The Allen-Cahn equation with dynamic boundary conditions and mass constraints

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Abstract

The Allen-Cahn equation, coupled with dynamic boundary conditions, has recently received a good deal of attention. The new issue of this paper is the setting of a rather general mass constraint which may involve either the solution inside the domain or its trace on the boundary. The system of nonlinear partial differential equations can be formulated as variational inequality. The presence of the constraint in the evolution process leads to additional terms in the equation and the boundary condition containing a suitable Lagrange multiplier. A well-posedness result is proved for the related initial value problem.

Key words: Allen-Cahn equation, dynamic boundary condition, mass constraint, variational inequality, Lagrange multiplier.

AMS (MOS) subject classification: 35K86, 49J40, 80A22.

1 Introduction

The Allen-Cahn equation [4] is a famous equation aiming to describe the order-disorder phase transition in a process of phase separation in a binary alloy. It is applicable to several directions, for example, it is widely employed in the description of the solid-liquid phase transition (see the monograph [8] and references therein).

Let $0 < T < +\infty$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be the bounded smooth domain occupied by the material. Also the boundary Γ of Ω is supposed to be smooth enough. We recall the isothermal Allen-Cahn equation in the following form:

$$\frac{\partial u}{\partial t} - \Delta u + W'(u) = f \quad \text{a.e. in } Q := \Omega \times (0,T), \tag{1.1}$$

where the unknown $u := u(x,t)$ stands for the order parameter and $f := f(x,t)$ is a given source term. The nonlinear term W' plays an important role, it is the derivative of a function W usually referred as double well potential, with two minima and a local unstable maximum in between. The prototype model for the Allen-Cahn equation is provided by $W(r) = (1/4)(r^2-1)^2$, $r \in \mathbb{R}$, so that $W'(r) = r^3 - r$, $r \in \mathbb{R}$, is the sum of an increasing function with a power growth and another smooth (in particular, Lipschitz continuous) function which breaks the monotonicity properties of the former (and is related to the non-convex part of the potential W). In this paper, we treat more general cases for such a nonlinearity, that is, we assume that W' is the sum of a maximal monotone graph (it can be a graph with vertical segments too) defined in the whole of $\mathbb{R}$ and of a Lipschitz perturbation.

Usually, the Allen-Cahn equation is coupled with the homogeneous Neumann boundary condition, which means no flux exchange at the boundary. Recently, equation (1.1) has been investigated (see, e.g., [9, 10, 13, 14, 23] and references therein) when complemented by a dynamic boundary condition of the following form:

$$u_\Gamma = u_{|\Gamma}, \quad \partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \kappa \Delta_\Gamma u_\Gamma + W'_\Gamma(u_\Gamma) = f_\Gamma \quad \text{a.e. on } \Sigma := \Gamma \times (0,T). \tag{1.2}$$

Here, $u_{|\Gamma}$ denotes the trace of u and ∂_ν represents the outward normal derivative on Γ, $\kappa > 0$ is a physical coefficient, Δ_Γ stands for the Laplace-Beltrami operator on Γ (see, e.g., [19, Chapter 3]), W_Γ denotes a potential with some properties similar to those of W, and f_Γ represents a known datum on Σ.

About dynamic boundary conditions, the mathematical research for the various problems was already running in 1990's. Especially, the Stefan problem with the dynamic boundary condition in the case $\kappa = 0$ was treated in a series of papers by Aiki [1, 2, 3]; in particular, the existence of a weak solution was investigated. Then again, some recent papers dealing with a dynamic boundary condition of type (1.2) are, among others, [9, 10, 11, 13, 14, 15, 18, 20, 24].

If one considers the Allen-Cahn equation (1.1) with condition (1.2), the order parameter u is conserved neither in the bulk nor, as u_Γ, on the boundary. The new issue of this paper is the setting of a mass constraint which can involve either the solution inside the domain or its counterpart on the boundary (or both of them). More precisely, we require that the solution u satisfy

$$k_* \le \int_\Omega w u(t) dx + \int_\Gamma w_\Gamma u_\Gamma(t) d\Gamma \le k^* \quad \text{for all } t \in [0,T], \tag{1.3}$$

where k_*, k^* are given constants fulfilling $k_* \le k^*$, and w and w_Γ are prescribed weight functions on Ω and Γ, respectively. For example, in the case when $w \equiv 1$, $w_\Gamma \equiv 0$ and $k_* = k^*$, (1.3) represents the conservation of the volume $\int_\Omega u(x,t)dx = k_*$, for all $t \in [0,T]$, a condition which instead arises naturally from the problem in the framework of a Cahn-Hilliard system (see, e.g., [15, 22]).

The analysis of the abstract theory for this kind of constraint was developed in [12] and motivated from the generalization of concrete problems [16, 17, 26] (see also [22], where the essential structure of possible constraints has been discussed for Cahn-Hilliard equation). In the abstract approach by [12] the constraint, and in particular the barriers k_* and k^* in (1.3), are allowed to depend on time. On the other hand, the abstract framework of [12] does not cover a special problem like ours, and especially it does not match with the presence of the nonlinearities W and W_Γ. In our approach, the solutions of the system (1.1)–(1.2) are not completely free to develop their dynamics, but they should respect the constraint (1.3) on the selected mass values. We discuss and characterize the properties of the unique solution of the initial value problem for the gradient flow system related to (1.1)–(1.3).

A brief outline of the present paper is as follows. In Section 2, we present the main results, consisting in the well-posedness of the Allen-Cahn equation with dynamic boundary conditions and mass constraints. We write the system as an evolution inclusion and characterize the solution with the help of a Lagrange multiplier. In Section 3, we prove the existence result. For the proof, we construct an approximate solution by substituting the maximal monotone graphs with their Yosida regularizations. For the approximated problem we can apply the result in [12], by checking the validity of the assumptions. Then, after proving some uniform estimates, we pass to the limit and conclude the existence proof. In Section 4, we prove the continuous dependence: of course, this result entails a uniqueness property. A final Section 5 contains the proof of a density result, which is useful in our approach. By the way, here is a detailed index of sections and subsections.

2 Main results

In this section, we present our main result. It is the well-posedness of the Allen-Cahn equation with dynamic boundary conditions and mass constraints. We apply the abstract formulation of the evolution inclusion.

2.1 Setting and assumptions

Let $0 < T < +\infty$ and let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be the bounded domain with smooth boundary $\Gamma := \partial\Omega$. We use the notation: $H := L^2(\Omega)$, $V := H^1(\Omega)$, $H_\Gamma := L^2(\Gamma)$, $V_\Gamma := H^1(\Gamma)$, with usual norms $|\cdot|_H$, $|\cdot|_V$, $|\cdot|_{H_\Gamma}$ and $|\cdot|_{V_\Gamma}$, respectively. Then, we obtain $V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*$, where "$\hookrightarrow\hookrightarrow$" stands for the dense and compact embedding, namely (V, H, V^*) is a standard Hilbert triplet. The same considerations hold for V_Γ and H_Γ. Moreover, we set

$$\boldsymbol{H} := H \times H_\Gamma, \quad \boldsymbol{V} := \big\{(u, u_\Gamma) \in V \times V_\Gamma \ : \ u_{|_\Gamma} = u_\Gamma \ \text{ a.e. on } \Gamma\big\},$$

where $u_{|_\Gamma}$ denotes the trace of u. Observe that $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces with the inner products

$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}} := (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma),\ \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{H},$$
$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{V}} := (u, z)_V + (u_\Gamma, z_\Gamma)_{V_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma),\ \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}$$

and related norms

$$|\boldsymbol{u}|_{\boldsymbol{H}} = \Big(|u|_H^2 + |u_\Gamma|_{H_\Gamma}^2\Big)^{1/2} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma) \in \boldsymbol{H},$$
$$|\boldsymbol{u}|_{\boldsymbol{V}} = \Big(|u|_V^2 + \big|u_\Gamma\big|_{V_\Gamma}^2\Big)^{1/2} \quad \text{for all } \boldsymbol{u} := \big(u, u_\Gamma\big) \in \boldsymbol{V}.$$

Then, we obtain $\boldsymbol{V} \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{V}^*$, where "$\hookrightarrow$" stands for the dense and continuous embedding (the density is checked in the Appendix). By the way, the above embeddings are also compact, of course. As a remark, let us restate that if $\boldsymbol{u} = (u, u_\Gamma) \in \boldsymbol{V}$ then u_Γ is exactly the trace of u on Γ; while, if $\boldsymbol{u} = (u, u_\Gamma)$ is just in $\boldsymbol{H}$, then $u \in H$ and $u_\Gamma \in H_\Gamma$ are independent.

The initial-value problem for the Allen-Cahn equation with dynamic boundary conditions is expressed by the following system (2.1)–(2.3) (cf. [9, 10])

$$\frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) = f, \quad \text{for some} \ \ \xi \in \beta(u), \quad \text{in } Q, \tag{2.1}$$

$$u_\Gamma = u_{|_\Gamma}, \quad \partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) = f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma, \tag{2.2}$$

$$u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \tag{2.3}$$

where β, β_Γ are maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$. Here, we let β, β_Γ be the subdifferentials

$$\beta = \partial\widehat{\beta}, \quad \beta_\Gamma = \partial\widehat{\beta}_\Gamma$$

of some lower semicontinuous and convex functions $\widehat{\beta}$, $\widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty)$ with $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$; in particular, this implies that $D(\beta) = D(\beta_\Gamma) = \mathbb{R}$, $0 \in \beta(0)$ and $0 \in \beta_\Gamma(0)$. The given functions

$$\pi,\ \pi_\Gamma : \mathbb{R} \to \mathbb{R} \text{ are Lipschitz continuous with Lipschitz constants } L,\ L_\Gamma, \tag{2.4}$$

respectively. Moreover, let

$$\boldsymbol{f} := (f, f_\Gamma) \in L^2(0, T; \boldsymbol{H}) \quad \text{and} \quad \boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{V}. \tag{2.5}$$

Now, take an arbitrary $\boldsymbol{v} = (v, v_\Gamma)$ in, say, $L^2(0,T;\boldsymbol{V})$ and test (2.1) by v: then, with the help of the boundary condition (2.2) we formally obtain

$$\int_0^T\!\!\int_\Omega \frac{\partial u}{\partial t} v\,dxdt + \int_0^T\!\!\int_\Omega \nabla u \cdot \nabla v\,dxdt + \int_0^T\!\!\int_\Omega \bigl(\xi + \pi(u)\bigr) v dxdt$$
$$+ \int_0^T\!\!\int_\Gamma \frac{\partial u_\Gamma}{\partial t} v_\Gamma\,d\Gamma dt + \int_0^T\!\!\int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma v_\Gamma\,d\Gamma dt + \int_0^T\!\!\int_\Gamma \bigl(\xi_\Gamma + \pi_\Gamma(u_\Gamma)\bigr) v_\Gamma\,d\Gamma dt$$
$$= \int_0^T\!\!\int_\Omega f v\,dxdt + \int_0^T\!\!\int_\Gamma f_\Gamma v_\Gamma\,d\Gamma dt, \tag{2.6}$$

a variational equality holding for all $\boldsymbol{v} = (v, v_\Gamma) \in L^2(0,T;\boldsymbol{V})$ and yielding a weak formulation of (2.1)–(2.2). Here, ∇_Γ denotes the surface gradient on Γ (see, e.g., [19, Chapter 3]). Hence, we can argue that a suitable space for the solution $\boldsymbol{u} = (u, u_\Gamma)$ of (2.3)–(2.6) (in which ξ and ξ_Γ represent selections of $\beta(u)$ and $\beta_\Gamma(u_\Gamma)$ as in is (2.1)–(2.2)) is $H^1(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$. Under suitable conditions on β, β_Γ, $u_0, u_{0\Gamma}$ such a (unique) solution actually exists (see, e.g., [9]) and possesses further regularity properties: $\boldsymbol{\xi} = (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H})$ and, in particular, $\boldsymbol{u} := (u, u_\Gamma) \in L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^2(\Gamma))$.

On the other hand, in this paper we are interested to the variational inequality obtained by (2.6) when replacing the "=" sign by "$\le$" and taking, in place of the test element $\boldsymbol{v}$, the difference $\boldsymbol{u} - \boldsymbol{z}$, where both the solution $\boldsymbol{u} = (u, u_\Gamma)$ and the arbitrary $\boldsymbol{z} = (z, z_\Gamma) \in L^2(0,T;\boldsymbol{V})$ have to satisfy the constraint (written in terms of z and z_Γ)

$$k_* \le \int_\Omega w\, z(t) dx + \int_\Gamma w_\Gamma\, z_\Gamma(t) d\Gamma \le k^*, \quad t \in [0,T]. \tag{2.7}$$

Here, k_* and k^* are real constants with $k_* \le k^*$, and $\boldsymbol{w} := (w, w_\Gamma)$ is fixed in $\boldsymbol{H}$. We require that the weight functions w and w_Γ satisfy

$$w \ge 0 \quad \text{a.e. in } \Omega, \quad w_\Gamma \ge 0 \quad \text{a.e. on } \Gamma$$

and

$$\sigma_0 := \int_\Omega w dx + \int_\Gamma w_\Gamma d\Gamma > 0. \tag{2.8}$$

The constraint (2.7) entails a limitation on a specific averaged value of the solution $\boldsymbol{u} = (u, u_\Gamma)$ in the bulk and/or on the boundary. Inequality (2.8) can be seen as a non-degeneracy condition on the weight element $\boldsymbol{w} := (w, w_\Gamma)$.

Hence, let us term (P) the initial-value problem related to the variational inequality and to the constraint in (2.7). Now, we define precisely the notion of solution to the problem (P) by means of a Lagrange multiplier.

Definition 2.1. *The triplet* $(\boldsymbol{u}, \boldsymbol{\xi}, \lambda)$ *is called the solution of* (P) *if*

$$\boldsymbol{u} = (u, u_\Gamma) \quad \textit{with} \quad u \in H^1(0,T;H) \cap C([0,T];V) \cap L^2\bigl(0,T;H^2(\Omega)\bigr),$$
$$u_\Gamma \in H^1(0,T;H_\Gamma) \cap C([0,T];V_\Gamma) \cap L^2\bigl(0,T;H^2(\Gamma)\bigr),$$
$$\boldsymbol{\xi} = (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H}), \quad \lambda \in L^2(0,T)$$

and u, u_Γ, ξ, ξ_Γ, λ satisfy

$$\frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) + \lambda w = f \quad \textit{a.e. in } Q, \tag{2.9}$$

$$\xi \in \beta(u) \quad \textit{a.e. in } Q, \tag{2.10}$$

$$u_{|\Gamma} = u_\Gamma, \quad \partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) + \lambda w_\Gamma = f_\Gamma \quad \textit{a.e. on } \Sigma, \tag{2.11}$$

$$\xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \textit{a.e. on } \Sigma, \tag{2.12}$$

$$u(0) = u_0 \quad \textit{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \textit{a.e. on } \Gamma, \tag{2.13}$$

$$k_* \le \int_\Omega w u(t) dx + \int_\Gamma w_\Gamma u_\Gamma(t) d\Gamma \le k^* \quad \textit{for all } t \in [0,T], \tag{2.14}$$

$$\lambda(t)\left(\int_\Omega w\big(u(t) - z\big) dx + \int_\Gamma w_\Gamma \big(u_\Gamma(t) - z_\Gamma\big) d\Gamma \right) \ge 0 \quad \textit{for a.a. } t \in (0,T)$$
$$\textit{and for all } \boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{V} \textit{ such that } k_* \le \int_\Omega wz dx + \int_\Gamma w_\Gamma z_\Gamma d\Gamma \le k^*. \tag{2.15}$$

2.2 Well-posedness

The first result states the continuous dependence on the data. The uniqueness of the component $\boldsymbol{u}$ of the solution (see the later Remark 3.3) is also guaranteed by this theorem.

Theorem 2.1. *For $i = 1, 2$ let $(\boldsymbol{u}^{(i)}, \boldsymbol{\xi}^{(i)}, \lambda^{(i)})$, with $\boldsymbol{u}^{(i)} = (u^{(i)}, u_\Gamma^{(i)})$ and $\boldsymbol{\xi}^{(i)} = (\xi^{(i)}, \xi_\Gamma^{(i)})$, be a solution to* (P) *corresponding to the data $\boldsymbol{f}^{(i)} = (f^{(i)}, f_\Gamma^{(i)})$ and $\boldsymbol{u}_0^{(i)} = (u_0^{(i)}, u_{0\Gamma}^{(i)})$. Then, there exists a positive constant $C > 0$, depending only on L, L_Γ and T, such that*

$$\begin{aligned} &\big|u^{(1)}(t) - u^{(2)}(t)\big|_H^2 + \big|u_\Gamma^{(1)}(t) - u_\Gamma^{(2)}(t)\big|_{H_\Gamma}^2 \\ &\quad + 2\int_0^t \big|\nabla u^{(1)}(s) - \nabla u^{(2)}(s)\big|_{H^d}^2 ds + 2\int_0^t \big|\nabla_\Gamma u_\Gamma^{(1)}(s) - \nabla_\Gamma u_\Gamma^{(2)}(s)\big|_{H_\Gamma^d}^2 ds \\ &\le C\bigg\{ \big|u_0^{(1)} - u_0^{(2)}\big|_H^2 + \big|u_{0\Gamma}^{(1)} - u_{0\Gamma}^{(2)}\big|_{H_\Gamma}^2 + \int_0^T \big|f^{(1)}(s) - f^{(2)}(s)\big|_H^2 ds \\ &\quad + \int_0^T \big|f_\Gamma^{(1)}(s) - f_\Gamma^{(2)}(s)\big|_{H_\Gamma}^2 ds \bigg\} \quad \textit{for all } t \in [0,T]. \end{aligned} \tag{2.16}$$

The second result deals with the existence of the solution. To this aim, we further assume that there exist positive constants c_0, ϱ such that

$$|s| \le c_0\big(1 + \widehat{\beta}(r)\big) \quad \text{for all } r \in \mathbb{R} \text{ and } s \in \beta(r), \tag{2.17}$$

$$|s| \le c_0\big(1 + \widehat{\beta}_\Gamma(r)\big) \quad \text{for all } r \in \mathbb{R} \text{ and } s \in \beta_\Gamma(r), \tag{2.18}$$

$$\big|\beta^\circ(r)\big| \le \varrho\big|\beta_\Gamma^\circ(r)\big| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{2.19}$$

where the minimal section β° of β is specified by

$$\beta^\circ(r) := \big\{ r^* \in \beta(r) \ : \ |r^*| = \min_{s \in \beta(r)} |s| \big\}, \quad r \in \mathbb{R}$$

and the same definition holds for β_Γ° (and for any maximal monotone graph!). We also require compatibility conditions for the initial data, that are

$$k_* \le \int_\Omega w u_0 dx + \int_\Gamma w_\Gamma u_{0\Gamma} d\Gamma \le k^* \tag{2.20}$$

and

$$\widehat{\beta}(u_0) \in L^1(\Omega), \quad \widehat{\beta}_\Gamma(u_{0\Gamma}) \in L^1(\Gamma). \tag{2.21}$$

Theorem 2.2. *Under the above assumptions, there exists one solution of* (P).

2.3 Abstract formulation

In this subsection, we comment on the formulation of the problem and on our results. The first remark is related to the mathematical treatment by the evolution inclusion governed by subdifferential operators.

Our mass constraint (2.14) can be rewritten as

$$k_* \le \big(\boldsymbol{w}, \boldsymbol{u}(t)\big)_{\boldsymbol{H}} \le k^* \quad \text{for all } t \in [0,T].$$

Then, we define the convex constraint set $\boldsymbol{K}$ that plays an important role in this paper:

$$\boldsymbol{K} := \big\{ \boldsymbol{z} \in \boldsymbol{V} \ : \ k_* \le (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}} \le k^* \big\},$$

with the indicator function $I_{\boldsymbol{K}} : \boldsymbol{H} \to [0,+\infty]$ fulfilling $I_{\boldsymbol{K}}(\boldsymbol{z}) = 0$ if $\boldsymbol{z} \in \boldsymbol{K}$, $I_{\boldsymbol{K}}(\boldsymbol{z}) = +\infty$ if $\boldsymbol{z} \in \boldsymbol{H} \setminus \boldsymbol{K}$. Moreover, we introduce the proper, lower semicontinuous and convex functional $\varphi : \boldsymbol{H} \to [0,+\infty]$ by

$$\varphi(\boldsymbol{z}) := \begin{cases} \displaystyle \frac{1}{2}\int_\Omega |\nabla z|^2 dx + \int_\Omega \widehat{\beta}(z) dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \int_\Gamma \widehat{\beta}_\Gamma(z_\Gamma) d\Gamma \\ \qquad\qquad \text{if } \boldsymbol{z} \in \boldsymbol{V}, \widehat{\beta}(z) \in L^1(\Omega) \text{ and } \widehat{\beta}_\Gamma(z_\Gamma) \in L^1(\Gamma), \\ +\infty \quad \text{otherwise.} \end{cases}$$

Therefore, it is possible to check that our problem enters the following abstract form of an evolution inclusion with a Lipschitz perturbation:

$$\boldsymbol{u}'(t) + \partial(\varphi + I_{\boldsymbol{K}})\big(\boldsymbol{u}(t)\big) + \boldsymbol{\pi}\big(\boldsymbol{u}(t)\big) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}, \ \text{ for a.a. } t \in (0,T), \tag{2.22}$$

where $\boldsymbol{\pi}(\boldsymbol{z}) := (\pi(z), \pi_\Gamma(z_\Gamma))$ for all $\boldsymbol{z} \in \boldsymbol{H}$. This kind of evolution inclusion is well known as a gradient flow equation including a Lipschitz perturbation, and it has been treated, in particular, in [6].

Thus, from this point of view the existence and uniqueness of the solution to the Cauchy problem for (2.22) is perfectly known. On the other hand, what is important here is to characterize the suitable selection from $\partial(\varphi + I_{\boldsymbol{K}})\big(\boldsymbol{u}(t)\big)$ for a.a. $t \in (0,T)$, which is our main concern. Now, one can check that (see, e.g., [5, p. 59] or [9]) the subdifferential operator $\partial\varphi$ can be expressed in a formal way as

$$\begin{aligned} &\boldsymbol{z}^* := (z^*, z_\Gamma^*) \in \partial\varphi(\boldsymbol{z}) \text{ is in } \boldsymbol{H} \text{ if and only if} \\ &(z^*, z_\Gamma^*) = \big(-\Delta z + \beta(z), \partial_\nu z - \Delta_\Gamma z_\Gamma + \beta_\Gamma(z_\Gamma)\big). \end{aligned}$$

However, when one adds the indicator function $I_{\boldsymbol{K}}$ to φ, then the subdifferential $\partial(\varphi+I_{\boldsymbol{K}})$ must take into account the constraint given by $\boldsymbol{K}$. Then, the point of emphasis of Theorem 2.2 is the (further) characterization with the help of the Lagrange multiplier λ in (2.9) and (2.11). Indeed, our analysis shows in particular that $\boldsymbol{z}^* := (z^*, z^*_\Gamma) \in \partial(\varphi + I_{\boldsymbol{K}})(\boldsymbol{z})$ lies in $\boldsymbol{H}$ if and only if there is a scalar $\lambda_{\boldsymbol{z}}$ such that

$$(z^*, z^*_\Gamma) = \bigl(-\Delta z + \beta(z) + \lambda_{\boldsymbol{z}} w, \partial_\nu z - \Delta_\Gamma z_\Gamma + \beta_\Gamma(z_\Gamma) + \lambda_{\boldsymbol{z}} w_\Gamma\bigr).$$

We point out that such a characterization problem was already treated in [12], in an abstract framework with appropriate assumptions for the abstract functions and tools. However, in our concrete problem for the Allen-Cahn equation with dynamic boundary conditions we cannot ensure the validity of [12, Assumption (A2)] for φ. Therefore, in the next section we consider an approximating problem to which we can (more or less) apply the abstract result of [12]. In this sense, our results turn out to be an extension of [12] for our concrete problem.

3 Existence

This section is devoted to the proof of Theorem 2.2. We make use of the Yosida approximations for maximal monotone operators and of well-known results of this theory (see, e.g., [5, 6, 21]). For each $\varepsilon \in (0, 1]$, we define $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$, along with the associated resolvent operators $J_\varepsilon, J_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$, by

$$\beta_\varepsilon(r) := \frac{1}{\varepsilon}\bigl(r - J_\varepsilon(r)\bigr) := \frac{1}{\varepsilon}\bigl(r - (I + \varepsilon\beta)^{-1}(r)\bigr),$$

$$\beta_{\Gamma,\varepsilon}(r) := \frac{1}{\varepsilon\varrho}\bigl(r - J_{\Gamma,\varepsilon}(r)\bigr) := \frac{1}{\varepsilon\varrho}\bigl(r - (I + \varepsilon\varrho\beta_\Gamma)^{-1}(r)\bigr) \quad \text{for all } r \in \mathbb{R},$$

where $\varrho > 0$ is same constant as in (2.19). Note that the two definitions are not symmetric since in the second it is $\varrho\varepsilon$ and not directly ε to be used as approximation parameter. Anyway, we easily have $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$. Moreover, the related Moreau-Yosida regularizations $\widehat{\beta}_\varepsilon$, $\widehat{\beta}_{\Gamma,\varepsilon}$ of $\widehat{\beta}$, $\widehat{\beta}_\Gamma : \mathbb{R} \to \mathbb{R}$ fulfill

$$\widehat{\beta}_\varepsilon(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|r - s|^2 + \widehat{\beta}(s) \right\} = \frac{1}{2\varepsilon}\bigl|r - J_\varepsilon(r)\bigr|^2 + \widehat{\beta}\bigl(J_\varepsilon(r)\bigr) = \int_0^r \beta_\varepsilon(s)ds,$$

$$\widehat{\beta}_{\Gamma,\varepsilon}(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon\varrho}|r - s|^2 + \widehat{\beta}_\Gamma(s) \right\} = \int_0^r \beta_{\Gamma,\varepsilon}(s)ds \quad \text{for all } r \in \mathbb{R}.$$

It is well known that β_ε is Lipschitz continuous with Lipschitz constant $1/\varepsilon$ and $\beta_{\Gamma,\varepsilon}$ is also Lipschitz continuous with constant $1/(\varepsilon\varrho)$. In addition, we have the standard properties

$$\bigl|\beta_\varepsilon(r)\bigr| \le \bigl|\beta^\circ(r)\bigr|, \quad \bigl|\beta_{\Gamma,\varepsilon}(r)\bigr| \le \bigl|\beta^\circ_\Gamma(r)\bigr| \quad \text{for all } r \in \mathbb{R},$$

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \quad 0 \le \widehat{\beta}_{\Gamma,\varepsilon}(r) \le \widehat{\beta}_\Gamma(r) \quad \text{for all } r \in \mathbb{R}.$$

We emphasize that (2.17)–(2.18) entail

$$\bigl|\beta_\varepsilon(r)\bigr| \le c_0\bigl(1 + \widehat{\beta}_\varepsilon(r)\bigr) \quad \text{for all } r \in \mathbb{R}, \tag{3.1}$$

$$\big|\beta_{\Gamma,\varepsilon}(r)\big| \le c_0\big(1+\widehat{\beta}_{\Gamma,\varepsilon}(r)\big) \quad \text{for all } r \in \mathbb{R}, \tag{3.2}$$

with the same constant c_0 as in (2.17)–(2.18). Indeed, arguing for instance for β_ε, it suffices to notice that for all $r \in \mathbb{R}$ there exists $s_\varepsilon \in \beta(J_\varepsilon(r))$ such that

$$\big|\beta_\varepsilon(r)\big| = |s_\varepsilon| \le c_0\Big(1+\widehat{\beta}\big(J_\varepsilon(r)\big)\Big) = c_0\Big(1+\widehat{\beta}_\varepsilon(r)\big)\Big),$$

thanks to (2.17). Moreover, owing to the assumption (2.19) and [9, Lemma 4.4], the inequality

$$\big|\beta_\varepsilon(r)\big| \le \varrho\big|\beta_{\Gamma,\varepsilon}(r)\big| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{3.3}$$

holds for β_ε and $\beta_{\Gamma,\varepsilon}$ as well.

3.1 Approximation of the problem

Let us consider an approximation of (P) which is stated as the following initial-value problem for a gradient flow equation: for each $\varepsilon \in (0,1]$ let $\boldsymbol{u}_\varepsilon$ solve the abstract Cauchy problem

$$\boldsymbol{u}_\varepsilon'(t) + \partial(\varphi_\varepsilon + I_{\boldsymbol{K}})\big(\boldsymbol{u}_\varepsilon(t)\big) + \boldsymbol{\pi}\big(\boldsymbol{u}_\varepsilon(t)\big) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T), \tag{3.4}$$

$$\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 \quad \text{in } \boldsymbol{H}, \tag{3.5}$$

with $\boldsymbol{u}_0 = (u_0, u_{0\Gamma}) \in \boldsymbol{K}$ satisfying (2.21) and $\varphi_\varepsilon : \boldsymbol{H} \to [0,+\infty]$ being defined by

$$\varphi_\varepsilon(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \int_\Omega \widehat{\beta}_\varepsilon(z) dx + \frac{\varepsilon}{2}\int_\Omega |z|^2 dx \\ \quad + \dfrac{1}{2}\displaystyle\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma) d\Gamma + \frac{\varepsilon}{2}\int_\Gamma |z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}, \\ +\infty \quad \text{if } \boldsymbol{z} \in \boldsymbol{H} \setminus \boldsymbol{V}; \end{cases}$$

moreover, it is understood that

$$\boldsymbol{\pi}(\boldsymbol{z}) := \big(\pi(z), \pi_\Gamma(z_\Gamma)\big) \quad \text{for all } \boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{H}.$$

For a proper convex lower semicontinuous function $\psi : \boldsymbol{V} \to (-\infty,+\infty]$, we denote by $\partial_*\psi$ its subdifferential operator acting from $\boldsymbol{V}$ to $\boldsymbol{V}^*$. In the next statement we point out the following characterization of $\partial_*\varphi_\varepsilon$.

Lemma 3.1. *The function $\varphi_\varepsilon : \boldsymbol{H} \to [0,+\infty]$ is convex and lower semicontinuous, with domain $D(\varphi_\varepsilon) = \boldsymbol{V}$. Moreover, φ_ε is lower semicontinuous in $\boldsymbol{V}$ as well and the subdifferential $\partial_*\varphi_\varepsilon$ is single-valued and specified by the following form:*

$$\begin{aligned} \big\langle \partial_*\varphi_\varepsilon(\boldsymbol{z}), \bar{\boldsymbol{z}} \big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} &= \big(\nabla z, \nabla \bar{z}\big)_{H^d} + \big(\beta_\varepsilon(z), \bar{z}\big)_H + \varepsilon\big(z, \bar{z}\big)_H \\ &\quad + \big(\nabla_\Gamma z_\Gamma, \nabla_\Gamma \bar{z}_\Gamma\big)_{H_\Gamma^d} + \big(\beta_{\Gamma,\varepsilon}(z_\Gamma), \bar{z}_\Gamma\big)_{H_\Gamma} + \varepsilon\big(z_\Gamma, \bar{z}_\Gamma\big)_{H_\Gamma} \\ &\qquad \textit{for all } \boldsymbol{z} = (z, z_\Gamma),\ \bar{\boldsymbol{z}} = (\bar{z}, \bar{z}_\Gamma) \in \boldsymbol{V}. \end{aligned} \tag{3.6}$$

Finally, there exists a positive constant $C_\varepsilon > 0$ depending on $\varepsilon > 0$ such that

$$\big|\partial_*\varphi_\varepsilon(\boldsymbol{z})\big|_{\boldsymbol{V}^*} \le C_\varepsilon\big(1 + \varphi_\varepsilon(\boldsymbol{z})\big) \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{V}. \tag{3.7}$$

Proof. The function φ_ε is convex and assumes finite value on all elements of $\boldsymbol{V}$; in addition, it is straightforward to check that φ_ε is strongly, whence also weakly, lower semicontinuous in $\boldsymbol{V}$. Now, let $\boldsymbol{z}_n \to \boldsymbol{z}$ strongly in $\boldsymbol{H}$ as $n \to \infty$, and assume that $\varphi_\varepsilon(\boldsymbol{z}_n) \le \alpha$ for some $\alpha \ge 0$ and all $n \in \mathbb{N}$. Then, as $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$ are non-negative, we easily conclude that $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{V}$ and consequently $\boldsymbol{z}_n$ weakly converges to $\boldsymbol{z}$ in $\boldsymbol{V}$ as $n \to \infty$. Then, it turns out that $\varphi_\varepsilon(\boldsymbol{z}) \le \alpha$ and the weak lower semicontinuity of φ_ε in $\boldsymbol{H}$ follows. Next, let $\boldsymbol{z}^* \in \partial_*\varphi_\varepsilon(\boldsymbol{z})$ in $\boldsymbol{V}^*$. Then, from the definition of the subdifferential, we have

$$
\begin{aligned}
&\langle \boldsymbol{z}^*, \delta\bar{\boldsymbol{z}}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \\
&\le \delta \int_\Omega \nabla z \cdot \nabla \bar{z} dx + \frac{\delta^2}{2} \int_\Omega |\nabla \bar{z}|^2 dx + \int_\Omega \Big\{ \widehat{\beta}_\varepsilon(z+\delta\bar{z}) - \widehat{\beta}_\varepsilon(z) \Big\} dx \\
&\quad + \delta\varepsilon \int_\Omega z\bar{z} dx + \frac{\delta^2\varepsilon}{2} \int_\Omega |\bar{z}|^2 dx + \delta \int_\Gamma \nabla_\Gamma z_\Gamma \cdot \nabla_\Gamma \bar{z}_\Gamma d\Gamma + \frac{\delta^2}{2} \int_\Gamma |\nabla_\Gamma \bar{z}_\Gamma|^2 d\Gamma \\
&\quad + \int_\Gamma \Big\{ \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma + \delta\bar{z}_\Gamma) - \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma) \Big\} d\Gamma + \delta\varepsilon \int_\Gamma z_\Gamma \bar{z}_\Gamma d\Gamma + \frac{\delta^2\varepsilon}{2} \int_\Gamma |\bar{z}_\Gamma|^2 d\Gamma, \qquad (3.8)
\end{aligned}
$$

for all $\bar{\boldsymbol{z}} \in \boldsymbol{V}$ and $\delta > 0$. Here, from the Lipschitz continuity of β_ε and $\beta_{\Gamma,\varepsilon}$ we infer that

$$
\left| \frac{\widehat{\beta}_\varepsilon(z+\delta\bar{z}) - \widehat{\beta}_\varepsilon(z)}{\delta} \right| \le \big|\beta_\varepsilon(\zeta) - \beta_\varepsilon(0)\big| |\bar{z}| \le \frac{1}{\varepsilon}\big(|z| + \delta|\bar{z}|\big)|\bar{z}| \quad \text{a.e. in } \Omega,
$$

$$
\left| \frac{\widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma+\delta\bar{z}_\Gamma) - \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma)}{\delta} \right| \le \big|\beta_{\Gamma,\varepsilon}(\zeta_\Gamma) - \beta_{\Gamma,\varepsilon}(0)\big| |\bar{z}_\Gamma| \le \frac{1}{\varepsilon\varrho}\big(|z_\Gamma| + \delta|\bar{z}_\Gamma|\big)|\bar{z}_\Gamma| \quad \text{a.e. on } \Gamma,
$$

for some intermediate functions $\zeta : \Omega \to \mathbb{R}$, between z and $\bar{z}$, and $\zeta_\Gamma : \Gamma \to \mathbb{R}$, between z_Γ and $\bar{z}_\Gamma$. Therefore, dividing (3.8) by δ and letting $\delta \to 0$, we obtain

$$
\begin{aligned}
\big\langle \boldsymbol{z}^*, \bar{\boldsymbol{z}} \big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \le\ & \big(\nabla z, \nabla \bar{z}\big)_{H^d} + \big(\beta_\varepsilon(z), \bar{z}\big)_H + \varepsilon\big(z, \bar{z}\big)_H + \big(\nabla_\Gamma z_\Gamma, \nabla_\Gamma \bar{z}_\Gamma\big)_{H_\Gamma^d} \\
& + \big(\beta_{\Gamma,\varepsilon}(z_\Gamma), \bar{z}_\Gamma\big)_{H_\Gamma} + \varepsilon\big(z_\Gamma, \bar{z}_\Gamma\big)_{H_\Gamma} \quad \text{for all } \bar{\boldsymbol{z}} := (\bar{z}, \bar{z}_\Gamma) \in \boldsymbol{V}.
\end{aligned}
$$

The opposite inequality can be shown as well, by taking $-\delta$ in place of δ. Thus, $\partial_*\varphi_\varepsilon$ is single-valued and the characterization (3.6) of $\partial_*\varphi_\varepsilon$ follows. Finally, we see that

$$
\begin{aligned}
&\int_\Omega \big|\beta_\varepsilon(z)\big|^2 dx = \int_\Omega \frac{1}{\varepsilon^2} \big|z - J_\varepsilon(z)\big|^2 dx \\
&\le \int_\Omega \frac{2}{\varepsilon} \left(\frac{1}{2\varepsilon} \big|z - J_\varepsilon(z)\big|^2 + \widehat{\beta}\big(J_\varepsilon(z)\big) \right) dx = \int_\Omega \frac{2}{\varepsilon} \widehat{\beta}_\varepsilon(z) dx
\end{aligned}
$$

and

$$
\int_\Gamma \big|\beta_{\Gamma,\varepsilon}(z_\Gamma)\big|^2 d\Gamma \le \int_\Gamma \frac{2}{\varrho\varepsilon} \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma) d\Gamma.
$$

Therefore, the boundedness property in (3.7) is also true. $\square$

Remark 3.1. The estimate (3.7) is somehow important in order to apply the abstract result in [12]. That was a reason for us to introduce the Moreau-Yosida regularizations $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$, otherwise with $\widehat{\beta}$ and $\widehat{\beta}_\Gamma$ instead (3.7) may not hold.

Now, we recall the fact that

$$\overline{\boldsymbol{K}} = \boldsymbol{K}_{\boldsymbol{H}} := \bigl\{ \boldsymbol{z} \in \boldsymbol{H} \ : \ k_* \le (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}} \le k^* \bigr\},$$

is a closed convex subset of $\boldsymbol{H}$. Then, the following result holds.

Proposition 3.1. *For each $\varepsilon \in (0,1]$, there exist a unique*

$$\boldsymbol{u}_\varepsilon \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$$

and a pair of functions $\boldsymbol{u}_\varepsilon^* \in L^2(0,T;\boldsymbol{H})$ and $\lambda_\varepsilon \in L^2(0,T)$ such that

$$\boldsymbol{u}_\varepsilon(t) \in \boldsymbol{K}_{\boldsymbol{H}} \quad \textit{for all } t \in [0,T]$$

and

$$\boldsymbol{u}_\varepsilon'(t) + \boldsymbol{u}_\varepsilon^*(t) + \lambda_\varepsilon(t)\boldsymbol{w} + \boldsymbol{\pi}\bigl(\boldsymbol{u}_\varepsilon(t)\bigr) = \boldsymbol{f}(t) \quad \textit{in } \boldsymbol{H}, \ \textit{for a.a. } t \in (0,T), \tag{3.9}$$

$$\boldsymbol{u}_\varepsilon^*(t) \in \partial\varphi_\varepsilon\bigl(\boldsymbol{u}_\varepsilon(t)\bigr) \quad \textit{in } \boldsymbol{H}, \ \textit{for a.a. } t \in (0,T), \tag{3.10}$$

$$\lambda_\varepsilon(t)\boldsymbol{w} \in \partial I_{\boldsymbol{K}_{\boldsymbol{H}}}\bigl(\boldsymbol{u}_\varepsilon(t)\bigr) \quad \textit{in } \boldsymbol{H}, \ \textit{for a.a. } t \in (0,T), \tag{3.11}$$

$$\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 \quad \textit{in } \boldsymbol{H}. \tag{3.12}$$

Moreover, λ_ε is given by

$$\lambda_\varepsilon(t) := \Bigl(\boldsymbol{f}(t) - \boldsymbol{u}_\varepsilon'(t) - \boldsymbol{\pi}\bigl(\boldsymbol{u}_\varepsilon(t)\bigr), \boldsymbol{z}_c \Bigr)_{\boldsymbol{H}} - \bigl(\boldsymbol{u}_\varepsilon^*(t), \boldsymbol{z}_c \bigr)_{\boldsymbol{H}} \quad \textit{for a.a. } t \in (0,T), \tag{3.13}$$

where $\boldsymbol{z}_c := (1/\sigma_0, 1/\sigma_0) \in \boldsymbol{V}$.

Proof. We sketch the basic steps.

1. For a given $\bar{\boldsymbol{u}} \in C([0,T];\boldsymbol{H})$, there is a unique function $\boldsymbol{u} \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$ solving

$$\boldsymbol{u}'(t) + \partial(\varphi_\varepsilon + I_{\boldsymbol{K}})\bigl(\boldsymbol{u}(t)\bigr) \ni \boldsymbol{f}(t) - \boldsymbol{\pi}\bigl(\bar{\boldsymbol{u}}(t)\bigr) \quad \text{in } \boldsymbol{H}, \ \text{for a.a. } t \in (0,T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \boldsymbol{H}.$$

Indeed, recalling that $\boldsymbol{f} - \boldsymbol{\pi}\bigl(\bar{\boldsymbol{u}}\bigr) \in L^2(0,T;\boldsymbol{H})$ and $\boldsymbol{u}_0 \in D(\varphi_\varepsilon + I_{\boldsymbol{K}})$ (cf. (2.5) and (2.20)), it suffices to apply, e.g., [6, Thm. 3.6, p. 72] for the existence, uniqueness and regularity of the solution $\boldsymbol{u}$. Thus, we construct the map

$$\Psi : \bar{\boldsymbol{u}} \mapsto \boldsymbol{u},$$

from $C([0,T];\boldsymbol{H})$ into itself.

2. For a given pair $\bar{\boldsymbol{u}}^{(1)}, \bar{\boldsymbol{u}}^{(2)} \in C([0,T];\boldsymbol{H})$, we can use the estimate (see, e.g., [6, Lemme 3.1, p. 64])

$$\left|\boldsymbol{u}^{(1)}(t)-\boldsymbol{u}^{(2)}(t)\right|_{\boldsymbol{H}}^2 \le C_{\boldsymbol{\pi}} \int_0^t \left|\bar{\boldsymbol{u}}^{(1)}(s)-\bar{\boldsymbol{u}}^{(2)}(s)\right|_{\boldsymbol{H}}^2 ds \quad \text{for all } t \in [0,T],$$

where $\boldsymbol{u}^{(i)} = \Psi(\bar{\boldsymbol{u}}^{(i)})$, $i=1,2$, and $C_{\boldsymbol{\pi}} > 0$ is a positive constant depending only on L and L_Γ (cf. (2.4)). Then, by recurrence one shows that there exists a suitable $k \in \mathbb{N}$ such that Ψ^k is a contraction mapping in $C([0,T];\boldsymbol{H})$, and consequently there exists a unique solution $\boldsymbol{u}_\varepsilon$ of the problem (3.4)–(3.5).

3. Now, in order to conclude the proof we can just apply Theorem 2.3 and Remark 3 of [12]. In fact, in view of Lemma 3.1, it is not difficult to check the validity of the assumptions (A1)–(A5) of [12] in our case. In particular, let us point out that the coercivity property stated in [12, (A5)] comes from the definition of φ_ε. However, one important point regards the density of $\boldsymbol{V}$ in $\boldsymbol{H}$, for which we refer the reader to the Appendix. Finally, we use the fact that $\overline{\boldsymbol{K}} = \boldsymbol{K}_{\boldsymbol{H}}$. □

Thanks to Proposition 3.1 and Lemma 3.1, we arrive at the following weak formulation of (3.9):

$$\begin{aligned}
&\int_\Omega g_\varepsilon(t) z dx + \int_\Gamma g_{\Gamma,\varepsilon}(t) z_\Gamma d\Gamma \\
&= \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla z dx + \int_\Omega \beta_\varepsilon\big(u_\varepsilon(t)\big) z dx + \varepsilon \int_\Omega u_\varepsilon(t) z dx + \int_\Gamma \nabla_\Gamma u_{\Gamma,\varepsilon}(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma \\
&\quad + \int_\Gamma \beta_{\Gamma,\varepsilon}\big(u_{\Gamma,\varepsilon}(t)\big) z_\Gamma d\Gamma + \varepsilon \int_\Gamma u_{\Gamma,\varepsilon}(t) z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V},
\end{aligned} \tag{3.14}$$

where

$$g_\varepsilon := f_\varepsilon - u_\varepsilon' - \lambda_\varepsilon w - \pi(u_\varepsilon) \in L^2(0,T;H),$$

$$g_{\Gamma,\varepsilon} := f_{\Gamma,\varepsilon} - u_{\Gamma,\varepsilon}' - \lambda_\varepsilon w_\Gamma - \pi_\Gamma(u_{\Gamma,\varepsilon}) \in L^2(0,T;H_\Gamma).$$

Moreover, we point out the following regularity properties for the solution.

Proposition 3.2. *For each $\varepsilon \in (0,1]$ we have that $u_\varepsilon \in L^2(0,T;H^2(\Omega))$ and $u_{\Gamma,\varepsilon} \in L^2(0,T;H^2(\Gamma))$.*

Proof. First, we take $z \in \mathcal{D}(\Omega)$, which entails that $z_\Gamma = 0$, in (3.14) and get

$$-\Delta u_\varepsilon(t) = g_\varepsilon(t) - \beta_\varepsilon\big(u_\varepsilon(t)\big) - \varepsilon u_\varepsilon(t) \quad \text{in } \mathcal{D}'(\Omega), \quad \text{for a.a. } t \in (0,T).$$

This implies that $-\Delta u_\varepsilon \in L^2(0,T;H)$ due to the regularity of the right hand side. On the other hand, we already know that $u_\varepsilon \in L^\infty(0,T;V)$ and $u_{\Gamma,\varepsilon} \in L^\infty(0,T;V_\Gamma)$. Then, we infer that (see, e.g., [7, Thm. 3.2, p. 1.79])

$$u_\varepsilon \in L^2\big(0,T;H^{3/2}(\Omega)\big)$$

and consequently, by a trace theorem [7, Thm. 2.27, p. 1.64], $\partial_\nu u_\varepsilon \in L^2(0,T;H_\Gamma)$. At this point, from the variational equality (3.14) we can obtain the characterization on the boundary

$$-\Delta_\Gamma u_{\Gamma,\varepsilon} = g_{\Gamma,\varepsilon} - \partial_\nu u_\varepsilon - \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) - \varepsilon u_{\Gamma,\varepsilon} \quad \text{a.e. on } \Sigma,$$

and the information that $\Delta_\Gamma u_{\Gamma,\varepsilon} \in L^2(0,T;H_\Gamma)$ implies (see, e.g., [19, p. 104])

$$u_{\Gamma,\varepsilon} \in L^2\bigl(0,T;H^2(\Gamma)\bigr).$$

Finally, this yields in particular that $u_{\Gamma,\varepsilon} \in L^2(0,T;H^{3/2}(\Gamma))$, whence (quoting again [7, Thm. 3.2, p. 1.79])

$$u_\varepsilon \in L^2\bigl(0,T;H^2(\Omega)\bigr).$$

□

By virtue of this lemma, our approximate problem can be written as

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon + \pi(u_\varepsilon) + \lambda_\varepsilon w = f \quad \text{a.e. in } Q, \tag{3.15}$$

$$\partial_\nu u_\varepsilon + \frac{\partial u_{\Gamma,\varepsilon}}{\partial t} - \Delta_\Gamma u_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) + \varepsilon u_{\Gamma,\varepsilon} + \pi_\Gamma(u_{\Gamma,\varepsilon}) + \lambda_\varepsilon w_\Gamma = f_\Gamma \quad \text{a.e. on } \Sigma, \tag{3.16}$$

$$u_{\Gamma,\varepsilon} = u_{\varepsilon|_\Gamma} \quad \text{a.e. on } \Sigma, \tag{3.17}$$

$$u_\varepsilon(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_{\Gamma,\varepsilon}(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma, \tag{3.18}$$

$$k_* \le k_\varepsilon(t) := \int_\Omega w u_\varepsilon(t) dx + \int_\Gamma w_\Gamma u_{\Gamma,\varepsilon}(t) d\Gamma \le k^* \quad \text{for all } t \in [0,T], \tag{3.19}$$

$$\lambda_\varepsilon(t) \in \partial I_{[k_*,k^*]}\bigl(k_\varepsilon(t)\bigr) \quad \text{for a.a. } t \in (0,T). \tag{3.20}$$

Remark 3.2. As $\boldsymbol{u}_\varepsilon(t) \in \boldsymbol{K_H}$ for all $t \in [0,T]$, we claim that the last condition is equivalent to (3.11). Actually, let us assume (3.20). For each $\boldsymbol{z} \in \boldsymbol{K_H}$, there exist uniquely $\alpha \in \mathbb{R}$ with $k_* \le \alpha \le k^*$ and $\boldsymbol{z}_N \in \boldsymbol{H}$ with $(\boldsymbol{w}, \boldsymbol{z}_N)_{\boldsymbol{H}} = 0$ such that

$$\boldsymbol{z} = \alpha \boldsymbol{z}_c + \boldsymbol{z}_N, \quad (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}} = \alpha.$$

Therefore, from the definition of subdifferential it follows that $\lambda_\varepsilon(t)\cdot((\boldsymbol{w},\boldsymbol{z})_{\boldsymbol{H}} - k_\varepsilon(t)) \le 0$, namely

$$\bigl(\lambda_\varepsilon(t)\boldsymbol{w}, \boldsymbol{z} - \boldsymbol{u}_\varepsilon(t)\bigr)_{\boldsymbol{H}} \le 0 \quad \text{for all } \boldsymbol{z} \in \boldsymbol{K_H}. \tag{3.21}$$

Thus, (3.11) holds. On the other hand, let us assume (3.11). Then, we can take $\boldsymbol{z} := r\boldsymbol{z}_c$, $r \in [k_*, k^*]$, as test function in (3.21), so that we obtain (3.20) by recalling that $k_\varepsilon(t) = (\boldsymbol{w}, \boldsymbol{u}_\varepsilon(t))_{\boldsymbol{H}}$ from (3.19).

3.2 A priori estimates

In this subsection, we obtain the uniform estimates independent of $\varepsilon > 0$.

Lemma 3.2. *There exist a positive constant M_1, independent of $\varepsilon \in (0,1]$, such that*

$$|u_\varepsilon|_{H^1(0,T;H)} + |u_\varepsilon|_{L^\infty(0,T;V)} + \sup_{t\in(0,T)} \int_\Omega \widehat{\beta}_\varepsilon\bigl(u_\varepsilon(t)\bigr) dx$$
$$+ |u_{\Gamma,\varepsilon}|_{H^1(0,T;H_\Gamma)} + |u_{\Gamma,\varepsilon}|_{L^\infty(0,T;V_\Gamma)} + \sup_{t\in(0,T)} \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(t)\bigr) d\Gamma \le M_1. \tag{3.22}$$

Proof. We can add u_ε to both sides of (3.15) and $u_{\Gamma,\varepsilon}$ to both sides of (3.16), then test (3.15) by $(\partial u_\varepsilon/\partial t) \in L^2(0,T;H)$ and use boundary conditions (3.16)–(3.17). Then, we deduce that

$$\begin{aligned}
&\int_0^t \bigl|u_\varepsilon'(s)\bigr|_H^2 ds + \frac{1}{2}\bigl|u_\varepsilon(t)\bigr|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon\bigl(u_\varepsilon(t)\bigr)dx + \frac{\varepsilon}{2}\bigl|u_\varepsilon(t)\bigr|_H^2 \\
&\quad + \int_0^t \bigl|u_{\Gamma,\varepsilon}'(s)\bigr|_{H_\Gamma}^2 ds + \frac{1}{2}\bigl|u_{\Gamma,\varepsilon}(t)\bigr|_{V_\Gamma}^2 + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(t)\bigr)d\Gamma + \frac{\varepsilon}{2}\bigl|u_{\Gamma,\varepsilon}(t)\bigr|_{H_\Gamma}^2 \\
&\quad + \int_0^t \lambda_\varepsilon(s)\left\{\int_\Omega w u_\varepsilon'(s)dx + \int_\Gamma w_\Gamma u_{\Gamma,\varepsilon}'(s)d\Gamma\right\}ds \\
\le\ & \frac{1}{2}|u_0|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(u_0)dx + \frac{\varepsilon}{2}|u_0|_H^2 + \frac{1}{2}|u_{0\Gamma}|_{V_\Gamma}^2 + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_{0\Gamma})d\Gamma + \frac{\varepsilon}{2}|u_{0\Gamma}|_{H_\Gamma}^2 \\
&\quad + \int_0^t \Bigl(f(s) - \pi\bigl(u_\varepsilon(s)\bigr), u_\varepsilon'(s)\Bigr)_H ds + \int_0^t \Bigl(f_\Gamma(s) - \pi_\Gamma\bigl(u_{\Gamma,\varepsilon}(s)\bigr), u_{\Gamma,\varepsilon}'(s)\Bigr)_{H_\Gamma} ds,
\end{aligned}$$

for all $t \in [0,T]$. We point out that (cf. (2.21))

$$\int_\Omega \widehat{\beta}_\varepsilon(u_0)dx \le \int_\Omega \widehat{\beta}(u_0)dx < +\infty, \quad \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_{0\Gamma})d\Gamma \le \int_\Gamma \widehat{\beta}_\Gamma(u_{0\Gamma})d\Gamma < +\infty.$$

Also note that, from (3.19), (3.20) and the chain rule differentiation lemma (see, e.g., [5, Lemma 4.4, p. 158] or [6, Lemme 3.3, p. 73]), the last term on the left hand side is exactly

$$\int_0^t \lambda_\varepsilon(s)k_\varepsilon'(s)ds = I_{[k_*,k^*]}\bigl(k_\varepsilon(t)\bigr) - I_{[k_*,k^*]}(k_0) \equiv 0 \quad \text{for all } t \in [0,T],$$

where $k_0 := (\boldsymbol{w}, \boldsymbol{u}_0)_{\boldsymbol{H}}$. Moreover, there exists a positive constant $\tilde{M}_1$, depending only on L, L_Γ, $|u_0|_H$, $|u_{0\Gamma}|_{H_\Gamma}$ and T, such that

$$\int_0^t \Bigl(f(s) - \pi\bigl(u_\varepsilon(s)\bigr), u_\varepsilon'(s)\Bigr)_H ds \le \frac{1}{2}\int_0^t \bigl|u_\varepsilon'(s)\bigr|_H^2 ds + \tilde{M}_1 \int_0^t \Bigl(1 + \bigl|f(s)\bigr|_H^2 + \bigl|u_\varepsilon(s)\bigr|_H^2\Bigr)ds,$$

and

$$\begin{aligned}
&\int_0^t \Bigl(f_\Gamma(s) - \pi_\Gamma\bigl(u_{\Gamma,\varepsilon}(s)\bigr), u_{\Gamma,\varepsilon}'(s)\Bigr)_{H_\Gamma} ds \\
&\le \frac{1}{2}\int_0^t \bigl|u_{\Gamma,\varepsilon}'(s)\bigr|_{H_\Gamma}^2 ds + \tilde{M}_1 \int_0^t \Bigl(1 + \bigl|f_\Gamma(s)\bigr|_{H_\Gamma}^2 + \bigl|u_{\Gamma,\varepsilon}(s)\bigr|_{H_\Gamma}^2\Bigr)ds
\end{aligned}$$

for all $t \in [0,T]$. Collecting the estimates and applying the Gronwall inequality, we easily get (3.22). □

Thanks to the growth conditions (2.17)–(2.18) (see also (3.1)–(3.2)), we obtain the following bound.

Lemma 3.3. *There exist a positive constant M_2, independent of $\varepsilon \in (0,1]$, such that*

$$|\lambda_\varepsilon|_{L^2(0,T)} \le M_2.$$

Proof. From the expression of λ_ε, given at (3.13), we have that

$$\lambda_\varepsilon(t) = \frac{1}{\sigma_0}\int_\Omega \Bigl(f(t) - u_\varepsilon'(t) - \pi\bigl(u_\varepsilon(t)\bigr)\Bigr)dx + \frac{1}{\sigma_0}\int_\Gamma \Bigl(f_\Gamma(t) - u_{\Gamma,\varepsilon}'(t) - \pi_\Gamma\bigl(u_{\Gamma,\varepsilon}(t)\bigr)\Bigr)d\Gamma - \bigl(\boldsymbol{u}_\varepsilon^*(t), \boldsymbol{z}_c\bigr)_{\boldsymbol{H}} \quad \text{for a.a. } t \in (0,T).$$

As $\boldsymbol{z}_c = (1/\sigma_0, 1/\sigma_0) \in \boldsymbol{V}$ and $\boldsymbol{u}_\varepsilon^*(t) = \partial\varphi_\varepsilon(\boldsymbol{u}_\varepsilon(t))$ in $\boldsymbol{H}$, using (3.6) we obtain

$$\bigl(\boldsymbol{u}_\varepsilon^*(t), \boldsymbol{z}_c\bigr)_{\boldsymbol{H}} = \frac{1}{\sigma_0}\int_\Omega \Bigl(\beta_\varepsilon\bigl(u_\varepsilon(t)\bigr) + \varepsilon u_\varepsilon(t)\Bigr)dx + \frac{1}{\sigma_0}\int_\Gamma \Bigl(\beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(t)\bigr) + \varepsilon u_{\Gamma,\varepsilon}(t)\Bigr)d\Gamma$$

for a.a. $t \in (0,T)$. Then, we can estimate λ_ε as follows:

$$\begin{aligned} &|\lambda_\varepsilon|_{L^2(0,T)}^2 \\ &\le \tilde{M}_2\Bigl(1 + |f|_{L^2(0,T;H)}^2 + |u_\varepsilon|_{H^1(0,T;H)}^2\Bigr) + \tilde{M}_2\Bigl(1 + |f_\Gamma|_{L^2(0,T;H_\Gamma)}^2 + |u_{\Gamma,\varepsilon}|_{H^1(0,T;H_\Gamma)}^2\Bigr) \\ &\quad + \tilde{M}_2 \sup_{t\in(0,T)}\left(\left|\int_\Omega \beta_\varepsilon\bigl(u_\varepsilon(t)\bigr)dx\right|^2 + \left|\int_\Gamma \beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(t)\bigr)d\Gamma\right|^2\right), \end{aligned}$$

where $\tilde{M}_2$ is a positive constant, depending on σ_0, $|\Omega|$, $|\Gamma|$, L, L_Γ, $|u_0|_H$, $|u_{0\Gamma}|_{H_\Gamma}$ and T. Now, we use the properties (3.1)–(3.2) along with the estimate (3.22) to conclude. □

Lemma 3.4. *There exist two positive constants M_3 and M_4, independent of $\varepsilon \in (0,1]$, such that*

$$\bigl|\beta_\varepsilon(u_\varepsilon)\bigr|_{L^2(0,T;H)} + \bigl|\beta_\varepsilon(u_{\Gamma,\varepsilon})\bigr|_{L^2(0,T;H_\Gamma)} \le M_3,$$

$$|u_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} + |\partial_\nu u_\varepsilon|_{L^2(0,T;H_\Gamma)} \le M_4.$$

Proof. Testing (3.15) by $\beta_\varepsilon(u_\varepsilon) \in L^2(0,T;V)$ and using (3.16)–(3.17), we infer that

$$\begin{aligned} &\int_\Omega \widehat{\beta}_\varepsilon\bigl(u_\varepsilon(t)\bigr)dx + \int_0^t\!\!\int_\Omega \beta_\varepsilon'\bigl(u_\varepsilon(s)\bigr)\bigl|\nabla u_\varepsilon(s)\bigr|^2 dxds + \int_0^t \Bigl|\beta_\varepsilon\bigl(u_\varepsilon(s)\bigr)\Bigr|_H^2 ds \\ &\quad + \varepsilon\int_0^t\!\!\int_\Omega u_\varepsilon(s)\beta_\varepsilon\bigl(u_\varepsilon(s)\bigr)dxds + \int_\Gamma \widehat{\beta}_\varepsilon\bigl(u_{\Gamma,\varepsilon}(t)\bigr)d\Gamma \\ &\quad + \int_0^t\!\!\int_\Gamma \beta_\varepsilon'\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\bigl|\nabla_\Gamma u_{\Gamma,\varepsilon}(s)\bigr|^2 d\Gamma ds + \int_0^t\!\!\int_\Gamma \beta_\varepsilon\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)d\Gamma ds \\ &\quad + \varepsilon\int_0^t\!\!\int_\Gamma u_{\Gamma,\varepsilon}(s)\beta_\varepsilon\bigl(u_{\Gamma,\varepsilon}(s)\bigr)d\Gamma ds \\ &\le \int_\Omega \widehat{\beta}_\varepsilon(u_0)dx + \int_\Gamma \widehat{\beta}_\varepsilon(u_{0\Gamma})d\Gamma + \int_0^t \Bigl(f(s) - \pi\bigl(u_\varepsilon(s)\bigr) - \lambda_\varepsilon w, \beta_\varepsilon\bigl(u_\varepsilon(s)\bigr)\Bigr)_H ds \\ &\quad + \int_0^t \Bigl(f_\Gamma(s) - \pi_\Gamma\bigl(u_{\Gamma,\varepsilon}(s)\bigr) - \lambda_\varepsilon w_\Gamma, \beta_\varepsilon\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\Bigr)_{H_\Gamma} ds \end{aligned}$$

for all $t \in [0,T]$. Now, we use the property (3.3) to deduce that

$$\begin{aligned}
&\int_0^t\!\!\int_\Gamma \beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\beta_{\Gamma,\varepsilon}\big(u_{\Gamma,\varepsilon}(s)\big)d\Gamma ds \\
&\quad = \int_0^t\!\!\int_\Gamma \Big|\beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\Big|\Big|\beta_{\Gamma,\varepsilon}\big(u_{\Gamma,\varepsilon}(s)\big)\Big| d\Gamma ds \\
&\quad \geq \frac{1}{\varrho}\int_0^t\!\!\int_\Gamma \Big|\beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\Big|^2 d\Gamma ds - \frac{c_0}{\varrho}\int_0^t\!\!\int_\Gamma \Big|\beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\Big| d\Gamma ds
\end{aligned}$$

for all $t \in [0,T]$, because $\beta_\varepsilon(r)$ and $\beta_{\Gamma,\varepsilon}(r)$ have the same sign for all $r \in \mathbb{R}$. We also note that

$$\int_0^t\!\!\int_\Omega \beta'_\varepsilon\big(u_\varepsilon(s)\big)\big|\nabla u_\varepsilon(s)\big|^2 dxds \geq 0, \qquad \varepsilon\int_0^t\!\!\int_\Omega u_\varepsilon(s)\beta_\varepsilon\big(u_\varepsilon(s)\big)dxds \geq 0,$$

$$\int_0^t\!\!\int_\Gamma \beta'_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\big|\nabla_\Gamma u_{\Gamma,\varepsilon}(s)\big|^2 d\Gamma ds \geq 0, \qquad \varepsilon\int_0^t\!\!\int_\Gamma u_{\Gamma,\varepsilon}(s)\beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)d\Gamma ds \geq 0$$

for all $t \in [0,T]$ and, in view of (2.5) and (2.21),

$$\begin{aligned}
&\int_\Gamma \widehat{\beta}_\varepsilon(u_{0\Gamma})d\Gamma = \int_\Gamma\int_0^{u_{0\Gamma}} \beta_\varepsilon(r)drd\Gamma \leq \int_\Gamma\int_0^{u_{0\Gamma}} \varrho\beta_{\Gamma,\varepsilon}(r)drd\Gamma + \int_\Gamma c_0|u_{0\Gamma}|d\Gamma \\
&\leq \varrho\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_{0\Gamma})d\Gamma + c_0|u_{0\Gamma}|_{L^1(\Gamma)} \leq \varrho\int_\Gamma \widehat{\beta}_\Gamma(u_{0\Gamma})d\Gamma + c_0|u_{0\Gamma}|_{L^1(\Gamma)} < +\infty.
\end{aligned}$$

Moreover, there exists a positive constant $\tilde{M}_3$, depending on ϱ, L, L_Γ, $|u_0|_H$, $|u_{0\Gamma}|_{H_\Gamma}$, T and independent of $\varepsilon \in (0,1]$, such that

$$\begin{aligned}
&\int_0^t \Big(f(s) - \pi\big(u_\varepsilon(s)\big) - \lambda_\varepsilon(s)w, \beta_\varepsilon\big(u_\varepsilon(s)\big)\Big)_H ds \\
&\quad \leq \frac{1}{2}\int_0^t \Big|\beta_\varepsilon\big(u_\varepsilon(s)\big)\Big|_H^2 ds + \tilde{M}_3\left(1 + |f|^2_{L^2(0,T;H)} + |u_\varepsilon|^2_{L^2(0,T;H)} + |\lambda_\varepsilon|^2_{L^2(0,T)}|w|^2_H\right), \\
&\int_0^t \Big(f_\Gamma(s) - \pi_\Gamma\big(u_{\Gamma,\varepsilon}(s)\big) - \lambda_\varepsilon(s)w_\Gamma, \beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\Big)_{H_\Gamma} ds \\
&\quad \leq \frac{1}{2\varrho}\int_0^t \Big|\beta_\varepsilon\big(u_{\Gamma,\varepsilon}(s)\big)\Big|_{H_\Gamma}^2 ds \\
&\qquad + \tilde{M}_3\left(1 + |f_\Gamma|^2_{L^2(0,T;H_\Gamma)} + |u_{\Gamma,\varepsilon}|^2_{L^2(0,T;H)} + |\lambda_\varepsilon|^2_{L^2(0,T)}|w_\Gamma|^2_{H_\Gamma}\right),
\end{aligned}$$

for all $t \in [0,T]$. Thus, we deduce that there is a positive constant M_3, which depends only on $|f|_{L^2(0,T;H)}$, $|f_\Gamma|_{L^2(0,T;H_\Gamma)}$, $|u_0|_H$, $|u_{0\Gamma}|_{H_\Gamma}$, M_1 and M_2, such that

$$\big|\beta_\varepsilon(u_\varepsilon)\big|_{L^2(0,T;H)} + \big|\beta_\varepsilon(u_{\Gamma,\varepsilon})\big|_{L^2(0,T;H_\Gamma)} \leq M_3.$$

Now, we can compare the terms in (3.15) and conclude that

$$|\Delta u_\varepsilon|_{L^2(0,T;H)} \leq M_4,$$

whence, recalling (3.22) and applying the theory of the elliptic regularity (see, e.g., [7, Thm. 3.2, p. 1.79]), we have that

$$|u_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} \le M_4$$

and, owing to the trace theory (see, e.g., [7, Thm. 2.25, p. 1.62]), that

$$|\partial_\nu u_\varepsilon|_{L^2(0,T;H_\Gamma)} \le M_4$$

for some constant M_4 independent of $\varepsilon \in (0,1]$. □

Lemma 3.5. *There exist three positive constants M_5, M_6 and M_7, independent of $\varepsilon \in (0,1]$, such that*

$$\bigl|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})\bigr|_{L^2(0,T;H_\Gamma)} \le M_5,$$
$$|u_{\Gamma,\varepsilon}|_{L^2(0,T;H^2(\Gamma))} \le M_6,$$
$$|u_\varepsilon|_{L^2(0,T;H^2(\Omega))} \le M_7.$$

Proof. We test (3.16) by $\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) \in L^2(0,T;H_\Gamma)$ and integrate on the boundary, deducing that

$$\begin{aligned} &\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(t)\bigr)d\Gamma + \int_0^t\!\!\int_\Gamma \beta'_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\bigl|\nabla_\Gamma u_\varepsilon(s)\bigr|^2 d\Gamma ds \\ &\quad + \int_0^t \Bigl|\beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\Bigr|_{H_\Gamma}^2 ds + \varepsilon \int_0^t\!\!\int_\Gamma u_{\Gamma,\varepsilon}(s)\beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)d\Gamma ds \\ &\le \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(u_{0\Gamma}\bigr)d\Gamma \\ &\quad + \int_0^t \Bigl(f_\Gamma(s) - \partial_\nu u_{\Gamma,\varepsilon}(s) - \pi_\Gamma\bigl(u_{\Gamma,\varepsilon}(s)\bigr) - \lambda_\varepsilon w_\Gamma, \beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\Bigr)_{H_\Gamma} ds \end{aligned} \tag{3.23}$$

for all $t \in [0,T]$. We note that

$$\int_0^t\!\!\int_\Gamma \beta'_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)\bigl|\nabla_\Gamma u_\varepsilon(s)\bigr|^2 d\Gamma ds \ge 0, \quad \varepsilon \int_0^t\!\!\int_\Gamma u_{\Gamma,\varepsilon}(s)\beta_{\Gamma,\varepsilon}\bigl(u_{\Gamma,\varepsilon}(s)\bigr)d\Gamma ds \ge 0$$

due to the properties of $\beta_{\Gamma,\varepsilon}$. Then, recalling that

$$\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}\bigl(u_{0\Gamma}\bigr)d\Gamma \le \int_\Gamma \widehat{\beta}_\Gamma\bigl(u_{0\Gamma}\bigr)d\Gamma < +\infty$$

by virtue of (2.21), and applying the Young inequality in the last term of (3.23), we see that there exists a positive constant $\tilde{M}_5$, depending only on M_1, M_2, M_3, M_4, L_Γ, $|u_{0\Gamma}|_{H_\Gamma}$ and T, such that

$$\bigl|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})\bigr|_{L^2(0,T;H_\Gamma)} \le \tilde{M}_5.$$

Hence, by comparison in (3.16) we also infer that

$$|\Delta_\Gamma u_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma)} \le \tilde{M}_5$$

and consequently (see, e.g., [19, Section 4.2]])

$$|u_{\Gamma,\varepsilon}|_{L^2(0,T;H^2(\Gamma))} \leq \Big(|u_{\Gamma,\varepsilon}|^2_{L^2(0,T;V_\Gamma)} + |\Delta_\Gamma u_{\Gamma,\varepsilon}|^2_{L^2(0,T;H_\Gamma)} \Big)^{1/2} \leq \big(M_1^2 T + \tilde{M}_5^2 \big)^{1/2} =: M_6.$$

In view of Lemma 3.4, using the theory of the elliptic regularity (see, e.g., [7, Thm. 3.2, p. 1.79]) along with the estimate $|u_{\Gamma,\varepsilon}|_{L^2(0,T;H^{3/2}(\Gamma))} \leq M_6$, it turns out that

$$|u_\varepsilon|_{L^2(0,T;H^2(\Omega))} \leq M_7$$

for some constant M_7 independent of $\varepsilon \in (0,1]$. □

3.3 Passage to the limit

In this subsection, we conclude the existence proof by passing to the limit on the sequence of approximate solutions. Indeed, owing to the estimates stated in the Lemmas from 3.2 to 3.5, there exist a subsequence of ε (not relabeled) and some limit functions u, u_Γ, ξ, ξ_Γ, λ such that

$$u_\varepsilon \to u \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2\big(0,T;H^2(\Omega)\big), \tag{3.24}$$

$$u_{\Gamma,\varepsilon} \to u_\Gamma \quad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2\big(0,T;H^2(\Gamma)\big), \tag{3.25}$$

$$\beta_\varepsilon(u_\varepsilon) \to \xi \quad \text{weakly in } L^2(0,T;H), \tag{3.26}$$

$$\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) \to \xi_\Gamma \quad \text{weakly in } L^2(0,T;H_\Gamma), \tag{3.27}$$

$$\lambda_\varepsilon \to \lambda \quad \text{weakly in } L^2(0,T) \tag{3.28}$$

as $\varepsilon \to 0$. From (3.24) and (3.25), due to strong compactness results (see, e.g., [25, Sect. 8, Cor. 4]) we infer that

$$u_\varepsilon \to u \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V), \tag{3.29}$$

$$u_{\Gamma,\varepsilon} \to u_\Gamma \quad \text{strongly in } C([0,T];H_\Gamma) \cap L^2(0,T;V_\Gamma). \tag{3.30}$$

Moreover, on account of (3.19) it is a standard matter to deduce that

$$k_\varepsilon \to k \quad \text{weakly in } H^1(0,T) \text{ and strongly in } C([0,T]), \tag{3.31}$$

where

$$k_* \leq k(t) := \int_\Omega w u(t) dx + \int_\Gamma w_\Gamma u_\Gamma(t) d\Gamma \leq k^* \quad \text{for all } t \in [0,T].$$

We point out that (3.17), (3.24) and (3.25) imply that $u_\Gamma = u_{|_\Gamma}$ a.e. on Σ, while (3.18), (3.29) and (3.30) entail

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

Now, (3.28), (3.31) and the maximal monotonicity of $\partial I_{[k_*,k^*]}$ allow us to conclude that

$$\lambda \in \partial I_{[k_*,k^*]}(k) \quad \text{a.e. in } (0,T),$$

while (3.29), (3.30) and the Lipschitz continuity of π and π_Γ imply that

$$\pi(u_\varepsilon) \to \pi(u) \quad \text{strongly in } C([0,T];H),$$
$$\pi_\Gamma(u_{\Gamma,\varepsilon}) \to \pi_\Gamma(u_\Gamma) \quad \text{strongly in } C([0,T];H_\Gamma)$$

as $\varepsilon \to 0$. At this point, we can pass to the limit in (3.15) and (3.16) obtaining

$$\frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) + \lambda w = f \quad \text{a.e. in } Q,$$
$$\partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) + \lambda w_\Gamma = f_\Gamma \quad \text{a.e. on } \Sigma.$$

Let us comment that $\partial_\nu u_\varepsilon \to \partial_\nu u$ weakly in $L^2(0,T;H^{1/2}(\Gamma))$ as $\varepsilon \to 0$, due to (3.24) and the linearity and continuity of the trace operator $u \mapsto \partial_\nu u$. Moreover, by applying [5, p. 42, Proposition 1.1] and using (3.26)–(3.27) with (3.29)–(3.30), we obtain

$$\xi \in \beta(u) \quad \text{a.e. in } Q, \qquad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma.$$

Thus, it turns out that the pair $\boldsymbol{u} = (u, u_\Gamma)$ is a solution of the limit problem, which can be stated exactly as in (2.9)–(2.15). Also, we note the regularities $u \in C([0,T];V)$ and $u_\Gamma \in C([0,T];V_\Gamma)$ for the solution as a consequence of (3.24) and (3.25). Morever, $\boldsymbol{u} = (u, u_\Gamma)$ solves the abstract problem:

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{H}) \cap C([0,T];\boldsymbol{V}), \tag{3.32}$$
$$\boldsymbol{u}^* = (-\Delta u + \xi, \partial_\nu u - \Delta_\Gamma u_\Gamma + \xi_\Gamma) \in L^2(0,T;\boldsymbol{H}), \tag{3.33}$$
$$\lambda \in L^2(0,T), \tag{3.34}$$
$$\boldsymbol{u}'(t) + \boldsymbol{u}^*(t) + \lambda(t)\boldsymbol{w} + \boldsymbol{\pi}\big(\boldsymbol{u}(t)\big) = \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T), \tag{3.35}$$
$$\boldsymbol{u}^*(t) \in \partial\varphi\big(\boldsymbol{u}(t)\big) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T), \tag{3.36}$$
$$\lambda(t)\boldsymbol{w} \in \partial I_{\boldsymbol{K}_{\boldsymbol{H}}}\big(\boldsymbol{u}(t)\big) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T), \tag{3.37}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \boldsymbol{H}. \tag{3.38}$$

Remark 3.3. Let us point out that

$$\boldsymbol{u}^*(t) + \lambda(t)\boldsymbol{w} \in \partial(\varphi + I_{\boldsymbol{K}})\big(\boldsymbol{u}(t)\big) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T).$$

Therefore, (3.32)–(3.38) imply that $\boldsymbol{u}$ is the solution of the Cauchy problem expressed by the abstract equation

$$\boldsymbol{u}'(t) + \partial(\varphi + I_{\boldsymbol{K}})\big(\boldsymbol{u}(t)\big) + \boldsymbol{\pi}\big(\boldsymbol{u}(t)\big) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}, \text{ for a.a. } t \in (0,T)$$

along with the initial condition (3.38). Then, we emphasize that although the solution $\boldsymbol{u}$ of this problem is uniquely determined, the auxiliary quantities $\boldsymbol{u}^*$ and λ are not unique in general, except in special cases like, e.g., the case in which β and β_Γ are single-valued (cf. [12, Remark 2]).

4 Continuous dependence

In this section, we prove Theorem 2.1.

Proof of Theorem 2.1. Let $(\boldsymbol{u}^{(1)}, \boldsymbol{\xi}^{(1)}, \lambda^{(1)})$ and $(\boldsymbol{u}^{(2)}, \boldsymbol{\xi}^{(2)}, \lambda^{(2)})$ be two different solutions of (P), corresponding to the data $(f^{(1)}, f_\Gamma^{(1)}, u_0^{(1)}, u_{0\Gamma}^{(1)})$ and $(f^{(2)}, f_\Gamma^{(2)}, u_0^{(2)}, u_{0\Gamma}^{(2)})$, respectively. We take the difference between (3.35) written for $\boldsymbol{u}^{(1)}(s) = (u^{(1)}(s), u_\Gamma^{(1)}(s))$ and (3.35) written for $\boldsymbol{u}^{(2)}(s) = (u^{(2)}(s), u_\Gamma^{(2)}(s))$ at the time $t = s$ (note that the abstract equation (3.35) comprehends both (2.9) and (2.11)). Then, we take the inner product with $\boldsymbol{u}^{(1)}(s) - \boldsymbol{u}^{(2)}(s)$ in $\boldsymbol{H}$. Using the monotonicity of β, β_Γ and $\partial I_{\boldsymbol{K_H}}$, we obtain

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{ds}\bigl|u^{(1)}(s) - u^{(2)}(s)\bigr|_H^2 + \frac{1}{2}\frac{d}{ds}\bigl|u_\Gamma^{(1)}(s) - u_\Gamma^{(2)}(s)\bigr|_H^2 \\
& \quad + \bigl|\nabla u^{(1)}(s) - \nabla u^{(2)}(s)\bigr|_{H^d}^2 + \bigl|\nabla_\Gamma u_\Gamma^{(1)}(s) - \nabla_\Gamma u_\Gamma^{(2)}(s)\bigr|_{H_\Gamma^d}^2 \\
\leq \; & \bigl(f^{(1)}(s) - f^{(2)}(s), u^{(1)}(s) - u^{(2)}(s)\bigr)_H + \bigl(f_\Gamma^{(1)}(s) - f_\Gamma^{(2)}(s), u_\Gamma^{(1)}(s) - u_\Gamma^{(2)}(s)\bigr)_{H_\Gamma} \\
& \quad - \Bigl(\pi\bigl(u^{(1)}(s)\bigr) - \pi\bigl(u^{(2)}(s)\bigr), u^{(1)}(s) - u^{(2)}(s)\Bigr)_H \\
& \quad - \Bigl(\pi_\Gamma\bigl(u_\Gamma^{(1)}(s)\bigr) - \pi_\Gamma\bigl(u_\Gamma^{(2)}(s)\bigr), u_\Gamma^{(1)}(s) - u_\Gamma^{(2)}(s)\Bigr)_H \\
\leq \; & \bigl|u^{(1)}(s) - u^{(2)}(s)\bigr|_H^2 + \bigl|u_\Gamma^{(1)}(s) - u_\Gamma^{(2)}(s)\bigr|_{H_\Gamma}^2 + \frac{1}{2}\bigl|f^{(1)}(s) - f^{(2)}(s)\bigr|_H^2 \\
& \quad + \frac{1}{2}\bigl|f_\Gamma^{(1)}(s) - f_\Gamma^{(2)}(s)\bigr|_{H_\Gamma}^2 + \frac{L^2}{2}\bigl|u^{(1)}(s) - u^{(2)}(s)\bigr|_H^2 + \frac{L_\Gamma^2}{2}\bigl|u_\Gamma^{(1)}(s) - u_\Gamma^{(2)}(s)\bigr|_{H_\Gamma}^2,
\end{aligned}
$$

for all $t \in [0, T]$. Then, by integrating with respect to s and applying the Gronwall lemma, it is straightforward to find a constant $C > 0$, depending only on L, L_Γ and T, such that the continuous dependence estimate (2.16) holds. □

5 Appendix

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. We use the same notation as in Section 2 for $\boldsymbol{H}$ and $\boldsymbol{V}$.

Proposition 5.1. *$\boldsymbol{V}$ is dense in $\boldsymbol{H}$.*

Proof. For a fixed $\boldsymbol{u} = (u, u_\Gamma) \in \boldsymbol{H}$ and for $n \in \mathbb{N}$, consider the following elliptic problem:

$$v_n - \frac{1}{n}\Delta v_n = u \quad \text{a.e. in } \Omega, \tag{5.1}$$

$$\frac{1}{n}\partial_\nu v_n + (v_n)_{|\Gamma} = u_\Gamma \quad \text{a.e. on } \Gamma. \tag{5.2}$$

Then, let us write a variational formulation of (5.1)–(5.2)

$$
\begin{aligned}
& \int_\Omega v_n \eta \, dx + \frac{1}{n}\int_\Omega \nabla v_n \cdot \nabla \eta \, dx + \int_\Gamma (v_n)_{|\Gamma} \eta_{|\Gamma} d\Gamma \\
& \quad = \int_\Omega u \, \eta \, dx + \int_\Gamma u_\Gamma \, \eta_{|\Gamma} d\Gamma \quad \text{ for all } \eta \in V.
\end{aligned} \tag{5.3}
$$

By applying the Lax-Milgram lemma, it is not difficult to see that for any $n \in \mathbb{N}$ there is a unique $v_n \in V$ that solves (5.3), i.e., satisfies the above problem (5.1)–(5.2) with $\Delta v_n \in L^2(\Omega)$ and $\partial_\nu v_n \in L^2(\Gamma)$. From the elliptic regularity for a Neumann boundary condition (see, e.g., [7, Thm. 3.2, p. 1.79]), we infer that $v_n \in H^{3/2}(\Omega)$, and this implies $(v_n)_\Gamma := (v_n)_{|\Gamma} \in H^1(\Gamma)$. Then, we have $\boldsymbol{v}_n = (v_n, (v_n)_\Gamma) \in \boldsymbol{V}$ for all $n \in \mathbb{N}$.

Now, we take $\eta = v_n$ in (5.3) and apply the elementary Young inequality to deduce that

$$\frac{1}{2}\int_\Omega |v_n|^2 dx + \frac{1}{n}\int_\Omega |\nabla v_n|^2 dx + \frac{1}{2}\int_\Gamma \bigl|(v_n)_\Gamma\bigr|^2 d\Gamma \\ \le \frac{1}{2}\int_\Omega |u|^2 dx + \frac{1}{2}\int_\Gamma |u_\Gamma|^2 d\Gamma =: M. \tag{5.4}$$

Hence, it turns out that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and $\{(1/\sqrt{n})\nabla v_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)^d$, with

$$\left|\frac{1}{n}\nabla v_n\right|_{L^2(\Omega)^d} \le \sqrt{\frac{M}{n}} \quad \text{for all } n \in \mathbb{N}.$$

Then, there exist a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ of $\{v_n\}_{n\in\mathbb{N}}$ and $v \in L^2(\Omega)$ such that

$$v_{n_k} \to v \quad \text{weakly in } L^2(\Omega), \qquad \frac{1}{n_k}\nabla v_{n_k} \to 0 \quad \text{strongly in } L^2(\Omega)^N \quad \text{as } k \to +\infty. \tag{5.5}$$

Next, choosing $\eta \in H^1_0(\Omega)$ in (5.3), we obtain

$$\int_\Omega (v_{n_k} - u)\eta \, dx = -\int_\Omega \frac{1}{n_k}\nabla v_{n_k} \cdot \nabla \eta \, dx \to 0 \quad \text{as } k \to +\infty,$$

namely, $v_{n_k} \to u$ in $H^{-1}(\Omega)$ as $k \to +\infty$; this means that the weak limit v in (5.5) should coincide with u and the entire sequence

$$v_n \text{ converges to } u \text{ weakly in } L^2(\Omega) \text{ as } n \to +\infty. \tag{5.6}$$

Now, from (5.4) it follows that $\{(v_n)_\Gamma\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Gamma)$; on the other hand, passing to the limit in (5.3) we realize that

$$\lim_{n\to+\infty}\int_\Gamma \bigl((v_n)_\Gamma - u_\Gamma\bigr)\eta_{|\Gamma} d\Gamma = 0 \quad \text{for all } \eta \in H^1(\Omega),$$

whence

$$(v_n)_\Gamma \to u_\Gamma \quad \text{weakly in } L^2(\Gamma) \text{ as } n \to +\infty. \tag{5.7}$$

Moreover, (5.4) implies

$$\limsup_{n\to+\infty}\left\{\int_\Omega |v_n|^2 dx + \int_\Gamma \bigl|(v_n)_\Gamma\bigr|^2 d\Gamma\right\} \le \int_\Omega |u|^2 dx + \int_\Gamma |u_\Gamma|^2 d\Gamma, \tag{5.8}$$

that entails the convergence of the norms $|v_n|_H$ and $|(v_n)_\Gamma|_{H_\Gamma}$ to $|u|_H$ and $|u_\Gamma|_{H_\Gamma}$, respectively. Thus, (5.6), (5.7) and (5.8) enable us to conclude that

$$v_n \to u \quad \text{strongly in } H, \quad (v_n)_\Gamma \to u_\Gamma \quad \text{strongly in } H_\Gamma,$$

that is,

$$\boldsymbol{v}_n = \bigl(v_n, (v_n)_\Gamma\bigr) \to \boldsymbol{u} = (u, u_\Gamma) \quad \text{strongly in } \boldsymbol{H} \text{ as } n \to +\infty,$$

which completes the proof. □

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