INEQUALITIES OF DIRICHLET EIGENVALUES FOR DEGENERATE ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

NA HUANG AND JINGJING XUE

ABSTRACT. Let $X_j, Y_j (j=1, \cdots, n)$ be vector fields satisfying Hörmander's condition and $\Delta_L = \sum_{j=1}^n (X_j^2 + Y_j^2)$. In this paper, we establish some inequalities of Dirichlet eigenvalues for degenerate elliptic partial differential operator

ities of Dirichlet eigenvalues for degenerate elliptic partial differential operator Δ_L and Δ_L^2 . These inequalities extend Yang's inequalities for Dirichlet eigenvalues of Laplacian to the settings here and the forms of inequalities are more general than Yang's inequalities. To obtain them, we give a generalization of the inequality by Chebyshev.

1. Introduction

Estimates of Dirichlet eigenvalues for Laplacian in the Euclidean space have been extensively studied. For the following Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n , Payne, Pólya and Weinberger in [11] obtained the inequality (now called the PPW inequality)

$$\lambda_{k+1} - \lambda_k \le \frac{4}{nk} \sum_{r=1}^k \lambda_r.$$

Hile and Protter in [4] proved the inequality (now called the HP inequality)

$$\sum_{r=1}^{k} \frac{\lambda_r}{\lambda_{k+1} - \lambda_r} \ge \frac{nk}{4}.$$

Recently, Yang in [13] established some important eigenvalue estimates including Yang's first inequality

$$\sum_{r=1}^{k} (\lambda_{k+1} - \lambda_r)^2 \le \frac{4}{n} \sum_{r=1}^{k} (\lambda_{k+1} - \lambda_r) \lambda_r$$

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and Yang's second inequality

$$\lambda_{k+1} \le \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{r=1}^{k} \lambda_r.$$

Some estimates for Dirichlet eigenvalues of sub-Laplacian on the Heisenberg group was deduced. Niu and Zhang in [10] obtained the PPW type inequality:

$$\lambda_{k+1} - \lambda_k \le \frac{2}{nk} \left(\sum_{r=1}^k \lambda_r \right).$$

Ilias and Makhoul in [5] gave the Yang type inequalities.

In the paper, we consider the following two Dirichlet problems:

(1.1)
$$\begin{cases} -\Delta_L u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and

(1.2)
$$\begin{cases} (-\Delta_L)^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega. \end{cases}$$

where $\Omega \subset R^{2n+1}$ is a bounded domain, the boundary $\partial\Omega$ is smooth and not characteristic, ν is the outward unit normal on $\partial\Omega$; Δ_L is the degenerate elliptic partial differential operator constituted by vector fields $X_j, Y_j (j = 1, \dots, n)$ satisfying Hörmander's condition,

(1.3)
$$\Delta_L = \sum_{j=1}^n (X_j^2 + Y_j^2),$$

where $X_j = \frac{\partial}{\partial x_j} + 2\sigma y_j |z|^{2\sigma - 2} \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - 2\sigma x_j |z|^{2\sigma - 2} \frac{\partial}{\partial t}, j = 1, \dots, n, x, y \in \mathbb{R}^n,$

$$t \in R, z = x + \sqrt{-1}y \in C, |z| = \left[\sum_{j=1}^{n} (x_j^2 + y_j^2)\right]^{\frac{1}{2}}, \sigma \text{ is any natural number. When }$$

 $\sigma = 1$, Δ_L is the sub-Laplacian on the Heisenberg group; when $\sigma = 2, 3, \dots, \Delta_L$ is the operators discussed by Greiner (see [3, 8]). We note that compared with sub-Laplacian on the Heisenberg group, those operators by Greiner do not have properties of group structure and translation. Some related papers see [9, 14].

From [7], we know that the eigenvalues of (1.1) and (1.2) exist and satisfy

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \rightarrow +\infty$$
.

The corresponding orthogonal normalized eigenfunctions $u_1, u_2, \dots, u_k, \dots$ satisfy $\langle u_i, u_l \rangle = \delta_{il}, i, l = 1, 2, \dots$. Since the boundary $\partial \Omega$ is not characteristic, the eigenfunctions are smooth by using the results in [12].

For convenience, we denote $L = -\Delta_L$ in the sequel. The main results of this paper are the following:

Theorem 1.1. Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then

$$(1.4) \qquad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \le \sqrt{\frac{2}{n}} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i \right)^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$ and $\alpha^2 \leq 2\beta$.

Inequality (1.4) is the generalization of Yang Type inequalities. Using Theorem 1, it follows some interesting corollaries.

Corollary 1.2. Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Yang type first inequality

(1.5)
$$\sum_{k=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{2}{n} \sum_{k=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$

Corollary 1.3. Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Payne-Pólya-Weinberger Type inequality

(1.6)
$$\lambda_{k+1} - \lambda_k \le \frac{2}{nk} \sum_{i=1}^k \lambda_i.$$

Corollary 1.4. Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Yang type second inequality

(1.7)
$$\lambda_{k+1} \le \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} \lambda_i\right).$$

Theorem 1.5. Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then

$$(1.8) \qquad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \\ \leq \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$, and $\alpha^2 \leq 2\beta$.

Corollary 1.6. Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then

$$(1.9) \quad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}}.$$

Corollary 1.7. Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then

(1.10)
$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \le \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \right]^{\frac{1}{2}} \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i \right]^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$, and $\alpha^2 \leq 2\beta$.

Corollary 1.8. Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then we have

(1.11)
$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4(n+1)}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$

Corollary 1.9. Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then we have

(1.12)
$$\lambda_{k+1} - \lambda_k \le \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}}\right)^2.$$

These results are new even for Laplacian on the Euclidean space and sub-Laplacian on the Heisenberg group.

This paper is arranged as follows. In Section 2 the definition of function couple χ_{λ} and its properties are given; two elementary inequalities (see Lemmas 2.6 and 2.8) are proved and examples of noncharacteristics and characteristics domains for vector fields are listed. The proofs of Theorem 1.1 and Corollaries 1.2-1.4 are put in Section 3. The proofs of Theorem 1.5 and Corollaries 1.6-1.9 are given in Section 4

2. Preliminary results

Definition 2.1. (see [5]) A couple (f, g) of functions on the interval $(0, \lambda)$ $(\lambda > 0)$ is said to belong to χ_{λ} provided that

- (i) f and g are positive.
- (ii) f and g satisfy

$$\left(\frac{f(x)-f(y)}{x-y}\right)^2 + \left(\frac{\left(f(x)\right)^2}{g(x)(\lambda-x)} + \frac{\left(f(y)\right)^2}{g(y)(\lambda-y)}\right) \left(\frac{g(x)-g(y)}{x-y}\right) \leq 0,$$

for any $x, y \in (0, \lambda), x \neq y$.

Lemma 2.2. Let $(f,g) \in \chi_{\lambda}$, then g must be nonincreasing; if $f(x) = (\lambda - x)^{\alpha}$, $g(x) = (\lambda - x)^{\beta}$, then $\alpha^2 \leq 2\beta$.

Proof. From Definition 2.1 we see that g must be nonincreasing. Because f and g satisfy

$$\left(\frac{f(x)-f(y)}{x-y}\right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda-x)} + \frac{(f(y))^2}{g(y)(\lambda-y)}\right) \left(\frac{g(x)-g(y)}{x-y}\right) \le 0,$$

letting $y \to x$, we have

$$(f'(x))^2 + \frac{2(f(x))^2}{g(x)(\lambda - x)}g'(x) \le 0$$

and then

$$\left(\frac{f'(x)}{f(x)}\right)^2 + \frac{2}{(\lambda - x)} \frac{g'(x)}{g(x)} \le 0.$$

Taking $f(x) = (\lambda - x)^{\alpha}$, $g(x) = (\lambda - x)^{\beta}$, it follows $\alpha^2 \le 2\beta$.

Definition 2.3. (see [5]) For any two operators A and B, their commutator [A, B] is defined by [A, B] = AB - BA.

Lemma 2.4. For $p = 1, 2, \dots, n$, we have

$$(2.1) L(x_p u_i) = x_p L u_i - 2X_p u_i,$$

$$[L, x_p] u_i = -2X_p u_i.$$

Proof. A direct calculation gives

$$X_{j}(x_{p}u_{i}) = (X_{j}x_{p})u_{i} + x_{p}(X_{j}u_{i}),$$

$$X_{j}^{2}(x_{p}u_{i}) = X_{j}((X_{j}x_{p})u_{i} + x_{p}(X_{j}u_{i}))$$

$$= 2(X_{j}x_{p})(X_{j}u_{i}) + x_{p}(X_{j}^{2}u_{i}).$$

and

$$Y_j(x_p u_i) = x_p(Y_j u_i).$$

$$Y_j^2(x_p u_i) = x_p(Y_j^2 u_i).$$

So

$$L(x_p u_i) = -\sum_{j=1}^n (X_j^2 + Y_j^2)(x_p u_i)$$

$$= -\sum_{j=1}^n \left[2(X_j x_p)(X_j u_i) + x_p(X_j^2 u_i) + x_p(Y_j^2 u_i) \right]$$

$$= x_p L u_i - 2X_p u_i,$$

and (2.1) is proved. Noting

$$[L, x_p] u_i = L(x_p u_i) - x_p L u_i = x_p L u_i - 2X_p u_i - x_p L u_i = -2X_p u_i,$$
(2.2) is proved. \Box

Lemma 2.5. (see [5]) Let $A: D \subset H \to H$ be a self-adjoint operator defined on a dense domain D, which is semibounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$. Let $\{T_p: D \to H\}_{p=1}^n$ be a collection of skew-symmetric operators, and $\{B_p: T_p(D) \to H\}_{p=1}^n$ be a collection of symmetric operators, leaving D invariant. We denote by $\{u_i\}_{i=1}^n$ a basis of orthonormal eigenvectors of A, u_i corresponding to λ_i . Let $k \geq 1$ and assume $\lambda_{k+1} \geq \lambda_k$. Then for any (f,g) in $\chi_{\lambda_{k+1}}$, it follows

$$(2.3) \left(\sum_{i=1}^{k} \sum_{p=1}^{n} f(\lambda_{i}) \left\langle [T_{p}, B_{p}] u_{i}, u_{i} \right\rangle \right)^{2}$$

$$\leq 4 \left(\sum_{i=1}^{k} \sum_{p=1}^{n} g(\lambda_{i}) \left\langle [A, B_{p}] u_{i}, B_{p} u_{i} \right\rangle \right) \left(\sum_{i=1}^{k} \sum_{p=1}^{n} \frac{(f(\lambda_{i}))^{2}}{g(\lambda_{i})(\lambda_{k+1} - \lambda_{i})} \|T_{p} u_{i}\|^{2} \right).$$

Lemma 2.6. For $\gamma \geq 1$, $s_i \geq 0$, $i = 1, \dots, k$, we have

$$\left(\sum_{i=1}^k s_i\right)^{\gamma} \le k^{\gamma - 1} \sum_{i=1}^k s_i^{\gamma}.$$

Proof. Let $\theta(s) = s^{\gamma}$, $s \ge 0$, $\gamma \ge 1$, so $\theta'(s) = \gamma s^{\gamma - 1} \ge 0$, $\theta''(s) = \gamma(\gamma - 1)s^{\gamma - 2} \ge 0$. Noting that $\theta(s)$ is a convex function on $(0, +\infty)$, we have that for $s_i > 0$, $i = 1, \dots, k$, it holds

$$\theta\left(\sum_{i=1}^{k} s_i \middle/ k\right) \le \frac{\sum_{i=1}^{k} (\theta(s_i))}{k},$$

and yields

$$\left(\frac{\sum_{i=1}^k s_i}{k}\right)^{\gamma} \le \frac{\sum_{i=1}^k s_i^{\gamma}}{k}.$$

The required inequality is proved.

Lemma 2.7. (Chebyshev's inequality, [6]) If $(a_k - a_j)(b_k - b_j) \leq 0$ for any nonnegative k, g, then

$$\sum_{i=1}^{n} a_i b_i \le \frac{1}{n} \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right).$$

A key preliminary inequality in the paper is the following which enable us to obtain estimates of eigenvalues more general than Yang's.

Lemma 2.8. If $A_1 \ge A_2 \ge \cdots \ge A_k \ge 0$, $0 \le B_1 \le B_2 \le \cdots \le B_k$, $0 \le C_1 \le C_2 \le \cdots \le C_k$, $i = 1, \dots, k$, then for $\alpha^2 \le 2\beta$, we have

(2.4)
$$\sum_{i=1}^{k} A_i^{\beta} B_i \sum_{i=1}^{k} A_i^{2\alpha-\beta-1} C_i \le \sum_{i=1}^{k} A_i^{\beta} \sum_{i=1}^{k} A_i^{2\alpha-\beta-1} B_i C_i.$$

Proof. When k=1, we see that (2.4) is true, since $A_1^{\beta}B_1A_1^{2\alpha-\beta-1}C_1-A_1^{\beta}A_1^{2\alpha-\beta-1}B_1C_1=0$. Now suppose that the conclusion is true for k-1, then

$$\begin{split} &\sum_{i=1}^{k} A_{i}^{\beta} B_{i} \sum_{i=1}^{k} A_{i}^{2\alpha-\beta-1} C_{i} - \sum_{i=1}^{k} A_{i}^{\beta} \sum_{i=1}^{k} A_{i}^{2\alpha-\beta-1} B_{i} C_{i} \\ &= \sum_{i=1}^{k-1} A_{i}^{\beta} B_{i} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} C_{i} - \sum_{i=1}^{k-1} A_{i}^{\beta} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} B_{i} C_{i} \\ &\quad + A_{k}^{2\alpha-1} B_{k} C_{k} - A_{k}^{2\alpha-1} B_{k} C_{k} \\ &\quad + A_{k}^{2\alpha-\beta-1} C_{k} \sum_{i=1}^{k-1} A_{i}^{\beta} B_{i} + A_{k}^{\beta} B_{k} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} C_{i} \\ &\quad - A_{k}^{2\alpha-\beta-1} B_{k} C_{k} \sum_{i=1}^{k-1} A_{i}^{\beta} - A_{k}^{\beta} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} B_{i} C_{i}. \end{split}$$

$$(2.5) \qquad \sum_{i=1}^{k} A_{i}^{\beta} B_{i} \sum_{i=1}^{k} A_{i}^{2\alpha-\beta-1} C_{i} - \sum_{i=1}^{k} A_{i}^{\beta} \sum_{i=1}^{k} A_{i}^{2\alpha-\beta-1} B_{i} C_{i}$$

$$\leq A_{k}^{2\alpha-\beta-1} C_{k} \sum_{i=1}^{k-1} A_{i}^{\beta} (B_{i} - B_{k}) - A_{k}^{\beta} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} C_{i} (B_{i} - B_{k})$$

$$= A_{k}^{2\alpha-\beta-1} \sum_{i=1}^{k-1} A_{i}^{2\alpha-\beta-1} \left(A_{i}^{2\beta-2\alpha+1} C_{k} - A_{k}^{2\beta-2\alpha+1} C_{i} \right) (B_{i} - B_{k}).$$

Noting $\alpha^2 \leq 2\beta$ and $2\alpha - 1 \leq \alpha^2$ implies $2\beta - 2\alpha + 1 \geq 0$. If $A_1 \geq A_2 \geq \cdots \geq A_k > 0$, $0 < B_1 \leq B_2 \leq \cdots \leq B_k$, $0 < C_1 \leq C_2 \leq \cdots \leq C_k$, then for $i = 1, \dots, k$,

$$\frac{A_i^{2\beta-2\alpha+1}C_k}{A_k^{2\beta-2\alpha+1}C_i} = \left(\frac{C_k}{C_i}\right) \left(\frac{A_i}{A_k}\right)^{2\beta-2\alpha+1} \ge 1, \frac{B_i}{B_k} \le 1.$$

and

$$(2.6) A_i^{2\beta - 2\alpha + 1} C_k - A_k^{2\beta - 2\alpha + 1} C_i \ge 0, B_i - B_k \le 0.$$

If A_i, B_i and $C_i, i = 1, \dots, k$, are nonnegative, then (2.6) is also true. Hence from (2.6),

$$\sum_{i=1}^{k} A_i^{\beta} B_i \sum_{i=1}^{k} A_i^{2\alpha-\beta-1} C_i - \sum_{i=1}^{k} A_i^{\beta} \sum_{i=1}^{k} A_i^{2\alpha-\beta-1} B_i C_i \le 0,$$

and (2.4) is proved.

By Lemma 2.8, we immediately have the following result proved in [1].

Corollary 2.9. If $A_1 \ge A_2 \ge \cdots \ge A_k \ge 0$, $0 \le B_1 \le B_2 \le \cdots \le B_k$, $0 \le C_1 \le C_2 \le \cdots \le C_k$, $i = 1, \dots, k$, then we have

$$\sum_{i=1}^{n} A_i^2 B_i \sum_{i=1}^{n} A_i C_i \le \sum_{i=1}^{n} A_i^2 \sum_{i=1}^{n} A_i B_i C_i.$$

Now let us describe some characteristic and noncharacteristic domains with respect to vector fields and give some such domains.

Definition 2.10. Let $\phi(z,t)$ be the boundary function of a domain Ω . We call that a point (z,t) on $\partial\Omega$ is a characteristic point with respect to vector fields X_j, Y_j $(j=1,\dots,n)$, if it satisfies $|\nabla_L\phi(z,t)|=0$, where $\nabla_L=(X_1,\dots,X_n,Y_1,\dots,Y_n)$. A domain with characteristic points is called a characteristic domain. If the boundary $\partial\Omega$ does not have any characteristic point, then Ω is said a noncharacteristic domain.

Proposition 2.11. The sets $\Omega_m = \{(z,t) \in C^{2n} \times R \mid (|z|-a)^2 + (t-b)^2 < m^2 \}$, $m = 1, 2, \dots$, are noncharacteristic domains with respect to $X_j, Y_j (j = 1, \dots, n)$, where a > 0, b is any real number.

Proof. Fix m and denote $\psi(z,t) = (|z|-a)^2 + (t-b)^2 - m^2$, then

$$X_{j}\psi(z,t) = \frac{\partial}{\partial x_{j}} \left((|z| - a)^{2} + (t - b)^{2} - m^{2} \right) + 2\sigma y_{j}|z|^{2\sigma - 2} \frac{\partial}{\partial t} \left((|z| - a)^{2} + (t - b)^{2} - m^{2} \right)$$

$$= \frac{2(|z| - a)x_{j}}{|z|} + 4\sigma y_{j}|z|^{2\sigma - 2} (t - b),$$

$$Y_{j}\psi(z,t) = \frac{\partial}{\partial y_{j}} \left((|z| - a)^{2} + (t - b)^{2} - m^{2} \right) - 2\sigma x_{j} |z|^{2\sigma - 2} \frac{\partial}{\partial t} \left((|z| - a)^{2} + (t - b)^{2} - m^{2} \right)$$

$$= \frac{2(|z| - a)y_{j}}{|z|} - 4\sigma x_{j} |z|^{2\sigma - 2} (t - b)$$

and

$$|\nabla_L \psi(z,t)|^2 = \sum_{j=1}^n \left(|X_j \psi(z,t)|^2 + |Y_j \psi(z,t)|^2 \right)$$

$$= \sum_{j=1}^n \left(\frac{4(|z|-a)^2 \left(x_j^2 + y_j^2\right)}{|z|^2} + 16\sigma^2 |z|^{4\sigma-4} (t-b)^2 \left(x_j^2 + y_j^2\right) \right)$$

$$= 4(|z|-a)^2 + 16\sigma^2 |z|^{4\sigma-2} (t-b)^2.$$

If $|\nabla_L \psi(z,t)| = 0$, then |z| = a, t = b. But points satisfying these conditions do not be on the boundary $\partial \Omega_m$, $m = 1, 2, \dots$, so Ω_m , $m = 1, 2, \dots$, are noncharacteristic.

If we take a = 2, b = 0, then (see [2] for the case of Heisenberg groups)

Corollary 2.12. The sets $\Omega_m = \{(z,t) \in C^{2n} \times R \, \Big| (|z|-2)^2 + t^2 < m^2 \}, \ m = 1, 2, \cdots$, are noncharacteristic domains with respect to vector fields $X_j, Y_j (j = 1, \cdots, n)$.

Proposition 2.13. The set $\Omega = \{(z,t) \in C^{2n} \times R \mid |z|^{4\sigma} + t^2 < 1 \}$ is a characteristic domain with respect to vector fields $X_i, Y_i (j = 1, \dots, n)$.

Proof. Let
$$\varphi(z,t) = |z|^{4\sigma} + t^2 - 1$$
, then

$$X_{j}\varphi(z,t) = \frac{\partial}{\partial x_{j}} \left(|z|^{4\sigma} + t^{2} - 1 \right) + 2\sigma y_{j} |z|^{2\sigma - 2} \frac{\partial}{\partial t} \left(|z|^{4\sigma} + t^{2} - 1 \right)$$
$$= 4\sigma |z|^{4\sigma - 2} x_{j} + 4\sigma y_{j} |z|^{2\sigma - 2} t;$$

$$Y_{j}\varphi(z,t) = \frac{\partial}{\partial y_{j}} \left(|z|^{4\sigma} + t^{2} - 1 \right) - 2\sigma x_{j} |z|^{2\sigma - 2} \frac{\partial}{\partial t} \left(|z|^{4\sigma} + t^{2} - 1 \right)$$
$$= 4\sigma |z|^{4\sigma - 2} y_{j} - 4\sigma x_{j} |z|^{2\sigma - 2} t.$$

Hence

$$\begin{aligned} |\nabla_L \varphi(z,t)|^2 &= \sum_{j=1}^n \left(|X_j \varphi(z,t)|^2 + |Y_j \varphi(z,t)|^2 \right) \\ &= \sum_{j=1}^n \left(16\sigma^2 |z|^{8\sigma - 4} \left(x_j^2 + y_j^2 \right) + 16\sigma^2 |z|^{4\sigma - 4} t^2 \left(x_j^2 + y_j^2 \right) \right) \\ &= 16\sigma^2 |z|^{8\sigma - 2} + 16\sigma^2 |z|^{4\sigma - 2} t^2. \end{aligned}$$

If $|\nabla_L \phi(z,t)| = 0$, then |z| = 0. We see that two points satisfying $z = 0, t = \pm 1$ are on the boundary $\partial \Omega$ and they are characteristic points.

Corollary 2.14. The sets $\Omega_r = \left\{ (z,t) \in C^{2n} \times R \left| |z|^{4\sigma} + t^2 < r^{4\sigma} \right. \right\} (r > 0)$ are characteristic domains with characteristic points $(0, \pm r^{2\sigma})$.

3. The proofs of Theorem 1.1 and Corollaries 1.2-1.4

Proof of Theorem 1.1. We apply (2.3) with $A = L = -\Delta_L$, $B_1 = x_1, \dots, B_n = x_n, B_{n+1} = y_1, \dots, B_{2n} = y_n, T_1 = X_1, \dots, T_n = X_n, T_{n+1} = Y_1, \dots, T_{2n} = Y_n, f(x) = (\lambda - x)^{\alpha}, g(x) = (\lambda - x)^{\beta}, \text{ and obtain}$

$$(3.1) \qquad \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\alpha} \left(\langle [X_{p}, x_{p}] u_{i}, u_{i} \rangle_{L^{2}} + \langle [Y_{p}, y_{p}] u_{i}, u_{i} \rangle_{L^{2}} \right) \right)^{2}$$

$$\leq 4 \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\beta} \left(\langle [L, x_{p}] u_{i}, x_{p} u_{i} \rangle_{L^{2}} + \langle [L, y_{p}] u_{i}, y_{p} u_{i} \rangle_{L^{2}} \right) \right)$$

$$\times \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{2\alpha - \beta - 1} \left(\|X_{p} u_{i}\|_{L^{2}}^{2} + \|Y_{p} u_{i}\|_{L^{2}}^{2} \right) \right).$$

Since

$$[X_p, x_p] u_i = [Y_p, y_p] u_i = u_i,$$

and

$$\langle [L, x_p] u_i, x_p u_i \rangle_{L^2} = 2 \int_{\Omega} u_i^2 - \langle [L, x_p] u_i, x_p u_i \rangle_{L^2}$$

from (2.2), it follows

(3.2)
$$\langle [L, x_p] u_i, x_p u_i \rangle_{L^2} = \int_{\Omega} u_i^2 = 1.$$

In a similar way, we obtain

(3.3)
$$\langle [L, y_p] u_i, y_p u_i \rangle_{L^2} = \int_{\Omega} u_i^2 = 1.$$

On the other hand, it yields

$$(3.4) \sum_{p=1}^{n} \|X_{p}u_{i}\|_{L^{2}}^{2} + \sum_{p=1}^{n} \|Y_{p}u_{i}\|_{L^{2}}^{2} = \int_{\Omega} \nabla_{L}u_{i}\nabla_{L}u_{i} = \int_{\Omega} Lu_{i}u_{i} = \int_{\Omega} \lambda_{i}u_{i}u_{i} = \lambda_{i}.$$

Instituting (3.2), (3.3) and (3.4) into (3.1), it deduces (1.4). \square Proof of Corollary 1.2. To obtain (1.5), we only need to take $\alpha = \beta = 2$ in (1.4). \square Proof of Corollary 1.3. When $\alpha = \beta$, we have from Theorem 1.1 that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \le \frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i.$$

Using Lemmas 2.6 and 2.7, it implies

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \ge \frac{1}{k^{\alpha - 1}} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \right)^{\alpha} \ge \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \right)^{\alpha - 1} (\lambda_{k+1} - \lambda_k)$$

and

$$\frac{2}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i \le \frac{2}{nk} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \right) \left(\sum_{i=1}^{k} \lambda_i \right),$$

hence

$$\left(\sum_{i=1}^{k} \left(\lambda_{k+1} - \lambda_{i}\right)\right)^{\alpha - 1} \left(\lambda_{k+1} - \lambda_{k}\right) \leq \frac{2}{nk} \left(\sum_{i=1}^{k} \left(\lambda_{k+1} - \lambda_{i}\right)^{\alpha - 1}\right) \left(\sum_{i=1}^{k} \lambda_{i}\right).$$

Since

$$\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\right)^{\alpha - 1} \ge \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1},$$

it shows (1.6).

Proof of Corollary 1.4. When $1 \le \alpha = \beta \le 2$, we have from Theorem 1.1 that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \le \frac{2}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i,$$

then

$$\lambda_{k+1} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} - \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i$$

$$= \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} (\lambda_{k+1} - \lambda_i)$$

$$\leq \frac{2}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i,$$

or

$$\lambda_{k+1} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1}$$

$$\leq \left(1 + \frac{2}{n}\right) \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \lambda_i$$

$$\leq \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1}\right) \left(\sum_{i=1}^{k} \lambda_i\right),$$

where Lemma 2.7 is used. Therefore

$$\left(\lambda_{k+1} - \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} \lambda_i\right)\right) \left(\sum_{i=1}^{k} \left(\lambda_{k+1} - \lambda_i\right)^{\alpha - 1}\right) \le 0.$$

Since $\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1}\right) \ge 0$, it follows

$$\lambda_{k+1} - \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} \lambda_i\right) \le 0$$

and (1.7) is proved.

4. Proofs of Theorem 1.5 and Corollaries 1.6-1.9

Proof of Theorem 1.5. Applying (2.3) with $A = L^2 = (-\Delta_L)^2$, $B_1 = x_1, \dots, B_n = x_n, B_{n+1} = y_1, \dots, B_{2n} = y_n, T_1 = X_1, \dots, T_n = X_n, T_{n+1} = Y_1, \dots, T_{2n} = Y_n, f(x) = (\lambda - x)^{\alpha}, g(x) = (\lambda - x)^{\beta}$, it follows

$$(4.1) \qquad \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\alpha} \left(\left\langle [X_{p}, x_{p}] u_{i}, u_{i} \right\rangle_{L^{2}} + \left\langle [Y_{p}, y_{p}] u_{i}, u_{i} \right\rangle_{L^{2}} \right) \right)^{2}$$

$$\leq 4 \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\beta} \left(\left\langle [L^{2}, x_{p}] u_{i}, x_{p} u_{i} \right\rangle_{L^{2}} + \left\langle [L^{2}, y_{p}] u_{i}, y_{p} u_{i} \right\rangle_{L^{2}} \right) \right)$$

$$\times \left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{2\alpha - \beta - 1} \left(\|X_{p} u_{i}\|_{L^{2}}^{2} + \|Y_{p} u_{i}\|_{L^{2}}^{2} \right) \right).$$

Since

$$\begin{split} & \sum_{p=1}^{n} \left\| X_{p} u_{i} \right\|_{L^{2}}^{2} + \sum_{p=1}^{n} \left\| Y_{p} u_{i} \right\|_{L^{2}}^{2} \\ & = \int_{\Omega} \nabla_{L} u_{i} \nabla_{L} u_{i} = \int_{\Omega} L u_{i} \cdot u_{i} \\ & \leq \left(\int_{\Omega} u_{i}^{2} \right)^{\frac{1}{2}} \left(\int_{\Omega} \left(L u_{i} \right)^{2} \right)^{\frac{1}{2}} = \lambda_{i}^{\frac{1}{2}}, \end{split}$$

it implies

(4.2)
$$\left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \left(\|X_p u_i\|_{L^2}^2 + \|Y_p u_i\|_{L^2}^2 \right) \right)$$
$$= \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i^{\frac{1}{2}} \right).$$

Recalling (3.2) and (3.3), we have

(4.3)
$$\left(\sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\alpha} \left(\langle [X_{p}, x_{p}] u_{i}, u_{i} \rangle_{L^{2}} + \langle [Y_{p}, y_{p}] u_{i}, u_{i} \rangle_{L^{2}} \right) \right)^{2}$$

$$= 4n^{2} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\alpha} \right)^{2}.$$

On the other hand, it obtains by (2.2) that

$$[L^{2}, x_{p}] u_{i} = L^{2} (x_{p}u_{i}) - x_{p}L^{2}u_{i}$$
$$= -2X_{p}Lu_{i} - 2L (X_{p}u_{i}),$$

and

$$\left[L^{2}, y_{p}\right] u_{i} = -2Y_{p}Lu_{i} - 2L\left(Y_{p}u_{i}\right).$$

Hence, we have

$$\langle \left[L^{2}, x_{p}\right] u_{i}, x_{p} u_{i} \rangle_{L^{2}} = 2 \int_{\Omega} L u_{i} \cdot X_{p} \left(x_{p} u_{i}\right) - 2 \int_{\Omega} x_{p} X_{p} u_{i} \cdot L u_{i} - 4 \int_{\Omega} X_{p}^{2} u_{i} \cdot u_{i}$$
$$= 2 \int_{\Omega} L u_{i} \cdot u_{i} - 4 \int_{\Omega} X_{p}^{2} u_{i} \cdot u_{i}$$

and

$$\begin{split} \left\langle \left[L^{2},y_{p}\right]u_{i},y_{p}u_{i}\right\rangle _{L^{2}}&=2\int_{\Omega}Lu_{i}\cdot Y_{p}\left(y_{p}u_{i}\right)-2\int_{\Omega}y_{p}Y_{p}u_{i}\cdot Lu_{i}-4\int_{\Omega}Y_{p}^{2}u_{i}\cdot u_{i}\\ &=2\int_{\Omega}Lu_{i}\cdot u_{i}-4\int_{\Omega}Y_{p}^{2}u_{i}\cdot u_{i}. \end{split}$$

Noting

$$-\sum_{p=1}^{n} \int_{\Omega} X_{p}^{2} u_{i} \cdot u_{i} - \sum_{p=1}^{n} \int_{\Omega} Y_{p}^{2} u_{i} \cdot u_{i} = \sum_{p=1}^{n} \left\| X_{p} u_{i} \right\|_{L^{2}}^{2} + \sum_{p=1}^{n} \left\| Y_{p} u_{i} \right\|_{L^{2}}^{2} = \int_{\Omega} L u_{i} \cdot u_{i},$$

$$(4.4) \qquad \sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\beta} \left(\left\langle \left[L^{2}, x_{p} \right] u_{i}, x_{p} u_{i} \right\rangle_{L^{2}} + \left\langle \left[L^{2}, y_{p} \right] u_{i}, y_{p} u_{i} \right\rangle_{L^{2}} \right)$$

$$= \sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\beta} \left(2 \int_{\Omega} L u_{i} \cdot u_{i} - 4 \int_{\Omega} X_{p}^{2} u_{i} \cdot u_{i} \right)$$

$$+ \sum_{i=1}^{k} \sum_{p=1}^{n} (\lambda_{k+1} - \lambda_{i})^{\beta} \left(2 \int_{\Omega} L u_{i} \cdot u_{i} - 4 \int_{\Omega} Y_{p}^{2} u_{i} \cdot u_{i} \right)$$

$$= 4 (n+1) \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\beta} \int_{\Omega} L u_{i} \cdot u_{i}$$

$$\leq 4 (n+1) \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\beta} \lambda_{i}^{\frac{1}{2}}.$$

Taking (4.2), (4.3) and (4.4) into (4.1), we obtain (1.8). \square Proof of Corollary 1.6. To obtain (1.9), take $\alpha = \beta = 2$ in (1.8). \square Proof of Corollary 1.7. From Theorem 1.5, we have

$$\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha}\right)^2 \le \frac{4(n+1)}{n^2} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{1}{2}}\right) \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i^{\frac{1}{2}}\right).$$

Applying Lemma 2.8 with $A_i = \lambda_{k+1} - \lambda_i$ and $B_i = C_i = \lambda_i^{\frac{1}{2}}$, it deduces (1.10). \square *Proof of Corollary 1.8.* To obtain (1.11), we only need to take $\alpha = \beta = 2$ in Corollary 1.7. \square

Proof of Corollary 1.9. We have from (1.8) that

$$\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha}\right)^{2} \leq \frac{4(n+1)}{n^{2}} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{1}{2}}\right) \times \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i^{\frac{1}{2}}\right).$$

Applying Lemma 2.7 to $\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{1}{2}}\right)$ and $\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{2\alpha - \beta - 1} \lambda_i^{\frac{1}{2}}\right)$, it follows

$$\left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\alpha}\right)^{2} \\
\leq \frac{4(n+1)}{n^{2}k^{2}} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\beta}\right) \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{2\alpha - \beta - 1}\right) \left(\sum_{i=1}^{k} \lambda_{i}^{\frac{1}{2}}\right)^{2} \\
= \frac{4(n+1)}{n^{2}k^{2}} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\alpha}\right) \times \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i})^{\alpha - 1}\right) \left(\sum_{i=1}^{k} \lambda_{i}^{\frac{1}{2}}\right)^{2},$$

where we have used $1 \le \alpha = \beta \le 2$. It implies

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha} \le \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \right) \left(\sum_{i=1}^{k} \lambda_i^{\frac{1}{2}} \right)^2,$$

then

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{\alpha - 1} \left((\lambda_{k+1} - \lambda_k) - \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^{k} \lambda_i^{\frac{1}{2}} \right)^2 \right) \le 0,$$

since $\lambda_i \leq \lambda_k$ for all $i \leq k$. Hence

$$(\lambda_{k+1} - \lambda_k) - \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}}\right)^2 \le 0,$$

and (1.12) is proved.

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DEPARTMENT OF APPLIED MATHEMATICS, NORTHWESTERN POLYTECHNICAL UNIVERSITY, XI'AN, SHAANXI, 710129, P. R. CHINA

E-mail address: huangna7@126.com