Currents on locally conformally Kähler manifolds

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Abstract We characterize the existence of a locally conformally Kähler metric on a compact complex manifold in terms of currents, adapting the celebrated result of Harvey and Lawson for Kähler metrics.

1 Introduction

A locally conformally Kähler manifold (LCK for short) is a Hermitian manifold (M, J, g) for which the fundamental two-form $\omega(X, Y) = g(JX, Y)$ satisfies

$$d\omega = \theta \wedge \omega, \qquad d\theta = 0 \tag{1.1}$$

for some one-form θ called the Lee form.

There are many examples of compact LCK and non-Kähler manifolds, among them the Hopf manifolds, see [DO], [OV].

As $d\theta = 0$, the twisted differential $d_\theta := d - \theta\wedge$ defines a twisted cohomology which is the Morse-Novikov cohomology of X. The LCK condition simply means that the fundamental form of (X, J, g) is d_θ-closed.

The aim of this note is to obtain an analogue of the intrinsic characterization in [HL] for Kähler manifolds in the context of LCK geometry.

2 LCK condition in terms of currents

Our main result is the following:

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Theorem 2.1: *Let X be a compact, complex manifold of complex dimension $n \geqslant 2$, and let θ be a closed one-form on X. Then X admits a LCK metric with Lee form θ if and only if there are no non-trivial positive currents which are $(1,1)$ components of d_θ-boundaries.*

Remark 2.2: Suppose X is a compact complex manifold, admitting a LCK metric, ω, with Lee form θ. Then any closed 1-form $\eta \in [\theta]_{dR}$ will be a Lee form for a conformal metric of ω and moreover, any conformal change of ω will be LCK with a Lee form in the same de Rham cohomology class as θ. Therefore, we need not fix θ, we can directly use its cohomology class, $[\theta]_{dR}$. By this observation, the theorem above can be stated as:

Let X be a compact, complex manifold of complex dimension $n \geqslant 2$, and let $[\theta]_{dR}$ a cohomology class in $H^1_{dR}(X)$. Then X admits a LCK metric with Lee form θ if and only if there are no non-trivial positive currents which are $(1,1)$-components of d_η-boundaries, for any closed one-form η belonging to $[\theta]_{dR}$.

The rest of Section 2 is devoted to the proof, which follows the lines in [HL]. We use the same results and intermediate steps as [HL], the difficult part being that of finding some proper analogues in LCK geometry for the Kähler notions used in the original article. Each following subsection is a step of the proof.

2.1 Range of d_θ is closed

Associated with d_θ are the following operators:

$$\partial_\theta = \partial - \theta^{1,0}\wedge, \qquad \overline{\partial}_\theta = \overline{\partial} - \theta^{0,1}\wedge, \qquad d^c_\theta = \mathrm{i}(\partial_\theta - \overline{\partial}_\theta)$$

Definition 2.3: A smooth function is called θ-pluriharmonic if it is locally the real part of a smooth $\overline{\partial}_\theta$-closed function.

We let $\mathcal{H}_\theta$ be the sheaf of germs of θ-pluriharmonic functions on X.

Lemma 2.4: *$\mathcal{H}_\theta$ is the kernel of the sheaves morphism $\mathcal{E}_{\mathbb{R}} \xrightarrow{d_\theta d^c_\theta} \mathcal{E}^{1,1}_{\mathbb{R}}$, where the subscript $\mathbb{R}$ denotes the germs of real valued forms.*
Proof: The proof is based on the following easy observation

$$\overline{\partial}_\theta f = 0 \Leftrightarrow \frac{1}{2\mathrm{i}}\left(\overline{\partial}_\theta f - \partial_\theta \overline{f}\right) = 0$$

Let now $f = u + \mathrm{i}v$. One obviously has

$$\overline{\partial}_\theta f = 0 \Leftrightarrow \frac{\overline{\partial}_\theta(u+\mathrm{i}v) - \partial_\theta(u-\mathrm{i}v)}{2\mathrm{i}} = 0 \Leftrightarrow d_\theta v + d^c_\theta u = 0 \tag{2.1}$$

Let g for which a f' exists such that $f = g + \mathrm{i}f'$ is $\overline{\partial}_\theta$-closed. It follows from (2.1) that $d_\theta f' + d^c_\theta g = 0$, which implies $d_\theta d^c_\theta g = 0$.

Conversely, if g satisfies $d_\theta d^c_\theta g = 0$, finding a f' such that $f = g + \mathrm{i}f'$ is $\overline{\partial}_\theta$-closed is equivalent to solving the equation $d_\theta f' = -d^c_\theta g$.

Since θ is locally exact, let $\theta = dh$ on a contractible open set. Then $e^{-h}d^c_\theta g$ is closed and by Poincaré lemma there exists a function h' such that $e^{-h}d^c_\theta g = dh'$. Then $f' = -e^h h'$ which completes the proof.

■

The above result shows that the following is an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{H}_\theta \longrightarrow \mathcal{E}_{\mathbb{R}} \xrightarrow{d_\theta d^c_\theta} \mathcal{E}^{1,1}_{\mathbb{R}} \xrightarrow{d_\theta} \left[\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}\right]_{\mathbb{R}} \xrightarrow{d_\theta} \cdots$$

Since $\left[\mathcal{E}^{p,q}\right]_{\mathbb{R}}$ are acyclic, the above is a resolution which computes the cohomology groups of $\mathcal{H}_\theta$.

We now prove that $H^i(X, \mathcal{H}_\theta)$ are finite dimensional for all $i \geqslant 0$.

Let $\mathcal{O}_\theta$ denote the sheaf of germs of smooth functions satisfying $\overline{\partial}_\theta f = 0$ and let $\mathcal{F}$ be the kernel of the sheaves morphism $\mathrm{Re} : \mathcal{O}_\theta \to \mathcal{H}_\theta$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_\theta \xrightarrow{\mathrm{Re}} \mathcal{H}_\theta \longrightarrow 0 \tag{2.2}$$

Proposition 2.5: *$\mathcal{O}_\theta$ is locally free of rank 1 over the sheaf of germs of holomorphic functions, $\mathcal{O}_X$, and $\mathcal{F}$ is locally constant.*

Proof: To prove that $\mathcal{F}$ is locally constant, we characterize the non-zero $\overline{\partial}_\theta$-closed real valued functions.

Let h be a (unique up to addition with constants) real valued smooth function on a contractible neighbourhood such that $\theta^{0,1} = \overline{\partial}h$. Then

$$\overline{\partial}_\theta f = 0 \Leftrightarrow \overline{\partial} f - f\theta^{0,1} = 0 \Leftrightarrow \overline{\partial} f = f\overline{\partial}h$$

Since both f and h are real valued, the above last equality gives, by conjugation, $\partial f = f\partial h$.

Summing up, we obtain $df = f dh$, which yields

$$d\log f = dh, \quad \text{and hence} \quad f = e^h \cdot c, \quad c \in \mathbb{R}$$

This proves that on the neighbourhood where $\theta^{0,1}$ is $\overline{\partial}$-exact, the sheaves $\mathcal{F}$ and $\underline{\mathbb{R}}$ are isomorphic.

We use a similar argument for $\mathcal{O}_\theta$. Let h be as above. Then e^h is $\overline{\partial}_\theta$-closed. Let $f \in \mathcal{O}_{\theta x,X}$ defined on an open set contained in the domain of h. Let $\lambda := fe^{-h}$. As $\overline{\partial}_\theta f = 0$, we have $\overline{\partial}\lambda \cdot e^h + \lambda\overline{\partial}h \cdot e^h - \lambda e^h \cdot \theta^{0,1} = 0$. Since e^h is nowhere vanishing, we conclude that $\overline{\partial}\lambda = 0$ which is equivalent to $\lambda \in \mathcal{O}_{x,X}$. Hence $\mathcal{O}_{\theta x,X} \cong \mathcal{O}_{x,X}$, proving that $\mathcal{O}_\theta$ is locally free of rank 1. ■

Corollary 2.6: *$\mathcal{F}$ and $\mathcal{O}_\theta$ have finite dimensional cohomology groups.*

Proof: By proving that O_θ is locally free of rank 1, we have proved its coherence. Using now the Cartan-Serre theorem for coherent sheaves on compact complex manifolds [T], we obtain the finite dimension of its cohomology groups. As for $\mathcal{F}$, the compactness of X assures the existence of a finite covering of contractible sets, on which $\mathcal{F}$ is isomorphic to $\underline{\mathbb{R}}$. However, $\underline{\mathbb{R}}$ has vanishing cohomology groups on contractible sets. Thus, by Leray theorem [D], we find a covering which computes via the Čech complex the cohomogy of $\mathcal{F}$. But every term in the Čech complex associated to this covering is a real finite dimension vector space, hence the finite cohomological dimension of $\mathcal{F}$ is obvious. ■

Splitting the long exact sequence in cohomology asociated to (2.2) into short exact sequences and using the above Corollary proves:

Corollary 2.7: *$\mathcal{H}_\theta$ has finite dimensional cohomology groups.*

Remark 2.8: One usually proves the finite dimensionality of the cohomology groups of a complex by means of elliptic operators, the most famous example being that of the Hodge isomorphism theorem stating that $H^\bullet(X, \mathbb{R}) \simeq \mathrm{Ker}\Delta$. Examples of elliptic operators dealing with twisted differentials such as d_θ are given in [AK]. However, we would be interested in an elliptic operator such that its kernel is given by the cohomology groups of the sheaf $\mathcal{H}_\theta$ and this case is not covered by the results in [AK]. More specifically, we are interested only in the second cohomology group of this sheaf, as we will see in the following sections.

Corollary 2.9: *The operator $d_\theta : \mathcal{E}^{1,1}_{\mathbb{R}} \longrightarrow \left[\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}\right]_{\mathbb{R}}$ has closed range.*

Proof: Since $H^2(X, \mathcal{H}_\theta)$ is finite dimensional, $\mathrm{Im}\, d_\theta$ has finite codimension in $Z(X) = \{\psi \in \left[\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}\right]_{\mathbb{R}},\, d_\theta\psi = 0\}$. $Z(X)$ remains a Fréchet space, as every closed subset of a Fréchet space does, so d_θ is a continuous linear function between two Fréchet spaces, whose codimension is finite (i.e.

$Z(X)/\operatorname{Im} d_\theta$ is finite dimensional). We will invoke now the open mapping theorem for Fréchet spaces [Tr, p.170], which states that every surjective continuous and linear map between two Fréchet spaces is open, in order to prove the following general result, which is an elementary functional analysis lemma. *If a liniar continuous map between two Fréchet spaces has finite codimension, its range is closed.*

We include its proof for the sake of completeness.

Let $T : A \to B$ be such a map. Let $\omega_1, \ldots, \omega_n$ be elements in B that give a basis for $B/\operatorname{Im} T$ and let $C = \langle \omega_1, \ldots, \omega_n \rangle \subset B$. We consider now the map $F : A \oplus C \to B$ given by $F(x + y) = T(x) + y$. It is a simple observation that F is surjective. F is also continuous and linear, hence by the open mapping theorem an open map. We may assume that T is actually injective, otherwise we factorize by its kernel and thus, F becomes a bijective open continuous map, hence a homeomorphism. $T(A) = F(A \oplus \{0\})$, which is a closed set.

So the open mapping theorem was crucial for proving that $\operatorname{Im} d_\theta$ is closed in $Z(X)$ and hence closed in $\left[\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}\right]_{\mathbb{R}}$.

■

2.2 Extension of d_θ to currents

We follow the definitions and conventions in [D] for currents.

Let $[\mathcal{E}'_i(X)]_{\mathbb{R}}$ denote the dual space of $[\mathcal{E}^i(X)]_{\mathbb{R}}$. Recall that the differential $d : [\mathcal{E}'_i(X)]_{\mathbb{R}} \to [\mathcal{E}'_{i-1}(X)]_{\mathbb{R}}$ acts by

$$\langle dT, \eta \rangle := \langle T, d\eta \rangle, \qquad \eta \in \mathcal{E}^{i-1}(X)$$

and the exterior product of a current and a 1-form $\cdot \wedge \xi : [\mathcal{E}'_i(X)]_{\mathbb{R}} \to [\mathcal{E}'_{i-1}(X)]_{\mathbb{R}}$ is defined by

$$\langle T \wedge \xi, \eta \rangle = \langle T, \xi \wedge \eta \rangle$$

We then define $d_\theta : [\mathcal{E}'_i(X)]_{\mathbb{R}} \to [\mathcal{E}'_{i-1}(X)]_{\mathbb{R}}$ as follows:

$$\langle d_\theta T, \eta \rangle = \langle T, d_\theta \eta \rangle, \qquad \eta \in \mathcal{E}^{i-1}(X) \tag{2.3}$$

Let $T \in [\mathcal{E}'_{p,q}(X)]_{\mathbb{R}}$. In particular, T is a $p+q$ current which vanishes on all (i,j) forms with $(i,j) \neq (p,q)$), $d_\theta T \in \mathcal{E}'_{p+q-1}$ and decomposes as:

$$d_\theta T = \sum_{i+j=p+q-1} (d_\theta T)_{i,j}$$

where

$$\langle (d_\theta T)_{i,j}, \eta\rangle = \langle T, d_\theta \eta\rangle, \qquad \eta \in \mathcal{E}^{i,j}(X), i+j = p+q-1$$

But since

$$\langle T, \eta\rangle = \sum_{i+j=p+q-1} \langle T_{i,j}, \eta_{ij}\rangle, \quad \eta_{ij} = \text{the}\,(i,j)\,\text{part of}\,\eta,$$

we obtain

$$(d_\theta T)_{i,j} = 0 \text{ for all } (i,j) \notin \{(p,q-1),(p-1,q)\}$$

Let now $T \in [\mathcal{E}'_{p,q+1}(X) \oplus \mathcal{E}'_{p+1,q}(X)]_{\mathbb{R}}$. Then $d_\theta T \in \mathcal{E}'_{p+q}(X)$. By (2.3), the only possibly non-zero components $d_\theta T$ are $(d_\theta T)_{p,q}$, $(d_\theta T)_{p+1,q-1}$, $(d_\theta T)_{p-1,q+1}$, as only the differential of (p,q), $(p+1,q-1)$ and $(p-1,q+1)$ forms can have non-trivial $(p,q+1)$ and $(p+1,q)$ parts. We have proved:

Claim 2.10: $\langle d_\theta T, \eta\rangle = \langle (d_\theta T)_{p,q}, \eta\rangle$, *for any* $\eta \in \mathcal{E}^{p,q}(X)$.

As in [HL], let $\Pi_{p,q} : \mathcal{E}'_{p+q}(X) \longrightarrow \mathcal{E}'_{p,q}(X)$ be the projector associating the (p,q) part of a $p+q$ current. Let also

$$(d^\theta_{p,q} T) \overset{\text{not.}}{=} (d_\theta T)_{p,q} = \Pi_{p,q} \circ d_\theta|_{[\mathcal{E}'_{p,q+1}(X)\oplus\mathcal{E}'_{p+1,q}(X)]_{\mathbb{R}}}(T)$$

Denote $B^\theta_{p,q} = \mathrm{Im}(d^\theta_{p,q})$. We prove:

Lemma 2.11: *Let* $\eta \in \mathcal{E}^{1,1}(X)$. *Then* $d\eta = \theta \wedge \eta$ *if and only if* $\langle T, \eta\rangle = 0$ *for any* $T \in B^\theta_{1,1}$.

Proof: If η is d_θ - closed, then $\langle T, d_\theta \eta\rangle = 0$ for all $T \in [\mathcal{E}'_{2,1}(X) \oplus \mathcal{E}'_{1,2}(X)]_{\mathbb{R}}$ and hence $\langle d^\theta_{1,1} T, \eta\rangle = 0$, yielding $\langle T, \eta\rangle = 0$ for all $T \in B^\theta_{1,1}$.

Conversely, if $\langle d^\theta_{1,1} T, \eta\rangle = 0$ for $T \in [\mathcal{E}'_{2,1}(X) \oplus \mathcal{E}'_{1,2}(X)]_{\mathbb{R}}$, then $\langle T, d_\theta \eta\rangle = 0$, equality which is attained even for all $T \in [\mathcal{E}'_3(X)]_{\mathbb{R}}$, since a $(3,0)$ current vanishes on $d_\theta \eta$, and thus $d_\theta \eta = 0$.

■

We finally prove:

Proposition 2.12: *The operator* $d^\theta_{1,1} : [\mathcal{E}'_{1,2}(X) \oplus \mathcal{E}'_{2,1}(X)]_{\mathbb{R}} \longrightarrow [\mathcal{E}'_{1,1}(X)]_{\mathbb{R}}$ *has closed range. In other words,* $B^\theta_{1,1}$ *is closed in* $[\mathcal{E}^{1,1}(X)]_{\mathbb{R}}$.

Proof: From (Claim 2.10) we know that $d^{\theta}_{1,1}$ is the adjoint of $d_\theta : [\mathcal{E}^{1,1}(X)]_{\mathbb{R}} \longrightarrow [\mathcal{E}^{1,2}(X) \oplus \mathcal{E}^{2,1}(X)]_{\mathbb{R}}$, which, by Corollary 2.9, has closed range. Since both $[\mathcal{E}^{1,1}(X)]_{\mathbb{R}}$ and $[\mathcal{E}^{1,2}(X) \oplus \mathcal{E}^{2,1}(X)]_{\mathbb{R}}$ are Fréchet spaces, we may apply the closed range theorem, as in [S, chap. IV, section 7.7] to conclude that $d^{\theta}_{1,1}$ has closed range too.

■

2.3 Positive currents

We collect here, mainly without proof, several facts we shall need about positive currents. The reference is [D].

Let T be a (p,p) current. It can be written locally as

$$T = \sum_{\substack{|I|=n-p \\ |J|=n-p}} T_{I,J} dz_I \wedge d\overline{z}_J,$$

where $T_{I,J}$ is a distribution.

For a positive current, $T_{I,J}$ is a complex measure that satisfies $\overline{T}_{I,J} = T_{J,I}$ and $T_{I,I} > 0$. We denote by $\|T\| := \sum |T_{I,J}|$ the mass measure of T.

Since $|T_{I,J}|$ is absolutely continuous with respect to $\|T\|$, Radon-Nykodim theorem applies and hence there exists a measurable function $f_{I,J}$ such that $T_{I,J} = \int f_{I,J} d\|T\|$. Letting $f := \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z}_J$, we may write

$$\langle T, \eta \rangle = \int_X \eta \wedge f d\|T\|, \qquad \eta \in \mathcal{E}^{p,p}(X) \tag{2.4}$$

If $\overrightarrow{T}_x$ is defined by $\eta_x(\overrightarrow{T}_x) = \eta_x \wedge f_x$, $x \in X$, (2.4) can be rewritten as

$$\langle T, \eta \rangle = \int_X \eta(\overrightarrow{T}) d\|T\|, \qquad \eta \in \mathcal{E}^{p,p}(X) \tag{2.5}$$

Proposition 2.13: *The current T is positive if and only if the function $f(x) = \sum_{I,J} f_{I,J}(x) dz_I \wedge d\overline{z}_J \in \Lambda^{n-p,n-p} T^*_x X$ is positive $\|T\|$ a.e.*

We apply the above considerations for a $(1,1)$ current T (in which case $\overrightarrow{T}$ is a bivector). Then

$$\begin{aligned} T \text{ is positive} &\Leftrightarrow f = \sum_{\substack{|I|=n-1 \\ |J|=n-1}} f_{I,J} dz_I \wedge d\overline{z}_J \text{ is positive } \|T\| \text{ a.e.} \\ &\Leftrightarrow \overrightarrow{T} \in \operatorname{conv}\big(G_{\mathbb{C}}(1, T_x X)\big) \end{aligned} \tag{2.6}$$

2.4 Wirtinger inequality on LCK manifolds

Theorem 2.14: *Let ω be a LCK form on X with Lee form θ. Then $\omega(\xi) \leqslant 1$ for any $\xi \in \operatorname{conv}\big(G_{\mathbb{R}}(2, T_x X)\big)$, with equality if and only if ξ lies in $\operatorname{conv}\big(G_{\mathbb{C}}(1, T_x X)\big)$.*

Remark 2.15: Although the inequality is often stated for Kähler forms, the condition $d\omega = 0$ is not used in the proof. The only property of the Kähler form which is used is that ω^n is a volume form, and this holds on LCK manifolds too (and more generally, whenever ω is strictly positive).

Using also (2.6), it then follows that for a positive $(1,1)$ current T,

$$\langle T, \omega\rangle = \int_X \omega_x(\overrightarrow{T}_x)\|T\| = \int_X \|T\|$$

But $\int_X \|T\| = \|T\|(X) > 0$ and hence $\langle T, \omega\rangle > 0$ for any positive non-zero $(1,1)$ current T.

2.5 Proof of Theorem 2.1

We adjust the proof in [HL].

Denote by $P_{1,1}(X)$ the space of positive currents on the compact LCK manifold X. Recall that we proved the following facts:

$$\langle T, \omega\rangle = 0, \text{ for } T \in B^{\theta}_{1,1} \text{ and } \langle T, \omega\rangle > 0, \text{ for } T \in P_{1,1}(X) \setminus \{0\}$$

and hence we have

$$B^{\theta}_{1,1} \cap P_{1,1} = \{0\} \tag{2.7}$$

The difficult task is to prove the converse.

We let X be complex, compact and fix a closed one form θ. Assuming (2.7), we look for a positive $(1,1)$ form which is d_θ-closed (it will define the LCK metric).

We choose an arbitrary Hermitian metric h on X and we let $\psi = -\operatorname{Im}(h)$. Then $\psi \in [\mathcal{E}^{1,1}(X)]_{\mathbb{R}}$. Using ψ we define the set $K = \{T \in P_{1,1}(X)\,;\, \langle T, \psi\rangle = 1\}$ which is a compact base for $P_{1,1}(X)$ and is weakly compact in $[\mathcal{E}'_{1,1}(X)]_{\mathbb{R}}$, as a consequence of Banach-Alaoglu theorem [D]. As $B^{\theta}_{1,1}(X)$ is closed, we may apply the Hahn-Banach separation theorem [S], stating there is a closed real hyperplane separating a closed set and a compact set in a locally convex space, as long as they are disjoint. The space of real (1,1)-currents, $\mathcal{E}^{1,1}_{\mathbb{R}}(X)$, is locally convex and so is the quotient space $\mathcal{E}^{1,1}_{\mathbb{R}}(X)/B^{\theta}_{1,1}(X)$ [Di].

Applying now the Hahn-Banach theorem for the locally convex space $\mathcal{E}^{1,1}_{\mathbb{R}}(X)/B^{\theta}_{1,1}(X)$, the closed set $\{0\}$ and the compact set K (which does not contain the 0 current), we get a hyperplane that separates K from 0. Thus, we obtain a continuous linear functional $f : \mathcal{E}^{1,1}_{\mathbb{R}}(X)/B^{\theta}_{1,1}(X) \to \mathbb{R}$, which takes only strictly positive or negative values on K and by a change of sign we can assume the values are strictly positive. f provides a functional $\tilde{f}$ on the whole $\mathcal{E}^{1,1}_{\mathbb{R}}(X)$, which vanishes on $B^{\theta}_{1,1}(X)$ and is positive on K.We define the real $(1,1)$-form, ω, as $\langle T, \omega \rangle = \tilde{f}(T)$, for any $(1,1)$-current T. This holds as definition since the pairing between a current and a form given by the evaluation $\langle T, \omega \rangle$ is nondegenerate. This real $(1,1)$-form will vanish on $B^{\theta}_{1,1}(X)$ and will be strictly positive on K.

Since the condition of vanishing on $B^{\theta}_{1,1}(X)$ is equivalent to $d_\theta \omega = 0$, we already obtain a d_θ-closed form. As ω is strictly positive on K and K is a compact base for $P_{1,1}(X)$, we also obtain the positivity on $P_{1,1}(X)$.

What remains is to show that ω is a non-degenerate, positive form.

We shall prove that

$$\omega_x(v \wedge \overline{v}) > 0 \text{ for any } v \in T^{1,0}_x X.$$

Let $\overrightarrow{T_x} = v \wedge \overline{v} \in G_{\mathbb{C}}(1,n) \subset \Lambda^{1,1} T_x X$.

By now, we asociated to each $(1,1)$ - current a smooth collection of bivectors $\{\overrightarrow{T_x}\}$ and now we go the other way around, by defining the $(1,1)$ current $T = \delta_x \overrightarrow{T}$, where δ_x is the Dirac measure concentrated in x. Then T is a positive current since $\overrightarrow{T}$ was chosen from $G_{\mathbb{C}}(1,n)$ and hence $\langle T, \omega \rangle > 0$. This is equivalent to $\omega(v \wedge \overline{v}) > 0$, concluding that ω is a positive d_θ-closed $(1,1)$ form, thus producing a LCK metric.

3 Transverse (p,p)-forms

In [AA], Alessandrini and Andreatta extend Theorem 14 in [HL, p. 176] to transverse closed (p,p)-forms. As a byproduct of adapting to d_θ the usual operations on currents, as presented in Section 2.2, we give an analogue of Theorem 1.17 in [AA, p. 188] by considering the existence of a transverse d_θ-closed (p,p)-form instead of a usual transverse closed (p,p)-form. The particular case $p = 1$ recovers precisely Theorem 2.1 of the present note.

Definition 3.1: A transverse (p,p)-form is a form which at any point belongs to the interior of the cone of strongly positive forms.

It is proved in [AA] that given a complex compact manifold X, there exists a transverse closed (p,p)-form if and only if there are no positive currents which are (p,p)-components of boundaries. The same steps and techniques can be used in order to prove the following result:

Proposition 3.2: *Let X be a complex, compact manifold and θ a real closed 1-form. There exists a transverse (p,p) d_θ-closed form if and only if there are no positive (p,p)- currents which are d_θ - boundaries.*

In order to prove this result we need first to present some intermediate facts.

Let $B^\theta_{p,p}$ denote the space of currents which are (p,p)-components of d_θ-boundaries and Ω^p_θ the kernel of the following sheaf morphism:

$$\overline{\partial}_\theta : \mathcal{E}^{p,0} \longrightarrow \mathcal{E}^{p,1}$$

We have this exact sequence of sheaves:

$$0 \longrightarrow \mathcal{H}_\theta \xrightarrow{f} \mathcal{L}^0_\theta \xrightarrow{f_0} \cdots \xrightarrow{f_{p-1}} \mathcal{L}^{p-1}_\theta \xrightarrow{g} \mathcal{B}^p_\theta \xrightarrow{g_p} \mathcal{B}^{p+1}_\theta \xrightarrow{g_{p+1}} \cdots$$

$$\cdots \xrightarrow{g_{2p-1}} \mathcal{B}^{2p-1}_\theta \xrightarrow{h} \mathcal{E}^{p,p}_\mathbb{R} \xrightarrow{d_\theta d^c_\theta} \mathcal{E}^{p+1,p+1}_\mathbb{R} \xrightarrow{d_\theta} \mathcal{E}^{p+1,p+2}_\mathbb{R} \oplus \mathcal{E}^{p+2,p+1}_\mathbb{R} \xrightarrow{d_\theta} \cdots$$

where:

$$\mathcal{L}^k_\theta = \overline{\Omega^{k+1}_\theta} \oplus \mathcal{E}^{0,k}_\mathbb{R} \oplus \mathcal{E}^{1,k-1}_\mathbb{R} \ldots \oplus \mathcal{E}^{k,0}_\mathbb{R} \oplus \Omega^{k+1}_\theta$$

for $0 \leqslant k \leqslant p-1$;

$$\mathcal{B}^k_\theta = \mathcal{E}^{k-p,p}_\mathbb{R} \oplus \ldots \oplus \mathcal{E}^{p,k-p}_\mathbb{R}$$

for $p \leqslant k \leqslant 2p-1$;

$$f : \mathcal{H}_\theta \to \mathcal{L}^0_\theta$$
$$f(\varphi) = (-\overline{\partial}_\theta\varphi, \varphi, -\partial_\theta\varphi)$$

$$f_k : \mathcal{L}^k_\theta \to \mathcal{L}^{k+1}_\theta$$

$$f_k(\varphi, a^{0,k}, a^{1,k-1} \ldots, a^{k-1,1}, a^{k,0}, \eta) =$$
$$(-\overline{\partial}_\theta\varphi, \varphi+\overline{\partial}_\theta a^{0,k}, \partial_\theta a^{0,k}+\overline{\partial} a^{1,k-1}, \ldots, \partial_\theta a^{i-1,j}+\overline{\partial}_\theta a^{i,j-1}, \ldots, \eta+\partial_\theta a^{k,0}, -\partial_\theta\eta)$$

$$g : \mathcal{L}^{p-1}_\theta \to \mathcal{B}^p_\theta$$

$$g(\varphi, a^{0,p-1}, a^{1,p-1} \ldots, a^{p-1,1}, a^{p,0}, \eta) =$$

$$(\varphi+\overline{\partial}_\theta a^{0,p}, \partial_\theta a^{0,p}+\overline{\partial} a^{1,p-1}, \ldots, \partial_\theta a^{i-1,j}+\overline{\partial}_\theta a^{i,j-1}, \ldots, \eta+\partial_\theta a^{p,0}, -\partial_\theta \eta)$$

$$g_k : \mathcal{B}_\theta^k \to \mathcal{B}_\theta^{k+1}$$

$$g_k(a^{k-p,p}+\ldots+a^{p,k-p}) =$$

$$(\partial_\theta a^{k-p,p}+\overline{\partial}_\theta a^{k-p+1,p-1}, \ldots, \partial_\theta a^{p-1.k-p}+\overline{\partial}_\theta a^{p,k-p-1})$$

$$h : \mathcal{B}_\theta^{2p-1} \to \mathcal{E}_\mathbb{R}^{p,p}$$

$$h(a^{p-1,p}, a^{p,p-1}) = \partial_\theta a^{p-1,p}+\overline{\partial}_\theta a^{p,p-1}$$

Remark 3.3: The sequence considered above is not a resolution for $\mathcal{H}_\theta$ since $\mathcal{L}_\theta^k$ are not acyclic.

Proposition 3.4: $\mathcal{L}_\theta^k$ has finite dimensional cohomology groups.

Proof: The sheaves $\mathcal{E}_\mathbb{R}^{k,q}$ are acyclic, therefore $H^i(\mathcal{L}_\theta^k) = H^i(\overline{\Omega}_\theta^k \oplus \Omega_\theta^k)$, for any $i > 0$. But Ω_θ^k and its conjugate have both finite dimensional cohomology groups, since Ω_θ^k is locally isomorphic to Ω^k via an argument similar to the coherence of the sheaf $\mathcal{O}_\theta$.

Let $\mathcal{Z}$ be the kernel of g_p. By splitting the sequence into short exact sequences and by using the proposition above, we obtain that the connecting morphism

$$H^k(X, \mathcal{Z}) \longrightarrow H^{k+p}(X, \mathcal{H}_\theta)$$

has finite domensional kernel and cokernel. We may now use the finite dimensionality of the cohomology of $\mathcal{H}_\theta$ and obtain the finite dimensionality of the cohomology groups of $\mathcal{Z}$.

Since the resolution:

$$0 \longrightarrow \mathcal{Z} \longrightarrow \mathcal{B}_\theta^p \xrightarrow{g_p} \mathcal{B}_\theta^{p+1} \xrightarrow{g_{p+1}} \cdots \xrightarrow{g_{2p-1}}$$

$$\xrightarrow{g_{2p-1}} \mathcal{B}_\theta^{2p-1} \xrightarrow{h} \mathcal{E}_\mathbb{R}^{p,p} \xrightarrow{d_\theta d_\theta^c} \mathcal{E}_\mathbb{R}^{p+1,p+1} \xrightarrow{d_\theta} \mathcal{E}_\mathbb{R}^{p+1,p} \oplus \mathcal{E}_\mathbb{R}^{p,p+1} \xrightarrow{d_\theta} \cdots$$

computes the cohomology of $\mathcal{Z}$, we conclude that $B_{p,p}^\theta$ is closed, since $H^{p+2}(X, \mathcal{Z})$ is finite dimensional.

Remark 3.5: It is easy to see that a (p,p)-form is d_θ-closed if and only if it vanishes on $B^\theta_{p,p}$.

The proof of Proposition 3.2 is now identical to the proof of Theorem 1.17 in [AA, p. 188], by replacing $B_{p,p}$ with $B^\theta_{p,p}$.

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References

[AA] L. Alessandrini, M. Andreatta, Closed transverse (p,p)-forms on compact complex manifolds, Compositio Mathematica **61** (1987) 181-200.

[AK] D. Angella, H. Kasuya, Hodge theory for twisted differentials, arXiv: 1407.3928

[D] J.-P. Demailly, Complex analytic and differential geometry, www-fourier.ujf-grenoble.fr/∼demailly/manuscripts/agbook.pdf

[Di] Jean Dieudonné, Treatise on analysis, vol. II, Academic Press, 1970.

[DO] S. Dragomir and L. Ornea, Locally conformal Kähler geometry, Progress in Math. **155**, Birkhäuser, Boston, Basel, 1998.

[HL] R. Harvey, H.B. Lawson Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), 169–198.

[OV] L. Ornea, M. Verbitsky, *A report on locally conformally Kähler manifolds*, in “Harmonic Maps and Differential Geometry” Contemporary Mathematics **542**, 135–150, 2011.

[S] H.H.Schaefer, Topological vector spaces, Springer, 1970, cap. II.

[T] J.L.Taylor, Several complex variables with connections to algebraic geometry and Lie groups, Amer Math Society, 2002.

[Tr] F. Tréves, Topological vector spaces, distributions and kernels, Academic Press, 1967

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