Measure Theoretic Trigonometric Functions

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We study the eigenvalues and eigenfunctions of the Laplacian $\Delta_{\mu} = \frac{d}{d\mu} \frac{d}{dx}$ for a Borel probability measure μ on the interval [0, 1] by a technique that follows the treatment of the classical eigenvalue equation $f'' = -\lambda f$ with homogeneous Neumann or Dirichlet boundary conditions. For this purpose we introduce generalized trigonometric functions that depend on the measure μ . In particular, we consider the special case where μ is a self-similar measure like e.g. the Cantor measure. We develop certain trigonometric identities that generalize the addition theorems for the sine and cosine functions. In certain cases we get information about the growth of the suprema of normalized eigenfunctions. For several special examples of μ we compute eigenvalues of Δ_{μ} and L_{∞} - and L_2 -norms of eigenfunctions numerically by applying the formulas we developed.

1. Introduction

Assume that μ is a Borel probability measure on the interval [0, 1]. We consider the Laplacian Δ_{μ} on [0, 1] for the measure μ and study the eigenvalue problem

$$\Delta_{\mu}f = -\lambda f$$

with either homogeneous Dirichlet boundary conditions

$$f(a) = f(b) = 0,$$

or homogeneous Neumann boundary conditions

$$f'(a) = f'(b) = 0$$

The definition of Δ_{μ} involves the derivative with respect to the measure μ . If a

function $g: [0,1] \to \mathbb{R}$ allows the representation

$$g(x) = g(a) + \int_{[a,x]} \frac{dg}{d\mu} d\mu$$
(1)

for all $x \in [0,1]$, then $\frac{dg}{d\mu}$ is unique in $L_2(\mu)$ and is called the μ -derivative of g. In Freiberg [7] an analytic calculus of the concept of μ derivatives is developed.

The operator Δ_{μ} is then given by

$$\Delta_{\mu}f = \frac{d}{d\mu}f'$$

for all $f \in L_2(\mu)$ for which f' and the μ -derivative of f' exist.

It is well known that if μ is a non-atomic Borel measure, Δ_{μ} has a pure point spectrum consisting only of eigenvalues with multiplicity one, that accumulate at infinity, see Freiberg [7, Lem. 5.1 and Cor. 6.9] or Bird, Ngai and Teplyaev [3, Th. 2.5]. Moreover, we have a pure point spectrum not only in the non-atomic case, see Vladimirov and Sheipak [29].

This operator and the resulting eigenvalue problem has been studied in numerous papers, for example in Feller [6], McKean and Ray [22], Kac and Krein [16], Fujita [13], Naimark and Solomyak [23], Freiberg and Zähle [12], Bird, Ngai and Teplyaev [3], Freiberg [7–10], Freiberg and Löbus [11], Hu, Lau and Ngai [14], Chen and Ngai [4], and Arzt and Freiberg [1].

In this paper we give a new technique of determining the eigenvalues and eigenfunctions of Δ_{μ} that involves a generalization of the sine and cosine functions.

In this we follow the classical case, where μ is the Lebesgue measure. There, the Dirichlet eigenvalue problem reads

$$f'' = -\lambda f$$

$$f(0) = f(1) = 0.$$

Then, for every non-negative λ , $f(x) = \sin(\sqrt{\lambda}x)$ satisfies the equation as well as the boundary condition on the left-hand side. On the right-hand side, the boundary condition is only met if $\sqrt{\lambda}$ is a zero point of the sine function, which are, indeed, very well known.

If we impose Neumann boundary conditions f'(0) = f'(1) = 0, we take $f(x) = \cos(\sqrt{\lambda}x)$, because this complies automatically with the left-hand side condition. The right-hand side condition again is satisfied if $\sqrt{\lambda}$ is a zero point of the sine function, which leads to the same eigenvalues as in the Dirichlet case (supplemented by zero). But here sine appears as the derivative of cosine, which will make a difference when we take more general measures.

Now let μ be an arbitrary Borel probability measure on [0, 1]. We construct functions

 $s_{\lambda,\mu}$ and $c_{\lambda,\mu}$ as a replacement for sin and cos by generalizing the series

$$\sin(zx) = \sum_{n=0}^{\infty} (-1)^n \frac{(zx)^{2n+1}}{(2n+1)!}$$

and

$$\cos(zx) = \sum_{n=0}^{\infty} (-1)^n \frac{(zx)^{2n}}{(2n)!}$$

There we replace $\frac{x^n}{n!}$ by appropriate functions $p_n(x)$ or $q_n(x)$, depending on whether we impose Neumann or Dirichlet boundary conditions. These functions fulfill the eigenvalue equation and meet the left-hand side Dirichlet and Neumann boundary condition, respectively.

Putting $p_n := p_n(1)$ and $q_n := q_n(1)$, we define

$$\sin^{D}_{\mu}(z) := \sum_{n=0}^{\infty} (-1)^{n} q_{2n+1} z^{2n+1}$$

and

$$\sin^N_\mu(z) := \sum_{n=0}^\infty (-1)^n p_{2n+1} z^{2n+1}.$$

For $s_{\lambda,\mu}(z,\cdot)$ and $c_{\lambda,\mu}(z,\cdot)$ to also match the right-hand side conditions, z has to be chosen as a zero point of \sin^{D}_{μ} in the Dirichlet case and \sin^{N}_{μ} in the Neumann case. All this is described in Section 3.

In Section 4 we show how to compute the norms in $L_2(\mu)$ of the eigenfunctions by using the sequences p_n and q_n .

The functions $c_{\lambda,\mu}(z,\cdot)$ and $s_{\lambda,\mu}(z,\cdot)$ satisfy an identity that generalizes the classical trigonometric identity. This is established in Section 5.

In Section 6 we consider symmetric measures and get some symmetry results.

The main results are established in Section 7. We outline these briefly here. Since the functions $p_n(x)$ and $q_n(x)$ are determined in a process of iterative integration alternately with respect to μ and the Lebesgue measure, the coefficients p_n and q_n are difficult to compute in general. But if μ is a self-similar measure with respect to the mappings

$$S_1(x) = r_1 x$$
 and $S_2(x) = r_2(x-1) + 1$

as well as the weight factors m_1 and m_2 , we develop a recursion formula for p_n and q_n .

To illustrate the structure of this recursion formula, we consider again the classical Lebesgue case. There we have

$$p_n = q_n = \frac{1}{n!}$$

which leads to $\sin^{N}_{\mu}(z) = \sin^{D}_{\mu}(z) = \sin(z)$. The sequence $p_n = \frac{1}{n!}$ can be viewed as the

solution of the problem

$$2^{n} p_{n} = \sum_{i=0}^{n} p_{i} p_{n-i},$$

$$p_{0} = p_{1} = 1,$$
(2)

which is derived from the equation $2^n = \sum_{i=0}^n {n \choose i}$. Our recursion formula for self-similar μ looks a little more involved, as it distinguishes between the two different kinds of boundary conditions. Additionally it is different for even and odd values of n, and it involves the parameters r_1, r_2, m_1, m_2 of the measure. However, it has the same basic structure as (2).

Moreover, we establish functional equations involving \sin^N_{μ} and \sin^D_{μ} that can be viewed as generalizations of the classical addition theorems.

In Section 8 we consider the especially interesting case where $r_1m_1 = r_2m_2$. Then the Neumann eigenvalues fulfill a renormalization formula

$$\lambda_{2n} = R \,\lambda_n,$$

where $1/R = r_1 m_1$. This property has been established in a special case by Volkmer [30] and in our setting by Freiberg [10]. This formula allows us to investigate the growth of subsequences

$$\left(\|\tilde{f}_{k2^n}\|_{\infty}\right)_{n\in\mathbb{N}},$$
 for odd k ,

where \tilde{f}_n denotes an eigenfunction to the *n*th Neumann eigenvalue that is normalized to one in $L_2(\mu)$.

We show in Section 9 that, if we assume $r_1 + r_2 = 1$ in addition to $r_1m_1 = r_2m_2$, the Dirichlet and Neumann eigenvalues coincide.

Finally, by using the formulas we developed in the course of our investigations, we compute approximations of eigenvalues for certain examples in Section 10.

Several remarks about possible further investigations are made in Section 11.

2. Derivatives and the Laplacian with respect to a measure

As in Freiberg [7,9], we define a derivative of a function with respect to a measure.

Definition 2.1. Let μ be a non-atomic Borel probability measure on [0, 1] and let $f: [0, 1] \to \mathbb{R}$. A function $h \in L_2([0, 1], \mu)$ is called the μ -derivative of f, if

$$f(x) = f(a) + \int_{a}^{x} h \, d\mu \qquad \text{for all } x \in [0, 1].$$

As can be easily seen, the μ -derivative in Definition 2.1 is unique in $L_2(\mu)$. We denote the μ -derivative of a function f by $\frac{df}{d\mu}$. The λ -derivative $\frac{df}{d\lambda}$ we denote by f', where λ denotes the Lebesgue measure on [0, 1].

We define $H^1([0,1],\mu) = H^1(\mu)$ to be the space of all $L_2(\mu)$ -functions whose μ derivative exists. According to our definition, if it exists, the μ -derivative is always in $L_2(\mu)$, and thus it is clear that, for every non-atomic measure μ , all functions in $H^1(\mu)$ are continuous. In case $\mu = \lambda$ is the Lebesgue measure, the definition of $H^1(\lambda)$ is equivalent to the usual definition of the Sobolev space $H^1 = W_2^1$.

The following useful lemma is an analogue to integration by parts and can be found in Freiberg [7, Prop. 3.1].

Lemma 2.2. For $c, d \in [0, 1]$ with c < d and functions $f \in H^1(\mu)$ and $g \in H^1(\lambda)$ we have

$$\int_{c}^{d} \frac{df}{d\mu}(t) g(t) d\mu(t) = f g \Big|_{c}^{d} - \int_{c}^{d} f(t) g'(t) dt$$

Let ν be another non-atomic Borel probability measure on [0, 1]. Then the space $H^2(\nu, \mu)$ is defined to be the collection of all functions in $H^1(\nu)$ whose ν -derivative belongs to $H^1(\mu)$.

Now we define the operator Δ_{μ} on which our investigations are focused as

$$\Delta_{\mu}f := \frac{d}{d\mu}f'$$

for all $f \in H^2(\lambda, \mu)$.

Remark 2.3. In Freiberg [7, Cor. 6.4] is shown that $H^2(\lambda, \mu)$ is dense in $L_2(\mu)$. Furthermore, it is well known (see e.g. [7, Cor. 3.2]) that Δ_{μ} is a negative symmetric operator on $L_2(\mu)$.

3. Generalized trigonometric functions

Let μ be an atomless Borel probability measure on [0, 1]. We construct sequences of functions $p_n(x)$ and $q_n(x)$ depending on μ .

Definition 3.1. For $x \in [0, 1]$ we set $p_0(x) = q_0(x) = 1$ and, for $n \in \mathbb{N}$,

$$p_n(x) := \begin{cases} \int_0^x p_{n-1}(t) \, d\mu(t) &, \text{ if } n \text{ is odd,} \\ \int_0^x p_{n-1}(t) \, dt &, \text{ if } n \text{ is even,} \end{cases}$$

and

$$q_n(x) := \begin{cases} \int_0^x q_{n-1}(t) \, dt & \text{, if } n \text{ is odd,} \\ \int_0^x q_{n-1}(t) \, d\mu(t) & \text{, if } n \text{ is even.} \end{cases}$$

Then, for $n \in \mathbb{N}_0$, we have by definition $p_{2n}, q_{2n+1} \in H^1(\lambda), p_{2n+1}, q_{2n} \in H^1(\mu)$ and

$$\frac{d}{d\mu}p_{2n+1} = p_{2n}, \quad q'_{2n+1} = q_{2n}, \quad p'_{2n} = p_{2n-1} \quad \text{and} \quad \frac{d}{d\mu}q_{2n} = q_{2n-1}.$$

Remark 3.2. (i) If we take μ to be the Lebesgue measure, then

$$p_n(x) = q_n(x) = \frac{x^n}{n!}$$

In the following, we will transfer classical concepts and techniques to a general measure μ by replacing $\frac{x^n}{n!}$ by $p_n(x)$ or $q_n(x)$. In this sense, we can look at $p_n(x)$ or $q_n(x)$ as a kind of generalized monomials.

(ii) It is easy to see that for all $n \in \mathbb{N}$ and $x \in [0, 1]$,

$$q_{n+1}(x) \le p_n(x)$$
 and $p_{n+1}(x) \le q_n(x)$.

To prove convergence of the series defined below, we will need the following lemma.

Lemma 3.3. For all $x \in [0,1]$, $z \in \mathbb{R}$ and $n \in \mathbb{N}_0$ holds

$$p_{2n+1}(x) \le \frac{1}{n!} q_2(x)^n, \qquad q_{2n+1}(x) \le \frac{1}{n!} p_2(x)^n, p_{2n}(x) \le \frac{1}{n!} p_2(x)^n, \qquad q_{2n}(x) \le \frac{1}{n!} q_2(x)^n.$$

Proof. The estimates for $q_{2n+1}(x)$ and $q_{2n}(x)$ are proved in Freiberg and Löbus [11, Lemma 2.3] with complete induction. The proof of the other estimates works analogously.

Definition 3.4. Using the functions $p_n(x)$ and $q_n(x)$ we now define for $x \in [0, 1]$ and $z \in \mathbb{R}$:

$$s_{\mu,\lambda}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n+1} p_{2n+1}(x), \qquad s_{\lambda,\mu}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n+1} q_{2n+1}(x)$$
$$c_{\lambda,\mu}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n}(x), \qquad c_{\mu,\lambda}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n} q_{2n}(x).$$

Note that for every $z \in \mathbb{R}$,

$$c_{\lambda,\mu}(z,\cdot), s_{\lambda,\mu}(z,\cdot) \in H^2(\lambda,\mu)$$
 and $s_{\mu,\lambda}(z,\cdot), c_{\mu,\lambda}(z,\cdot) \in H^2(\mu,\lambda).$

Remark 3.5. (i) If μ is the Lebesgue measure, then

$$\mathbf{s}_{\mu,\lambda}(z,x) = \mathbf{s}_{\lambda,\mu}(z,x) = \sin(zx), \qquad \mathbf{c}_{\lambda,\mu}(z,x) = \mathbf{c}_{\mu,\lambda}(z,x) = \cos(zx).$$

(ii) Functions corresponding to $s_{\lambda,\mu}(z,\cdot)$ and $c_{\mu,\lambda}(z,\cdot)$ have also been constructed in Freiberg and Löbus [11], where they are used to determine the number of zeros of Dirichlet eigenfunctions.

Lemma 3.6. For every $z \in \mathbb{R}$ the series in Definition 3.4 converge uniformly absolutely on [0, 1] and the following differentiation rules hold:

$$\frac{d}{d\mu} \mathbf{s}_{\mu,\lambda}(z,\cdot) = z \, \mathbf{c}_{\lambda,\mu}(z,\cdot), \qquad \mathbf{s}'_{\lambda,\mu}(z,\cdot) = z \, \mathbf{c}_{\mu,\lambda}(z,\cdot), \\ \mathbf{c}'_{\lambda,\mu}(z,\cdot) = -z \, \mathbf{s}_{\mu,\lambda}(z,\cdot), \qquad \frac{d}{d\mu} \, \mathbf{c}_{\mu,\lambda}(z,\cdot) = -z \, \mathbf{s}_{\lambda,\mu}(z,\cdot)$$

Proof. Let $z \in \mathbb{R}$. Since $q_2(x) \leq 1$ for $x \in [0, 1]$, we get by Lemma 3.3 for $N \in \mathbb{N}$

$$\sup_{x \in [0,1]} \sum_{n=N}^{\infty} |z|^{2n+1} p_{2n+1}(x) \le \sup_{x \in [0,1]} \sum_{n=N}^{\infty} \frac{|z|^{2n+1} q_2(x)^n}{n!} \le \sum_{n=N}^{\infty} \frac{|z|^{2n+1}}{n!}$$

Hence, for every $z \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} |z|^{2n+1} p_{2n+1}(x)$ converges uniformly in x. The proof for the other series works analogously with the estimates in Lemma 3.3. Thus, we can differentiate term by term and get the above rules.

Now we show the relation between $c_{\lambda,\mu}(z,\cdot)$ and $s_{\lambda,\mu}(z,\cdot)$ to the eigenvalue problem for Δ_{μ} . Consider the Neumann problem

$$\frac{d}{d\mu}f' = -\lambda f$$
$$f'(0) = f'(1) = 0.$$

It is well known that the eigenvalues can be sorted according to size such that

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{N,2} < \cdots,$$

where $\lambda_{N,0} = 0$ and $\lim_{m \to \infty} \lambda_{N,m} = \infty$.

Proposition 3.7. The Neumann eigenvalues $\lambda_{N,m}$, $m \in \mathbb{N}_0$, are the squares of the non-negative zeros of the function \sin^N_μ given by

$$\sin^{N}_{\mu}(z) := s_{\mu,\lambda}(z,1) = \sum_{n=0}^{\infty} (-1)^{n} p_{2n+1} z^{2n+1}, \qquad for \ z \in \mathbb{R}.$$

where we write p_n instead of $p_n(1)$ for simplicity. The corresponding eigenfunctions $f_{N,m}$ are given by

$$f_{N,m}(x) := c_{\lambda,\mu} \left(\sqrt{\lambda_{N,m}}, x \right) = \sum_{n=0}^{\infty} (-1)^n \lambda_{N,m}^n p_{2n}(x), \quad x \in [0,1]$$

Proof. Using the differentiation rules from Lemma 3.6 it is easy to see that $c_{\lambda,\mu}(z,\cdot)$ satisfies the eigenvalue equation if $\lambda = z^2$, while it also fulfills the left boundary condition $c'_{\lambda,\mu}(z,0) = -z s_{\mu,\lambda}(z,0) = 0$. Here, the dash refers to the second argument of $c_{\lambda,\mu}$. In order that $c_{\lambda,\mu}(z,\cdot)$ satisfies the right boundary condition, too, z has to be zero itself or it must be chosen such that $s_{\mu,\lambda}(z,1) = 0$. It is known (see Freiberg [7] p.40) that the solution of the above problem is unique up to a multiplicative constant. So z is a zero point of \sin^N_{μ} if and only if z^2 is a Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$.

Thus, for $m \in \mathbb{N}_0$, $f_{N,m} = c_{\lambda,\mu}(\sqrt{\lambda_{N,m}}, x)$ is an eigenfunction to the *m*th Neumann eigenvalue $\lambda_{N,m}$.

We treat the Dirichlet eigenvalue problem

$$\frac{d}{d\mu}f' = -\lambda f$$
$$f(0) = f(1) = 0$$

similarly. We denote the Dirichlet eigenvalues such that

$$\lambda_{D,1} < \lambda_{D,2} < \lambda_{D,3} < \cdots$$

where $\lambda_{D,1} > 0$ and $\lim_{n \to \infty} \lambda_{D,n} = \infty$.

Proposition 3.8. The Dirichlet eigenvalues $\lambda_{D,m}$, $m \in \mathbb{N}$, are the squares of the positive zeros of the function \sin^{D}_{μ} given by

$$\sin^{D}_{\mu}(z) := s_{\lambda,\mu}(z,1) = \sum_{n=0}^{\infty} (-1)^{n} q_{2n+1} z^{2n+1}, \qquad \text{for } z \in \mathbb{R}$$

where, as above, q_n stands for $q_n(1)$. The corresponding eigenfunctions $f_{D,m}$ are given by

$$f_{D,m}(x) = s_{\lambda,\mu} \left(\sqrt{\lambda_{D,m}}, x \right) = \sqrt{\lambda_{D,m}} \sum_{n=0}^{\infty} (-1)^n \lambda_{D,m}^n q_{2n+1}(x), \quad x \in [0,1]$$

Proof. The function $s_{\lambda,\mu}(z, \cdot)$ satisfies the equation if $\lambda = z^2$ and also the left boundary condition $s_{\lambda,\mu}(z,0) = 0$. The right boundary condition gives $s_{\lambda,\mu}(z,1) = 0$. So z^2 is a Dirichlet eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$ if and only if z is a zero point of \sin^D_{μ} and $z \neq 0$.

Thus, for $m \in \mathbb{N}$, the function $f_{D,m} = s_{\lambda,\mu} (\sqrt{\lambda_{D,m}}, x)$ is an eigenfunction to the *m*th Dirichlet eigenvalue $\lambda_{D,m}$.

So if we only know the sequences $(p_n(1))_n$ and $(q_n(1))_n$, we can determine the Neumann and Dirichlet eigenvalues by means of the functions \sin^N_μ and \sin^D_μ .

Remark 3.9. (i) As was pointed out to me only recently by V. Kravchenko, a construction analogous to that in Definitions 3.1 and 3.4 has also been done in [19] for Sturm-Liouville equations of the form

$$(pu')' + qu = z^2 u.$$

There, the corresponding spectral problem is also transformed to the problem of finding zeros of a power series as in Propositions 3.7 and 3.8. See also Kravchenko and Porter [20].

(ii) An eigenfunction is only unique up to a multiplicative constant. Throughout the chapter we will use the notations $f_{N,m}$ and $f_{D,m}$ for the eigenfunctions as constructed above. One would also get these by imposing the additional conditions $f_{N,m}(0) = 1$ and $f'_{D,m}(0) = \sqrt{\lambda_{D,m}}$.

Analogously to \sin^N_μ and \sin^D_μ we define

$$\cos^N_\mu(z) := c_{\lambda,\mu}(z,1) = \sum_{n=0}^{\infty} (-1)^n p_{2n} z^{2n}$$

and

$$\cos^{D}_{\mu}(z) := c_{\mu,\lambda}(z,1) = \sum_{n=0}^{\infty} (-1)^{n} q_{2n} z^{2n}$$

for $z \in \mathbb{R}$.

These functions are linked with the eigenvalue problems with mixed boundary conditions

(ND)
$$\frac{\frac{d}{d\mu}f' = -\lambda f}{f'(0) = 0, \quad f(1) = 0,}$$

and

(DN)
$$\frac{\frac{d}{d\mu}f' = -\lambda f}{f(0) = 0, \quad f'(1) = 0.}$$

We treat these problems as the problems in the above Propositions 3.7 and 3.8. If $\lambda > 0$ is chosen such that $\cos^{N}_{\mu}(\sqrt{\lambda}) = 0$, the solutions to (ND) are multiples of $c_{\lambda,\mu}(\sqrt{\lambda}, \cdot)$, because

$$c'_{\lambda,\mu}(\sqrt{\lambda},0) = -\sqrt{\lambda} s_{\mu,\lambda}(\sqrt{\lambda},0) = 0$$

and

$$c_{\lambda,\mu}(\sqrt{\lambda},1) = \cos^N_\mu(\sqrt{\lambda}).$$

Similarly, if $\lambda > 0$ satisfies $\cos^{D}_{\mu}(\sqrt{\lambda}) = 0$, the solutions to (DN) are multiples of $s_{\lambda,\mu}(\sqrt{\lambda}, \cdot)$, because

$$\mathbf{s}_{\lambda,\mu}(\sqrt{\lambda},0) = 0$$

and

$$s_{\lambda,\mu}'(\sqrt{\lambda},1) = \sqrt{\lambda} c_{\mu,\lambda}(\sqrt{\lambda},1) = \sqrt{\lambda} \cos^{D}_{\mu}(\sqrt{\lambda}),$$

where the derivative refers to the second argument of $s_{\lambda,\mu}$. Therefore, the (ND) eigenvalues are the squares of the zeros of \cos^N_{μ} and the (DN) eigenvalues are the squares of the zeros of \cos^D_{μ} .

In the course of the following sections we will often use some of the following easy to prove multiplication formulas which we state here for easy reference. For absolutely summable sequences $(a_n)_n$ and $(b_n)_n$ holds

$$\left(\sum_{j=0}^{\infty} a_{2j}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k} b_{2n-2k},\tag{3}$$

$$\left(\sum_{j=0}^{\infty} a_{2j}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k+1}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k} b_{2n+1-2k},\tag{4}$$

$$\left(\sum_{j=0}^{\infty} a_{2j+1}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k+1}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k+1} b_{2n+1-2k}.$$
(5)

4. Calculation of L_2 -norms

It turns out that by knowing the sequences $(p_n)_n$ and $(q_n)_n$ we can not only determine the Neumann and Dirichlet eigenvalues, but also the $L_2(\mu)$ -norms of the eigenfunctions $f_{N,m}$ and $f_{D,m}$. We will need the following lemma to achieve that.

Lemma 4.1. For $k, n \in \mathbb{N}_0$ with $k \leq n$ and for all $x \in [0, 1]$ we have

$$\int_0^x p_{2k}(t) \, p_{2n-2k}(t) \, d\mu(t) = \sum_{j=0}^{2k} (-1)^j p_j(x) \, p_{2n+1-j}(x) \tag{6}$$

and

$$\int_0^x q_{2k+1}(t) q_{2n+1-2k}(t) d\mu(t) = \sum_{j=0}^{2k+1} (-1)^{j+1} q_j(x) q_{2n+3-j}(x).$$
(7)

Proof. We prove (6) by induction on k. If k = 0 and $n \ge 0$, we have

$$\int_0^x p_0(t) \, p_{2n}(t) \, d\mu(t) = p_{2n+1}(x)$$

and so the assertion holds. Now, take $k \in \mathbb{N}_0$ and assume that the assertion holds for k

and all $n \ge k$. Then, for all $n \ge k+1$,

$$\int_0^x p_{2k+2}(t) p_{2n-2k-2}(t) d\mu(t) = p_{2k+2}(x) p_{2n-2k-1}(x) - \int_0^x p_{2k+1}(t) p_{2n-2k-1}(t) dt$$
$$= p_{2k+2}(x) p_{2n-2k-1}(x) - p_{2k+1}(x) p_{2n-2k}(x) + \int_0^x p_{2k}(t) p_{2n-2k}(t) d\mu(t),$$

by Lemma 2.2. Thus, by the induction hypothesis, we have for all $n \ge k+1$

$$\int_0^x p_{2k+2}(t) \, p_{2n-2k-2}(t) \, d\mu(t) = \sum_{j=0}^{2k+2} (-1)^j p_j(x) \, p_{2n+1-j}(x),$$

which proves (6).

The proof of (7) works the same way.

Proposition 4.2. Let $z \in \mathbb{R}$. Then

$$\|c_{\lambda,\mu}(z,\cdot)\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^n (n+1-2k) \, p_{2k} \, p_{2n+1-2k},\tag{8}$$

and

$$\|\mathbf{s}_{\lambda,\mu}(z,\cdot)\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty} (-1)^n z^{2n+2} \sum_{k=0}^{n+1} (n+1-2k) \, q_{2k+1} \, q_{2n+2-2k},\tag{9}$$

where $p_j = p_j(1)$ and $q_j = q_j(1)$.

Proof. First we prove (8). Using (3) we get for all $x \in [0, 1]$ and $z \in \mathbb{R}$

$$c_{\lambda,\mu}(z,x)^2 = \left(\sum_{j=0}^{\infty} (-1)^j z^{2j} p_{2j}(x)\right) \left(\sum_{k=0}^{\infty} (-1)^k z^{2k} p_{2k}(x)\right)$$
$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^n p_{2k}(x) p_{2n-2k}(x).$$

Consequently, applying (6),

$$\begin{aligned} \|\mathbf{c}_{\lambda,\mu}(z,\cdot)\|_{L_{2}(\mu)}^{2} &= \int_{0}^{1} \mathbf{c}_{\lambda,\mu}(z,t)^{2} \, d\mu(t) \\ &= \sum_{n=0}^{\infty} (-1)^{n} \, z^{2n} \, \sum_{k=0}^{n} \int_{0}^{1} p_{2k}(t) \, p_{2n-2k}(t) \, d\mu(t) \\ &= \sum_{n=0}^{\infty} (-1)^{n} \, z^{2n} \, \sum_{k=0}^{n} \sum_{j=0}^{2k} (-1)^{j} p_{j} \, p_{2n+1-j}. \end{aligned}$$

Note that for any sequence $a = (a_j)_{j \in \mathbb{N}_0}$ holds

$$\sum_{k=0}^{n} \sum_{j=0}^{2k} a_j = \sum_{k=0}^{n} \sum_{j=0}^{k} a_{2j} + \sum_{k=1}^{n} \sum_{j=0}^{k-1} a_{2j+1}$$
$$= \sum_{j=0}^{n} \sum_{k=j}^{n} a_{2j} + \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} a_{2j+1}$$
$$= \sum_{j=0}^{n} (n-j+1) a_{2j} + \sum_{j=1}^{n} (n-j+1) a_{2j-1}$$

and thus,

$$\sum_{k=0}^{n} \sum_{j=0}^{2k} (-1)^{j} p_{j} p_{2n+1-j} = \sum_{k=0}^{n} (n-k+1) p_{2k} p_{2n+1-2k} - \sum_{k=1}^{n} (n-k+1) p_{2k-1} p_{2n+2-2k}$$
$$= \sum_{k=0}^{n} (n-k+1) p_{2k} p_{2n+1-2k} - \sum_{k=1}^{n} k p_{2k} p_{2n+1-2k}$$
$$= \sum_{k=0}^{n} (n+1-2k) p_{2k} p_{2n+1-2k},$$

which proves (8).

The proof of (9) works analogously.

We put $z = \sqrt{\lambda_{N,m}}$ and $z = \sqrt{\lambda_{D,m}}$ to get the following corollary.

Corollary 4.3. The $L_2(\mu)$ -norm of the Neumann eigenfunction $f_{N,m}$ is given by

$$||f_{N,m}||^2_{L_2(\mu)} = \sum_{n=0}^{\infty} (-1)^n \lambda_{N,m}^n \sum_{k=0}^n (n+1-2k) \, p_{2k} \, p_{2n+1-2k}$$

and of the Dirichlet eigenfunction $f_{D,m}$ by

$$\|f_{D,m}\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty} (-1)^n \lambda_{D,m}^{n+1} \sum_{k=0}^{n+1} (n+1-2k) q_{2k+1} q_{2n+2-2k}.$$

5. A trigonometric identity

As in the previous section, we consider an atomless Borel probability measure μ on [0,1]. We prove a formula that links the functions $c_{\lambda,\mu}, c_{\mu,\lambda}, s_{\mu,\lambda}$, and $s_{\lambda,\mu}$ generalizing the trigonometric identity $\sin^2 + \cos^2 = 1$. For this we need the following lemma.

Lemma 5.1. For $k, n \in \mathbb{N}$ with $k \leq n$ and for all $x \in [0, 1]$ we have

$$\int_0^x q_{2k-1}(t) \, p_{2n-2k}(t) \, d\mu(t) = \sum_{j=0}^{2k-1} (-1)^{j+1} q_j(x) \, p_{2n-j}(x).$$

Proof. We prove this by induction on k. For k = 1 and $n \ge 1$, we get by Lemma 2.2

$$\int_0^x q_1(t) \, p_{2n-2}(t) \, d\mu(t) = q_1(x) \, p_{2n-1}(x) - \int_0^x p_{2n-1}(t) \, dt = q_1(x) \, p_{2n-1}(x) - p_{2n}(x),$$

and so the assertion holds. Now, take $k \in \mathbb{N}$, and assume that the assertion holds for k and all $n \geq k$. Then, again by using Lemma 2.2, we get

$$\int_0^x q_{2k+1}(t) p_{2n-2k-2}(t) d\mu(t) = q_{2k+1}(x) p_{2n-2k-1}(x) - \int_0^x q_{2k}(t) p_{2n-2k-1}(t) dt$$
$$= q_{2k+1}(x) p_{2n-2k-1}(x) - q_{2k}(x) p_{2n-2k}(x) + \int_0^x q_{2k-1}(t) p_{2n-2k}(t) d\mu(t).$$

Thus, by the induction hypothesis, for all $n \ge k + 1$,

$$\int_0^x q_{2k+1}(t) \, p_{2n-2k-2}(t) \, d\mu(t) = \sum_{j=0}^{2k+1} (-1)^{j+1} q_j(x) \, p_{2n-j}(x),$$

which finishes the proof.

Corollary 5.2. If we set n = k in Lemma 5.1, we get the formula

$$\sum_{j=0}^{2n} (-1)^j q_j(x) p_{2n-j}(x) = 0,$$

which holds for all $n \in \mathbb{N}$ and $x \in [0, 1]$.

With the above corollary we can prove the following theorem.

Theorem 5.3. For all $x \in [0, 1]$ and $z \in \mathbb{R}$ holds

$$c_{\mu,\lambda}(z,x) c_{\lambda,\mu}(z,x) + s_{\lambda,\mu}(z,x) s_{\mu,\lambda}(z,x) = 1.$$

Proof. Take $x \in [0, 1]$ and $z \in \mathbb{R}$. Then, by Corollary 5.2,

$$\begin{aligned} \mathbf{c}_{\mu,\lambda}(z,x) \ \mathbf{c}_{\lambda,\mu}(z,x) + \mathbf{s}_{\lambda,\mu}(z,x) \ \mathbf{s}_{\mu,\lambda}(z,x) \\ &= \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^n q_{2k}(x) \ p_{2n-2k}(x) + \sum_{n=0}^{\infty} (-1)^n z^{2n+2} \sum_{k=0}^n q_{2k+1}(x) \ p_{2n+1-2k}(x) \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n z^{2n} \left[\sum_{k=0}^n q_{2k}(x) \ p_{2n-2k}(x) - \sum_{k=0}^{n-1} q_{2k+1}(x) \ p_{2n-(2k+1)}(x) \right] \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n z^{2n} \sum_{k=0}^{2n} (-1)^k \ q_k(x) \ p_{2n-k}(x) \\ &= 1. \end{aligned}$$

6. Symmetric measures

In this section we consider symmetric measures μ on [0, 1], meaning that, additionally to being an atomless Borel probability measure, μ shall satisfy

$$\mu\bigl([0,x]\bigr) = \mu\bigl([1-x,1]\bigr)$$

for all $x \in [0, 1]$.

Proposition 6.1. Let μ be symmetric and let $x \in [0, 1]$. Then, for $n \in \mathbb{N}_0$ holds

$$p_{2n+1}(x) = \sum_{k=0}^{n} p_{2k+1} q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k} p_{2n-2k+1}(x) - p_{2n+1}(1-x), \quad (10)$$

and for $n \in \mathbb{N}$

$$p_{2n}(x) = \sum_{k=0}^{n-1} p_{2k+1} q_{2n-2k-1}(x) - \sum_{k=1}^{n} p_{2k} p_{2n-2k}(x) + p_{2n}(1-x).$$
(11)

Proof. For $p_1(x)$ the formula reduces to $p_1(x) = p_1 - p_1(1-x)$. This holds since

$$p_1(x) = \mu([0,x]) = \mu([1-x,1]) = \int_0^1 d\mu - \int_0^{1-x} d\mu = p_1 - p_1(1-x).$$

Assume $p_{2n+1}(x)$ satisfies the above formula for some $n \in \mathbb{N}_0$. Then

$$p_{2n+2}(x) = \int_0^x p_{2n+1}(t) dt$$

$$= \sum_{k=0}^n p_{2k+1} \int_0^x q_{2n-2k}(t) dt - \sum_{k=1}^n p_{2k} \int_0^x p_{2n-2k+1}(t) dt - \int_0^x p_{2n+1}(1-t) dt$$

$$= \sum_{k=0}^n p_{2k+1} q_{2n-2k+1}(x) - \sum_{k=1}^n p_{2k} p_{2n-2k+2}(x) - \int_{1-x}^1 p_{2n+1}(t) dt$$

$$= \sum_{k=0}^n p_{2k+1} q_{2n-2k+1}(x) - \sum_{k=1}^n p_{2k} p_{2n-2k+2}(x) - p_{2n+2}(1) + p_{2n+2}(1-x)$$

$$= \sum_{k=0}^n p_{2k+1} q_{2n-2k+1}(x) - \sum_{k=1}^{n+1} p_{2k} p_{2n-2k+2}(x) + p_{2n+2}(1-x).$$

Now, let the assertion be true for some $2n, n \in \mathbb{N}$. Since μ is symmetric, we have that $d\mu(t) = d\mu(1-t)$. Thus,

$$p_{2n+1}(x) = \int_0^x p_{2n}(t) d\mu(t)$$

$$= \sum_{k=0}^{n-1} p_{2k+1} \int_0^x q_{2n-2k-1}(t) d\mu(t) - \sum_{k=1}^n p_{2k} \int_0^x p_{2n-2k}(t) d\mu(t) + \int_0^x p_{2n}(1-t) d\mu(t)$$

$$= \sum_{k=0}^{n-1} p_{2k+1} q_{2n-2k}(x) - \sum_{k=1}^n p_{2k} p_{2n-2k+1}(x) + \int_{1-x}^1 p_{2n}(t) d\mu(t)$$

$$= \sum_{k=0}^{n-1} p_{2k+1} q_{2n-2k}(x) - \sum_{k=1}^n p_{2k} p_{2n-2k+1}(x) + p_{2n+1}(1) - p_{2n+1}(1-x)$$

$$= \sum_{k=0}^n p_{2k+1} q_{2n-2k}(x) - \sum_{k=1}^n p_{2k} p_{2n-2k+1}(x) - p_{2n+1}(1-x).$$

Corollary 6.2. Let μ be symmetric. Then, for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} p_{2k} p_{2n-2k+1} = \sum_{k=0}^{n} p_{2k+1} q_{2n-2k}.$$
(12)

Proof. This follows from Proposition 6.1 by putting x = 1 in (10).

Remark 6.3. In the special case where μ is the Lebesgue measure, the above formula reduces to $\sum_{k=0}^{n} (-1)^k {n \choose k} = 0.$

Corollary 6.4. Let μ be symmetric. Then the following statements hold.

- (i) $p_{2n} = q_{2n}$ for all $n \in \mathbb{N}$.
- (ii) $\cos^N_\mu(z) = \cos^D_\mu(z)$ for all $z \in \mathbb{R}$.
- (iii) $\cos^{N}_{\mu}(z)^{2} + \sin^{N}_{\mu}(z) \sin^{D}_{\mu}(z) = 1 \text{ for all } z \in \mathbb{R}.$
- (iv) We have the recursion formula

$$p_{2n} = \frac{1}{2} \sum_{k=1}^{2n-1} (-1)^{k+1} p_k q_{2n-k}.$$
 (13)

Proof. We prove (i) by induction. By putting n = 1 in (12), we find that

$$p_3 + p_2 p_1 = p_1 q_2 + p_3,$$

which implies $p_2 = q_2$. Assume that $p_{2k} = q_{2k}$ for all k smaller than some $n \in \mathbb{N}$, $n \ge 2$. We reverse the order of the summands in the second sum of (12) to get

$$\sum_{k=0}^{n-1} p_{2k} p_{2n-2k+1} + p_{2n} p_1 = \sum_{k=0}^{n-1} p_{2n-2k+1} q_{2k} + p_1 q_{2n}.$$

Now it follows from the induction hypothesis that $p_{2n} = q_{2n}$. Then, (ii) follows immediately and by Proposition 5.3 also (iii).

Clearly, (iv) follows from (i) and Corollary 5.2.

Proposition 6.5. Let μ be symmetric. Then, for all $z \in \mathbb{R}$ and $x \in [0, 1]$,

$$c_{\lambda,\mu}(z,1-x) = \cos^N_\mu(z) c_{\lambda,\mu}(z,x) + \sin^N_\mu(z) s_{\lambda,\mu}(z,x).$$

Proof. Rearranging (11) gives

$$p_{2n}(1-x) = \sum_{k=0}^{n} p_{2k} p_{2n-2k}(x) - \sum_{k=0}^{n-1} p_{2k+1} q_{2n-2k-1}(x).$$

We multiply the equation with $(-1)^n z^{2n}$ and sum from n = 0 to infinity to get

$$c_{\lambda,\mu}(z, 1-x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} p_{2k} \cdot (iz)^{2n-2k} p_{2n-2k}(x) - \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (iz)^{2k+1} p_{2k+1} \cdot (iz)^{2n-2k-1} q_{2n-2k-1}(x) = \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n} \cdot \sum_{k=0}^{\infty} (-1)^k z^{2k} p_{2k}(x) + \sum_{n=1}^{\infty} (-1)^n z^{2n+1} p_{2n+1} \cdot \sum_{k=0}^{\infty} (-1)^k z^{2k+1} q_{2k+1}(x) = \cos_{\mu}^N(z) c_{\lambda,\mu}(z, x) + \sin_{\mu}^N(z) s_{\lambda,\mu}(z, x).$$

Corollary 6.6. Let μ be symmetric. Then the Neumann eigenfunctions $f_{N,m}$ are either symmetric or antisymmetric, that is, either

$$f_{N,m}(x) = f_{N,m}(1-x)$$
 or $f_{N,m}(x) = -f_{N,m}(1-x)$

for all $x \in [0, 1]$.

Proof. Let z^2 be a Neumann eigenvalue. Then, by Proposition 3.7, $\sin^N_{\mu}(z) = 0$ and hence, by Corollary 6.4 (iii), $|\cos^N_{\mu}(z)| = 1$. Thus, by Proposition 6.5, we get

$$c_{\lambda,\mu}(z,1-x) = \pm c_{\lambda,\mu}(z,x)$$

Since $c_{\lambda,\mu}(z, \cdot) = f_{N,m}$ for $z^2 = \lambda_m$ the corollary is proved.

Analogous to (10) there is a formula relating $q_{2n+1}(x)$ to $q_{2n+1}(1-x)$, namely

$$q_{2n+1}(x) = \sum_{k=0}^{n} q_{2k+1} p_{2n-2k}(x) - \sum_{k=1}^{n} q_{2k} q_{2n+2k+1}(x) - q_{2n+1}(1-x).$$
(14)

The proof is exactly like the proof of Proposition 6.1. As in the proof of Proposition 6.5, we rearrange, multiply with $(-1)^n z^{2n+1}$, and sum up to get

$$\mathbf{s}_{\lambda,\mu}(z,1-x) = \sin^D_\mu(z) \, \mathbf{c}_{\lambda,\mu}(z,x) - \cos^D_\mu(z) \, \mathbf{s}_{\lambda,\mu}(z,x).$$

If now z^2 is a Dirichlet eigenvalue, then $\sin^D_\mu(z) = 0$ and $\cos^D_\mu(z) = \cos^N_\mu(z) = \pm 1$ and it follows that

$$\mathbf{s}_{\lambda,\mu}(z,1-x) = \mp \mathbf{s}_{\lambda,\mu}(z,x).$$

Thus, we have the following proposition.

Proposition 6.7. Let μ be symmetric. Then the Dirichlet eigenfunctions $f_{D,m}$ are either symmetric or antisymmetric, that is, either

$$f_{D,m}(x) = f_{D,m}(1-x)$$
 or $f_{D,m}(x) = -f_{D,m}(1-x)$

for all $x \in [0, 1]$.

7. Self-similar measures

In this section we impose that the measure μ has a self-similar structure. For definitions of the concept of iterated function systems and self-similar measures, see Hutchinson [15]. For reasons of simplicity, we take an IFS consisting only of two mappings, but it should not raise considerable problems to generalize this to an arbitrary number.

Let r_1, r_2, m_1 and m_2 be positive numbers satisfying $r_1 + r_2 \leq 1$ and $m_1 + m_2 = 1$. Let $\mathcal{S} = (S_1, S_2)$ be the IFS given by

$$S_1(x) = r_1 x$$
 and $S_2(x) = r_2 x + 1 - r_2, \quad x \in [0, 1].$

By K we denote the invariant set of S and by μ its invariant measure with vector of weights (m_1, m_2) .

In this case we are able to prove several properties of the functions $p_n(x)$ and $q_n(x)$ that resemble corresponding ones of $\frac{x^n}{n!}$. These we will employ to examine the Neumann and Dirichlet eigenfunctions and eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$. In particular, we will develop a recursion law for $p_n(1)$ and $q_n(1)$.

The self-similar structure of the measure can be used in integral transformations to receive derivation rules like the following.

Lemma 7.1. Let $F \in H^1(\mu)$ and $f = \frac{dF}{d\mu}$. Then

$$\frac{d}{d\mu}F(r_1x) = m_1f(r_1x)$$

and

$$\frac{d}{d\mu}F(1-r_2+r_2x) = m_2f(1-r_2+r_2x).$$

Proof. Since $F \in H^1(\mu)$, it can be written as

$$F(r_1 x) = F(0) + \int_0^{r_1 x} f(t) \, d\mu(t).$$

The measure μ is invariant with respect to S_1 and S_2 which means that

$$\mu = m_1(S_1\mu) + m_2(S_2\mu).$$

Consequently, if restricted to $[0, r_1]$, we have

$$\mu = m_1(S_1\mu),$$

and hence

$$F(r_1x) = F(0) + \int_0^{r_1x} m_1 f(t) \, d(S_1\mu)(t) = F(0) + \int_0^x m_1 f(r_1t) \, d\mu(t).$$

Thus, the first assertion follows.

Analogously, it follows that on $[1 - r_2, 1]$ we have

$$\mu = m_2(S_2\mu)$$

and thus,

$$F(1 - r_2 + r_2 x) = F(0) + \int_0^{1 - r_2} f(t) \, d\mu(t) + \int_{1 - r_2}^{1 - r_2 + r_2 x} f(t) \, d\mu(t)$$

= $F(1 - r_2) + \int_{1 - r_2}^{1 - r_2 + r_2 x} m_2 f(t) \, d(S_2 \mu)(t)$
= $F(1 - r_2) + \int_0^x m_2 f(1 - r_2 + r_2 t) \, d\mu(t),$

which proves the second assertion.

In the following proposition we present a formula that can be viewed as an analogue of the binomial theorem, adapted to the self-similar measure μ . It relates values on the left part of K, contained in $[0, r_1]$, to values on the right part, contained in $[1 - r_2, 1]$.

Proposition 7.2. For $x \in [0,1]$ and $n \in \mathbb{N}_0$,

$$p_{2n+1}(1 - r_2 + r_2 x) = \sum_{i=0}^{n} p_{2i+1}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i} q_{2n-2i}(r_1 x) + \sum_{i=0}^{n} p_{2i}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i} \left(\frac{m_2}{m_1}\right)^{n-i+1} p_{2n-2i+1}(r_1 x)$$
(15)
$$+ \left[1 - (r_1 + r_2)\right] \sum_{i=0}^{n-1} p_{2i+1}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i-1} \left(\frac{m_2}{m_1}\right)^{n-i} p_{2n-2i-1}(r_1 x),$$

where a sum from 0 to -1 is regarded as zero, and, for $n \in \mathbb{N}$,

$$p_{2n}(1 - r_2 + r_2 x) = \sum_{i=0}^{n} p_{2i}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i} p_{2n-2i}(r_1 x) + \sum_{i=0}^{n-1} p_{2i+1}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i} \left(\frac{m_2}{m_1}\right)^{n-i-1} q_{2n-2i-1}(r_1 x)$$
(16)
$$+ \left[1 - (r_1 + r_2)\right] \sum_{i=0}^{n-1} p_{2i+1}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i-1} p_{2n-2i-2}(r_1 x).$$

Remark 7.3. If $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$ (and hence, μ is the Lebesgue measure), the above formulas reduce to

$$\left(r_1 + r_2 x\right)^n = \sum_{i=0}^n \binom{n}{i} r_1^i (r_2 x)^{n-i}, \qquad n \in \mathbb{N}.$$

Proof. We prove the proposition by induction. As seen in the proof of Lemma 7.1 we have $\mu = m_1(S_1\mu)$ on $[0, r_1]$ and $\mu = m_2(S_2\mu)$ on $[1 - r_2, 1]$. Therefore,

$$p_{1}(1 - r_{2} + r_{2}x) = \int_{0}^{1 - r_{2} + r_{2}x} d\mu = \int_{0}^{r_{1}} d\mu + \int_{1 - r_{2}}^{1 - r_{2} + r_{2}x} d\mu$$

$$= p_{1}(r_{1}) + m_{2} \int_{1 - r_{2}}^{1 - r_{2} + r_{2}x} d(S_{2}\mu) = p_{1}(r_{1}) + m_{2} \int_{0}^{x} d\mu$$

$$= p_{1}(r_{1}) + m_{2} \int_{0}^{r_{1}x} d(S_{1}\mu) = p_{1}(r_{1}) + \frac{m_{2}}{m_{1}} \int_{0}^{r_{1}x} d\mu$$

$$= p_{1}(r_{1}) + \frac{m_{2}}{m_{1}} p_{1}(r_{1}x),$$

which proves the assertion for p_1 .

Assume that the formula for p_{2n+1} holds for some $n \in \mathbb{N}_0$. Then

$$p_{2n+2}(1-r_2+r_2x) = \int_0^{r_1} p_{2n+1}(t) dt + \int_{r_1}^{1-r_2} p_{2n+1}(t) dt + \int_{1-r_2}^{1-r_2+r_2x} p_{2n+1}(t) dt$$
$$= p_{2n+2}(r_1) + [1-(r_1+r_2)]p_{2n+1}(r_1) + r_2 \int_0^x p_{2n+1}(1-r_2+r_2t) dt.$$

Applying the induction hypothesis, we get

$$\begin{split} p_{2n+2}(1-r_2+r_2x) &= p_{2n+2}(r_1) + [1-(r_1+r_2)]p_{2n+1}(r_1) \\ &+ \sum_{i=0}^n p_{2i+1}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i}r_2 \int_0^x q_{2n-2i}(r_1t) \, dt \\ &+ \sum_{i=0}^n p_{2i}(r_1)(\frac{r_2}{r_1})^{n-i}(\frac{m_2}{m_1})^{n-i+1}r_2 \int_0^x p_{2n-2i+1}(r_1t) \, dt \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\frac{r_2}{r_1})^{n-i-1}(\frac{m_2}{m_1})^{n-i}r_2 \int_0^x p_{2n-2i-1}(r_1t) \, dt \\ &= p_{2n+2}(r_1) + [1-(r_1+r_2)]p_{2n+1}(r_1) \\ &+ \sum_{i=0}^n p_{2i+1}(r_1)(\frac{r_2}{r_1})^{n-i+1}(\frac{m_2}{m_1})^{n-i}q_{2n-2i+1}(r_1x) \\ &+ \sum_{i=0}^n p_{2i}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i+1}p_{2n-2i+2}(r_1x) \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i}p_{2n-2i}(r_1x) \\ &= \sum_{i=0}^{n+1} p_{2i}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i+1}p_{2n-2i+2}(r_1x) + \sum_{i=0}^n p_{2i+1}(r_1)(\frac{r_2}{r_1})^{n-i+1}(\frac{m_2}{m_1})^{n-i}q_{2n-2i+1}(r_1x) \\ &+ [1-(r_1+r_2)] \sum_{i=0}^n p_{2i+1}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i}p_{2n-2i}(r_1x), \end{split}$$

which is the formula for p_{2n+2} .

Furthermore, suppose that the assertion is true for p_{2n} for some $n \in \mathbb{N}$. Then, transforming μ as in the proof of the initial step and applying the induction hypothesis in

the same way as above,

$$p_{2n+1}(1 - r_2 + r_2 x) = \int_0^{r_1} p_{2n}(t) d\mu(t) + \int_{r_1}^{1 - r_2} p_{2n}(t) d\mu(t) + \int_{1 - r_2}^{1 - r_2 + r_2 x} p_{2n}(t) d\mu(t)$$

$$= p_{2n+1}(r_1) + m_2 \int_0^x p_{2n}(1 - r_2 + r_2 t) d\mu(t)$$

$$= p_{2n+1}(r_1) + \sum_{i=0}^n p_{2i}(r_1)(\frac{r_2}{r_1})^{n-i}(\frac{m_2}{m_1})^{n-i+1} p_{2n-2i+1}(r_1 x) + \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i} q_{2n-2i}(r_1 x)$$

$$+ [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\frac{r_2}{r_1})^{n-i-1}(\frac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1 x)$$

$$= \sum_{i=0}^n p_{2i+1}(r_1)(\frac{r_2m_2}{r_1m_1})^{n-i} q_{2n-2i}(r_1 x) + \sum_{i=0}^n p_{2i}(r_1)(\frac{r_2}{r_1})^{n-i}(\frac{m_2}{m_1})^{n-i+1} p_{2n-2i+1}(r_1 x)$$

$$+ [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\frac{r_2}{r_1})^{n-i-1}(\frac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1 x),$$

which is the formula for p_{2n+1} .

Analogous formulas hold for the functions q_n .

Proposition 7.4. For $x \in [0, 1]$ and $n \in \mathbb{N}_0$,

$$q_{2n+1}(1 - r_2 + r_2 x) = \sum_{i=0}^{n} q_{2i+1}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i} p_{2n-2i}(r_1 x) + \sum_{i=0}^{n} q_{2i}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i+1} \left(\frac{m_2}{m_1}\right)^{n-i} q_{2n-2i+1}(r_1 x)$$
(17)
$$+ \left[1 - (r_1 + r_2)\right] \sum_{i=0}^{n} q_{2i}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i} p_{2n-2i}(r_1 x),$$

and, for $n \in \mathbb{N}$,

$$q_{2n}(1 - r_2 + r_2 x) = \sum_{i=0}^{n} q_{2i}(r_1) \left(\frac{r_2 m_2}{r_1 m_1}\right)^{n-i} q_{2n-2i}(r_1 x) + \sum_{i=0}^{n-1} q_{2i+1}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i-1} \left(\frac{m_2}{m_1}\right)^{n-i} p_{2n-2i-1}(r_1 x)$$
(18)
$$+ \left[1 - (r_1 + r_2)\right] \sum_{i=0}^{n-1} q_{2i}(r_1) \left(\frac{r_2}{r_1}\right)^{n-i-1} \left(\frac{m_2}{m_1}\right)^{n-i} p_{2n-2i-1}(r_1 x).$$

Proof. The proof works by induction analogously to that of Proposition 7.2. \Box

We translate the formulas about the functions $p_n(x)$ and $q_n(x)$ into formulas about $c_{\lambda,\mu}(z,x)$ and $s_{\lambda,\mu}(z,x)$. In the Lebesgue case, these are the usual addition theorems for $\cos(r_1z + r_2xz)$ and $\sin(r_1z + r_2xz)$.

Corollary 7.5. Let $z \in \mathbb{R}$ and $x \in [0,1]$. With the abbreviation $\overline{z} := \sqrt{\frac{r_2 m_2}{r_1 m_1}} z$ we get

$$c_{\lambda,\mu}(z, 1 - r_2 + r_2 x) = c_{\lambda,\mu}(z, r_1) c_{\lambda,\mu}(\bar{z}, r_1 x) - \sqrt{\frac{r_2 m_1}{r_1 m_2}} s_{\mu,\lambda}(z, r_1) s_{\lambda,\mu}(\bar{z}, r_1 x) - [1 - (r_1 + r_2)] z s_{\mu,\lambda}(z, r_1) c_{\lambda,\mu}(\bar{z}, r_1 x)$$
(19)

and

$$s_{\lambda,\mu}(z, 1 - r_2 + r_2 x) = s_{\lambda,\mu}(z, r_1) c_{\lambda,\mu}(\bar{z}, r_1 x) + \sqrt{\frac{r_2 m_1}{r_1 m_2}} c_{\mu,\lambda}(z, r_1) s_{\lambda,\mu}(\bar{z}, r_1 x) + [1 - (r_1 + r_2)] z c_{\mu,\lambda}(z, r_1) c_{\lambda,\mu}(\bar{z}, r_1 x).$$
(20)

Proof. We prove (20). We multiply (17) with $(-1)^n z^{2n+1} = \frac{1}{i} (iz)^{2n+1}$, sum from n = 0 to infinity and get

$$\begin{split} \mathbf{s}_{\lambda,\mu}(z,1-r_{2}+r_{2}x) &= \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k+1} q_{2k+1}(r_{1}) \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2n-2k} p_{2n-2k}(r_{1}x) \\ &+ \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}} \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_{1}) \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2n-2k+1} q_{2n-2k+1}(r_{1}x) \\ &+ [1-(r_{1}+r_{2})]z \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_{1}) \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2n-2k} p_{2n-2k}(r_{1}x) \\ &= \frac{1}{i} \left(\sum_{n=0}^{\infty} (iz)^{2n+1} q_{2n+1}(r_{1})\right) \left(\sum_{k=0}^{\infty} \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2k} p_{2k}(r_{1}x)\right) \\ &+ \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}} \frac{1}{i} \left(\sum_{n=0}^{\infty} (iz)^{2n} q_{2n}(r_{1})\right) \left(\sum_{k=0}^{\infty} \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2k+1} q_{2k+1}(r_{1}x)\right) \\ &+ [1-(r_{1}+r_{2})]z \left(\sum_{n=0}^{\infty} (iz)^{2n} q_{2n}(r_{1})\right) \left(\sum_{k=0}^{\infty} \left(i\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z\right)^{2k} p_{2k}(r_{1}x)\right) \\ &= \mathbf{s}_{\lambda,\mu}(z,r_{1}) \mathbf{c}_{\lambda,\mu}(\bar{z},r_{1}x) + \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}} \mathbf{c}_{\mu,\lambda}(z,r_{1}) \mathbf{s}_{\lambda,\mu}(\bar{z},r_{1}x) + [1-(r_{1}+r_{2})]z \mathbf{c}_{\mu,\lambda}(z,r_{1}) \mathbf{c}_{\lambda,\mu}(\bar{z},r_{1}x). \end{split}$$

By multiplying (16) with $(-1)^n z^{2n}$ and summing up, (19) is proved in the same way. \Box

The following scaling properties hold that are a replacement of the property $(\frac{1}{2}x)^n = \frac{1}{2^n}x^n$ for p_n and q_n .

Proposition 7.6. For $x \in [0,1]$ and $n \in \mathbb{N}_0$ we have

$$p_{2n+1}(r_1x) = r_1^n m_1^{n+1} p_{2n+1}(x), \qquad q_{2n+1}(r_1x) = r_1^{n+1} m_1^n q_{2n+1}(x),$$

and, for $n \in \mathbb{N}$,

$$p_{2n}(r_1x) = (r_1m_1)^n p_{2n}(x),$$
 $q_{2n}(r_1x) = (r_1m_1)^n q_{2n}(x).$

Proof. We prove the asserted property for p_n by induction on $n \in \mathbb{N}$. Since μ satisfies $\mu(B) = m_1(S_1\mu)(B)$ for all Borel sets $B \subseteq [0, r_1]$, we have

$$p_1(r_1x) = \int_0^{r_1x} d\mu = m_1 \int_0^{r_1x} d(S_1\mu) = m_1 \int_0^x d\mu = m_1 p_1(x).$$

Suppose the assertion is true for p_{2n+1} for some $n \in \mathbb{N}_0$. Then

$$p_{2n+2}(r_1x) = \int_0^{r_1x} p_{2n+1}(t) \, dt = r_1 \int_0^x p_{2n+1}(r_1t) \, dt = (r_1m_1)^{n+1} p_{2n+2}(x).$$

If we assume that the formula holds for p_{2n} for some $n \in \mathbb{N}$, then, transforming μ as above,

$$p_{2n+1}(r_1x) = \int_0^{r_1x} p_{2n}(t) \, d\mu(t) = m_1 \int_0^x p_{2n}(r_1t) \, d\mu(t) = r_1^n m_1^{n+1} p_{2n+1}(x).$$

The formula for q_n is proved analogously.

Next, we deduce formulas corresponding to those in Proposition 7.6 that relate values of $c_{\lambda,\mu}(z,\cdot)$ and $s_{\lambda,\mu}(z,\cdot)$ at $S_1(x) = r_1 x$ to values of $c_{\lambda,\mu}(\sqrt{r_1 m_1} z,\cdot)$ and $s_{\lambda,\mu}(\sqrt{r_1 m_1} z,\cdot)$ at x.

Proposition 7.7. For all $x \in [0,1]$ and $z \in \mathbb{R}$ we have

$$c_{\lambda,\mu}(z, S_1(x)) = c_{\lambda,\mu}(\sqrt{r_1 m_1} z, x)$$
(21)

and

$$\mathbf{s}_{\lambda,\mu}\left(z,S_1(x)\right) = \sqrt{\frac{r_1}{m_1}} \mathbf{s}_{\lambda,\mu}\left(\sqrt{r_1m_1}z,x\right).$$
(22)

Furthermore, we have

$$\mathbf{s}_{\mu,\lambda}(z, S_1(x)) = \sqrt{\frac{m_1}{r_1}} \mathbf{s}_{\mu,\lambda}(\sqrt{r_1 m_1} z, x)$$

and

$$c_{\mu,\lambda}(z, S_1(x)) = c_{\lambda,\mu}(\sqrt{r_1 m_1} z, x).$$

Proof. With Proposition 7.6 we get

$$c_{\lambda,\mu}(z,r_1x) = \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n}(r_1x) = \sum_{n=0}^{\infty} (-1)^n (\sqrt{r_1m_1}z)^{2n} p_{2n}(x) = c_{\lambda,\mu} (\sqrt{r_1m_1}z,x)$$

and

$$s_{\lambda,\mu}(z,r_1x) = \sum_{n=0}^{\infty} (-1)^n z^{2n+1} q_{2n+1}(r_1x) = \sqrt{\frac{r_1}{m_1}} \sum_{n=0}^{\infty} (-1)^n (\sqrt{r_1m_1}z)^{2n+1} q_{2n+1}(x)$$
$$= \sqrt{\frac{r_1}{m_1}} s_{\lambda,\mu} (\sqrt{r_1m_1}z, x).$$

The other two equations are obtained by deriving.

The counterparts of (21) and (22) are the following formulas for $c_{\lambda,\mu}(z, S_2(x))$ and $s_{\lambda,\mu}(z, S_2(x))$.

Proposition 7.8. For all $x \in [0,1]$ and $z \in \mathbb{R}$ we have

$$c_{\lambda,\mu}(z, S_2(x)) = \cos^N_{\mu}(\sqrt{r_1m_1}z) c_{\lambda,\mu}(\sqrt{r_2m_2}z, x) - \sqrt{\frac{r_2m_1}{r_1m_2}} \sin^N_{\mu}(\sqrt{r_1m_1}z) s_{\lambda,\mu}(\sqrt{r_2m_2}z, x) - [1 - (r_1 + r_2)]\sqrt{\frac{m_1}{r_1}} z \sin^N_{\mu}(\sqrt{r_1m_1}z) c_{\lambda,\mu}(\sqrt{r_2m_2}z, x)$$
(23)

and

$$s_{\lambda,\mu}\left(z, S_{2}(x)\right) = \sqrt{\frac{r_{1}}{m_{1}}} \sin^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) c_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z, x\right) + \sqrt{\frac{r_{2}}{m_{2}}} \cos^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) s_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z, x\right) + \left[1 - (r_{1} + r_{2})\right] z \cos^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) c_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z, x\right).$$
(24)

Furthermore, we have

$$s_{\mu,\lambda}(z, S_2(x)) = \sqrt{\frac{m_1}{r_1}} \sin^N_{\mu}(\sqrt{r_1m_1}z) c_{\mu,\lambda}(\sqrt{r_2m_2}z, x) + \sqrt{\frac{m_2}{r_2}} \cos^N_{\mu}(\sqrt{r_1m_1}z) s_{\mu,\lambda}(\sqrt{r_2m_2}z, x) - [1 - (r_1 + r_2)] \sqrt{\frac{m_1m_2}{r_1r_2}} z \sin^N_{\mu}(\sqrt{r_1m_1}z) s_{\mu,\lambda}(\sqrt{r_2m_2}z, x)$$

and

$$c_{\mu,\lambda}(z, S_2(x)) = \cos^D_{\mu}(\sqrt{r_1m_1}z) c_{\mu,\lambda}(\sqrt{r_2m_2}z, x) - \sqrt{\frac{r_1m_2}{r_2m_1}} \sin^D_{\mu}(\sqrt{r_1m_1}z) s_{\mu,\lambda}(\sqrt{r_2m_2}z, x) - [1 - (r_1 + r_2)] z \sqrt{\frac{m_2}{r_2}} \cos^D_{\mu}(\sqrt{r_1m_1}z) s_{\mu,\lambda}(\sqrt{r_2m_2}z, x).$$

Proof. By (19) and Proposition 7.7 we get

$$\begin{aligned} \mathbf{c}_{\lambda,\mu}(z,1-r_{2}+r_{2}x) &= \mathbf{c}_{\lambda,\mu}(z,r_{1})\,\mathbf{c}_{\lambda,\mu}\left(\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z,r_{1}x\right) - \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}}\,\mathbf{s}_{\mu,\lambda}(z,r_{1})\,\mathbf{s}_{\lambda,\mu}\left(\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z,r_{1}x\right) \\ &- \left[1 - (r_{1}+r_{2})\right]z\,\mathbf{s}_{\mu,\lambda}(z,r_{1})\,\mathbf{c}_{\lambda,\mu}\left(\sqrt{\frac{r_{2}m_{2}}{r_{1}m_{1}}}z,r_{1}x\right) \\ &= \cos^{N}_{\mu}(\sqrt{r_{1}m_{1}}z)\,\mathbf{c}_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z,x\right) - \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}}\sin^{N}_{\mu}(\sqrt{r_{1}m_{1}}z)\,\mathbf{s}_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z,x\right) \\ &- \left[1 - (r_{1}+r_{2})\right]\sqrt{\frac{m_{1}}{r_{1}}}z\sin^{N}_{\mu}(\sqrt{r_{1}m_{1}}z)\,\mathbf{c}_{\lambda,\mu}\left(\sqrt{r_{2}m_{2}}z,x\right).\end{aligned}$$

Analogously, (24) is proved using (20).

The other two equations are obtained by deriving.

If the functions \cos_{μ}^{N} , \sin_{μ}^{N} and \sin_{μ}^{D} are assumed to be known, then equations (21) and (23) allow to compute basically all relevant values of the function $c_{\lambda,\mu}(z,\cdot)$. If, namely, x is a point in the invariant set K, then there is a sequence $(x_n)_n$ that converges to x and takes only values of the form

$$S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_n}(0)$$
 or $S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_n}(1)$,

where $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in \{1, 2\}$. For each of these values, (21) and (23) can be applied n times to get a formula containing only values of \cos^N_{μ} , \sin^N_{μ} and \sin^D_{μ} . For example

$$\begin{aligned} \mathbf{c}_{\lambda,\mu} \left(z, S_2(S_1(1)) \right) &= \cos^N_\mu (\sqrt{r_1 m_1} z) \cos^N_\mu (\sqrt{r_2 m_2 r_1 m_1} z) \\ &- \sqrt{\frac{r_2 m_1}{r_1 m_2}} \sin^N_\mu (\sqrt{r_1 m_1} z) \sin^D_\mu (\sqrt{r_2 m_2 r_1 m_1} z) \\ &- \left[1 - (r_1 + r_2) \right] \sqrt{\frac{m_1}{r_1}} z \sin^N_\mu (\sqrt{r_1 m_1} z) \cos^N_\mu (\sqrt{r_2 m_2 r_1 m_1} z). \end{aligned}$$

The same holds for $s_{\lambda,\mu}$ and formulas (22) and (24). This procedure we will use to compute approximate values of the maxima and to give plots of eigenfunctions in Section 10.

Therefore we are interested in the functions \sin_{μ}^{D} , \sin_{μ}^{N} , \cos_{μ}^{N} , and \cos_{μ}^{D} . These have power series representations with coefficients $p_n = p_n(1)$ and $q_n = q_n(1)$. For these numerical sequences we prove a recursion formula in the following. **Proposition 7.9.** (i) For $n \in \mathbb{N}_0$,

$$p_{2n+1} = \sum_{i=0}^{n} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} p_{2i+1} q_{2n-2i} + \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} p_{2i} p_{2n-2i+1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} p_{2i+1} p_{2n-2i-1}.$$
(25)

(ii) For $n \in \mathbb{N}$,

$$p_{2n} = \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} p_{2i} p_{2n-2i} + \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1} p_{2i+1} q_{2n-2i-1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i-1} p_{2i+1} p_{2n-2i-2}.$$
(26)

(iii) For $n \in \mathbb{N}_0$,

$$q_{2n+1} = \sum_{i=0}^{n} r_1^{i+1} m_1^i (r_2 m_2)^{n-i} q_{2i+1} p_{2n-2i} + \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i+1} m_2^{n-i} q_{2i} q_{2n-2i+1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i} p_{2n-2i}.$$
(27)

(iv) For $n \in \mathbb{N}$,

$$q_{2n} = \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i} q_{2n-2i} + \sum_{i=0}^{n-1} r_1^{i+1} m_1^i r_2^{n-i-1} m_2^{n-i} q_{2i+1} p_{2n-2i-1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} (r_1 m_1)^i r_2^{n-i-1} m_2^{n-i} q_{2i} p_{2n-2i-1}.$$
(28)

Remark 7.10. If we take $r_1 = m_1$ and $r_2 = m_2$ (and thus $r_1 + r_2 = 1$ and μ is the

Lebesgue measure), the above formulas reduce to $\sum_{i=0}^{n} {n \choose i} r_1^i r_2^{n-i} = 1$.

Proof. We put x = 1 in Propositions 7.2, 7.4, and 7.6. Then we eliminate all terms of the form $p_n(r_1)$ and $q_n(r_1)$ to obtain formulas that contain only the members of the sequences $(p_n)_n$ and $(q_n)_n$ (as well as r_1, r_2, m_1 and m_2).

To get the desired recursion formulas, we solve the above formulas for the highest order terms.

Corollary 7.11. (i) For $n \in \mathbb{N}$,

$$p_{2n+1} = \frac{1}{1 - r_1^n m_1^{n+1} - r_2^n m_2^{n+1}} \bigg(\sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} p_{2i+1} q_{2n-2i} \\ + \sum_{i=1}^n (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} p_{2i} p_{2n-2i+1} \\ + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} p_{2i+1} p_{2n-2i-1} \bigg).$$

$$(29)$$

(ii) For $n \in \mathbb{N}$,

$$p_{2n} = \frac{1}{1 - (r_1 m_1)^n - (r_2 m_2)^n} \left(\sum_{i=1}^{n-1} (r_1 m_1)^i (r_2 m_2)^{n-i} p_{2i} p_{2n-2i} + \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1} p_{2i+1} q_{2n-2i-1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i-1} p_{2i+1} p_{2n-2i-2} \right).$$
(30)

(iii) For $n \in \mathbb{N}$,

$$q_{2n+1} = \frac{1}{1 - r_1^{n+1} m_1^n - r_2^{n+1} m_2^n} \left(\sum_{i=0}^{n-1} r_1^{i+1} m_1^i (r_2 m_2)^{n-i} q_{2i+1} p_{2n-2i} \right. \\ \left. + \sum_{i=1}^n (r_1 m_1)^i r_2^{n-i+1} m_2^{n-i} q_{2i} q_{2n-2i+1} \right. \\ \left. + \left[1 - (r_1 + r_2) \right] \sum_{i=0}^n (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i} p_{2n-2i} \right).$$

$$(31)$$

(iv) For $n \in \mathbb{N}$,

$$q_{2n} = \frac{1}{1 - (r_1 m_1)^n - (r_2 m_2)^n} \left(\sum_{i=1}^{n-1} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i} q_{2n-2i} + \sum_{i=0}^{n-1} r_1^{i+1} m_1^i r_2^{n-i-1} m_2^{n-i} q_{2i+1} p_{2n-2i-1} + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} (r_1 m_1)^i r_2^{n-i-1} m_2^{n-i} q_{2i} p_{2n-2i-1} \right).$$
(32)

Remark 7.12. Consider two self-similar measures μ and μ^* on [0, 1], where μ^* is the reflection of μ with respect to the point $\frac{1}{2}$. Thus, μ^* is described as invariant measure by interchanging the parameters r_1 , m_1 and r_2 , m_2 in the IFS defining μ . Then the above recursive formulas show that the associated p- and q-sequences satisfy $p_{2n}^* = q_{2n}$, $q_{2n}^* = p_{2n}$, $p_{2n+1}^* = p_{2n+1}$ and $q_{2n+1}^* = q_{2n+1}$ for all $n \in \mathbb{N}$. Hence, $\cos_{\mu^*}^N = \cos_{\mu}^D$, $\cos_{\mu^*}^D = \cos_{\mu}^N$, $\sin_{\mu^*}^N = \sin_{\mu}^N$ and $\sin_{\mu^*}^D = \sin_{\mu}^D$. This is consistent with the physical intuition that the Neumann as well as the Dirichlet eigenfrequencies do not change when the vibrating string producing them is reversed.

Example 7.13. We take $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then, K is the middle third Cantor set and μ is the normalized $\frac{\log 2}{\log 3}$ -dimensional Hausdorff measure restricted to K. We calculate the first members of the sequences $(p_n)_n$ and $(q_n)_n$ using formulas (29) and (31) for p_{2n+1} and q_{2n+1} , which simplify to

$$p_{2n+1} = \frac{1}{2 \cdot 6^n - 2} \left(\sum_{i=1}^{2n} p_i \, p_{2n+1-i} + \sum_{i=0}^{n-1} p_{2i+1} \, p_{2n-2i-1} \right)$$
$$q_{2n+1} = \frac{1}{3 \cdot 6^n - 2} \left(\sum_{i=1}^{2n} q_i \, q_{2n+1-i} + \sum_{i=0}^n q_{2i} \, q_{2n-2i} \right).$$

Since μ is symmetric, we can use for p_{2n} and q_{2n} the simpler formula (13)

$$p_{2n} = q_{2n} = \frac{1}{2} \sum_{i=1}^{2n-1} (-1)^{i+1} p_i q_{2n-i}$$

from Corollary 6.4. Then,

$$p_{1} = 1, \qquad q_{1} = 1, \qquad p_{2} = \frac{1}{2},$$

$$p_{3} = \frac{1}{5}, \qquad q_{3} = \frac{1}{8}, \qquad p_{4} = \frac{3}{80},$$

$$p_{5} = \frac{27}{2800}, \qquad q_{5} = \frac{21}{4240}, \qquad p_{6} = \frac{311}{296800},$$

$$p_{7} = \frac{6383}{31\,906\,000}, \qquad q_{7} = \frac{33\,253}{383\,465\,600}, \qquad p_{8} = \frac{4\,716\,349}{329\,780\,416\,000}$$

and therefore

$$\sin^{N}_{\mu}(z) = z - \frac{6}{5} \frac{z^{3}}{3!} + \frac{81}{70} \frac{z^{5}}{5!} - \frac{57\,447}{56\,975} \frac{z^{7}}{7!} + \cdots$$
$$\sin^{D}_{\mu}(z) = z - \frac{3}{4} \frac{z^{3}}{3!} + \frac{63}{106} \frac{z^{5}}{5!} - \frac{299\,277}{684\,760} \frac{z^{7}}{7!} + \cdots$$

and

$$\cos^{N}_{\mu}(z) = \cos^{D}_{\mu}(z) = 1 - \frac{z^{2}}{2!} + \frac{9}{10}\frac{z^{4}}{4!} - \frac{2\,799}{3\,710}\frac{z^{6}}{6!} + \frac{42\,447\,141}{73\,611\,700}\frac{z^{8}}{8!} - \dots$$

More values of the sequences p_n and q_n , plots of \sin^N_{μ} and \sin^D_{μ} as well as further examples can be found in Section 10. The functions \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} can be characterized by the following system of functional equations.

Theorem 7.14. For $z \in \mathbb{R}$ we have

$$\sin^{N}_{\mu}(z) = \sqrt{\frac{m_{1}}{r_{1}}} \sin^{N}_{\mu}(\sqrt{r_{1}m_{1}}z) \cos^{D}_{\mu}(\sqrt{r_{2}m_{2}}z)
+ \sqrt{\frac{m_{2}}{r_{2}}} \cos^{N}_{\mu}(\sqrt{r_{1}m_{1}}z) \sin^{N}_{\mu}(\sqrt{r_{2}m_{2}}z)
- [1 - (r_{1} + r_{2})] \sqrt{\frac{m_{1}m_{2}}{r_{1}r_{2}}} z \sin^{N}_{\mu}(\sqrt{r_{1}m_{1}}z) \sin^{N}_{\mu}(\sqrt{r_{2}m_{2}}z)
\sin^{D}_{\mu}(z) = \sqrt{\frac{r_{1}}{m_{1}}} \sin^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) \cos^{N}_{\mu}(\sqrt{r_{2}m_{2}}z)
+ \sqrt{\frac{r_{2}}{m_{2}}} \cos^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) \sin^{D}_{\mu}(\sqrt{r_{2}m_{2}}z)
+ [1 - (r_{1} + r_{2})] z \cos^{D}_{\mu}(\sqrt{r_{1}m_{1}}z) \cos^{N}_{\mu}(\sqrt{r_{2}m_{2}}z)$$
(33)
(34)

$$\cos^{N}_{\mu}(z) = \cos^{N}_{\mu}(\sqrt{r_{1}m_{1}z})\cos^{N}_{\mu}(\sqrt{r_{2}m_{2}z})
- \sqrt{\frac{r_{2}m_{1}}{r_{1}m_{2}}}\sin^{N}_{\mu}(\sqrt{r_{1}m_{1}z})\sin^{D}_{\mu}(\sqrt{r_{2}m_{2}z})
- [1 - (r_{1} + r_{2})]\sqrt{\frac{m_{1}}{r_{1}}}z\sin^{N}_{\mu}(\sqrt{r_{1}m_{1}z})\cos^{N}_{\mu}(\sqrt{r_{2}m_{2}z})
\cos^{D}_{\mu}(z) = \cos^{D}_{\mu}(\sqrt{r_{1}m_{1}z})\cos^{D}_{\mu}(\sqrt{r_{2}m_{2}z})
- \sqrt{\frac{r_{1}m_{2}}{r_{2}m_{1}}}\sin^{D}_{\mu}(\sqrt{r_{1}m_{1}z})\sin^{N}_{\mu}(\sqrt{r_{2}m_{2}z})
- [1 - (r_{1} + r_{2})]\sqrt{\frac{m_{2}}{r_{2}}}z\cos^{D}_{\mu}(\sqrt{r_{1}m_{1}z})\sin^{N}_{\mu}(\sqrt{r_{2}m_{2}z}).$$
(35)
(36)

Furthermore, the functions \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} are the only analytic functions that solve the above system of functional equations and satisfy the conditions that \sin^N_{μ} and \sin^D_{μ} are odd, \cos^N_{μ} and \cos^D_{μ} are even, and

$$\lim_{z \to 0} \frac{\sin^N_{\mu}(z)}{z} = \lim_{z \to 0} \frac{\sin^D_{\mu}(z)}{z} = 1$$

and

$$\cos^N_\mu(0) = \cos^D_\mu(0) = 1.$$

Remark 7.15. If we would know all the values of all four functions on a given interval, say, [0, a], then, using the formulas above, we could calculate all values of all four functions on $[0, (\max_i \sqrt{r_i m_i})^{-1}a]$. Then, iteratively, we get the values on $[0, (\max_i \sqrt{r_i m_i})^{-2}a]$ and so on. So, the functions are determined on $[0, \infty)$ by their values on an arbitrary small interval [0, a].

Furthermore, the theorem describes a kind of "self-similarity" of our four functions.

Proof. To show that \sin^{N}_{μ} , \sin^{D}_{μ} , \cos^{N}_{μ} and \cos^{D}_{μ} satisfy the equations, put x = 1 in Proposition 7.8.

Suppose that f_1, f_2, g_1 and g_2 are real analytic functions that satisfy the above equations, and that f_1, f_2 are odd, g_1, g_2 are even, $\lim_{z\to 0} \frac{f_1(z)}{z} = \lim_{z\to 0} \frac{f_2(z)}{z} = 1$, and $g_1(0) = g_2(0) = 1$. Then, power series representations exist, that is, there are real sequences $(a_n), (b_n), (c_n)$ and (d_n) such that for all $z \in \mathbb{R}$ holds

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n+1}, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^{2n+1}, \quad g_1(z) = \sum_{n=0}^{\infty} c_n z^{2n}, \quad g_2(z) = \sum_{n=0}^{\infty} d_n z^{2n},$$

where $a_0 = b_0 = c_0 = d_0 = 1$. Since these functions satisfy (33), we get for all $z \in \mathbb{R}$

$$\begin{split} \sum_{n=0}^{\infty} a_n z^{2n+1} &= \sqrt{\frac{m_1}{r_1}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^n a_k \sqrt{r_1 m_1}^{2k+1} d_{n-k} \sqrt{r_2 m_2}^{2n-2k} \\ &+ \sqrt{\frac{m_2}{r_2}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^n c_k \sqrt{r_1 m_1}^{2k} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k} \\ &- [1 - (r_1 + r_2)] \sqrt{\frac{m_1 m_2}{r_1 r_2}} \sum_{n=0}^{\infty} z^{2n+3} \sum_{k=0}^n a_k \sqrt{r_1 m_1}^{2k+1} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k}. \end{split}$$

If we derive this equation 2j + 1 times and put z = 0, we receive formula (25) for a_j . Analogously, one can show that b_j satisfies (27), c_j satisfies (26) and d_j satisfies (28). Together with the initial condition $a_0 = b_0 = c_0 = d_0 = 1$ it follows that $a_j = p_{2j+1}$, $b_j = q_{2j+1}, c_j = p_{2j}$ and $d_j = q_{2j}$ for all $j \in \mathbb{N}$. Thus, $f_1 = \sin^N_{\mu}, f_2 = \sin^D_{\mu}, g_1 = \cos^N_{\mu}$ and $g_2 = \cos^D_{\mu}$.

Example 7.16. (i) If we take $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$, then K is the unit interval and μ the Lebesgue measure. The functions \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} equal the usual sine and cosine functions, and the formulas in Theorem 7.14 simplify to

$$\sin(z) = \sin(r_1 z + r_2 z) = \sin(r_1 z) \cos(r_2 z) + \cos(r_1 z) \sin(r_2 z),$$

$$\cos(z) = \cos(r_1 z + r_2 z) = \cos(r_1 z) \cos(r_2 z) - \sin(r_1 z) \sin(r_2 z).$$

(ii) Let $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then μ is the Cantor measure and the formulas in Theorem 7.14 can be rewritten as

$$\sin^{N}_{\mu}(\sqrt{6}z) = \frac{\sqrt{6}}{2} \sin^{N}_{\mu}(z) \left(2\cos^{N}_{\mu}(z) - z\sin^{N}_{\mu}(z) \right)$$
(37)

$$\sin^{D}_{\mu}(\sqrt{6}z) = \frac{\sqrt{6}}{3}\cos^{N}_{\mu}(z) \left(2\sin^{D}_{\mu}(z) + z\cos^{N}_{\mu}(z)\right)$$
(38)

$$\cos^{N}_{\mu}(\sqrt{6}z) = \cos^{N}_{\mu}(z)^{2} - \sin^{N}_{\mu}(z)\sin^{D}_{\mu}(z) - z\cos^{N}_{\mu}(z)\sin^{N}_{\mu}(z).$$
(39)

Since K is symmetric, $\cos^N_\mu = \cos^D_\mu$.

Observe that Theorem 7.14 in combination with the recursive rules in Corollary 7.11 supply a technique for investigation of further properties of the eigenvalues. On a given interval [0, a] we can approximate the functions \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} arbitrarily exact by polynomials consisting of sufficiently many members of the corresponding power series. Then, by Theorem 7.14, we can extend all four functions successively to larger intervals.

8. Self-similar measures with $r_1m_1 = r_2m_2$

In this section we suppose μ is a self-similar measure as in the last section but with parameters additionally satisfying $r_1m_1 = r_2m_2$. This case is particularly interesting because there we have the following property.

Theorem 8.1. Let $r_1m_1 = r_2m_2$. If λ is the mth Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$, then $\frac{1}{r_1m_1}\lambda$ is the 2mth Neumann eigenvalue, that is, for all $m \in \mathbb{N}$,

$$r_1 m_1 \lambda_{N,2m} = \lambda_{N,m}.$$

This Theorem has been proved with the method of Prüfer angles by Volkmer [30] for the case $r_1 = r_2 = \frac{1}{3}$, $m_1 = m_2 = \frac{1}{2}$ and by Freiberg [10] in a more general setting. It delivers the foundation for the statements in this section. An analogous property for Dirichlet eigenvalues does not seem to hold. However, in the symmetric case there is a similar relation between Dirichlet eigenvalues and eigenvalues of the problems (DN) or (ND) posed in Section 3. Remember, (DN) has boundary conditions f(0) = f'(1) = 0and (ND) has f'(0) = f(1) = 0.

Proposition 8.2. Let μ be symmetric, that is $r := r_1 = r_2$ and $m_1 = m_2 = \frac{1}{2}$ and let λ be an eigenvalue of (DN) or (ND). Then $\frac{2}{r}\lambda$ is a Dirichlet eigenvalue and if f is a $\frac{2}{r}\lambda$ -Dirichlet eigenfunction, then $f \circ S_1$ is a λ -(DN) eigenfunction, and $f \circ S_2$ is a λ -(ND) eigenfunction.

Proof. In Corollary 6.4 we showed that since μ is symmetric, we have $\cos^N_{\mu} = \cos^D_{\mu}$. Then we can factorize (34) and get

$$\sin^{D}_{\mu}(\sqrt{\frac{2}{r}}z) = \cos^{N}_{\mu}(z) \cdot \left[2\sqrt{2r}\sin^{D}_{\mu}(z) + (1-2r)z\cos^{N}_{\mu}(z)\right]$$

Since λ is an eigenvalue of the (DN) and the (ND) problem, $\cos^N_{\mu}(\sqrt{\lambda}) = 0$. Then, $\sin^D_{\mu}(\sqrt{\frac{2}{r}\lambda}) = 0$ and thus, $\frac{2}{r}\lambda$ is a Dirichlet eigenvalue. From Propositions 7.7 and 7.8 we get for $x \in [0, 1]$

$$s_{\lambda,\mu}\left(\sqrt{\frac{2}{r}\lambda}, S_1(x)\right) = \sqrt{2r} s_{\lambda,\mu}(\sqrt{\lambda}, x)$$

and

$$s_{\lambda,\mu}\left(\sqrt{\frac{2}{r}\lambda}, S_2(x)\right) = \sqrt{2r} \sin^D_\mu\left(\sqrt{\lambda}\right) c_{\lambda,\mu}\left(\sqrt{\lambda}, x\right),$$

which proves the proposition.

In the following we treat only the Neumann eigenvalue problem for a (not necessarily symmetric) measure μ using Theorem 8.1. With the formula

$$\cos^{D}_{\mu}(z) \, \cos^{N}_{\mu}(z) + \sin^{D}_{\mu}(z) \, \sin^{N}_{\mu}(z) = 1, \tag{40}$$

which follows from Theorem 5.3 by setting x = 1, we rearrange the functional equations from Theorem 7.14. With the abbreviation

$$h(z) := r_1 \cos^N_\mu(z) + r_2 \cos^D_\mu(z) - \left[1 - (r_1 + r_2)\right] z \sin^N_\mu(z)$$
(41)

we can write

$$\sin^{N}_{\mu}(z) = \frac{\sqrt{r_{1}m_{1}}}{r_{1}r_{2}} \sin^{N}_{\mu} \left(\sqrt{r_{1}m_{1}}z\right) h\left(\sqrt{r_{1}m_{1}}z\right), \tag{42}$$

$$\cos^{N}_{\mu}(z) = -\frac{r_{2}}{r_{1}} + \frac{1}{r_{1}} \cos^{N}_{\mu} \left(\sqrt{r_{1}m_{1}}z\right) h\left(\sqrt{r_{1}m_{1}}z\right), \tag{43}$$

and

$$\sin^{D}_{\mu}(z) = \left[1 - (r_1 + r_2)\right]z + \frac{1}{\sqrt{r_1 m_1}} \sin^{D}_{\mu}\left(\sqrt{r_1 m_1}z\right) h\left(\sqrt{r_1 m_1}z\right), \tag{44}$$

$$\cos^{D}_{\mu}(z) = -\frac{r_{1}}{r_{2}} + \frac{1}{r_{2}}\cos^{D}_{\mu}\left(\sqrt{r_{1}m_{1}}z\right)h\left(\sqrt{r_{1}m_{1}}z\right).$$
(45)

Employing the above formulas we can calculate the values of \cos^N_{μ} , \cos^D_{μ} and \sin^D_{μ} at the zero points of \sin^N_{μ} .

Lemma 8.3. Let $m \in \mathbb{N}$ and let v(m) be the multiplicity of the prime factor 2 in m. Let $z_m := \sqrt{\lambda_{N,m}}$ be the square root of the mth Neumann eigenvalue, that is, the mth zero point of \sin^N_{μ} . Then

$$\cos^{N}_{\mu}(z_{m}) = \left(-\frac{r_{2}}{r_{1}}\right)^{2^{\nu(m)}} \tag{46}$$

$$\cos^{D}_{\mu}(z_{m}) = \left(-\frac{r_{1}}{r_{2}}\right)^{2^{\nu(m)}}$$
(47)

$$\sin^D_\mu(z_m) = a_{\nu(m)} \cdot z_m \tag{48}$$

where $(a_k)_k$ is determined by

$$a_0 = 1 - (r_1 + r_2)$$

and, for $k \in \mathbb{N}$,

$$a_{k} = 1 - (r_{1} + r_{2}) + a_{k-1} \left(r_{1} \left(-\frac{r_{2}}{r_{1}} \right)^{2^{k-1}} + r_{2} \left(-\frac{r_{1}}{r_{2}} \right)^{2^{k-1}} \right).$$

Proof. Suppose m is odd. Then $\sin^N_{\mu}(z_m) = 0$ and $\sin^N_{\mu}(\sqrt{r_1m_1}z_m) \neq 0$. To see this, suppose $\sin^N_{\mu}(\sqrt{r_1m_1}z_m) = 0$. Then $r_1m_1z_m^2$ would be a Neumann eigenvalue, say $r_1m_1z_m^2 = \lambda_{N,l}$ for some $l \in \mathbb{N}$, and because of Theorem 8.1, z_m^2 would be the eigenvalue $\lambda_{N,2l}$. Thus, m = 2l, which is a contradiction.

Hence, it follows by (42) that $h(\sqrt{r_1m_1}z_m) = 0$. Then, by (43), $\cos^N_\mu(z_m) = -\frac{r_2}{r_1}$.

By (35) follows that, for all $z \in \mathbb{R}$, if

$$\sin^N_\mu \left(\sqrt{r_1 m_1} z\right) = 0,$$

then

$$\cos^N_\mu(z) = \cos^N_\mu \left(\sqrt{r_1 m_1} z\right)^2.$$

Thus, if m = 2l for some odd l, then $\sqrt{r_1m_1}z_m = z_l$ and hence,

$$\cos^{N}_{\mu}(z_m) = \cos^{N}_{\mu}(z_l)^2 = \left(-\frac{r_2}{r_1}\right)^2.$$

Iteratively, we get that, if $m = 2^k l$ for some odd l,

$$\cos^N_\mu(z_m) = \left(-\frac{r_2}{r_1}\right)^{2^k},$$

which proves (46).

Since $\sin^N_{\mu}(z_m) = 0$ for all $m \in \mathbb{N}$ we get by (40) that

$$\cos^{D}_{\mu}(z_m) = \frac{1}{\cos^{N}_{\mu}(z_m)} = \left(-\frac{r_1}{r_2}\right)^{2^{\nu(m)}},$$

which is (47).

Now we show (48). At first, suppose v(m) = 0, that is, m is odd. Then, as above, $h(\sqrt{r_1m_1}z_m) = 0$ and thus, by (44),

$$\sin^{D}_{\mu}(z_{m}) = \left[1 - (r_{1} + r_{2})\right] z_{m}$$

Observe that we have for all m

$$h(z_m) = r_1 \left(-\frac{r_2}{r_1}\right)^{2^{\nu(m)}} + r_2 \left(-\frac{r_1}{r_2}\right)^{2^{\nu(m)}}.$$
(49)

Suppose $v(m) \ge 1$. Then $\sqrt{r_1 m_1} z_m = z_{\frac{m}{2}}$ and thus,

$$\frac{\sin^{D}_{\mu}(z_{m})}{z_{m}} = 1 - (r_{1} + r_{2}) + \frac{\sin^{D}_{\mu}(\sqrt{r_{1}m_{1}}z_{m})}{\sqrt{r_{1}m_{1}}z_{m}}h(\sqrt{r_{1}m_{1}}z_{m})$$
$$= 1 - (r_{1} + r_{2}) + \frac{\sin^{D}_{\mu}(z_{\frac{m}{2}})}{z_{\frac{m}{2}}}h(z_{\frac{m}{2}})$$
$$= 1 - (r_{1} + r_{2}) + \frac{\sin^{D}_{\mu}(z_{\frac{m}{2}})}{z_{\frac{m}{2}}}\left(r_{1}\left(-\frac{r_{2}}{r_{1}}\right)^{2^{\nu(m)-1}} + r_{2}\left(-\frac{r_{1}}{r_{2}}\right)^{2^{\nu(m)-1}}\right).$$

Hence, $\frac{\sin_{\mu}^{D}(z_{m})}{z_{m}}$ depends only on v(m) and so, with $a_{v(m)} = \frac{\sin_{\mu}^{D}(z_{m})}{z_{m}}$, we get $a_{v(m)} = 1 - (r_{1} + r_{2}) + a_{v(m)-1} \left(r_{1} \left(-\frac{r_{2}}{r_{1}} \right)^{2^{v(m)-1}} + r_{2} \left(-\frac{r_{1}}{r_{2}} \right)^{2^{v(m)-1}} \right),$

which proves the assertion.

We use the above computed values of $\cos^{N}_{\mu}(z_{m})$ and Propositions 7.7 and 7.8 to get a relation between the *m*th and the 2*m*th Neumann eigenfunction.

Proposition 8.4. Let $m \in \mathbb{N}$ and v(m) be the 2-multiplicity of m. We denote the mth Neumann eigenfunction by $f_m := c_{\lambda,\mu}(z_m, \cdot)$. Then, for all $x \in [0, 1]$,

$$f_{2m}(S_1(x)) = f_m(x) \tag{50}$$

and

$$f_{2m}(S_2(x)) = \left(-\frac{m_1}{m_2}\right)^{2^{\nu(m)}} f_m(x).$$
(51)

Proof. Because of Theorem 8.1 we have $\lambda_m = r_1 m_1 \lambda_{2m}$ and thus,

$$\sin^N_\mu \left(\sqrt{r_1 m_1} z_{2m} \right) = 0.$$

Since $f_m = c_{\lambda,\mu}(z_m, \cdot)$, Propositions 7.7 and 7.8 give for $x \in [0, 1]$

$$f_{2m}(S_1(x)) = f_m(x)$$

and

$$f_{2m}(S_2(x)) = \cos^N_\mu(z_m) f_m(x).$$

Noting that $\frac{r_2}{r_1} = \frac{m_1}{m_2}$, we get with (46) that

$$f_{2m}(S_2(x)) = \left(-\frac{m_1}{m_2}\right)^{2^{v(m)}} f_m(x)$$

The above proposition can be employed to work out the relationship between the suprema and the $L_2(\mu)$ norms of f_m and f_{2m} .

Proposition 8.5. Let $m \in \mathbb{N}$ and v(m) the 2-multiplicity of m. Then

$$||f_{2m}||^2_{L_2(\mu)} = \left(m_1 + m_2 \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)+1}}\right) ||f_m||^2_{L_2(\mu)}$$
(52)
and

$$||f_{2m}||_{\infty} = \max\left\{1, \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)}}\right\}||f_m||_{\infty}.$$
(53)

Proof. At first we prove (52). For $m \in \mathbb{N}$ we have

$$\|f_{2m}\|_{L_{2}(\mu)}^{2} = \int_{S_{1}(0)}^{S_{1}(1)} f_{2m}(t)^{2} d\mu(t) + \int_{S_{2}(0)}^{S_{2}(1)} f_{2m}(t)^{2} d\mu(t)$$

$$= m_{1} \int_{S_{1}(0)}^{S_{1}(1)} f_{2m}(t)^{2} d(S_{1}\mu)(t) + m_{2} \int_{S_{2}(0)}^{S_{2}(1)} f_{2m}(t)^{2} d(S_{2}\mu)(t)$$

$$= m_{1} \int_{0}^{1} f_{2m}(S_{1}(t))^{2} d\mu(t) + m_{2} \int_{0}^{1} f_{2m}(S_{2}(t))^{2} d\mu(t).$$

By (50) and (51) we get

$$\|f_{2m}\|_{L_2(\mu)}^2 = m_1 \int_0^1 f_m(t)^2 d\mu(t) + m_2 \left(-\frac{m_1}{m_2}\right)^{2^{\nu(m)+1}} \int_0^1 f_m(t)^2 d\mu(t)$$
$$= \left[m_1 + m_2 \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)+1}}\right] \|f_m\|_{L_2(\mu)}^2.$$

Now we show (53). With (50) and (51) we have

$$\sup_{x \in [S_1(0), S_1(1)]} |f_{2m}(x)| = \sup_{x \in [0, 1]} |f_{2m}(S_1(x))| = \sup_{x \in [0, 1]} |f_m(x)| = ||f_m||_{\infty}$$

and

$$\sup_{x \in [S_2(0), S_2(1)]} |f_{2m}(x)| = \sup_{x \in [0,1]} \left| f_{2m}(S_2(x)) \right| = \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)}} ||f_m||_{\infty}.$$

Therefore, since f_{2m} is linear on $[S_1(1), S_2(0)]$ and continuous,

$$\sup_{x \in [0,1]} |f_{2m}(x)| = \max\left\{1, \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)}}\right\} ||f_m||_{\infty}.$$

Now we consider the normalized Neumann eigenfunctions. For $m \in \mathbb{N}_0$ we set

$$\tilde{f}_m := \|f_m\|_{L_2(\mu)}^{-1} f_m.$$

We are interested in the asymptotic behaviour of the sequence $(\|\tilde{f}_m\|_{\infty})_m$. With Proposition 8.5 we get some information about certain subsequences stated in the following theorem.

Theorem 8.6. Let μ be a self-similar measure with $r_1m_1 = r_2m_2$. Then, for all $m \in \mathbb{N}_0$,

$$\|\tilde{f}_{2m}\|_{\infty} = \frac{\max\left\{1, \left(\frac{m_1}{m_2}\right)^{2^{\nu(m)}}\right\}}{\sqrt{m_1 + m_2\left(\frac{m_1}{m_2}\right)^{2^{\nu(m)+1}}}} \|\tilde{f}_m\|_{\infty}.$$
(54)

Suppose $m_1 \leq m_2$ and let l be an odd number. Then, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^{k}l}\|_{\infty} = m_{1}^{-\frac{k}{2}} \prod_{j=1}^{k} \left(1 + \left(\frac{m_{1}}{m_{2}}\right)^{2^{j}-1}\right)^{-\frac{1}{2}} \|\tilde{f}_{l}\|_{\infty}.$$
(55)

Proof. (54) follows directly from (52) and (53). Suppose $m_1 \leq m_2$ and $l \in \mathbb{N}$ is odd. Then iterative application of (54) gives

$$\|\tilde{f}_{2l}\|_{\infty} = \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \frac{m_1}{m_2}}} \|\tilde{f}_l\|_{\infty},$$
$$\|\tilde{f}_{2^2l}\|_{\infty} = \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \left(\frac{m_1}{m_2}\right)^3}} \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \frac{m_1}{m_2}}} \|\tilde{f}_l\|_{\infty},$$

and so on, and therefore (55) holds.

Corollary 8.7. Let $l \in \mathbb{N}$ be odd. Then the following statements hold.

(i) If $m_1 = m_2$, then for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_{\infty} = \|\tilde{f}_l\|_{\infty}.$$

(ii) If $m_1 < m_2$, then $C := \frac{1}{\sqrt{m_1(1 + \frac{m_1}{m_2})}} > 1$, and we have for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_{\infty} \ge C^k \|\tilde{f}_l\|_{\infty}.$$

Additionally, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^{k}l}\|_{\infty} \le m_{1}^{-\frac{k}{2}} \left(\frac{m_{2}}{m_{1}}\right)^{\frac{k}{2}(2^{k}-1)} \|\tilde{f}_{l}\|_{\infty}.$$

Proof. (i) follows directly from (55) by putting $m_1 = m_2 = \frac{1}{2}$. If $m_1 < m_2$, then, for all $j \in \mathbb{N}$,

$$1 + \left(\frac{m_1}{m_2}\right)^{2^j - 1} \le 1 + \frac{m_1}{m_2}.$$

Then,

$$\|\tilde{f}_{2^{k}l}\|_{\infty} \ge m_{1}^{-\frac{k}{2}} \left(1 + \frac{m_{1}}{m_{2}}\right)^{-\frac{k}{2}} \|\tilde{f}_{l}\|_{\infty},$$

and since $m_1 < m_2$ implies $m_1 < \frac{1}{2}$, we have $m_1(1 + \frac{m_1}{m_2}) < 1$.

For the upper estimate, we write

$$\prod_{j=1}^{k} \left(1 + \left(\frac{m_1}{m_2}\right)^{2^j - 1} \right) \ge \left(1 + \left(\frac{m_1}{m_2}\right)^{2^k - 1} \right)^k \ge \left(\frac{m_1}{m_2}\right)^{k(2^k - 1)},$$

which proves (ii).

9. Self-similar measures with $r_1m_1 = r_2m_2$ and $r_1 + r_2 = 1$

As in the previous section we have the condition $r_1m_1 = r_2m_2$. We treat the special case where $r_1 + r_2 = 1$ from which follows that $r_1 = m_2$ and $r_2 = m_1$. Such measures have been investigated e.g. by Sabot [25] and [26].

Theorem 9.1. Let μ be a self-similar measure where $r_1 = m_2$ and $r_2 = m_1$ (and therefore $r_1 + r_2 = 1$). Then the positive eigenvalues of $-\frac{d}{d\mu}\frac{d}{x}$ with Neumann boundary conditions coincide with those with Dirichlet boundary conditions.

Proof. Since the eigenvalues are the squares of the zeros of \sin^N_{μ} and \sin^D_{μ} , respectively, it is sufficient to show that $\sin^N_{\mu} = \sin^D_{\mu}$. To do that we show that for all $n \in \mathbb{N}_0$

$$p_{2n+1} = q_{2n+1}$$

We do this by complete induction using the recursion formulas from Corollary 7.11. By Definition 3.1 we have

$$p_1 = \int_0^1 d\mu = 1$$

and

$$q_1 = \int_0 dt = 1.$$

Now, let $n \in \mathbb{N}$ and suppose that for $i = 0, \ldots, n-1$ holds $p_{2i+1} = q_{2i+1}$. By (29) and

rearrangement of the order of the terms in the sums we get

$$p_{2n+1} = \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \left(\sum_{i=0}^{n-1} m_2^i r_2^{i+1} (r_1 m_1)^{n-i} p_{2i+1} q_{2n-2i} \right. \\ \left. + \sum_{i=1}^n (r_2 m_2)^i m_1^{n-i} r_1^{n-i+1} p_{2i} p_{2n-2i+1} \right) \\ = \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \left(\sum_{i=1}^n m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i p_{2n+1-2i} q_{2i} \right. \\ \left. + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i} p_{2i+1} \right).$$

Then, by the induction hypothesis and (31),

$$p_{2n+1} = \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \left(\sum_{i=1}^n m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i q_{2n+1-2i} q_{2i} + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i} q_{2i+1} \right)$$
$$= q_{2n+1}.$$

With the above theorem we can reformulate Theorem 5.3 to get a property of the Wronskian of $f_{N,m}$ and $f_{D,m}$.

Corollary 9.2. Let μ be as above, let λ_m be the mth eigenvalue, let $f_{N,m} = c_{\lambda,\mu}(\sqrt{\lambda_m}, \cdot)$ and $f_{D,m} = s_{\lambda,\mu}(\sqrt{\lambda_m}, \cdot)$ be the corresponding Neumann and Dirichlet eigenfunctions constructed in Section 3. Then, for all $x \in [0, 1]$,

$$f_{N,m}(x) f'_{D,m}(x) - f_{D,m}(x) f'_{N,m}(x) = \sqrt{\lambda_m}.$$

Proof. We put $z = \sqrt{\lambda_m}$ in Theorem 5.3 and observe that

$$f'_{N,m} = c'_{\lambda,\mu}(\sqrt{\lambda_m}, \cdot) = -\sqrt{\lambda_m} s_{\mu,\lambda}(\sqrt{\lambda_m}, \cdot)$$

and

$$f'_{D,m} = \mathbf{s}'_{\lambda,\mu}(\sqrt{\lambda_m}, \cdot) = \sqrt{\lambda_m} \, \mathbf{c}_{\mu,\lambda}(\sqrt{\lambda_m}, \cdot).$$

Since eigenfunctions can be multiplied with any non-zero number, the above equation states basically that the Wronskian is constant. A similar property of a different Wronskian has been established in Freiberg [7, p. 41].

10. Figures and numbers

In this section we give some explicit results and figures calculated by using formulas we developed in the preceding sections for several examples of self-similar measures. For the calculations we used Sagemath cloud [28]. The program code that we used can be found in the appendix.

Example 10.1. Tables 1, 2 and 3 collect the first few values of the sequences $(p_n)_n$ and $(q_n)_n$ for the classical Cantor set with evenly distributed measure, that is, for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. We computed these values with the recursion formulas in Corollary 7.11 that we implemented for that purpose in Sagemath.

Figures 1 and 2 show plots of the functions \sin^N_{μ} and \sin^D_{μ} for $x \in (0, 50)$ and for $x \in (0, 120)$, respectively, where the first 100 terms of the series are taken into account. In the figures, \sin^D_{μ} is drawn in a dash-dot line and \sin^N_{μ} in a solid line. The zero points of these functions squared give the Dirichlet and Neumann eigenvalues, respectively. Observe that the pictures suggest that the eigenvalues are in the order

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \dots$$

Table 4 contains the first 32 positive Neumann eigenvalues correct to 15 decimal places (rounded down). These values have been calculated as zero points of the polynomial

$$\sum_{n=0}^{a} (-1)^n p_{2n+1} z^n \quad \left(\approx \frac{\sin^N_\mu(\sqrt{z})}{\sqrt{z}}\right)$$

For that we used the command findroot from the mpmath library in Sagemath cloud ([28]) with a starting value that we took from a plot in each case. This way, we computed each zero point of the above polynomial with an accuracy of 100 digits, where we chose a in each case such that the first 15 decimals of the zero point remain fixed against any further increase of a. Note that by Lemma 3.3 we have

$$p_{2n+1} \le \frac{1}{n!} q_2(1)^n = \frac{1}{n! \cdot 2^n}$$

from which a more detailed error estimate can be obtained.

Observe that, as stated in Theorem 8.1, we have that $\lambda_{N,2m} = 6 \cdot \lambda_{N,m}$ for all m. The distances between eigenvalues differ very much, there are several groups that lie very close together while there are big gaps as well.

In Table 5 we give approximate values of the $L_2(\mu)$ norms and the sup norms of the eigenfunctions $f_{N,m} = c_{\lambda,\mu}(\sqrt{\lambda_m}, \cdot)$.

The L_2 norms have been calculated with the formula in Corollary 4.3 where we put in the values for λ from Table 4. The number of summands had to be chosen higher with bigger eigenvalues, so that the limit value could be approximated with sufficient accuracy.

For the supremum norms we calculated $f_{N,m}(S_w(0))$ and $f_{N,m}(S_w(1))$ for all words $w \in \{1,2\}^n$ for a certain iteration level n and determined the biggest of these values.

We varied *n* between 5 and 8 to get the values. These calculations were made with the formulas in Proposition 7.8. For that, the eigenvalue λ_m and values of the functions \sin^N_{μ} , \sin^D_{μ} and \cos^N_{μ} were needed. Note that the sup norm values are rather rough approximations.

Then we determined the sup norm of the normalized eigenfunctions

$$\|\tilde{f}_{N,m}\|_{\infty} = \frac{\|f_{N,m}\|_{\infty}}{\|f_{N,m}\|_{L_2(\mu)}}$$

Observe that, as stated in Equation (54), the values for even m are the same as for $\frac{m}{2}$, respectively.

Figure 3 shows plots of $f_{1,N}$ to $f_{4,N}$ and $f_{1,D}$ to $f_{4,D}$. These were done by iterative use of the formulas in Propositions 7.7 and 7.8 as for the calculation of the sup-norms.

In Table 6 we state the first 32 eigenvalues with Dirichlet boundary conditions exact to 15 decimals. The procedure for the calculations is the same as with the Neumann eigenvalues explained above. Note that two values at a time lie close together, namely $\lambda_{D,2m-1}$ and $\lambda_{D,2m}$. Especially close together are pairs of the form $\lambda_{D,2n-1}$ and $\lambda_{D,2n}$. Therefore we had to increase the accuracy of $\lambda_{D,31}$ and $\lambda_{D,32}$ to 25 digits to make the difference visible.

Estimates of the Dirichlet eigenvalues have also been obtained by Vladimirov and Sheipak in [29] and by Etienne [5] with completely different methods.

As in the Neumann case, we calculated norms of Dirichlet eigenfunctions, see Table 7.

n	p_{2n+1}
0	1
1	$\frac{1}{5}$
2	$\frac{27}{2800}$
3	$\frac{6383}{31906000}$
4	$\frac{928046087}{427065638720000}$
5	$\frac{18312146532699}{1290321173531252800000}$
6	$\frac{36205626974761334065053}{595390835517679574442022016000000}$
7	$\frac{4976934962986304441117658183}{27444983400881701904144720110742041600000}$
8	$\frac{9554109968352546557662907330504773561465623}{24293779244421488801231482393897413175652507508121600000000}$
9	$\frac{146991787616583137720984325054111289057094244281881523497}{228839658236344563453452927437095017291959177590164358527465655296000000000}$

Table 1: The first ten odd members of (p_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

n	q_{2n+1}
0	1
1	$\frac{1}{8}$
2	$\frac{21}{4240}$
3	$\frac{33253}{383465600}$
4	$\frac{76118969}{91537621184000}$
5	$\frac{20165083798890939}{4103397246999022891520000}$
6	$\frac{129726498389261896497}{6714982210971717632658867200000}$
7	$\frac{2413673468793966201825434809368471}{45210174990342427454327995801851920608256000000}$
8	$\frac{1194381655935980000421990244022269580561517}{11036319046998816108771342849627021590229476137440051200000}$
9	$\frac{126866175828333349955887526100988154691317901447037378112773}{762232235417372510271600164875680211782266161937386279477493896522956800000000}$

Table 2: The first ten odd members of (q_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.



Table 3: The first ten even members of (p_n) and (q_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.



Figure 1: \sin^N_{μ} (solid) and \sin^D_{μ} (dash-dot) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.



Figure 2: \sin^N_{μ} (solid) and \sin^D_{μ} (dash-dot) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.



Figure 3: The first four Neumann (left) and Dirichlet (right) eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{N,m}$	a	m	$\lambda_{N,m}$	a
1	7.097431098141122	12	17	9211.739397756251229	86
2	42.584586588846733	19	18	9288.334945327771442	85
3	61.344203922701662	19	19	9316.347022410075024	85
4	255.507519533080403	28	20	9827.408549489299413	87
5	272.983570819147205	28	21	9847.990083199668501	87
6	368.065223536209975	30	22	9975.764600539458261	87
7	383.552883127693176	31	23	9994.037352597068208	87
8	1533.045117198482423	47	24	13250.348047303559112	97
9	1548.055824221295240	47	25	13260.716598784444965	96
10	1637.901424914883235	48	26	13324.616699806778407	97
11	1662.627433423243043	48	27	13342.227668891503102	97
12	2208.391341217259852	53	28	13807.903792596954355	97
13	2220.769449967796401	53	29	13816.727250920634538	98
14	2301.317298766159059	53	30	13875.492724365506427	98
15	2312.582120727584404	53	31	13883.672380356518424	97
16	9198.270703190894542	85	32	55189.624219145367256	160
					•

Table 4: Neumann eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\ f_{N,m}\ _2$	$\ f_{N,m}\ _{\infty}$	$\ \tilde{f}_{N,m}\ _{\infty}$	m	$\ f_{N,m}\ _2$	$\ f_{N,m}\ _{\infty}$	$\ \tilde{f}_{N,m}\ _{\infty}$
1	0.801	1.000	1.248	17	0.666	1.001	1.503
2	0.801	1.000	1.248	18	0.687	1.007	1.467
3	0.966	1.261	1.306	19	0.829	1.307	1.577
4	0.801	1.000	1.248	20	0.746	1.049	1.405
5	0.746	1.049	1.405	21	0.688	1.093	1.588
6	0.966	1.261	1.306	22	0.897	1.356	1.512
7	1.145	1.604	1.401	23	1.057	1.703	1.612
8	0.801	1.000	1.248	24	0.966	1.261	1.306
9	0.687	1.007	1.467	25	0.826	1.262	1.529
10	0.746	1.049	1.405	26	0.886	1.306	1.474
11	0.897	1.356	1.512	27	1.063	1.694	1.594
12	0.966	1.261	1.306	28	1.145	1.604	1.401
13	0.886	1.306	1.474	29	1.049	1.656	1.579
14	1.145	1.604	1.401	30	1.346	2.029	1.508
15	1.346	2.029	1.508	31	1.579	2.563	1.625
16	0.801	1.000	1.248	32	0.801	1.000	1.248

Table 5: Norms of Neumann eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{D,m}$	$\mid a$	m	$ $ $\lambda_{D,m}$	a
1	14.435240512053874	13	17	9233.86793 80086 63779	84
2	35.26023 80242 77225	16	18	9271.628792721274161	83
3	140.78105 33845 56059	24	19	9589.268396141598781	85
4	151.29061 60550 19631	23	20	9598.240412584912727	85
5	326.057328357753770	29	21	9923.464452585818608	85
6	353.41692 07675 57756	29	22	9957.065202153829857	87
7	876.27445 96020 73755	39	23	12190.285583570241470	93
8	876.50531 85096 60313	39	24	12190.292419099534112	94
9	1581.17702 42871 45662	46	25	13284.126824873170732	94
10	1619.40072 91584 24238	46	26	13311.274484046062950	95
11	2029.61356 34510 19039	51	27	13668.536903946319748	96
12	2033.85281 30577 61437	51	28	13671.268166872611762	96
13	2268.79163 36445 60767	53	29	13851.839512664376419	96
14	2289.604069442469130	52	30	13866.937824133173771	96
15	5258.339396921217309	71	31	31550.0364002815218746422325788	139
16	5258.33940 31726 23308	71	32	31550.0364002815218748968965410	139

Table 6: Dirichlet eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\ f_{D,m}\ _2$	$\ f_{D,m}\ _{\infty}$	$\ \tilde{f}_{D,m}\ _{\infty}$	m	$\ f_{D,m}\ _2$	$\ f_{D,m}\ _{\infty}$	$\ \widetilde{f}_{D,m}\ _{\infty}$
1	0.627	0.920	1.469	17	8.492	15.635	1.841
2	0.711	0.985	1.387	18	7.115	12.394	1.742
3	0.446	0.790	1.770	19	1.685	3.996	2.372
4	0.457	0.793	1.734	20	1.679	3.985	2.374
5	1.115	1.628	1.461	21	5.110	10.266	2.009
6	1.273	2.105	1.654	22	5.787	11.252	1.944
7	0.262	0.646	2.469	23	0.415	1.195	2.883
8	0.262	0.646	2.468	24	0.415	1.306	3.151
9	2.798	5.034	1.799	25	9.565	18.428	1.927
10	2.656	4.694	1.767	26	8.944	16.597	1.856
11	0.719	1.460	2.032	27	1.950	4.852	2.489
12	0.717	1.602	2.233	28	1.945	4.863	2.501
13	3.048	5.661	1.857	29	8.777	15.403	1.755
14	3.481	6.296	1.809	30	9.995	19.451	1.946
15	0.151	0.509	3.369	31	0.087	0.416	4.765
16	0.151	0.528	3.491	32	0.087	0.416	4.765

Table 7: Norms of Dirichlet eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.



Figure 4: \sin_{μ}^{N} (solid) and \sin_{μ}^{D} (dash-dot) for $r_{1} = 1/3$, $r_{2} = 1/4$, $m_{1} = \frac{1}{3^{d_{H}}}$ and $m_{2} = \frac{1}{4^{d_{H}}}$

Example 10.2. For the next example, we take the asymmetric self-similar measure with $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{1}{3^{d_H}}$ and $m_2 = \frac{1}{4^{d_H}}$ where d_H is the Hausdorff dimension of the invariant set. That is, d_H is the solution of the equation

$$\frac{1}{3^{d_H}} + \frac{1}{4^{d_H}} = 1$$

For the calculations we used 0.56049886522386387883902233 for d_H . Variation of this value led to no change in the first 15 digits of the eigenvalues. Plots of \sin^N_{μ} and \sin^D_{μ} are shown in Figure 4 and the first eigenvalues exact to 15 decimal places are displayed in Table 8. Note that here $m_1r_1 \neq m_2r_2$. There seem to be no fixed order of Neumann and Dirichlet eigenvalues as in Example 10.1 and there are no clear pairings of the values.

Example 10.3. Figure 5 shows plots of \sin^N_{μ} and \sin^D_{μ} for $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{4}$ and $m_1 = \frac{3}{7}$, $m_2 = \frac{4}{7}$. The invariant set is geometrically the same as in Example 10.2, but m_1 and m_2 are chosen such that $r_1m_1 = r_2m_2 = \frac{1}{7}$ and thus, $\lambda_{N,2m} = 7 \cdot \lambda_{N,m}$.

Comparing with Example 10.1, we observe that the Neumann eigenvalues behave qualitatively similar, but the Dirichlet eigenvalues do not appear in such close pairs. However, it seems to hold again, that two Neumann and two Dirichlet eigenvalues appear in turns.

Example 10.4. We choose the measure with $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$. This measure is supported on the whole interval [0, 1], yet is singular to the Lebesgue measure. In Theorem 9.1 we showed that \sin^N_{μ} and \sin^D_{μ} and thus the Dirichlet and

m	$\lambda_{N,m}$	a	m	$\lambda_{D,m}$	a
1	6.567037965687942	11	1	16.107849410419070	12
2	41.632795946820830	16	2	35.907601066462638	15
3	66.822767372091789	19	3	128.330447556120622	21
4	233.355013145153884	24	4	236.463676343561213	24
5	365.584215801794021	27	5	373.701929431216995	27
6	389.945618826510339	28	6	423.638157028808414	28
7	582.138208794906725	30	7	713.786986198043209	31
8	1295.888937033626505	37	8	2013.164883016581104	44

Table 8: Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{1}{3^{d_H}}$ and $m_2 = \frac{1}{4^{d_H}}$.



Figure 5: \sin^{N}_{μ} (solid) and \sin^{D}_{μ} (dash-dot) for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

m	$\lambda_{N,m}$	a	m	$\lambda_{D,m}$	a
1	6.752284245618646	10	1	16.452512161464721	12
2	47.265989719330522	16	2	36.904245287406090	15
3	62.066872795561511	18	3	154.577520453343494	21
4	330.861928035313659	26	4	212.376524344704458	23
5	345.194670941772007	27	5	395.526819249411977	27
6	434.468109568930577	28	6	417.532700806716224	27
7	446.407999438501248	28	7	1083.253271255975735	34
8	2316.033496247195616	45	8	1485.470110503836517	37
9	2332.825185220436900	46	9	2360.481274606702758	44
10	2416.362696592404055	46	10	2397.801276276128236	44
11	2434.484694248270572	46	11	2830.491432008378221	47
12	3041.276766982514042	50	12	2850.987710468049166	47
13	3051.736543145083444	50	13	3093.525406096403347	48
14	3124.855996069508739	50	14	3111.593713450879200	48
15	3133.914016082441210	49	15	7582.772906434721944	58
16	16212.234473730369315	82	16	10398.290767742394136	64

Table 9: Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

Neumann eigenvalues coincide. In Figure 6 a plot of \sin_{μ}^{N} is displayed. It is comparable to the sine function, which we would get for $r_1 = r_2 = m_1 = m_2 = 0.5$. Table 10 contains the first 16 eigenvalues.

Example 10.5. We take $r_1 = 0.9$, $r_2 = 0.1$, $m_1 = 0.1$ and $m_2 = 0.9$. The resulting measure is supported on [0, 1] as in Example 10.4, but in Figure 7 we see that \sin^N_{μ} looks very different from the sine function. Table 11 contains the first 16 eigenvalues up to 15 decimal places.

m	λ_m	a	m	λ_m	a
1	11.113238313123921	13	9	1012.173153820335730	54
2	46.305159638016340	19	10	1194.689209582619744	57
3	97.600761284513435	24	11	1396.721102656337624	60
4	192.938165158401419	29	12	1694.457661189469363	65
5	286.725410299828738	34	13	1910.161155469398890	67
6	406.669838685472647	38	14	2157.157626864968439	70
7	517.717830447592425	41	15	2316.668360848575284	73
8	803.909021493339246	48	16	3349.620922888913526	83

Table 10: Neumann (and Dirichlet) eigenvalues for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.



Figure 6: \sin_{μ}^{N} (coincides with \sin_{μ}^{D}) for $r_{1} = 0.6$, $r_{2} = 0.4$, $m_{1} = 0.4$ and $m_{2} = 0.6$.



Figure 7: \sin_{μ}^{N} (coincides with \sin_{μ}^{D}) for $r_{1} = 0.9$, $r_{2} = 0.1$, $m_{1} = 0.1$ and $m_{2} = 0.9$.

m	λ_m	a	m	λ_m	a
1	111.021168159382246	9	9	164619.744662988877161	44
2	1233.568535104247184	15	10	165477.638319057818349	43
3	1403.454381590697316	15	11	166543.871983977116823	43
4	13706.317056713857601	24	12	173265.973035888557594	43
5	14892.987448715203651	24	13	176376.608221577384951	43
6	15593.937573229970183	25	14	177512.483919664028463	44
7	15976.123552769762561	25	15	178137.700783469958606	44
8	152292.411741265084466	40	16	1692137.908236278716292	72

Table 11: Neumann (and Dirichlet) eigenvalues for $r_1 = 0.9$, $r_2 = 0.1$, $m_1 = 0.1$ and $m_2 = 0.9$.

11. Remarks and outlook

In this section we state several remarks and thoughts that could be subject of future studies.

Conjecture 1. Due to the examination of several examples (see e.g. Examples 10.1, 10.3, 10.4 and 10.5) we conjecture that in case of a self-similar measure μ with $r_1m_1 = r_2m_2$ the Neumann and Dirichlet eigenvalues satisfy

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \dots$$

Remark 2. It would be very interesting to find out, if there was a relation between our sequences $(p_n)_n$ and $(q_n)_n$ to any known number sequences as e.g. Bernoulli or Euler numbers. Indeed, the definition of $p_n(x)$ or $q_n(x)$ (Definition 3.1) is reminiscent of the recursive definition of the Euler polynomials $E_n(x)$ by $E_0(x) := 1$ and

$$E_n(x) := \int_c^x n E_{n-1}(t) \, dt,$$

where $c = \frac{1}{2}$ if n is odd and c = 0 for even n. Then the nth Euler number is $E_n = 2^n E_n(1/2)$.

Furthermore, Equation (13) has a similar structure as the recursion rule

$$\alpha_n = \frac{1}{2n} \sum_{j=0}^{n-1} \alpha_j \, \alpha_{n-1-j}$$

with $\alpha_0 = \alpha_1 = 1$, where $\alpha_n = \frac{1}{n!} |E_{2n}|$.

Remark 3. One could investigate the functional equations in Theorem 7.14 further. In the simple case where $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$, for instance, we can transform them (after eliminating terms containing \sin^D_{μ} by using $\cos^N_{\mu}(z)^2 + \sin^N_{\mu}(z) \sin^D_{\mu}(z) = 1$)

with the abbreviations $u(z) = z \sin^N_{\mu}(z)$ and $v(z) = 2 \cos^N_{\mu}(z)$ to

$$u(\sqrt{6}z) = 3 u(z) v(z) - 3 u(z)^{2}$$
$$v(\sqrt{6}z) = v(z)^{2} - v(z) u(z) - 2.$$

From this one can derive recursion formulas for the sequence $(p_n)_n$ that contain only members of p_n and not, as in Corollary 7.11, both p_n and q_n .

Furthermore, it could be possible to somehow solve these functional equations to get a more direct representation of \sin_{μ}^{N} and \cos_{μ}^{N} .

Remark 4. We defined our functions \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} only for real arguments. However, one can just allow the argument to be complex. Then these power series can be treated with methods of complex analysis.

Remark 5. It is also interesting to consider the eigenvalue problem

$$\frac{d}{d\mu}\frac{d}{d\nu}f = -\lambda f$$

with appropriate boundary conditions, where both μ and ν are non-atomic finite Borel measures. This can be done by modifying the above considerations by replacing λ with ν .

The case where both derivatives are with respect to the same measure, that is, $\mu = \nu$ is much simpler. There we get

$$c_{\mu,\mu}(z,x) = \cos(z \, p_1(x))$$

and

$$\mathbf{s}_{\mu,\mu}(z,x) = \sin\bigl(z\,p_1(x)\bigr).$$

The eigenvalues are $\lambda_k = k^2 \pi^2$, $k \in \mathbb{N}$, as in the classical Lebesgue measure case. This is treated in Arzt and Freiberg [2]. See also Freiberg and Zähle [12].

Remark 6. Our recursion law for p_n and q_n works only for self-similar measures with $r_1 + r_2 \leq 1$. It would be interesting to develop similar formulas for measures with overlaps, i.e. with $r_1 + r_2 > 1$. Such measures are treated for example in Ngai [24] and Chen and Ngai [4], which contains, in particular, numerical solutions of the eigenvalue problem by the finite elements method.

Remark 7. In this work, we examined the eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ by following the basic lines of the treatment of the classical second derivative operator on the interval. In this classical case all eigenvalues are multiples of π^2 and have therefore direct representations in many forms, e.g. by using the series expansion of arctan. Maybe one can find a series representation of eigenvalues of the generalized operator, too, by using such functions as \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} .

Remark 8. In Corollary 8.7 we stated upper and lower estimates for subsequences $(\|\tilde{f}_{2^k l}\|_{\infty})_k$, l odd, of the suprema of the normed eigenfunctions. We have no information about the growth of the sequence $(\|\tilde{f}_{2k+1}\|_{\infty})_k$, though.

Such estimates could be used to prove estimates of the heat kernel

$$K(t, x, y) = \sum_{m=1}^{\infty} e^{-\lambda_m t} \tilde{f}_m(x) \, \tilde{f}_m(y)$$

for the corresponding quasi-diffusion process. This process has been investigated for example in Löbus [21] and Küchler [17, 18].

Remark 9. We used the functions $p_n(x)$ and $q_n(x)$, $x \in [0, 1]$, defined in Definition 3.1 to replace monomials $\frac{1}{n!}x^n$ in the classical case. One could use these functions to build a kind of generalized polynomials that are adjusted to the measure μ . For instance, we take the sequence

$$\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = q_1(x), \quad \tilde{P}_2(x) = p_2(x), \quad \tilde{P}_3(x) = q_3(x), \quad \dots$$

and orthogonalize it in $L_2(\mu)$ by using the Gram-Schmidt process. We take odd numbered $q_n(x)$ and even numbered $p_n(x)$, because they are the building blocks for the eigenfunctions $s_{\lambda,\mu}(z,\cdot)$ and $c_{\lambda,\mu}(z,\cdot)$ and they are continuously Lebesgue-differentiable, namely

$$q'_n(x) = q_{n-1}(x)$$
 for odd n

and

$$p'_n(x) = p_{n-1}(x)$$
 for even n .

Furthermore, we can μ -integrate them and get

$$\int_0^x q_n(t) \, d\mu(t) = q_{n+1}(x) \quad \text{for odd } n$$

and

$$\int_0^x p_n(t) \, d\mu(t) = p_{n+1}(x) \quad \text{for even } n.$$

With that we can apply the generalized integration by parts rule from Lemma 2.2 to do the calculations in the Gram-Schmidt algorithm. Note again that we use the notation $p_n := p_n(1)$ and $q_n := q_n(1)$ and assume that those numbers are given since we have a recursion rule in the self-similar case. Then we get

$$P_0(x) := 1,$$
$$P_1(x) := q_1(x) - \int_0^1 q_1(t) \, d\mu(t) = q_1(x) - q_2,$$

$$P_2(x) := p_2(x) - \int_0^1 p_2(t) \, d\mu(t) - \frac{\int_0^1 (q_1(t) - q_2) \, p_2(t) \, d\mu(t)}{\int_0^1 (q_1(t) - q_2)^2 \, d\mu(t)} (q_1(x) - q_2).$$

We calculate

$$\int_0^1 (q_1(t) - q_2) p_2(t) d\mu(t) = \int_0^1 q_1(t) p_2(t) d\mu(t) - \int_0^1 q_2 p_2(t) d\mu(t)$$
$$= q_1(t) p_3(t) \Big|_0^1 - \int_0^1 p_3(t) dt - q_2 p_3$$
$$= q_1 p_3 - p_4 - q_2 p_3$$

and

$$\int_0^1 (q_1(t) - q_2)^2 d\mu(t) = \int_0^1 q_1(t)^2 d\mu(t) - 2q_2 \int_0^1 q_1(t) d\mu(t) + q_2^2 \int_0^1 d\mu(t)$$
$$= q_1 q_2 - \int_0^1 q_2(t) dt - 2q_2^2 + q_2^2 p_1$$
$$= q_1 q_2 - q_3 - 2q_2^2 + q_2^2 p_1.$$

To simplify these expressions a bit we utilize $p_1 = q_1 = 1$ which follows from the definition and $p_2 + q_2 = 1$ which follows from Corollary 5.2 by putting n = 1. Then we get

$$P_2(x) = p_2(x) - \frac{p_4 - p_2 p_3}{q_3 - p_2 q_2} q_1(x) + \frac{q_2 p_4 - q_3 p_3}{q_3 - p_2 q_2}$$

In this fashion one can calculate a sequence of $L_2(\mu)$ -orthogonal "polynomials".

As an example we take the Lebesgue measure for μ and put $p_n(x) = q_n(x) = \frac{1}{n!}x^n$. Then

$$P_0(x) = 1$$
, $P_1(x) = x - \frac{1}{2}$, $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$

which are the first Legendre polynomials on [0, 1] (not normed).

If μ is the standard Cantor measure, then $p_2 = q_2 = \frac{1}{2}$, $p_3 = \frac{1}{5}$ and $q_3 = \frac{1}{8}$ and we get

$$P_0(x) = 1$$
, $P_1(x) = q_1(x) - \frac{1}{2}$, $P_2(x) = p_2(x) - \frac{1}{2}q_1(x) + \frac{1}{20}$.

Maybe one can use these functions for further analytical studies.

Remark 10. With the presented methods one could investigate not only the equation $\frac{d}{d\mu}f' = -\lambda f$, but maybe other differential equations on the interval [0, 1] that are generalized involving a self-similar measure μ .

Remark 11 (Fourier series). It is well known that the normed eigenfunctions $(\tilde{f}_{N,k})_{k=0}^{\infty}$ and $(\tilde{f}_{D,k})_{k=1}^{\infty}$ form orthonormal bases in $L_2(\mu)$ (see [7]). We denote $n_{N,k} := \|\mathbf{c}_{\lambda,\mu}(\sqrt{\lambda_{N,k}}, \cdot)\|_{L_2(\mu)}$ and $n_{D,k} := \|\mathbf{s}_{\lambda,\mu}(\sqrt{\lambda_{D,k}}, \cdot)\|_{L_2(\mu)}$ so that

$$\tilde{f}_{N,k} = \frac{1}{n_{N,k}} c_{\lambda,\mu}(\sqrt{\lambda_{N,k}}, \cdot)$$

and

$$\tilde{f}_{D,k} = \frac{1}{n_{D,k}} \mathbf{s}_{\lambda,\mu}(\sqrt{\lambda_{D,k}}, \cdot)$$

We decompose some functions $f \in L_2(\mu)$ into series of eigenfunctions (Fourier series), ignoring questions about convergence for the moment. Assume that for $x \in [0, 1]$

$$f(x) = \sum_{k=0}^{\infty} a_k \tilde{f}_{N,k}(x)$$

with

$$a_k = \int_0^1 f(t) \,\tilde{f}_{N,k}(t) \,d\mu(t).$$

For reasons of simplicity, we take μ to be a symmetric measure. Then $\cos^N_{\mu} = \cos^D_{\mu}$ and we have $\cos^N_{\mu}(z)^2 + \sin^N_{\mu}(z) \sin^D_{\mu}(z) = 1$. From that follows that $\cos^N_{\mu}(\sqrt{\lambda_{N,k}})^2 = 1$ and it is heuristically clear that $\cos^N_{\mu}(\sqrt{\lambda_{N,k}}) = (-1)^k$. Employing this fact and Lemma 2.2, the computations can be made explicitly, following the lines of the classical (Euclidean) case.

As a first example, take f(x) = x. Then, for $k \in \mathbb{N}$,

$$\begin{aligned} a_k &= \frac{1}{n_{N,k}} \int_0^1 t \cdot \mathbf{c}_{\lambda,\mu}(\sqrt{\lambda_k}, t) \, d\mu(t) \\ &= \frac{1}{n_{N,k}} \left[\frac{1}{\sqrt{\lambda_{N,k}}} t \, \mathbf{s}_{\mu,\lambda}\left(\sqrt{\lambda_{N,k}}, t\right) \Big|_0^1 - \frac{1}{\sqrt{\lambda_{N,k}}} \int_0^1 \mathbf{s}_{\mu,\lambda}\left(\sqrt{\lambda_{N,k}}, t\right) \, dt \right] \\ &= \frac{1}{n_{N,k}\lambda_{N,k}} \, \mathbf{c}_{\lambda,\mu}(\sqrt{\lambda_{N,k}}, t) \Big|_0^1 \\ &= \frac{1}{n_{N,k}\lambda_{N,k}} \left(\cos^N_\mu(\sqrt{\lambda_{N,k}}) - 1 \right). \end{aligned}$$

Thus, $a_k = 0$ for even $k \ge 1$ and $a_k = -\frac{2}{n_{N,k}\lambda_{N,k}}$ for odd k. Furthermore, we have

$$a_0 = \int_0^1 t \, d\mu(t) = q_2(1) = q_2.$$

Therefore, we have the decomposition into Neumann eigenfunctions

$$x = q_2 - 2\sum_{k=0}^{\infty} \frac{1}{c_{N,2k+1}\lambda_{N,2k+1}} \tilde{f}_{N,2k+1}(x).$$

Note that the required norms $n_{N,k}$ can be computed with Corollary 4.3.

We apply Parseval's identity to this series. This gives

$$\int_0^1 t^2 d\mu(t) = q_2^2 + \sum_{k=0}^\infty \frac{4}{c_{N,2k+1}^2 \lambda_{N,2k+1}^2},$$

and with

$$\int_0^1 t^2 d\mu(t) = t q_2(t) \Big|_0^1 - \int_0^1 q_2(t) dt = q_2 - q_3$$

and $1 - q_2 = p_2$ we get

$$\sum_{k=0}^{\infty} \frac{1}{c_{N,2k+1}^2 \lambda_{N,2k+1}^2} = \frac{1}{4} (p_2 q_2 - q_3).$$

If we choose the Lebesgue measure for μ (then $p_2 = q_2 = \frac{1}{2}$, $q_3 = \frac{1}{6}$ and $c_{N,2k+1}^2 = \frac{1}{2}$), the above equation becomes the well known identity

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

In the same fashion we compute the decomposition of some more examples (μ symmetric):

$$x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{n_{D,k}\sqrt{\lambda_{D,k}}} \tilde{f}_{D,k}(x)$$

$$1 = \sum_{k=0}^{\infty} \frac{2}{c_{D,2k+1}\sqrt{\lambda_{D,2k+1}}} \tilde{f}_{D,2k+1}(x)$$

$$f_{D,2n+1}(x) = \frac{2}{\sqrt{\lambda_{D,2n+1}}} - 2\sqrt{\lambda_{D,2n+1}} \sum_{k=1}^{\infty} \frac{1}{(\lambda_{N,2k} - \lambda_{D,2n+1})c_{N,2k}} \tilde{f}_{N,2k}(x),$$
for every $n \in \mathbb{N}_0$

which are plotted for the standard middle third Cantor measure in Figures 8 and 9. For the images we computed graphs of the eigenfunctions by iteratively applying the formulas in Propositions 7.7 and 7.8. For the normalization we used the norms $n_{N,k}$ and $n_{D,k}$ shown in Tables 5 and 7.



Figure 8: Left side: approximation of f(x) = x by the first 5 terms of the Dirichlet Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Right side: approximation of f(x) = 1 by the first 5 terms of the Dirichlet Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

Applying Parseval's identity to these decompositions leads, as above, to

$$\sum_{k=1}^{\infty} \frac{1}{n_{D,k}^2 \lambda_{D,k}} = q_2 - q_3$$
$$\sum_{k=0}^{\infty} \frac{1}{c_{D,2k+1}^2 \lambda_{D,2k+1}} = \frac{1}{4}$$
$$\sum_{k=1}^{\infty} \frac{1}{\left(\lambda_{N,2k} - \lambda_{D,2n+1}\right)^2 c_{N,2k}^2} = \frac{c_{D,2n+1}^2}{4\lambda_{D,2n+1}} - \frac{1}{\lambda_{D,2n+1}^2}$$

If we again take the Lebesgue measure for μ , we receive the well known identities

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$
$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$
$$\sum_{k=1}^{\infty} \frac{1}{\left(4k^2 - (2n+1)^2\right)^2} = \frac{\pi^2}{16(2n+1)^2} - \frac{1}{2(2n+1)^4}.$$

Remark 12. The definition of the operator $-\frac{d}{d\mu}\frac{d}{dx}$ can be extended to subsets of \mathbb{R}^d , $d \in \mathbb{N}$, see, for example, Solomyak and Verbitsky [27], Naimark and Solomyak [23] and Hu, Lau and Ngai [14]. This case, however, is substantially more difficult and the techniques presented here can probably not be readily extended to it.

Remark 13. Analogously to our measure trigonometric functions we can define measure



Figure 9: Left side: approximation of $f_{D,1}$ by the first 3 terms of the Neumann Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Right side: approximation of $f_{D,3}$ by the first 3 terms of the Neumann Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

theoretic exponential functions. For $x \in [0, 1]$ and $z \in \mathbb{C}$ put

$$e_{\lambda,\mu}(z,x) := \sum_{n=0}^{\infty} z^{2n} p_{2n}(x) + \sum_{n=0}^{\infty} z^{2n+1} q_{2n+1}(x)$$

and

$$e_{\mu,\lambda}(z,x) := \sum_{n=0}^{\infty} z^{2n} q_{2n}(x) + \sum_{n=0}^{\infty} z^{2n+1} p_{2n+1}(x)$$

Then, $e_{\lambda,\mu}(z,\cdot) \in H^2(\lambda,\mu)$ and $e_{\mu,\lambda}(z,\cdot) \in H^2(\mu,\lambda)$ for every $z \in \mathbb{C}$. Furthermore, for all $t \in \mathbb{R}$ and $x \in [0,1]$, we have Euler's formula

$$\mathbf{e}_{\lambda,\mu}(it,x) = \mathbf{c}_{\lambda,\mu}(t,x) + i\,\mathbf{s}_{\lambda,\mu}(t,x)$$

and

$$e_{\mu,\lambda}(it, x) = c_{\mu,\lambda}(t, x) + i s_{\mu,\lambda}(t, x).$$

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A. Program code

Here we present the code that was used to compute the values and graphics given in Section 10. It was written in Sagemath cloud [28], which is based on Python.

Each of the four programs run independently if copied into an empty Sage worksheet, but note that Python is sensitive to indentation.

Program 1 This is used to compute and print p_n and q_n .

```
# import mpmath for control over the accuracy in the calculations
# mp.dps determines the number of significant digits
# this is not needed if you give the parameters below as fractions, then
# the calculations are done symbolically
from mpmath import mp, mpf
mp.dps = 100
mp.pretty = True
# we set the parameters of the self-similar measure \mu
# r_1 and r_2 are the scaling factors,
\# m_1 and m_2 the weight factors, m_1 + m_2 =1,
# if the values are given as fraction like r1 = 1/3, the computation
# is done symbolically,
# if you give decimal values, write r1=mpf(0.333), then the calculation
# is done with
# the number of significant digits given by mp.dps above
r1 = 1/3
r_{2=1/3}
m1 = 1/2
m2 = 1/2
# we calculate p_n and q_n with the four formulas in Corollary 7.11
# the values are put in a list, the n-th entry is accessed
# by p[n] or q[n], n=0,1,2,...
p=[1,1]
q=[1,1]
for n in range(1,10):
    #p[2n]
    p.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* p[2*i] * p[2*n-2*i] for i in range(1,n)]) + sum([r1^i * m1^(i+1)
```

```
* r2^(n-i) * m2^(n-i-1) * p[2*i+1] * q[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([r1^i * m1^(i+1) * (r2 * m2)^(n-i-1) * p[2*i+1]
* p[2*n-2*i-2] for i in range(0,n)])))
    #q[2n]
    q.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* q[2*i] * q[2*n-2*i] for i in range(1,n)]) + sum([r1^(i+1) * m1^i
* r2^(n-i-1) * m2^(n-i) * q[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([(r1 * m1)^i * r2^(n-i-1) * m2^(n-i) * q[2*i]
* p[2*n-2*i-1] for i in range(0,n)])))
    #p[2n+1]
    p.append(1/(1-r1^n*m1^(n+1)-r2^n*m2^(n+1))*(sum([r1^i*m1^(i+1)
* (r2*m2)^(n-i) * p[2*i+1] * q[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i) * m2^(n-i+1) * p[2*i] * p[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^(i+1) * r2^(n-i-1))
* m2^(n-i) * p[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])))
    #q[2n+1]
    q.append(1/(1-r1^(n+1)*m1^n-r2^(n+1)*m2^n)*(sum([r1^(i+1)*m1^i
* (r2*m2)^(n-i) * q[2*i+1] * p[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i+1) * m2^(n-i) * q[2*i] * q[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^i * (r2 * m2)^(n-i)
* q[2*i] * p[2*n-2*i] for i in range(0,n+1)])))
#prints a table of the p_n from p_0 to p_19
print(' n p_n');
for n in range(0,20):
   print('{0:2} {1}'.format(n,p[n]));
#prints a table of the q_n from q_0 to q_19
print(' n q_n');
for n in range(0,20):
   print('{0:2} {1}'.format(n,q[n]));
```

Program 2 This first computes p_n and q_n for n = 0, ..., 2an - 1, then plots the function

$$\frac{\sin_{\mu}^{N}(\sqrt{z})}{\sqrt{z}} \approx \sum_{k=0}^{\mathrm{an}-1} (-1)^{k} p_{2k+1} z^{k}, \quad z > 0,$$

whose zero points are the Neumann eigenvalues. Then, with starting points read off the plot, it computes the Neumann eigenvalues numerically. The same is done with the Dirichlet eigenvalues.

```
# import mpmath for contol over the accuracy in the calculations
# mp.dps determines the number of significant digits
# findroot enables us to find roots with accuracy given by mp.dps
from mpmath import mp, mpf, findroot
mp.dps = 80
mp.pretty = True
# we set the parameters of the self-similar measure \mu
# r_1 and r_2 are the scaling factors,
\# m_1 and m_2 the weight factors, m_1 + m_2 =1,
# for faster computation, write r1=mpf(1/3), ..., this converts the
# fraction in the mpmath-float format with number of significant
# digits given by mp.dps
r1 = 1/3
r2=1/3
m1 = 1/2
m2 = 1/2
# we calculate p_n and q_n with the four formulas in Corollary 7.11
# the values are put in a list, the n-th entry is accessed by p[n]
# or q[n], n=0,1,2,...
# 'an' gives the number of terms we compute
# that is, we get p_0,..., p_{2*an-1} and q_0, ..., q_{2*an-1}
an = 50;
p = [1, 1]
q=[1,1]
for n in range(1,an):
    #p[2n]
    p.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* p[2*i] * p[2*n-2*i] for i in range(1,n)]) + sum([r1^i * m1^(i+1)])
* r2^(n-i) * m2^(n-i-1) * p[2*i+1] * q[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([r1^i * m1^(i+1) * (r2 * m2)^(n-i-1) * p[2*i+1]
```

```
* p[2*n-2*i-2] for i in range(0,n)])))
```

```
#q[2n]
    q.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* q[2*i] * q[2*n-2*i] for i in range(1,n)]) + sum([r1^(i+1) * m1^i
* r2^(n-i-1) * m2^(n-i) * q[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([(r1 * m1)^i * r2^(n-i-1) * m2^(n-i) * q[2*i]
* p[2*n-2*i-1] for i in range(0,n)])))
    #p[2n+1]
    p.append(1/(1-r1^n*m1^(n+1)-r2^n*m2^(n+1))*(sum([r1^i*m1^(i+1)
* (r2*m2)^(n-i) * p[2*i+1] * q[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i) * m2^(n-i+1) * p[2*i] * p[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^(i+1) * r2^(n-i-1))
* m2^(n-i) * p[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])))
    #q[2n+1]
    q.append(1/(1-r1^(n+1)*m1^n-r2^(n+1)*m2^n)*(sum([r1^(i+1)*m1^i
* (r2*m2)^(n-i) * q[2*i+1] * p[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i+1) * m2^(n-i) * q[2*i] * q[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^i * (r2 * m2)^(n-i)
* q[2*i] * p[2*n-2*i] for i in range(0,n+1)])))
# defines the function f(z) = \frac{\sum_{x \in \mathbb{N}} \frac{\sqrt{z}}{\sqrt{z}}}{\sqrt{z}},
# see the explanation in Example 10.1,
# 'an' denotes the number of considered summands of the series,
\# z=mpf(z) converts the argument to mpmath float format for higher
# precision,
# the zeros of this function are the Neumann eigenvalues
def f(z):
    z=mpf(z);
    return sum([(-1)^k * p[2*k+1]*z^k \text{ for } k \text{ in } range(0,an)])
# plots the function f on (0,400),
# if 'an' is big enough, you can read approximate values for the Neumann
# eigenvalues from the graph
plot(f,(0,400));
```

calculates an approximation of Neumann eigenvalue near the given

```
# starting point,
# starting points can be read off the plot,
# accuracy depends on 'an' and 'mp.dps'
findroot(f,10);
findroot(f,40);
findroot(f,60);
findroot(f,250);
findroot(f,270);
findroot(f,360);
findroot(f,380);
# defines the function g(z) = \frac{\sum_{x \in \mathbb{Z}}}{\frac{z}},
# analogously to the Neumann case
# the zeros of this function are the Dirichlet eigenvalues
def g(z):
    z=mpf(z);
    return sum([(-1)^k * q[2*k+1]*mpf(z)^k for k in range(0,an)])
# plots the function g on (0, 400)
# if 'an' is big enough, you can read approximate values for the
# Dirichlet eigenvalues from the graph
plot(g,(0,400));
# calculates an approximation of Dirichlet eigenvalue near the given
# starting point,
# starting points can be read off the plot,
# accuracy depends on 'an' and 'mp.dps'
findroot(g,15);
findroot(g,35);
findroot(g,140);
findroot(g,150);
findroot(g,320);
findroot(g,350);
```

Program 3 This plots \sin^N_{μ} , \sin^D_{μ} , \cos^N_{μ} and \cos^D_{μ} as well as the first Neumann and Dirichlet eigenfunctions. Furthermore, it gives approximate values for the suprema of these eigenfunctions.

```
from mpmath import mp, mpf, findroot
mp.dps = 80
mp.pretty = True
r1=mpf(1/3)
r2=mpf(1/3)
m1 = mpf(1/2)
m2=mpf(1/2)
an = 50;
p = [1, 1]
q = [1, 1]
for n in range(1,an):
    #p[2n]
   p.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* p[2*i] * p[2*n-2*i] for i in range(1,n)]) + sum([r1^i * m1^(i+1)])
* r2^(n-i) * m2^(n-i-1) * p[2*i+1] * q[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([r1^i * m1^(i+1) * (r2 * m2)^(n-i-1) * p[2*i+1]
* p[2*n-2*i-2] for i in range(0,n)])))
    #q[2n]
    q.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* q[2*i] * q[2*n-2*i] for i in range(1,n)]) + sum([r1^(i+1) * m1^i
* r2^(n-i-1) * m2^(n-i) * q[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([(r1 * m1)^i * r2^(n-i-1) * m2^(n-i) * q[2*i]
* p[2*n-2*i-1] for i in range(0,n)])))
    #p[2n+1]
    p.append(1/(1-r1^n*m1^(n+1)-r2^n*m2^(n+1))*(sum([r1^i*m1^(i+1)
* (r2*m2)^(n-i) * p[2*i+1] * q[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i) * m2^(n-i+1) * p[2*i] * p[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^(i+1) * r2^(n-i-1)
* m2^(n-i) * p[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])))
    #q[2n+1]
    q.append(1/(1-r1^(n+1)*m1^n-r2^(n+1)*m2^n)*(sum([r1^(i+1)*m1^i
* (r2*m2)^(n-i) * q[2*i+1] * p[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i+1) * m2^(n-i) * q[2*i] * q[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^i * (r2 * m2)^(n-i))
```

```
* q[2*i] * p[2*n-2*i] for i in range(0,n+1)])))
```

```
# defines \sin_{\mu}^N, \sin_{\mu}^D, \cos_{mu}^N and \cos_{mu}^D,
# again, only the first 'an' terms are considered
# z=mpf(z) converts the argument to mpmath float format for higher
# precision
def sinN(z):
   z = mpf(z)
   return sum([(-1)^k * p[2*k+1]*z^(2*k+1) for k in range(0,an)])
def sinD(z):
   z = mpf(z)
   return sum([(-1)^k * q[2*k+1]*z^(2*k+1) for k in range(0,an)])
def cosN(z):
   z = mpf(z)
   return sum([(-1)^k * p[2*k]*z^(2*k) for k in range(0,an)])
def cosD(z):
   z = mpf(z)
   return sum([(-1)^k * q[2*k]*z^(2*k) for k in range(0,an)])
# plots the above defined functions
plot(sinN, (0, 21))
plot(sinD,(0,21))
plot(cosN,(0,21))
plot(cosD,(0,21))
# next we plot the eigenfunctions (c_{\lambda,\mu}(z,\cdot) and
# s_{\lambda,\mu}(z,\cdot) where z is the square root of an eigenvalue)
# we do this by iterative use of Propositions 7.7 and 7.8
# we define the IFS S_1, S_2
def S1(x):
   return r1*x
def S2(x):
   return r2*x-r2+1
# 'it' gives the number of iterations
it=4
```

```
# x is the list of 'corner points' of the self-similar set, that is,
# of the 'it'-th iteration
x = [0, 1]
for i in range(it):
    x = map(S1,x) + map(S2,x)
# with auxiliary function f2, g1, g2, zit we construct values of
# c_{\lambda, \mu} and s_{\lambda, \mu} iteratively at the points given
# in 'x'
def f2(a,b,z):
    return (cosN( sqrt(r1*m1)*z ) - (1-(r1+r2))*sqrt(m1/r1)*z
*sinN(sqrt(r1*m1)*z))*a - sqrt(r2*m1/(r1*m2))*sinN(sqrt(r1*m1)*z)*b
def g1(a):
    return sqrt(r1/m1)*a
def g2(a,b,z):
    return (sqrt(r1/m1)*sinD( sqrt(r1*m1)*z ) + (1-(r1+r2))*z
*cosD(sqrt(r1*m1)*z))*a + sqrt(r2/m2)*cosD(sqrt(r1*m1)*z) * b
def zit(z,i):
    if i == 0:
        return [z,z]
    else:
        return zit(z,i-1) + zit(z, i-1)
def clm(z,i):
    if i == 0:
        return [1, \cos N(z)]
    else:
        return clm(sqrt(r1*m1)*z,i-1) + map(f2,clm(sqrt(r2*m2)*z,i-1),
slm( sqrt(r2*m2)*z, i-1 ),zit(z,i-1) )
def slm(z,i):
    if i == 0:
        return [0, sinD(z)]
    else:
        return map(g1, slm(sqrt(r1*m1)*z,i-1)) + map(g2,
```

clm(sqrt(r2*m2)*z,i-1), slm(sqrt(r2*m2)*z, i-1), zit(z,i-1))

sets the first Neumann and Dirichlet eigenvalues for the standard # Cantor measure, calculated in 'determination of eigenvalues'

```
1N1 = mpf(7.097431098141122)
1N2 = mpf(42.584586588846733)
1N3 = mpf(61.344203922701662)
1N4 = mpf(255.507519533080403)
1N5 = mpf(272.983570819147205)
1N6 = mpf(368.065223536209975)
1N7 = mpf(383.552883127693176)
1D1 = mpf(14.435240512053874)
1D2 = mpf(35.260238024277225)
1D3 = mpf(140.781053384556059)
1D4 = mpf(151.290616055019631)
1D5 = mpf(326.057328357753770)
1D6 = mpf(353.416920767557756)
```

plots the first seven Neumann eigenfunctions (not normed)
the points at x ('corner points') are joined by straight lines,
this is alright because these 'gap intervals' do not belong to the
support of the measure \mu

```
list_plot(zip(x,clm(sqrt(lN1),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN2),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN3),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN4),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN5),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN6),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,clm(sqrt(lN7),it)),plotjoined=true,thickness=0.5)
```

plots the first six Dirichlet eigenfunctions (not normed)

```
list_plot(zip(x,slm(sqrt(lD1),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,slm(sqrt(lD2),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,slm(sqrt(lD3),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,slm(sqrt(lD4),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,slm(sqrt(lD5),it)),plotjoined=true,thickness=0.5)
list_plot(zip(x,slm(sqrt(lD6),it)),plotjoined=true,thickness=0.5)
```

we compute the sup-norm of the eigenfunctions
max(clm(sqrt(lN1),it))
max(clm(sqrt(lN2),it))
max(slm(sqrt(lD1),it))
max(slm(sqrt(lD2),it))

Program 4 This computes the $L2(\mu)$ -norms of the Neumann eigenfunctions

 $c_{\lambda,\mu}(\sqrt{\lambda_{N,k}},\cdot)$

and Dirichlet eigenfunctions

```
s_{\lambda,\mu}(\sqrt{\lambda_{D,k}},\cdot)
```

with the formulas from Corollary 4.3.

```
from mpmath import mp, mpf, findroot
mp.dps = 80
mp.pretty = True
r1=mpf(1/3)
r2=mpf(1/3)
m1=mpf(1/2)
m2=mpf(1/2)
an = 50;
p=[1,1]
q=[1,1]
for n in range(1,an):
    #p[2n]
    p.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* p[2*i] * p[2*n-2*i] for i in range(1,n)]) + sum([r1^i * m1^(i+1)])
* r2^(n-i) * m2^(n-i-1) * p[2*i+1] * q[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([r1^i * m1^(i+1) * (r2 * m2)^(n-i-1) * p[2*i+1]
* p[2*n-2*i-2] for i in range(0,n)])))
    #q[2n]
    q.append(1/(1-(r1*m1)^n-(r2*m2)^n)*(sum([(r1*m1)^i * (r2*m2)^(n-i)
* q[2*i] * q[2*n-2*i] for i in range(1,n)]) + sum([r1^(i+1) * m1^i
```

```
* r2^(n-i-1) * m2^(n-i) * q[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])
+ (1-r1-r2)* sum([(r1 * m1)^i * r2^(n-i-1) * m2^(n-i) * q[2*i]
* p[2*n-2*i-1] for i in range(0,n)])))
    #p[2n+1]
    p.append(1/(1-r1^n*m1^(n+1)-r2^n*m2^(n+1))*(sum([r1^i*m1^(i+1)
* (r2*m2)^(n-i) * p[2*i+1] * q[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i) * m2^(n-i+1) * p[2*i] * p[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^(i+1) * r2^(n-i-1)
* m2^(n-i) * p[2*i+1] * p[2*n-2*i-1] for i in range(0,n)])))
   #q[2n+1]
    q.append(1/(1-r1^(n+1)*m1^n-r2^(n+1)*m2^n)*(sum([r1^(i+1)*m1^i
* (r2*m2)^(n-i) * q[2*i+1] * p[2*n-2*i] for i in range(0,n)])
+ sum([r1^i * m1^i * r2^(n-i+1) * m2^(n-i) * q[2*i] * q[2*n-2*i+1]
for i in range(1,n+1)]) + (1-r1-r2)* sum([r1^i * m1^i * (r2 * m2)^(n-i)
* q[2*i] * p[2*n-2*i] for i in range(0,n+1)])))
# sets the first Neumann and Dirichlet eigenvalues for the standard
# Cantor measure, calculated in 'determination of eigenvalues'
lN1 = mpf(7.097431098141122)
1N2 = mpf(42.584586588846733)
1N3 = mpf(61.344203922701662)
1N4 = mpf(255.507519533080403)
1N5 = mpf(272.983570819147205)
1N6 = mpf(368.065223536209975)
1N7 = mpf(383.552883127693176)
1D1 = mpf(14.435240512053874)
1D2 = mpf(35.260238024277225)
1D3 = mpf(140.781053384556059)
1D4 = mpf(151.290616055019631)
1D5 = mpf(326.057328357753770)
1D6 = mpf(353.416920767557756)
# L2(\mu)-norms of the Neumann eigenfunctions
# c_{\lambda, mu}(\sqrt{lNk}, \cdot) for k=1, ..., 7
# with Corollary 4.3
sqrt(sum([(-1)^n * 1N1^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N2^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
```
```
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N3^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N4^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N5^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N6^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
sqrt(sum([(-1)^n * 1N7^n * sum([(n+1-2*k)*p[2*k]*p[2*n+1-2*k]
for k in range(0,n+1) ]) for n in range(0,an) ]))
# L2(\mu)-norms of the Dirichlet eigenfunctions
# s_{lambda,\mu}(sqrt{lDk}, cdot) for k = 1, ..., 6
sqrt(sum([(-1)^n * lD1^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
sqrt(sum([(-1)^n * lD2^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
sqrt(sum([(-1)^n * lD3^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
sqrt(sum([(-1)^n * 1D4^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
sqrt(sum([(-1)^n * 1D5^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
sqrt(sum([(-1)^n * 1D6^(n+1) * sum([(n+1-2*k)*q[2*k+1]*q[2*n+2-2*k]
for k in range(0,n+2) ]) for n in range(0,an-1) ]))
```