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VALUE SHARING BY AN ENTIRE FUNCTION WITH ITS DERIVATIVES

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ABSTRACT. We prove a uniqueness theorem for an entire function, which shares certain values with its higher order derivatives.

1. INTRODUCTION, DEFINITIONS AND RESULTS

Let f be a non-constant meromorphic function in the open complex plane \mathbb{C} . We denote by $n(r, \infty; f)$ the number of poles of f lying in |z| < r, the poles are counted according to their multiplicities. The quantity

$$N(r,\infty;f) = \int_{0}^{r} \frac{n(t,\infty;f) - n(0,\infty;f)}{t} dt + n(0,\infty;f) \log r$$

is called the integrated counting function or simply the counting function of poles of f.

Also $m(r,\infty;f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} |f(re^{i\theta})| d\theta$ is called the proximity function of poles of f,

where $\log^+ x = \log x$ if $x \ge 1$ and $\log^+ x = 0$ if $0 \le x < 1$.

The sum $T(r, f) = m(r, \infty; f) + N(r, \infty; f)$ is called the Nevanlinna characteristic function of f. We denote by S(r, f) any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ except possibly a set of finite linear measure.

For
$$a \in \mathbb{C}$$
, we put $N(r, a; f) = N\left(r, \infty; \frac{1}{f-a}\right)$ and $m(r, a; f) = m\left(r, \infty; \frac{1}{f-a}\right)$.

Let us denote by $\overline{n}(r, a; f)$ the number of distinct *a*-points of f lying in |z| < r, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r,a;f) = \int_{0}^{r} \frac{\overline{n}(t,a;f) - \overline{n}(0,a;f)}{t} dt + \overline{n}(0,a;f) \log r$$

denotes the reduced counting function of a-points of f.

Also by $\overline{N}_{(2)}(r,a;f)$ we denote the reduced counting function of multiple *a*-points of *f*.

Let $A \subset \mathbb{C}$ and $n_A(r, a; f)$ be the number of *a*-points of *f* lying in $A \cap \{z : |z| < r\}$, where $a \in \mathbb{C} \cup \{\infty\}$ and the *a*-points are counted according to their multiplicities. We put

$$\overline{N}_A(r,a;f) = \int_0^r \frac{\overline{n}_A(t,a;f) - \overline{n}_A(0,a;f)}{t} dt + \overline{n}_A(0,a;f) \log r$$

For $a \in \mathbb{C} \cup \{\infty\}$ we denote by E(a; f) the set of *a*-points of *f* (counted with multiplicities) and by $\overline{E}(a; f)$ the set of distinct *a*-points of *f*.

For standard definitions and results of the value distribution theory the reader may consult [2] and [8].

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In 1977 L. A. Rubel and C. C. Yang [7] first investigated the uniqueness of entire functions sharing certain values with their derivatives. They proved the following result.

Theorem A. [7] Let f be a non-constant entire function. If $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$ for two distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.

In 1979, E. Mues and N. Steinmetz [6] improved Theorem A in the following manner.

Theorem B. [6] Let a, b be two distinct finite complex numbers and f be a non-constant entire function. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(b; f) = \overline{E}(b; f^{(1)})$, then $f \equiv f^{(1)}$.

In 1986, G. Jank, E. Mues and L. Volkmann [3] dealt with the case of a single shared value by the two derivatives of an entire function. Their result may be stated as follows.

Theorem C. [3] Let f be a non-constant entire function and $a \neq 0$ be a finite complex number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$ then $f \equiv f^{(1)}$.

In 2002 J. Chang and M. Fang [1] extended Theorem C in the following way.

Theorem D. [1] Let f be a non-constant entire function and a, b be two non-zero finite constants. If $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \subset \overline{E}(b; f^{(2)})$, then either $f = \lambda e^{\frac{bz}{a}} + \frac{ab - a^2}{b}$ or $f = \lambda e^{\frac{bz}{a}} + a$, where $\lambda \neq 0$ is a constant.

In Theorem C it is not possible to replace the second derivative by any higher order derivative. For, let $f(z) = e^{\omega z} + \omega - 1$, where $\omega^{n-1} = 1$, $\omega \neq 1$ and $n(\geq 3)$ is an integer. Then $\overline{E}(\omega; f) = \overline{E}(\omega; f^{(1)}) = \overline{E}(\omega; f^{(n)})$ but $f \neq f^{(1)}$.

Considering higher order derivatives, H. Zhong [10] proved the following result.

Theorem E. [10] Let f be a non-constant entire function and $a \neq 0, \infty$) be a complex number. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n \geq 1$, then $f \equiv f^{(n)}$.

P. Li and C. C. Yang [5] also considered the higher order derivatives and proved the following theorem.

Theorem F. [5] Let f be a non-constant entire function, a be a finite nonzero complex number and n be a positive integer. If $E(a; f) = E(a; f^{(n)}) = E(a; f^{(n+1)})$, then $f \equiv f^{(1)}$.

To state the next result we require the following definition. Let f and g be two nonconstant meromorphic functions defined in \mathbb{C} . For $a \in \mathbb{C} \cup \{\infty\}$ we put $B = \overline{E}(a; f) \Delta \overline{E}(a; g)$, where Δ denotes the symmetric difference of sets. The functions f and g are said to share the value a IMN if $N_B(r, a; f) = S(r, f)$ and $N_B(r, a; g) = S(r, g)$ {see [10]}.

In 1997 L. Z. Yang [9] improved a result of H. Zhong [10] and proved the following theorem.

Theorem G. [9] Let f be a non-constant entire function and $a \neq 0, \infty$ be a complex number. If f and $f^{(n)}$ $(n \geq 1)$ share the value a IMN and $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(n+1)})$, then $f = \lambda e^z$, where $\lambda \neq 0$ is a constant.

Recently Theorem G is improved in the following manner.

Theorem H. [4] Let f be a non-constant entire function and $a \neq 0, \infty$ be a complex value. Suppose that $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(n)})$ and $B = \overline{E}(a; f^{(n)}) \setminus \{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(n+1)})\}$. If $N_A(r, a; f) + N_B(r, a; f^{(n)}) = S(r, f)$, then either $f = \lambda e^z$ or $f = \lambda e^z + a$, where $\lambda \neq 0$ is a constant.

It seems to be an interesting problem to investigate the situation when an entire function f shares a nonzero finite value with three consecutive derivatives $f^{(n)}$, $f^{(n+1)}$ and $f^{(n+2)}$, where $n \ge 1$. In the paper we prove the following result in this direction.

Theorem 1.1. Let f be a non-constant entire function, $n(\geq 1)$ be an integer and a, b be two nonzero finite complex numbers. Further suppose that $A = \overline{E}(a; f) \setminus \overline{E}(b; f^{(n)})$ and $B = \overline{E}(b; f^{(n)}) \setminus \{\overline{E}(a; f^{(n+1)}) \cap \overline{E}(a; f^{(n+2)})\}$. If $N_A(r, a; f) + N_B(r, b; f^{(n)}) + \overline{N}_{(2}(r, a; f) = S(r, f)$, then a = b and either $f = \alpha e^z$ or $f = a + \alpha e^z$, where $\alpha \neq 0$ is a constant.

Putting $A = B = \emptyset$ we get the following corollary.

Corollary 1.1. Let f be a non-constant entire function, $n(\geq 1)$ be an integer and a, b be two nonzero finite complex numbers. If $\overline{E}(a; f) \subset \overline{E}(b; f^{(n)}) \subset \overline{E}(a; f^{(n+1)}) \cap \overline{E}(a; f^{(n+2)})$ and $\overline{N}_{(2}(r, a; f) = S(r, f)$, then a = b and either $f = \alpha e^z$ or $f = a + \alpha e^z$, where $\alpha \neq 0$ is a constant.

2. Lemmas

In this section we state necessary lemmas.

Lemma 2.1. $\{p.39 \ [8]\}$ Let f be a non-constant meromorphic function in \mathbb{C} and n be a positive integer. Then

$$N(r,0;f^{(n)}) \le N(r,0;f) + n\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.2. $\{p.57 [2]\}$ Let f be a non-constant meromorphic function in \mathbb{C} and a, b be finite nonzero complex numbers and n be a positive integer. Then

$$T(r,f) \leq \overline{N}(r,\infty;f) + N(r,a;f) + \overline{N}(r,b;f^{(n)}) + S(r,f).$$

Lemma 2.3. $\{p.47 [2]\}$ Let f be a non-constant meromorphic function in \mathbb{C} and a_1, a_2, a_3 be distinct meromorphic functions satisfying $T(r, a_{\nu}) = S(r, f)$ for $\nu = 1, 2, 3$. Then

$$T(r,f) \leq \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + \overline{N}(r,a_3;f) + S(r,f),$$

where $\overline{N}(r, a_{\nu}; f) = \overline{N}(r, 0; f - a_{\nu})$ for $\nu = 1, 2, 3$.

3. Proof of Theorem 1.1

Proof. We denote by $N_{(2}(r, a; f \mid f^{(n)} = b)$ the counting function (counted with multiplicities) of those multiple *a*-points of *f* which are *b* -points of $f^{(n)}$. We first note that

$$N_{(2}(r, a; f) \leq N_A(r, a; f) + N_{(2}(r, a; f \mid f^{(n)} = b) \\ \leq n \overline{N}_{(2}(r, a; f) + S(r, f) \\ = S(r, f).$$

Let $z_1 \notin A \cup B$ be a simple *a*-point of f. Then in some neighbourhood of z_1 we get by Taylor's expansion

$$f(z) = a + f^{(1)}(z_1)(z - z_1) + \dots + \frac{b}{n!}(z - z_1)^n + \frac{a}{(n+1)!}(z - z_1)^{n+1} + \frac{a}{(n+2)!}(z - z_1)^{n+2} + \frac{f^{(n+3)}(z_1)}{(n+3)!}(z - z_1)^{n+3} + O(z - z_1)^{n+4},$$

and so

$$f^{(n)}(z) = b + a(z - z_1) + \frac{a}{2!}(z - z_1)^2 + \frac{f^{(n+3)}(z_1)}{3!}(z - z_1)^3 + O(z - z_1)^4,$$

$$f^{(n+1)}(z) = a + a(z - z_1) + \frac{f^{(n+3)}(z_1)}{2!}(z - z_1)^2 + O(z - z_1)^3$$

and

$$f^{(n+2)}(z) = a + f^{(n+3)}(z_1)(z-z_1) + O(z-z_1)^2.$$

We note that $f^{(1)}(z_1) \neq 0$. We put $\phi = \frac{f^{(n+1)} - f^{(n+2)}}{f - a}$, $\psi = \frac{f^{(n+1)} - f^{(n+2)}}{f^{(n)} - b}$ and $H = \frac{bf^{(n+1)} - af^{(n)}}{f - a}$. Then by the hypothesis we see that $T(r, \phi) + T(r, \psi) + T(r, H) = S(r, f)$. Now from above we get

$$\phi(z) = \frac{a - f^{(n+3)}(z_1)}{f^{(1)}(z_1)} + O(z - z_1), \tag{3.1}$$

$$\psi(z) = 1 - \frac{f^{(n+3)}(z_1)}{a} + O(z - z_1), \qquad (3.2)$$

and

$$H(z) = \frac{ab - a^2}{f^{(1)}(z_1)} + O(z - z_1).$$
(3.3)

We now consider the following cases.

Case 1. Let $f^{(n+1)} \equiv f^{(n+2)}$. Then on integration we get $f^{(n+1)}(z) = \alpha e^z$, where $\alpha \neq 0$ is a constant. By successive integration we obtain

$$f(z) = \alpha e^{z} + P(z) = f^{(n+1)}(z) + P(z), \qquad (3.4)$$

where P is a polynomial of degree $p(\leq n)$.

First we suppose that P is non-constant. Then by Lemma 2.3 we get

$$T(r,f) = \overline{N}(r,a;f) + S(r,f).$$
(3.5)

Now from (3.4) we see that every *a*-point of f, which does not belong to $A \cup B$, is a zero of P. This shows that

$$\overline{N}(r,a;f) \leq N(r,0;P) + N_A(r,a;f) + \overline{N}_B(r,a;f)$$

$$\leq N_B(r,b;f^{(n)}) + S(r,f)$$

$$= S(r,f),$$

which contradicts (3.5). Therefore $P(z) \equiv \beta$, a constant. Then from (3.4) we get

$$f(z) = \alpha e^z + \beta \tag{3.6}$$

and so

$$f^{(n)}(z) \equiv f^{(n+1)}(z) \equiv f^{(n+2)}(z) = \alpha e^z.$$
 (3.7)

We see that $\overline{N}(r,b;f^{(n)}) \neq S(r,f)$ and $\overline{N}(r,a;f^{(n+1)}) \neq S(r,f)$. So by the hypothesis $\overline{E}(b;f^{(n)}) \cap \overline{E}(a;f^{(n+1)}) \neq \emptyset$. Hence from (3.7) we get a = b.

Let $\beta \neq a$. Since f does not assume the values β and ∞ , we see that $\overline{N}(r, a; f) = T(r, f) + S(r, f)$. Again we have from (3.7) $\overline{N}(r, b; f^{(n)}) \neq S(r, f)$. Since $N_A(r, a; f) + N_B(r, b; f^{(n)}) = S(r, f)$, we get $\overline{E}(a; f) \cap \overline{E}(a; f^{(n+1)}) \neq \emptyset$. So from (3.6) and (3.7) we get $\beta = 0$. Therefore

 $f = \alpha e^z$. The other possibility is $\beta = a$ and so $f = a + \alpha e^z$.

Case 2. Let $f^{(n+1)} \not\equiv f^{(n+2)}$. By the hypothesis we get

$$\overline{N}(r,b;f^{(n)}) \leq N(r,1;\frac{f^{(n+2)}}{f^{(n+1)}}) + N_B(r,b;f^{(n)})
\leq T(r,\frac{f^{(n+2)}}{f^{(n+1)}}) + S(r,f)
= \overline{N}(r,0;f^{(n+1)}) + S(r,f).$$
(3.8)

By Lemma 2.1 we get from (3.8)

$$\overline{N}(r,b;f^{(n)}) \le N(r,0;f^{(n)}) + S(r,f).$$
(3.9)

On the other hand,

$$\begin{aligned} m(r,a;f) &\leq m(r,0;f^{(n)}) + S(r,f) \\ &= T(r,f^{(n)}) - N(r,0;f^{(n)}) + S(r,f) \\ &\leq T(r,f) - N(r,0;f^{(n)}) + S(r,f) \end{aligned}$$

and so

$$N(r,0;f^{(n)}) \le N(r,a;f) + S(r,f).$$
(3.10)

Since $N_B(r,b; f^{(n)}) = S(r,f)$, we have $N(r,b; f^{(n)}) = \overline{N}(r,b; f^{(n)}) + S(r,f)$ and so from (3.9) and (3.10) we get, because $N_A(r,a;f) = S(r,f)$,

$$N(r,a;f) = N(r,b;f^{(n)}) + S(r,f).$$
(3.11)

By Lemma 2.2 we obtain from (3.11)

$$T(r, f) \le 2N(r, a; f) + S(r, f).$$
 (3.12)

First we suppose that $a \neq b$. We put $L = \phi - \frac{\psi H}{b-a}$. Then T(r,L) = S(r,f). If possible, let $L \equiv 0$. Then we get $f^{(n+1)} - f^{(n)} = a - b$. Solving the differential equation we get $f(z) = \alpha e^z + P(z)$, where P is a polynomial of degree n with leading coefficient $\frac{b-a}{n!}$ and α is a constant. By the hypothesis we see that f cannot be a polynomial and so $\alpha \neq 0$.

Since P is non-constant, by Lemma 2.3 we get

$$\overline{N}(r,a;f) = T(r,f) + S(r,f).$$
(3.13)

Since $N_A(r, a; f) = S(r, f)$, by (3.13) we get $\overline{E}(a; f) \cap \overline{E}(b; f^{(n)}) \neq \emptyset$. If $z_0 \in \overline{E}(a; f) \cap \overline{E}(b; f^{(n)})$, we see that $P(z_0) = 0$. Therefore, from (3.13) we get

$$T(r, f) = N(r, a; f) + S(r, f)$$

$$\leq N_A(r, a; f) + N(r, 0; P) + S(r, f)$$

$$= S(r, f),$$

a contradiction. Hence $L \not\equiv 0$.

Let z_1 be a simple *a*-point of f such that $z_1 \notin A \cup B$. Then by (3.1), (3.2) and (3.3) we get $L(z_1) = 0$. Therefore

$$N(r,a;f) \leq N_A(r,a;f) + N_B(r,a;f) + N(r,0;L) + N_{(2}(r,a;f) \leq N_B(r,b;f^{(n)}) + S(r,f) = S(r,f),$$

which contradicts (3.13). Therefore a = b.

Let $H \neq 0$. If $z_1 \notin A \cup B$ is a simple *a*-point of *f*, then from (3.3) we get $H(z_1) = 0$. Hence

$$\overline{N}(r,a;f) \leq N_A(r,a;f) + \overline{N}_B(r,a;f) + N(r,0;H) + N_{(2}(r,a;f)$$
$$\leq N_B(r,b;f^{(n)}) + S(r,f) = S(r,f),$$

and so $N(r,a;f) \leq \overline{N}(r,a;f) + N_{(2}(r,a;f) = S(r,f)$, which contradicts (3.12). Therefore $H \equiv 0$ and so $f^{(n)} \equiv f^{(n+1)}$. This implies $f^{(n+1)} \equiv f^{(n+2)}$, which contradicts the basic assumption of Case 2. This proves the theorem.

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