21] .

2.2 A classification of classical r-matrices

According to the construction of the deformed string action, one may expect the correspondence between integrable deformations of $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and classical r-matrices, called the gravity/CYBE correspondence [29] . To study along this direction, it would be valuable to classify some typical class of skew-symmetric solutions of CYBE.

There are three types of classical r-matrices: i) Jordanian, ii) abelian, and iii) abelian Jordanian. In particular, the third class will play a crucial role in the next section.

In order to study deformations of AdS_5 later, let us consider the case of $\mathfrak{su}(2,2)$.

i) Jordanian r-matrix.

The first class is classical r-matrices of Jordanian type,

$$r_{\mathrm{Jor}} = E_{ij} \wedge (E_{ii} - E_{jj}) - 2 \sum_{i<k<j} E_{ik} \wedge E_{kj} \qquad (1 \le i < j \le 4)\,, \tag{2.6}$$

where $(E_{ij})_{kl} \equiv \delta_{ik}\delta_{jl}$ are the fundamental representation of $\mathfrak{su}(2,2)$. The characteristic property of Jordanian type r-matrices is the nilpotency. Indeed, we could verify that the associated linear R-operator exhibits $(R_{\mathrm{Jor}})^n = 0$ for $n \ge 3$.

Jordanian deformations of the $\mathrm{AdS}_5{\times}\mathrm{S}^5$ superstring are considered in [21] . A simple example of the corresponding type IIB supergravity solution is presented in [25] . Only the AdS_5 part is deformed and it contains a three-dimensional Schrödinger spacetime as a subspace. Hence it may be regarded as a generalization of [30–32]. It seems likely that the resulting metric is closely related to a null Melvin twist [26].

ii) Abelian r-matrix.

The second class is abelian r-matrices composed of the Cartan generators as follows:

$$r_{\text{Abe}} = \sum_{1\leq i<j\leq 3} \mu_{ij}\,(E_{ii} - E_{i+1,i+1}) \wedge (E_{jj} - E_{j+1,j+1})\,, \tag{2.7}$$

where $\mu_{ij} = -\mu_{ji}$ are arbitrary parameters. Since these commute with each other and hence satisfy CYBE obviously. The abelian r-matrix is a particular example of the Drinfeld-Reshetikhin twists [15,16,22]. Note that abelian r-matrices are intrinsic to higher rank cases (rank ≥ 2).

It has been shown in [29] that abelian r-matrices lead to γ-deformed backgrounds [28], which include the Lunin-Maldacena background [27] as a particular case. In section 4, we will consider an abelian twist of AdS_5 with a single parameter. For multi-parameter cases, see Appendix C.

iii) Abelian Jordanian r-matrix.

The third class is composed of r-matrices which are nilpotent *and* abelian. These should be called *abelian Jordanian* r-matrices. A typical example takes the following form,

$$r_{\text{AJ}} = \sum_{\substack{i,k=1,2\\ j,l=3,4}} \nu_{(ij),(kl)}\, E_{ij} \wedge E_{kl}\,, \tag{2.8}$$

with arbitrary parameters $\nu_{(ij),(kl)} = -\nu_{(kl),(ij)}$. Because E_{ij} $(i = 1,2\,, j = 3,4)$ are the positive root generators and commute with each other, the square of the associated R-operator already vanishes like,

$$(R_{\text{AJ}})^2 = 0\,,$$

in comparison to Jordanian r-matrices (2.6) for which $(R_{\text{Jor}})^2 \neq 0$ and $(R_{\text{Jor}})^3 = 0$ in general.

In the next section, we will show that classical r-matrices of abelian Jordanian type correspond to the gravity duals of NC gauge theories [33,34].

3 Examples - gravity duals of NC gauge theories

Let us consider examples of classical r-matrices of abelian Jordanian type. These lead to the gravity duals of NC gauge theories [33,34]. Hereafter we will concentrate on the AdS_5 part and S^5 is not deformed.

A possible example is given by

$$r_{\text{AJ}} = \mu\, p_2 \wedge p_3 + \nu\, p_0 \wedge p_1\,, \tag{3.1}$$

where μ, ν are deformation parameters. Here p_μ ($\mu = 0,1,2,3$) are the upper triangular matrices defined as

$$p_\mu \equiv \frac{1}{2}\gamma_\mu - m_{\mu 5}\,. \tag{3.2}$$

For our convention of γ_μ and the $\mathfrak{su}(2,2)$ generators, see Appendix A. It should be emphasized that p_μ's are upper triangular and satisfy the following property:

$$p_\mu p_\nu = 0\,. \tag{3.3}$$

Thus the classical r-matrix (3.1) is of abelian Jordanian type and trivially satisfies CYBE.

To evaluate the Lagrangian (2.1), let us take the following coset parametrization [25]:

$$g_{\rm s} = \exp\left[p_0\, x^0 + p_1\, x^1 + p_2\, x^2 + p_3\, x^3\right] \exp\left[\frac{\gamma_5}{2}\log z\right] \quad \in\ SU(2,2)/SO(1,4)\,. \tag{3.4}$$

Then the AdS_5 part of (2.1) can be rewritten as

$$L = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{Tr}\left[A_\alpha P_2(J_\beta)\right] \tag{3.5}$$

$$\text{with} \qquad J_\beta \equiv \frac{1}{1 - 2\eta\left[R_{\rm AJ}\right]_g \circ P_2} A_\beta\,. \tag{3.6}$$

Here $A_\alpha = g^{-1}\partial_\alpha g$ is restricted to $\mathfrak{su}(2,2)$ and the associated R-operator $R_{\rm AJ}$ with (3.1) is determined by the relation (2.4).

It is convenient to divide the Lagrangian L into two parts like $L = L_G + L_B$, where L_G is the metric part and L_B is the coupling to an NS-NS two-form, respectively:

$$\begin{aligned} L_G &\equiv \frac{1}{2}[\mathrm{Tr}(A_\tau P_2(J_\tau)) - \mathrm{Tr}(A_\sigma P_2(J_\sigma))]\,, \\ L_B &\equiv \frac{1}{2}[\mathrm{Tr}(A_\tau P_2(J_\sigma)) - \mathrm{Tr}(A_\sigma P_2(J_\tau))]\,. \end{aligned} \tag{3.7}$$

To derive the explicit form of L, it is sufficient to compute the projected current $P_2(J_\alpha)$ rather than J_α itself. Hence the computation is reduced to solving the following equation,

$$\left(1 - 2\eta P_2 \circ \left[R_{\rm AJ}\right]_g\right) P_2(J_\alpha) = P_2(A_\alpha)\,. \tag{3.8}$$

Note that $P_2(A_\alpha)$ is expanded with γ matrices as follows:

$$P_2(A_\alpha) = \frac{\partial_\alpha x^0\,\gamma_0 + \partial_\alpha x^1\,\gamma_1 + \partial_\alpha x^2\,\gamma_2 + \partial_\alpha x^3\,\gamma_3 + \partial_\alpha z\,\gamma_5}{2z}\,. \tag{3.9}$$

Then, by combining (3.9) with (3.8), $P_2(J_\alpha)$ can be obtained as

$$\begin{aligned} P_2(J_\alpha) = &\frac{z(z^2\partial_\alpha x_0 + 2\eta\nu\partial_\alpha x_1)}{2(z^4 - 4\eta^2\nu^2)}\gamma_0 + \frac{z(z^2\partial_\alpha x_1 + 2\eta\nu\partial_\alpha x_0)}{2(z^4 - 4\eta^2\nu^2)}\gamma_1 \\ &+ \frac{z(z^2\partial_\alpha x_2 + 2\eta\mu\partial_\alpha x_3)}{2(z^4 + 4\eta^2\mu^2)}\gamma_2 + \frac{z(z^2\partial_\alpha x_3 - 2\eta\mu\partial_\alpha x_2)}{2(z^4 + 4\eta^2\mu^2)}\gamma_3 + \frac{\partial_\alpha z}{2z}\gamma_5\,. \end{aligned} \tag{3.10}$$

The resulting forms of L_G and L_B are given by, respectively,

$$L_G = -\frac{\gamma^{\alpha\beta}}{2}\Big[\frac{z^2(-\partial_\alpha x_0\partial_\beta x_0 + \partial_\alpha x_1\partial_\beta x_1)}{z^4 - 4\eta^2\nu^2} + \frac{z^2(\partial_\alpha x_2\partial_\beta x_2 + \partial_\alpha x_3\partial_\beta x_3)}{z^4 + 4\eta^2\mu^2} + \frac{\partial_\alpha z\partial_\beta z}{z^2}\Big], \quad (3.11)$$

$$L_B = \epsilon^{\alpha\beta}\Big[-\frac{2\eta\nu}{z^4 - 4\eta^2\nu^2}\partial_\alpha x_0\partial_\beta x_1 + \frac{2\eta\mu}{z^4 + 4\eta^2\mu^2}\partial_\alpha x_2\partial_\beta x_3\Big]. \quad (3.12)$$

Here two deformation parameters μ, ν and one normalization factor η are contained.

It is easy to see the metric and the NS-NS two-form from (3.11) and (3.12). By introducing new parameter a and a' through the identification,

$$2\eta\,\mu = a^2\,, \qquad 2\eta\,\nu = ia'^2\,, \quad (3.13)$$

one can find that the resulting metric and two-form exactly agree with the ones of the gravity duals of NC gauge theories presented in [33,34], up to the coordinate change $z = 1/u$ and the Wick rotation $x_0 \to ix_0$. This result shows that the gravity duals of NC gauge theories [33,34] are integrable deformation of AdS_5.

4 Abelian twists of AdS_5

As another kind of integrable deformation of AdS_5, we consider an abelian twist of AdS_5 with a single parameter[3]. For a three-parameter generalization, see Appendix C.

Let us consider an abelian r-matrix,

$$r^{(\mu)}_{\mathrm{Abe}} = \mu\, h_1 \wedge h_2\,, \quad (4.1)$$

with a deformation parameter μ. Here h_i $(i = 1, 2)$ are two of the Cartan generators of $\mathfrak{su}(2,2)$ and belong to the fundamental representation,

$$h_1 = \mathrm{diag}(-1, 1, -1, 1)\,, \quad h_2 = \mathrm{diag}(-1, 1, 1-, 1)\,. \quad (4.2)$$

Then, the AdS_5 part of the Lagrangian (2.1) is given by

$$L = L_G + L_B = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{Tr}\left[A_\alpha P_2(J_\beta)\right], \quad (4.3)$$

$$\text{with} \quad J_\beta \equiv \frac{1}{1 - 2\eta\big[R^{(\mu)}_{\mathrm{Abe}}\big]_g \circ P_2} A_\beta\,, \quad (4.4)$$

where the current A_α is $\mathfrak{su}(2,2)$-valued and the R-operator associated with (4.1) is defined by the rule (2.4).

[3] Abelian twists of S^5 have been studied in [29]. The resulting geometries are three-parameter γ-deformed S^5 [27,28].

The projected current $P_2(J_\alpha)$ is to be determined by solving the equation,

$$\left(1 - 2\eta P_2 \circ \left[R^{(\mu)}_{\text{Abe}}\right]_g\right) P_2(J_\alpha) = P_2(A_\alpha)\,. \tag{4.5}$$

By using the coset parameterization (C.5) , $P_2(A_\alpha)$ is expanded with respect to γ matrices,

$$\begin{aligned} P_2(A_\alpha) = \frac{1}{2}\Big[&-\partial_\alpha\rho\,\gamma_1 + i\cosh\rho\,\partial_\alpha\psi_3\,\gamma_5 \\ &- \sinh\rho\,(\cos\zeta\,\partial_\alpha\psi_1\,\gamma_2 + \partial_\alpha\zeta\,\gamma_3 - i\sin\zeta\,\partial_\alpha\psi_2\,\gamma_0)\Big]\,. \end{aligned} \tag{4.6}$$

Then, by plugging (4.6) with (4.5) , $P_2(J_\alpha)$ can be obtained as

$$P_2(J_\alpha) = j^0_\alpha\,\gamma_0 + j^1_\alpha\,\gamma_1 + j^2_\alpha\,\gamma_2 + j^3_\alpha\,\gamma_3 + j^5_\alpha\,\gamma_5\,, \tag{4.7}$$

with the coefficients

$$\begin{aligned} j^0_\alpha &= \frac{i}{2}\frac{\sin\zeta\sinh\rho}{1+16\eta^2\mu^2\sin^2 2\zeta\sinh^4\rho}\left(\partial_\alpha\psi_2 + 8\eta\mu\cos^2\zeta\sinh^2\rho\partial_\alpha\psi_1\right)\,, \\ j^1_\alpha &= -\frac{1}{2}\partial_\alpha\rho\,, \\ j^2_\alpha &= -\frac{1}{2}\frac{\cos\zeta\sinh\rho}{1+16\eta^2\mu^2\sin^2 2\zeta\sinh^4\rho}\left(\partial_\alpha\psi_1 - 8\eta\mu\sin^2\zeta\sinh^2\rho\partial_\alpha\psi_2\right)\,, \\ j^3_\alpha &= -\frac{1}{2}\sinh\rho\,\partial_\alpha\zeta\,, \\ j^5_\alpha &= \frac{i}{2}\cosh\rho\partial_\alpha\psi_3\,. \end{aligned} \tag{4.8}$$

Finally, the resulting expressions of L_G and L_B are given by, respectively,

$$\begin{aligned} L_G = -\frac{\gamma^{\alpha\beta}}{2}\Big[&\sinh^2\rho\partial_\alpha\zeta\partial_\beta\zeta + \partial_\alpha\rho\partial_\beta\rho - \cosh^2\rho\partial_\alpha\psi_3\partial_\beta\psi_3 \\ &+ \frac{\sinh^2\rho}{1+\hat{\gamma}^2\sin^2\zeta\cos^2\zeta\sinh^4\rho}(\cos^2\zeta\partial_\alpha\psi_1\partial_\beta\psi_1 + \sin^2\zeta\partial_\alpha\psi_2\partial_\beta\psi_2)\Big]\,, \end{aligned} \tag{4.9}$$

$$L_B = -\epsilon^{\alpha\beta}\frac{\hat{\gamma}\cos^2\zeta\sin^2\zeta\sinh^4\rho}{1+\hat{\gamma}^2\cos^2\zeta\sin^2\zeta\sinh^4\rho}\partial_\alpha\psi_1\partial_\beta\psi_2\,. \tag{4.10}$$

Here a new deformation parameter $\hat{\gamma}$ is defined as

$$\hat{\gamma} \equiv 8\eta\,\mu\,. \tag{4.11}$$

Now one can read off the metric and NS-NS two-form from (4.9) and (4.10) . By performing the coordinate transformation,

$$\rho_1 = \cos\zeta\sinh\rho\,, \qquad \rho_2 = \sin\zeta\sinh\rho\,, \qquad \rho_3 = i\cosh\rho\,, \tag{4.12}$$

the resulting metric and NS-NS two-form are given by

$$ds^2 = d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2 d\psi_1^2 + \rho_2^2 d\psi_2^2}{1+\hat{\gamma}^2\,\rho_1^2\rho_2^2} + \rho_3^2 d\psi_3^2 + ds^2_{\mathrm{S}_5}\,, \tag{4.13}$$

$$B_2 = \frac{\hat{\gamma}\,\rho_1^2\rho_2^2}{1+\hat{\gamma}^2\,\rho_1^2\rho_2^2} d\psi_1 \wedge d\psi_2\,. \tag{4.14}$$

Here there is a constraint $\sum_{i=1}^{3} \rho_i^2 = -1$.

These expressions are quite similar to a one-parameter γ-deformed S^5 [27, 28] and thus the solution with the metric (4.13) and the NS-NS two-form (4.14) may be regarded as a single parameter γ-deformation of AdS_5 .

5 Conclusion and discussion

We have shown that the gravity duals of NC gauge theories [33, 34] can be derived from the Yang-Baxter sigma model description of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring with classical r-matrices. The corresponding classical r-matrices are 1) solutions of CYBE, 2) skew-symmetric, 3) nilpotent and 4) abelian. These should be called *abelian Jordanian deformations*. As a result, the gravity duals are found to be integrable deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$. Then, abelian twists of AdS_5 have also been investigated. These results provide a support for the gravity/CYBE correspondence proposed in [29].

Our main result here is the integrability of $\mathcal{N}$=4 super Yang-Mills (SYM) theory on noncommutative (NC) spaces. Now there are an enormous amount of arguments on the integrability for scattering amplitudes of $\mathcal{N}$=4 SYM. Integrable deformations of it would be found on NC spaces. Our analysis has revealed a relation between classical r-matrices and deformations parameters of NC spaces. There may be a close connection to deformation quantization of Kontsevich [35]. Thus one may expect a deep mathematical structure behind the correspondence. We hope that our result could shed light on new fundamental aspects of integrable deformations.

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Appendix

A Notation and convention

We shall here summarize our notation and convention, which basically follow [36] .

An element of $\mathfrak{su}(2,2|4)$ is identified with an 8×8 supermatrix,

$$M = \begin{bmatrix} m & \xi \\ \zeta & n \end{bmatrix} . \qquad \text{(A.1)}$$

Here m and n are 4×4 matrices with Grassmann even elements, while ξ and ζ are 4×4 matrices with Grassmann odd elements. These matrices satisfy a reality condition. Then m and n belong to $\mathfrak{su}(2,2) = \mathfrak{so}(2,4)$ and $\mathfrak{su}(4) = \mathfrak{so}(6)$, respectively.

We are concerned with deformations of AdS_5 . An explicit basis of $\mathfrak{su}(2,2)$ is the following. The γ matrices are given by

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} , \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} , \quad \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ,$$

$$\gamma_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \text{(A.2)}$$

and satisfy the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} , \qquad \{\gamma_\mu, \gamma_5\} = 0 , \qquad (\gamma_5)^2 = 1 . \qquad \text{(A.3)}$$

The Lie algebra $\mathfrak{so}(1,4)$ is formed by the generators

$$m_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] , \qquad m_{\mu 5} = \frac{1}{4}[\gamma_\mu, \gamma_5] \qquad (\mu,\nu = 0,1,2,3) , \qquad \text{(A.4)}$$

and then $\mathfrak{so}(2,4) = \mathfrak{su}(2,2)$ is spanned by the following set:

$$m_{\mu\nu} , \quad m_{\mu 5} , \quad \gamma_\mu , \quad \gamma_5 . \qquad \text{(A.5)}$$

B Multi-parameter deformations of AdS_5

We present here multi-parameter deformations of AdS_5 by using the Yang-Baxter sigma model description with classical r-matrices. These may be regarded as a multi-parameter generalization of the gravity duals of NC gauge theories discussed in [33,34]. In the original construction [33,34] based on twisted T-dualities, it would be intricate to perform T-dualities many times. A technical advantage of the Yang-Baxter sigma model description is that a single r-matrix gives the corresponding metric and NS-NS two-form in a more direct way.

Let us consider the following classical r-matrix,

$$
\begin{aligned}
r_{\rm AJ} &= \mu_1\, p_2 \wedge p_3 + \mu_2\, p_3 \wedge p_1 + \mu_3\, p_1 \wedge p_2 \\
&\quad + \nu_1\, p_0 \wedge p_1 + \nu_2\, p_0 \wedge p_2 + \nu_3\, p_0 \wedge p_3\,,
\end{aligned} \tag{B.1}
$$

where μ_1, μ_2, μ_3 and ν_1, ν_2, ν_3 are six deformation parameters, and p_μ are defined in (3.2). By following the analysis in section 3, it is straightforward to get the deformed string action. For simplicity, we shall write down only the resulting metric and NS-NS two-form,

$$
\begin{aligned}
ds^2 = \frac{dz^2}{z^2} &+ z^2 G\Big[-(z^4 + 4\eta^2(\mu_1^2 + \mu_2^2 + \mu_3^2))dx_0^2 + (z^4 + 4\eta^2(\mu_1^2 - \nu_2^2 - \nu_3^2))dx_1^2 \\
&\qquad + (z^4 + 4\eta^2(\mu_2^2 - \nu_3^2 - \nu_1^2))dx_2^2 + (z^4 + 4\eta^2(\mu_3^2 - \nu_1^2 - \nu_2^2))dx_3^2 \\
&- 8\eta^2\big[(\mu_2\nu_3 - \mu_3\nu_2)dx_0 dx_1 + (\mu_3\nu_1 - \mu_1\nu_3)dx_0 dx_2 + (\mu_1\nu_2 - \mu_2\nu_1)dx_0 dx_3 \\
&\quad - (\mu_1\mu_2 + \nu_1\nu_2)dx_1 dx_2 - (\mu_2\mu_3 + \nu_2\nu_3)dx_2 dx_3 - (\mu_1\mu_3 + \nu_1\nu_3)dx_1 dx_3\big]\Big]\,,
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
B_2 = 2\eta\, G\big[&(z^4\mu_1 - \eta^2\nu_1 K)dx_2 \wedge dx_3 - (z^4\nu_1 + \eta^2\mu_1 K)dx_0 \wedge dx_1 \\
&+ (z^4\mu_2 - \eta^2\nu_2 K)dx_3 \wedge dx_1 - (z^4\nu_2 + \eta^2\mu_2 K)dx_0 \wedge dx_2 \\
&+ (z^4\mu_3 - \eta^2\nu_3 K)dx_1 \wedge dx_2 - (z^4\nu_3 + \eta^2\mu_3 K)dx_0 \wedge dx_3\big]\,.
\end{aligned} \tag{B.3}
$$

Here a scalar function G and a constant parameter K are defined as

$$
\begin{aligned}
G^{-1} &\equiv z^8 + 4\eta^2 z^4(\mu_1^2 + \mu_2^2 + \mu_3^2 - \nu_1^2 - \nu_2^2 - \nu_3^2) - \eta^4 K^2\,, \\
K &\equiv 4(\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3)\,.
\end{aligned}
$$

By taking the following identification of the parameters

$$
2\eta\,\mu_1 = a^2\,, \quad 2\eta\,\nu_1 = i a'^2\,, \quad \mu_2 = \mu_3 = \nu_2 = \nu_3 = 0\,, \tag{B.4}
$$

and performing a Wick rotation $x_0 \to i x_0$, one can reproduce the metric and NS-NS two-form of the two-parameter case [33,34].

Note that the metric (B.2) and NS-NS two-form (B.3) are complemented with the other field components and gives a complete solution of type IIB supergravity. It gives a consistent string background because it is basically obtained by performing a chain of (twisted) T-dualities for AdS_5.

C Three-parameter abelian twists of AdS_5

Let us consider here a three-parameter generalization of the abelian deformation of AdS_5 discussed in section 4 .

We will consider the following classical r-matrix,

$$r_{\mathrm{Abe}}^{(\mu_1,\mu_2,\mu_3)} = \mu_3\, h_1 \wedge h_2 + \mu_1\, h_2 \wedge h_3 + \mu_2\, h_3 \wedge h_1\,, \tag{C.1}$$

with deformation parameters μ_i . Here h_i are the three Cartan generators of $\mathfrak{su}(2,2)$ and belong to the fundamental representation,

$$h_1 = \mathrm{diag}(-1,1,-1,1)\,, \quad h_2 = \mathrm{diag}(-1,1,1-,1)\,, \quad h_3 = \mathrm{diag}(1,1,-1,-1)\,. \tag{C.2}$$

By using the r-matrix (C.1), the AdS_5 part of (2.1) can be rewritten as

$$L = L_G + L_B = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{Tr}\left[A_\alpha P_2(J_\beta)\right]\,, \tag{C.3}$$

$$\text{with} \quad J_\beta \equiv \frac{1}{1 - 2\eta\left[R_{\mathrm{Abe}}^{(\mu_1,\mu_2,\mu_3)}\right]_g \circ P_2} A_\beta\,, \tag{C.4}$$

where $A_\alpha = g^{-1}\partial_\alpha g$ is restricted to $\mathfrak{su}(2,2)$ and the R-operator associated with (C.1) is determined by the rule (2.4) .

To evaluate the Lagrangian (C.3) , let us adopt the following coset parametrization [18] :

$$g = \Lambda(\psi_1,\psi_2,\psi_3)\,\Xi(\zeta)\,\check{g}_\rho(\rho) \quad \in \; SU(2,2)/SO(1,4)\,. \tag{C.5}$$

Here the matrices Λ, Ξ and $\check{g}_\rho$ are defined as

$$\Lambda(\psi_1,\psi_2,\psi_3) \equiv \exp\left[\frac{i}{2}(\psi_1\, h_1 + \psi_2\, h_2 + \psi_3\, h_3)\right]\,,$$

$$\Xi(\zeta) \equiv \begin{pmatrix} \cos\frac{\zeta}{2} & \sin\frac{\zeta}{2} & 0 & 0 \\ -\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\zeta}{2} & -\sin\frac{\zeta}{2} \\ 0 & 0 & \sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{pmatrix}\,, \quad \check{g}_\rho(\rho) \equiv \begin{pmatrix} \cosh\frac{\rho}{2} & 0 & 0 & \sinh\frac{\rho}{2} \\ 0 & \cosh\frac{\rho}{2} & -\sinh\frac{\rho}{2} & 0 \\ 0 & -\sinh\frac{\rho}{2} & \cosh\frac{\rho}{2} & 0 \\ \sinh\frac{\rho}{2} & 0 & 0 & \cosh\frac{\rho}{2} \end{pmatrix}\,.$$

To find the projected current $P_2(J_\alpha)$, it is necessary to solve the following equation,

$$\left(1 - 2\eta P_2 \circ \left[R_{\mathrm{Abe}}^{(\mu_1,\mu_2,\mu_3)}\right]_g\right) P_2(J_\alpha) = P_2(A_\alpha)\,. \tag{C.6}$$

Note that $P_2(A_\alpha)$ is expanded with respect to the γ matrices,

$$\begin{aligned} P_2(A_\alpha) = \frac{1}{2}\Big[&-\partial_\alpha\rho\,\gamma_1 + i\cosh\rho\,\partial_\alpha\psi_3\,\gamma_5 \\ &- \sinh\rho(\cos\zeta\,\partial_\alpha\psi_1\,\gamma_2 + \partial_\alpha\zeta\,\gamma_3 - i\sin\zeta\,\partial_\alpha\psi_2\,\gamma_0)\Big]\,. \end{aligned} \tag{C.7}$$

Then, by combining (C.7) with (C.6), $P_2(J_\alpha)$ can be obtained as

$$P_2(J_\alpha) = j^0_\alpha\,\gamma_0 + j^1_\alpha\,\gamma_1 + j^2_\alpha\,\gamma_2 + j^3_\alpha\,\gamma_3 + j^5_\alpha\,\gamma_5\,, \tag{C.8}$$

with the coefficients

$$\begin{aligned}
j^0_\alpha &= -\frac{i}{2}\frac{\sin\zeta\sinh\rho}{1-16\eta^2[(\mu_1^2\sin^2\zeta+\mu_2^2\cos^2\zeta)\sinh^2 2\rho-\mu_3^2\sin^2 2\zeta\sinh^4\rho]} \\
&\quad\times\Big[(-1+16\eta^2\mu_2^2\cos^2\zeta\sinh^2 2\rho)\partial_\alpha\psi_2 \\
&\qquad -8\eta(\mu_1-8\eta\mu_2\mu_3\cos^2\zeta\sinh^2\rho)\cosh^2\rho\partial_\alpha\psi_3 \\
&\qquad -8\eta(\mu_3-8\eta\mu_1\mu_2\cosh^2\rho)\cos^2\zeta\sinh^2\rho\partial_\alpha\psi_1\Big]\,, \\
j^1_\alpha &= -\frac{1}{2}\partial_\alpha\rho\,, \\
j^2_\alpha &= \frac{1}{2}\frac{\cos\zeta\sinh\rho}{1-16\eta^2[(\mu_1^2\sin^2\zeta+\mu_2^2\cos^2\zeta)\sinh^2 2\rho-\mu_3^2\sin^2 2\zeta\sinh^4\rho]} \\
&\quad\times\Big[(-1+16\eta^2\mu_1^2\sin^2\zeta\sinh^2 2\rho)\partial_\alpha\psi_1 \\
&\qquad +8\eta(\mu_3+8\eta\mu_1\mu_2\cosh^2\rho)\sin^2\zeta\sinh^2\rho\partial_\alpha\psi_2 \\
&\qquad +8\eta(\mu_2+8\eta\mu_1\mu_3\sin^2\zeta\sinh^2\rho)\cosh^2\rho\partial_\alpha\psi_3\Big]\,, \\
j^3_\alpha &= -\frac{1}{2}\sinh\rho\partial_\alpha\zeta\,, \\
j^5_\alpha &= \frac{i}{2}\frac{\cosh\rho}{1-16\eta^2[(\mu_1^2\sin^2\zeta+\mu_2^2\cos^2\zeta)\sinh^2 2\rho-\mu_3^2\sin^2 2\zeta\sinh^4\rho]} \\
&\quad\times\Big[(1+16\eta^2\mu_3^2\sin^2 2\zeta\sinh^4\rho)\partial_\alpha\psi_3 \\
&\qquad -8\eta(\mu_2-8\eta\mu_1\mu_3\sin^2\zeta\sinh^2\rho)\cos^2\zeta\sinh^2\rho\partial_\alpha\psi_1 \\
&\qquad +8\eta(\mu_1+8\eta\mu_2\mu_3\cos^2\zeta\sinh^2\rho)\sin^2\zeta\sinh^2\rho\partial_\alpha\psi_2\Big]\,.
\end{aligned} \tag{C.9}$$

Finally, L_G and L_B are given by, respectively,

$$\begin{aligned}
L_G = -\frac{\gamma^{\alpha\beta}}{2}\Big[&-\sinh^2\rho\partial_\alpha\rho\partial_\beta\rho \\
&+(\sin\zeta\sinh\rho\partial_\alpha\zeta-\cos\zeta\cosh\rho\partial_\alpha\rho)(\sin\zeta\sinh\rho\partial_\beta\zeta-\cos\zeta\cosh\rho\partial_\beta\rho) \\
&+(\cos\zeta\sinh\rho\partial_\alpha\zeta+\sin\zeta\cosh\rho\partial_\alpha\rho)(\cos\zeta\sinh\rho\partial_\beta\zeta+\sin\zeta\cosh\rho\partial_\beta\rho) \\
&+\hat{G}\big[\sinh^2\rho(\cos^2\zeta\partial_\alpha\psi_1\partial_\beta\psi_1+\sin^2\zeta\partial_\alpha\psi_2\partial_\beta\psi_2)-\cosh^2\rho\partial_\alpha\psi_3\partial_\beta\psi_3 \\
&\quad -\cos^2\zeta\sin^2\zeta\cosh^2\rho\sinh^2\rho\big(\textstyle\sum_i\hat{\gamma}_i\partial_\alpha\psi_i\big)\big(\textstyle\sum_j\hat{\gamma}_j\partial_\beta\psi_j\big)\big]\Big]\,,
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
L_B = -\epsilon^{\alpha\beta}\hat{G}\Big[&\hat{\gamma}_3\cos^2\zeta\sin^2\zeta\sinh^4\rho\partial_\alpha\psi_1\partial_\beta\psi_2 \\
&-\sinh^2\rho\cosh^2\rho\left(\hat{\gamma}_2\cos^2\zeta\partial_\alpha\psi_3\partial_\beta\psi_1+\hat{\gamma}_1\sin^2\zeta\partial_\alpha\psi_2\partial_\beta\psi_3\right)\Big]\,.
\end{aligned} \tag{C.11}$$

Here a scalar function $\hat{G}$ is defined as

$$\hat{G}^{-1} \equiv 1-(\hat{\gamma}_1^2\sin^2\zeta+\hat{\gamma}_2^2\cos^2\zeta)\cosh^2\rho\sinh^2\rho+\hat{\gamma}_3^2\cos^2\zeta\sin^2\zeta\sinh^4\rho\,, \tag{C.12}$$

and new deformation parameters $\hat{\gamma}_i$ are

$$\hat{\gamma}_i \equiv 8\eta\,\mu_i \qquad (i=1,2,3)\,. \tag{C.13}$$

By performing the coordinate transformation (4.12), the metric and NS-NS two-form associated with (C.10) and (C.11) are written into a compact forms,

$$ds^2 = \sum_{i=1}^{3}(d\rho_i^2 + \hat{G}\rho_i^2 d\psi_i^2) + \hat{G}\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i\,d\psi_i\right)^2 + ds^2_{\mathrm{S}_5}\,, \tag{C.14}$$

$$B_2 = \hat{G}\left(\hat{\gamma}_3\,\rho_1^2\rho_2^2\,d\psi_1\wedge d\psi_2 + \hat{\gamma}_1\,\rho_2^2\rho_3^2\,d\psi_2\wedge d\psi_3 + \hat{\gamma}_2\,\rho_3^2\rho_1^2\,d\psi_3\wedge d\psi_1\right)\,. \tag{C.15}$$

Here there is a constraint $\sum_{i=1}^{3}\rho_i^2 = -1$ and $\hat{G}$ turns out to be

$$\hat{G}^{-1} = 1 + \hat{\gamma}_3^2\,\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\,\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\,\rho_3^2\rho_1^2\,. \tag{C.16}$$

These are quite similar to γ-deformed S^5 [27, 28] and hence the metric (C.14) and NS-NS two-form (C.15) may be regarded as γ-deformed AdS_5.

The one-parameter result in section 4 is reproduced by setting the parameters as

$$\hat{\gamma}_1 = \hat{\gamma}_2 = 0\,, \qquad \hat{\gamma}_3 = \hat{\gamma}\,. \tag{C.17}$$

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