Greybody factors for Myers–Perry black holes

Petarpa Boonserm [a)]

Department of Mathematics and Computer Science, Faculty of Science,
Chulalongkorn University, Phayathai Road, Patumwan, Bangkok 10330,
Thailand

Auttakit Chatrabhuti [b)] and Tritos Ngampitipan [c)]

Department of Physics, Particle Physics Research Laboratory, Faculty of Science,
Chulalongkorn University, Phayathai Road, Patumwan, Bangkok 10330,
Thailand

Matt Visser [d)]

School of Mathematics, Statistics, and Operations Research,
Victoria University of Wellington;
PO Box 600, Wellington 6140, New Zealand.

Abstract:

The Myers–Perry black holes are higher-dimensional generalizations of the usual (3+1)-dimensional rotating Kerr black hole. They are of considerable interest in Kaluza–Klein models, specifically within the context of brane-world versions thereof. In the present article we shall consider the greybody factors associated with scalar field excitations of the Myers–Perry spacetimes, and develop some rigorous bounds on these greybody factors. These bounds are of relevance for characterizing both the higher-dimensional Hawking radiation, and the super-radiance, that is expected for these spacetimes.

Keywords:

Hawking radiation, greybody factors, rigorous bounds, Myers–Perry black holes.

[a)] petarpa.boonserm@gmail.com

[b)] dma3ac2@gmail.com

[c)] tritos.ngampitipan@gmail.com

[d)] matt.visser@msor.vuw.ac.nz

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I.   INTRODUCTION.

Greybody factors modulate the absorption cross-sections of classical black holes, and alter the closely related Hawking emission[1,2] probabilities of semi-classical black holes.[3–6] Physically, the incoming or outgoing wave back-scatters off the gravitational field surrounding the black hole, leading to a non-trivial transmission coefficient. In the case of Hawking radiation, this modifies the naive Planckian spectrum by multiplying it with a frequency-dependent greybody factor. Explicitly evaluating these greybody factors is typically an impossible task, even for the simple case of the Schwarzschild black hole.[7] In view of this difficulty, techniques for placing analytic bounds on the greybody factors have now become of some interest.[7–11] (Alternatively one might seek to extract qualitative or numerical information.[12–14])

The bounds developed in references 7–11 apply to various black holes, (Schwarzschild, Reissner–Nordström, Kerr, Kerr–Newman, etcetera), and are all based on a very general technique for bounding one-dimensional barrier penetration probabilities; a technique that was first developed in reference 15, with later formal developments to be found in references 16–19, and additional related discussion in references 20–23. In the current article we shall apply the same sort of formalism to the Myers–Perry rotating black holes in $(3+1+n)$ dimensions.[24,25] The Myers–Perry black holes are particularly important in that they are the simplest of the higher-dimensional rotating black holes, being of particular interest in both Kaluza–Klein scenarios and in brane-world scenarios.

We first describe the Myers-Perry spacetime,[24,25] setting up the relevant Teukolsky equation for scalar field excitations.[26] An important part of the technical analysis is the fact that we can place positivity constraints on both the separation constant and on the effective potential; without such positivity constraints progress would be severely limited. We then analyze both the greybody factors and (when relevant) super-radiant emission as a function of the angular momentum quantum number m. While zero angular momentum ($m = 0$) serves as a good template for the other cases, there are some significant differences to take into account. After completing the analysis and summarizing the general case, we specialize to $(3+1)$ dimensions to verify compatibility with the usual Kerr black hole, and also consider the specific $(3+1+1)$ five-dimensional case which is perhaps most relevant to brane-world models. We conclude with a brief discussion of the significance of our results.

II.   TEUKOLSKY EQUATION FOR SCALAR FIELDS.

In setting up the formalism, it is best to first focus on the geometry of the specific spacetimes under consideration, and then analyse the technical steps involved in separation of variables, leading up to the development of the Teukolsky equation for scalar field excitations. With this in hand, one can then proceed to examination of the effective potential. For some general background on black hole perturbation theory see references 27–34.

A. Myers–Perry spacetime.

The Myers–Perry geometry (with only one of the angular momentum parameters being non-zero) is described by the metric[24,25]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\varphi^2$$
$$+\frac{\mu}{r^{n-1}\Sigma}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 + r^2\cos^2\theta\,\mathrm{d}\Omega_n^2. \tag{1}$$

Here

$$\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}}, \qquad\qquad \Sigma = r^2 + a^2\cos^2\theta, \tag{2}$$

and $\mathrm{d}\Omega_n^2$ is the line-element on the unit n-sphere S^n. We choose coordinates so that

$$\mathrm{d}\Omega_n^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\theta_2^2 + \sin^2\theta_1\sin^2\theta_2\,\mathrm{d}\theta_3^2 + \cdots + \left(\prod_{i=1}^{n-1}\sin^2\theta_i\right)\mathrm{d}\theta_n^2, \tag{3}$$

whence recursively

$$\mathrm{d}\Omega_n^2(\theta_1,\ldots,\theta_n) = \mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\Omega_{n-1}^2(\theta_2,\ldots,\theta_n). \tag{4}$$

(Several other coordinate conventions on the n-sphere are also relatively common.) This Myers–Perry spacetime has $4 + n$ dimensions, 4 of them "usual" and n "extra". This is sometimes phrased as $3 + 1 + n$ dimensions, (meaning 3 of space, 1 of time, and n "extra" Kaluza–Klein dimensions). The black hole mass M_{BH}, and angular momentum J, are defined as follows

$$M_{BH} = \frac{(n+2)\,A_{n+2}}{16\pi G}\mu, \qquad\qquad J = \frac{2a}{n+2}\,M_{BH}. \tag{5}$$

Here G denotes the gravitational constant in the $(4+n)$-dimensional space-time, and the quantity $A_{n+2} = 2\pi^{(n+3)/2}/\Gamma[(n+3)/2]$ is the area of a $(n+2)$-dimensional unit sphere. The location of the black hole horizon r_H is the solution of $\Delta(r_H) = 0$, such that $\mu = r_H^{n-1}(r_H^2 + a^2)$ is satisfied.

- In the specific case of $n = 0$ this spacetime reduces to the standard Kerr black hole, with the usual inner and outer horizons.
- In the specific case of $n = 1$, we have $\mu = r_H^2 + a^2$, so then $r_H = \sqrt{\mu - a^2}$, and the horizon exists only when $a < \sqrt{\mu}$; in fact the horizon shrinks to zero area in the extreme limit $a \to \sqrt{\mu}$. So the case $n = 1$ is somewhat different from $n > 1$.
- On the other hand, in the case of $n \geqslant 2$, for $\mu > 0$ a unique positive solution for r_H always exists for all a. Indeed $r_H \in \left(0, \mu^{1/(n+1)}\right]$.

B. Separation of variables.

In this article we will focus on scalar field emission from the Myers–Perry black hole. The relevant excitations can be described by the Klein–Gordon equation

$$\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\Phi\right) = 0. \tag{6}$$

Here the metric determinant factorizes nicely into 4-dimensional and n-dimensional pieces. Specifically, with conventions as in equation (3), we have

$$\sqrt{-g} = (\Sigma \sin\theta) \times (r^n \cos^n \theta) \times \left(\prod_{i=1}^{n-1} \sin^{n-i}\theta_i\right), \tag{7}$$

with the trailing factor arising from the unit n-sphere.

Similarly to the Kerr–Newman black hole in four dimensions, the Myers–Perry solution enjoys a hidden symmetry due to the existence of a Killing–Yano tensor.[35] In view of this, we can use the separation of variables ansatz[36]

$$\Phi(t, r, \theta, \varphi, \theta_1, \ldots, \theta_n) = e^{-i\omega t} \, e^{im\varphi} \, \tilde{R}_{j\ell m}(r) \, S_{\ell m}(\theta) \, Y_{jn}(\theta_1, \ldots, \theta_n). \tag{8}$$

Here the $Y_{jn}(\theta_1, \ldots, \theta_n)$ are the quite standard hyper-spherical harmonics defined on the unit n-sphere, which satisfy the differential equation[37]

$$\Delta_{S^n} Y_{jn}(\theta_1, \ldots, \theta_n) + j(j + n - 1)Y_{jn} = 0. \tag{9}$$

The important observation is that for the n-sphere the Laplacian eigenvalues are $-j(j+n-1)$. In 4 dimensions ($n \to 0$) these hyperspherical harmonics reduce to trivial constants, (and $j \to 0$). In 5 dimensions ($n \to 1$) they are simply sines and cosines. If one wishes an explicit rendition of the Laplacian on the n-sphere then, with coordinates as in equation (3), we have

$$\sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{n-1} \sin^{n-i}\theta_i} \partial_{\theta_k} \left[\left(\prod_{i=1}^{n-1} \sin^{n-i}\theta_i\right) \frac{\partial_{\theta_k} Y_{jn}}{\prod_{i=1}^{k-1} \sin^2\theta_i}\right] + j(j + n - 1)Y_{jn} = 0. \tag{10}$$

We mention in passing that when you choose coordinates to write the n-sphere metric recursively, as in equation (4), then the Laplacian can also be expressed recursively

$$\Delta_{S^n} X = \frac{1}{\sin^{n-1}\theta_1} \frac{\partial}{\partial\theta_1} \left(\sin^{n-1}\theta_1 \, \frac{\partial X}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1} \, \Delta_{S^{n-1}} X. \tag{11}$$

In contrast to the hyper-spherical harmonics defined on the hyper-sphere S^n, the spheroidal harmonics $S_{\ell m}(\theta) \, e^{im\varphi}$ are defined on the two angular variables associated with the "usual" 4-dimensional part of the spacetime. They are the appropriate generalization of the standard spherical harmonics $Y_{\ell m}(\theta, \phi)$. The spheroidal harmonics satisfy the differential equation[12]

$$\left\{\frac{1}{\sin\theta \cos^n\theta} \frac{d}{d\theta} \left[\sin\theta \cos^n\theta \frac{d}{d\theta}\right] - \left(\omega a \sin\theta - \frac{m}{\sin\theta}\right)^2 - \frac{j(j + n - 1)}{\cos^2\theta} + \lambda_{j\ell m}\right\} S_{\ell m}(\theta) = 0. \tag{12}$$

Note that going to 4 dimensions corresponds to setting $n \to 0$ *and* setting $j \to 0$, in which case this differential equation reduces to that for the Kerr (or Kerr–Newman) geometry as given in reference 11. These spheroidal harmonics are very closely related both to the Heun functions,[38–41] and to the hyper-spherical harmonics.[37,42]

The separation constant $\lambda_{j\ell m}$ in this spheroidal differential equation is positive. To see this let us define a new variable by $du = \sin\theta \, \cos^n\theta \, d\theta$, then

$$\frac{d}{d\theta} = \frac{du}{d\theta} \frac{d}{du} = \sin\theta \cos^n\theta \frac{d}{du}. \tag{13}$$

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Therefore

$$\frac{1}{\sin\theta\cos^n\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin\theta\cos^n\theta\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right] = \frac{\mathrm{d}}{\mathrm{d}u}\left[(\sin\theta\cos^n\theta)^2\frac{\mathrm{d}S(\theta)}{\mathrm{d}u}\right]. \tag{14}$$

Then the angular equation (12) for the spheroidal harmonics becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[(\sin\theta\cos^n\theta)^2\frac{\mathrm{d}S(\theta)}{\mathrm{d}u}\right] = \left[\left(\omega a\sin\theta - \frac{m}{\sin\theta}\right)^2 + \frac{j(j+n-1)}{\cos^2\theta} - \lambda_{j\ell m}\right]S(\theta). \tag{15}$$

Multiplying the above equation by $S(\theta)$ and integrating both sides over u yields

$$\int S(\theta)\frac{\mathrm{d}}{\mathrm{d}u}\left[(\sin\theta\cos^n\theta)^2\frac{\mathrm{d}S(\theta)}{\mathrm{d}u}\right]\mathrm{d}u$$
$$= \int\left[\left(\omega a\sin\theta - \frac{m}{\sin\theta}\right)^2 + \frac{j(j+n-1)}{\cos^2\theta} - \lambda_{j\ell m}\right]S^2(\theta)\,\mathrm{d}u. \tag{16}$$

Integrate the left hand side by parts, using periodicity to discard boundary terms, and then rearrange to obtain

$$\lambda_{j\ell m}\int S^2(\theta)\,\mathrm{d}u = \int\left[\left(\omega a\sin\theta - \frac{m}{\sin\theta}\right)^2 + \frac{j(j+n-1)}{\cos^2\theta}\right]S^2(\theta)\,\mathrm{d}u$$
$$+ \int\left[(\sin\theta\cos^n\theta)^2\left(\frac{\mathrm{d}S(\theta)}{\mathrm{d}u}\right)^2\right]\mathrm{d}u. \tag{17}$$

Now the right hand side of this equation is manifestly positive, as is the factor $\int S^2\,\mathrm{d}u$ on the left hand side. Therefore the separation constant $\lambda_{j\ell m}$ is guaranteed to be positive.

C. Effective potential.

We now construct the effective potential, starting from the radial part of the variable-separated Klein–Gordon equation.[12–14] We have

$$\left\{\frac{1}{r^n}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^n\Delta\frac{\mathrm{d}}{\mathrm{d}r}\right] + \frac{[(r^2+a^2)\omega - ma]^2}{\Delta} - \frac{j(j+n-1)a^2}{r^2} - \lambda_{j\ell m}\right\}\tilde{R}_{j\ell m}(r) = 0. \tag{18}$$

Let us now define a new radial mode function

$$\tilde{R}_{j\ell m}(r) = \frac{r^{-\frac{n}{2}}R_{j\ell m}(r)}{\sqrt{r^2+a^2}}. \tag{19}$$

It is now a quite standard calculation to show that the radial Teukolsky equation, (the Regge–Wheeler-like equation governing the radial modes), is given by[12–14]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} - U_{j\ell m}(r)\right\}R_{j\ell m}(r) = 0, \tag{20}$$

where r_* is the standard "tortoise coordinate"

$$\mathrm{d}r_* = \frac{r^2+a^2}{\Delta(r)}\,\mathrm{d}r. \tag{21}$$

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Note that the tortoise coordinates can be expressed as

$$r_* = \int_{r_H}^r \frac{r^2 + a^2}{\Delta(r)} dr \sim A_n \ln(r - r_H) + B_n(r), \tag{22}$$

where the exact expressions for the coefficients A_n and functions $B_n(r)$ depend on the number of extra dimensions n. However, we can quite generally observe that as $r \to r_H$ we have $r_* \to -\infty$, and as $r \to \infty$ we have $r_* \to \infty$. So the region $r > r_H$ outside the black hole, (the domain of outer communication), maps into the entire real line $-\infty \le r_* \le +\infty$ in terms of the tortoise coordinate.

The Teukolsky potential, (sometimes called the Regge–Wheeler–Teukolsky potential), is now seen to be

$$U_{j\ell m}(r) = \frac{\Delta(r)}{(r^2 + a^2)^2} \left[\lambda_{j\ell m} + \frac{j(j + n - 1)a^2}{r^2} + \frac{n(n - 2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} \right.$$
$$\left. - \frac{3r^2 \Delta(r)}{(r^2 + a^2)^2} + \frac{[r\Delta(r)]'}{r^2 + a^2} \right] - \left(\omega - \frac{ma}{r^2 + a^2} \right)^2. \tag{23}$$

Note that for $j = n = 0$ this reduces to the Teukolsky potential for the ordinary Kerr black hole in 4 dimensional space-time. (See reference 11.) For purposes of calculation, we now define quantities

$$\varpi(r) = \frac{a}{a^2 + r^2}, \tag{24}$$

and more specifically

$$\Omega_H = \frac{a}{a^2 + r_H^2}. \tag{25}$$

Here $\varpi(r)$ is related to frame dragging, while Ω_H is the "angular velocity" of the event horizon.[11] We can now re-express the Teukolsky potential as

$$U_{j\ell m}(r) = V_{j\ell m}(r) - (\omega - m\varpi)^2, \tag{26}$$

with

$$V_{j\ell m}(r) = \frac{\Delta(r)}{(r^2 + a^2)^2} \left[\lambda_{j\ell m} + \frac{j(j + n - 1)a^2}{r^2} \right.$$
$$\left. + \frac{n(n - 2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} - \frac{3r^2 \Delta(r)}{(r^2 + a^2)^2} + \frac{[r\Delta(r)]'}{r^2 + a^2} \right]. \tag{27}$$

D.   Positivity properties.

To show positivity of $V_{j\ell m}(r)$, we start by noting that $\Delta(r) > 0$ outside the horizon, (that is for $r > r_H$). This is standard for $n = 0$, and trivial for $n = 1$. For $n \ge 1$ we generically re-express $\Delta(r)$ as

$$\Delta(r) = r^2 + a^2 - r^{1-n}\mu$$
$$= r^2 + a^2 - (r/r_H)^{1-n}(r_H^2 + a^2)$$
$$\ge (r_H^2 + a^2)(1 - (r_H/r)^{n-1}). \tag{28}$$

Since $r \geq r_H$, we can see that $\Delta(r) \geq 0$ for $n \geq 1$. Using this result, we make the following observations. First, for $n \geq 1$ we have

$$\frac{[r\Delta(r)]'}{r^2 + a^2} - \frac{3r^2\Delta(r)}{(r^2 + a^2)^2} \propto [r\Delta(r)]'(r^2 + a^2) - 3r^2\Delta(r)$$

$$= a^2(r^2 + a^2) + \frac{\mu}{r^{n-1}}\left[(n+1)r^2 + (n-2)a^2\right]$$

$$= a^2\Delta(r) + \frac{\mu}{r^{n-1}}\left[(n+1)r^2 + (n-1)a^2\right]$$

$$\geq 0. \tag{29}$$

Note that the equivalent result for $n = 0$ was already derived in reference 11 for the Kerr–Newman spacetime. Second, for $n \geq 0$, we also have

$$\frac{n(n-2)\Delta(r)}{4r^2} + \frac{n\Delta'(r)}{2r} \propto n\{(n-2)\Delta(r) + 2r\Delta'(r)\}$$

$$= n\{(n+2)r^2 + (n-2)a^2 + n\mu r^{1-n}\}. \tag{30}$$

Now for $n \geq 2$ this quantity is certainly positive. For $n = 0$ this quantity is identically zero. For $n = 1$ this quantity reduces to $3r^2 - a^2 + \mu = 3r^2 - r_H^2 \geq 0$ (provided the horizon exists). In all situations the relevant quantity is non-negative. Thus, by now combining these results with the fact that $\lambda_{j\ell m} > 0$, and the fact that both $n \geq 0$ and $j \geq 0$, we can conclude that $V_{j\ell m}(r)$ is always positive for all values of j, ℓ, m, and r.

E.   Super-radiance.

Now note that the effective potential is

$$U_{j\ell m}(r) = V_{j\ell m}(r) - (\omega - m\varpi)^2; \qquad V_{j\ell m}(r) \geq 0. \tag{31}$$

However, the quantity $\omega - m\varpi$ can under suitable circumstances change sign. This is the harbinger of super-radiance. Some rather general analyses can be found in references 43–44, while a specific analysis closely related to the current situation can be found in reference 11. The key point is that super-radiance is a phenomenon in which the reflected wave is larger in its amplitude than the incident wave. From mathematical point of view, super-radiance is a phenomenon in which $|r| > 1$, where r is the reflection coefficient. Super-radiance will occur once $\omega - m\varpi$ changes sign in the domain of outer communication which, given the asymptotic behaviour of ϖ, occurs whenever $0 < \omega < m\Omega_H$, that is $m > m_* \equiv \omega/\Omega_H$. Once super-radiance occurs, the bound on the greybody factor becomes a bound on the spontaneous emission amplitude. A detailed discussion of this particular issue can be found in reference 11.

III.   ANALYTIC BOUND FOR SCALAR TRANSMISSION.

From reference 15, (see also references 16, 17, 18, and 19 for further developments and applications), we have the extremely general result that

$$T_{j\ell m} \geq \text{sech}^2\left(\int_{-\infty}^{\infty} \vartheta \, dr_*\right), \tag{32}$$

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where

$$\vartheta = \frac{\sqrt{[h'(r_*)]^2 + [U_{j\ell m}(r_*) + h^2(r_*)]^2}}{2h(r_*)}, \tag{33}$$

for any positive function $h(r_*)$. Equivalently

$$\vartheta = \frac{\sqrt{[h'(r_*)]^2 + [V_{j\ell m}(r_*) - (\omega - m\varpi)^2 + h^2(r_*)]^2}}{2h(r_*)}. \tag{34}$$

We shall now use the positivity properties of $\lambda_{j\ell m}$ and $V_{j\ell m}$, together with the super-radiant/non-super-radiant distinction, to systematically analyse this bound in various cases. In particular

- The modes $m < m_* \equiv \omega/\Omega_H$ are not super-radiant.

- The modes $m \geq m_* \equiv \omega/\Omega_H$ are super-radiant.

In situations where super-radiance occurs, in addition to the greybody factor $T_{j\ell m}$, there is a closely related spontaneous emission rate which satisfies the bound[11]

$$\Gamma_{j\ell m} \leq \omega \sinh^2\left(\int_{-\infty}^{\infty} \vartheta \, dr_*\right). \tag{35}$$

IV. NON-SUPER-RADIANT MODES $(m < m_*)$.

It is convenient to split the discussion of non-super-radiant modes into three sub-cases:

- $m = 0$ zero-angular-momentum modes: This is the most fundamental case, and most straightforward case to analyze. This case provides a useful template for the more complicated situations.

- $m \neq 0$ nonzero-angular-momentum modes: These are most conveniently further split into two sub-cases.

  - $m < 0$ negative-angular-momentum modes.
  - $m \in (0, m_*)$ low-lying positive-angular-momentum modes.

A. Zero angular momentum modes $(m = 0)$.

We choose $\tilde{h}(r_*) = \omega > 0$ and $m = 0$, then

$$U_{j\ell,m=0}(r) = \frac{\Delta(r)}{(r^2 + a^2)^2}\left[\lambda_{j\ell,m=0} + \frac{j(j+n-1)a^2}{r^2} + \frac{n(n-2)\Delta(r)}{4r^2}\right.$$
$$\left. + \frac{n\Delta'(r)}{2r} - \frac{3r^2\Delta(r)}{(r^2+a^2)^2} + \frac{[r\Delta(r)]'}{r^2+a^2}\right] - \omega^2. \tag{36}$$

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Then

$$
\begin{aligned}
T &\geq \operatorname{sech}^2\left(\frac{1}{2\omega}\int_{-\infty}^{\infty}|V|\,\mathrm{d}r_*\right)\\
&= \operatorname{sech}^2\left(\frac{1}{2\omega}\int_{r_h}^{\infty}|V(r)|\frac{r^2+a^2}{\Delta(r)}\mathrm{d}r\right)\\
&= \operatorname{sech}^2\left[\frac{1}{2\omega}\int_{r_h}^{\infty}\left|\frac{1}{r^2+a^2}\left\{\lambda_{j\ell,m=0}+\frac{j(j+n-1)a^2}{r^2}+\frac{n(n-2)\Delta(r)}{4r^2}\right.\right.\right.\\
&\qquad\left.\left.\left.-\frac{3r^2\Delta(r)}{(r^2+a^2)^2}+\frac{n\Delta'(r)}{2r}+\frac{[r\Delta(r)]'}{r^2+a^2}\right\}\right|\mathrm{d}r\right].
\end{aligned}
\tag{37}
$$

For $n\geq 1$ and $r\geq r_H$, in view of the positivity properties of the separation constant and effective potential, we can replace $\int|\cdots|\,\mathrm{d}r\to|\int\cdots\mathrm{d}r|$. Therefore

$$
\begin{aligned}
T\geq \operatorname{sech}^2\left|\frac{1}{2\omega}\int_{r_h}^{\infty}\frac{1}{r^2+a^2}\left\{\lambda_{j\ell,m=0}+\frac{j(j+n-1)a^2}{r^2}+\frac{n(n-2)\Delta(r)}{4r^2}\right.\right.\\
\left.\left.-\frac{3r^2\Delta(r)}{(r^2+a^2)^2}+\frac{n\Delta'(r)}{2r}+\frac{[r\Delta(r)]'}{r^2+a^2}\right\}\mathrm{d}r\right|.
\end{aligned}
\tag{38}
$$

We would like to integrate this equation term by term. Start by considering the first term:

$$
\int_{r_H}^{\infty}\frac{\lambda_{j\ell,m=0}}{r^2+a^2}\,\mathrm{d}r=\frac{\lambda_{j\ell,m=0}}{a}\arctan\frac{r}{a}\bigg|_{r_H}^{\infty}=\frac{\lambda_{j\ell,m=0}}{a}\arctan\frac{a}{r_H}.
\tag{39}
$$

For the last two integrals, we can show that they can be simplified as follows:

$$
\int_{r_H}^{\infty}\frac{1}{r^2+a^2}\left[-\frac{3r^2\Delta(r)}{(r^2+a^2)^2}+\frac{[r\Delta(r)]'}{r^2+a^2}\right]\mathrm{d}r=\int_{r_H}^{\infty}\frac{r^2\Delta(r)}{(r^2+a^2)^3}\,\mathrm{d}r.
\tag{40}
$$

This can be explicitly integrated (for instance by using Mathematica) and we arrive at

$$
\begin{aligned}
\int_{r_H}^{\infty}\frac{r^2\Delta(r)}{(r^2+a^2)^3}\mathrm{d}r=&\frac{n}{8r_H}-\frac{n(n-2)(r_H^2+a^2)}{8(n+2)r_H^3}\;{}_2F_1\left(1,\frac{n+2}{2},\frac{n+4}{2},-\frac{a^2}{r_H^2}\right)\\
&-\frac{a^2}{4r_H(r_H^2+a^2)}+\frac{1}{2a}\arctan\frac{a}{r_H}.
\end{aligned}
\tag{41}
$$

Here ${}_2F_1(z_1,z_2,z_3,z_4)$ is the hypergeometric function. Let us now consider the j-dependent integral:

$$
\int_{r_H}^{\infty}\frac{j(j+n-1)a^2}{r^2(r^2+a^2)}\mathrm{d}r=\frac{j(j+n-1)}{r_H}-\frac{j(j+n-1)}{a}\arctan\frac{a}{r_H}.
\tag{42}
$$

We can also integrate the n-dependent terms as

$$
\begin{aligned}
\int_{r_H}^{\infty}\frac{1}{r^2+a^2}\left[\frac{n(n-2)\Delta(r)}{4r^2}+\frac{n\Delta'(r)}{2r}\right]\mathrm{d}r=&\frac{n^2(r_H^2+a^2)}{4(n+2)r_H^3}\;{}_2F_1\left(1,\frac{n+2}{2},\frac{n+4}{2},-\frac{a^2}{r_H^2}\right)\\
&+\frac{n(n-2)}{4r_H}+\frac{n}{a}\arctan\frac{a}{r_H}.
\end{aligned}
\tag{43}
$$

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Finally, combining the results from equation (39), (41), (42), and (43), we obtain

$$T_{j\ell,m=0} \geq \mathrm{sech}^2 \left| \frac{1}{2\omega r_H} I_{j\ell,m=0} \right|, \tag{44}$$

where we define

$$\begin{aligned} I_{j\ell,m=0} = \ &\frac{n(2n-3)}{8} + j(j+n-1) + \frac{a^2}{4(r_H^2+a^2)} \\ &+ \left(\frac{2n+1}{2} - j(j+n-1) + \lambda_{j\ell,m=0} \right) \frac{r_H}{a} \arctan \frac{a}{r_H} \\ &+ \frac{n(r_H^2+a^2)}{8r_H^2} \ {}_2F_1 \left(1, \frac{n+2}{2}, \frac{n+4}{2}, -\frac{a^2}{r_H^2} \right). \end{aligned} \tag{45}$$

For a consistency check, consider the limit $a \to 0$ (with both $n = 0$ and $j = 0$),

$$\begin{aligned} \lim_{a \to 0} I_{j=0,\ell,m=0} &= \lim_{a \to 0} \left[-\frac{a^2}{4(r_H^2+a^2)} + \left(\frac{1}{2} + \lambda_{j=0,\ell,m=0} \right) \frac{r_H}{a} \arctan \frac{a}{r_H} \right] \\ &= \frac{1}{2} + \lambda_{j=0,\ell,m=0}. \end{aligned} \tag{46}$$

This is the same result as for the Kerr black hole, (the Kerr–Newman black hole for $Q = 0$), as is to be expected.

B. Non-zero angular momentum mode ($m \neq 0$).

From the basic inequality we have

$$T_{jlm} \geq \mathrm{sech}^2 \left[\int_{-\infty}^{\infty} \frac{\sqrt{[\tilde{h}'(r_*)]^2 + \left[\tilde{U}_{j\ell m}(r_*) + \tilde{h}^2(r_*) \right]^2}}{2\tilde{h}(r_*)} \mathrm{d}r_* \right], \tag{47}$$

for all $\tilde{h}(r_*) > 0$. By now using the triangle inequality

$$|a| + |b| \geq \sqrt{a^2 + b^2}, \tag{48}$$

we have

$$\begin{aligned} T_{jlm} &\geq \mathrm{sech}^2 \left[\int_{-\infty}^{\infty} \frac{\left| \tilde{h}'(r_*) \right| + \left| \tilde{U}_{j\ell m}(r_*) + \tilde{h}^2(r_*) \right|}{2\tilde{h}(r_*)} \mathrm{d}r_* \right] \\ &\geq \mathrm{sech}^2 \left[\int_{-\infty}^{\infty} \frac{\left| \tilde{h}'(r_*) \right|}{2\tilde{h}(r_*)} \mathrm{d}r_* + \int_{-\infty}^{\infty} \frac{\left| \tilde{U}_{j\ell m}(r_*) + \tilde{h}^2(r_*) \right|}{2\tilde{h}(r_*)} \mathrm{d}r_* \right]. \end{aligned} \tag{49}$$

Provided that $\tilde{h}'(r_*)$ is monotone, we have

$$\int_{-\infty}^{\infty} \frac{|\tilde{h}'(r_*)|}{2\tilde{h}(r_*)} \mathrm{d}r_* = \begin{cases} \frac{1}{2} \ln \frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)} & \text{for } \tilde{h}'(r_*) > 0; \\ \\ -\frac{1}{2} \ln \frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)} & \text{for } \tilde{h}'(r_*) < 0. \end{cases} \tag{50}$$

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Let us now rewrite the potential as

$$U_{j\ell m} = V_{j\ell,m} - (\omega - m\,\varpi(r))^2. \tag{51}$$

This form of potential is exactly the same as for the 4-dimensional Kerr–Newman black hole, and thus we simply choose

$$\tilde{h}(r_*) = h(r) = \omega - m\varpi(r). \tag{52}$$

Note that this choice for $h(r)$ is always monotonic as a function of r. However, we can see that $h(r)$ is positive if and only if $\omega > m\Omega_H$. This condition is satisfied for $m < \omega/\Omega_H$, (that is $m < m_*$), where the mode does not suffer from super-radiant instability.

1. Negative-angular-momentum modes ($m < 0$).

Note that in this case, for $h(r)$ defined in equation (52),

$$\frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)} = \frac{h(\infty)}{h(r_H)} = \frac{\omega}{\omega - m\Omega_H} = \frac{1}{1 - m\Omega_H/\omega} < 1. \tag{53}$$

Then

$$\frac{1}{2}\left|\ln\left[\frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)}\right]\right| = \frac{1}{2}\ln(1 - m\Omega_H/\omega). \tag{54}$$

Note also that in this case we have $\omega - m\Omega_H > h(r) > \omega$, so

$$\int_{-\infty}^{\infty}\frac{|U_{j\ell m} + h^2(r)|}{2h(r)}\mathrm{d}r_* = \int_{-\infty}^{\infty}\frac{|V_{j\ell m}|}{2h(r)}\mathrm{d}r_* < \int_{-\infty}^{\infty}\frac{V_{j\ell m}}{2\omega}\mathrm{d}r_*. \tag{55}$$

Then

$$T_{j\ell,m<0} \geq \mathrm{sech}^2\left\{\frac{1}{2}\ln(1 - m\Omega_H/\omega) + \int_{-\infty}^{\infty}\frac{V_{j\ell,m<0}}{2\omega}\mathrm{d}r_*\right\}, \tag{56}$$

$$\geq \mathrm{sech}^2\left\{\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2\omega r_H}I_{j\ell,m<0}\right\}. \tag{57}$$

It is easy to see that this result is very similar to the result we have for $m = 0$, with the replacement $\lambda_{j\ell,m=0} \to \lambda_{j\ell,m<0}$. We can write down $I_{j\ell m}$ explicitly as

$$
\begin{aligned}
I_{j\ell m} =\ & \frac{n(2n-3)}{8} + j(j+n-1) + \frac{a^2}{4(r_H^2 + a^2)} \\
& + \left(\frac{2n+1}{2} - j(j+n-1) + \lambda_{j\ell m}(a\omega)\right)\frac{r_H}{a}\arctan\frac{a}{r_H} \\
& + \frac{n(r_H^2 + a^2)}{8r_H^2}\,_2F_1\left(1, \frac{n+2}{2}, \frac{n+4}{2}, -\frac{a^2}{r_H^2}\right).
\end{aligned}
\tag{58}
$$

2. Low-lying positive-angular-momentum modes $(m \in (0, m_*))$.

Recall that for $m_* > m > 0$, $h(r)$ is positive and monotonic as a function of r, for this situation we first consider

$$\frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)} = \frac{h(\infty)}{h(r_H)} = \frac{\omega}{\omega - m\Omega_H} = \frac{1}{1 - m\Omega_H/\omega} > 1. \tag{59}$$

Then, we have

$$\frac{1}{2}\left|\ln\left[\frac{\tilde{h}(\infty)}{\tilde{h}(-\infty)}\right]\right| = -\frac{1}{2}\ln(1 - m\Omega_H/\omega). \tag{60}$$

Note also that in this case we have $\omega - m\Omega_H < h(r) < \omega$, so

$$\int_{-\infty}^{\infty} \frac{|U_{j\ell m} + h^2(r)|}{2h(r)} dr_* = \int_{-\infty}^{\infty} \frac{|V_{j\ell,m>0}|}{2h(r)} dr_* < \int_{-\infty}^{\infty} \frac{V_{j\ell,m>0}}{2(\omega - m\Omega_H)} dr_*. \tag{61}$$

Then, we arrive at the result

$$T_{j\ell,m>0} \geq \operatorname{sech}^2\left\{-\frac{1}{2}\ln(1 - m\Omega_H/\omega) + \int_{-\infty}^{\infty} \frac{V_{j\ell,m>0}}{2(\omega - m\Omega_H)} dr_*\right\}, \tag{62}$$

$$\geq \operatorname{sech}^2\left\{-\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2r_H\omega(1 - m/m_*)}I_{j\ell,m>0}\right\}, \tag{63}$$

where $I_{j\ell,m>0}$ is defined by equation (58).

V.   SUPER-RADIANT MODES $(m \geq m_*)$.

It is a good strategy to split the super-radiant modes into two sub-classes depending on the relative sizes of ω^2 and $(\omega - m\Omega_H)^2$. Note that $\omega^2 = (\omega - m\Omega_H)^2$ when $m = 2\omega/\Omega_H = 2m_*$. This suggests that it might be useful to split the super-radiant modes as follows:

- $m \in [m_*, 2m_*)$.

- $m \in [2m_*, \infty)$.

A.   Low-lying super-radiant modes $(m \in [m_*, 2m_*))$.

In this region we have $\omega^2 > (\omega - m\Omega_H)^2$ and we choose

$$h(r) = \max\{\omega - m\varpi(r), m\Omega_H - \omega\}. \tag{64}$$

We can see that $h(r) > 0$ and monotone decreasing as we move from spatial infinity to the horizon, and become a flat horizontal line near the horizon. Note that $h(r) \geq m\Omega_H - \omega$ everywhere. By using $h(r)$ as defined in equation (64), we have

$$\int_{-\infty}^{\infty} \frac{|h'(r)|}{h(r)} dr_* = |\ln h(r)|_{r_H}^{\infty} = \ln\left(\frac{\omega}{m\Omega_H - \omega}\right) = -\ln(m/m_* - 1). \tag{65}$$

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It is now straightforward to show that

$$\int_{-\infty}^{\infty} \frac{V_{j\ell m}}{2h(r)} \mathrm{d}r_* \leq \int_{-\infty}^{\infty} \frac{V_{j\ell m}}{2(m\Omega_H - \omega)} \mathrm{d}r'_* = \frac{I_{j\ell m}}{2(m\Omega_H - \omega)r_H} = \frac{I_{j\ell m}}{2\omega(m/m_* - 1)r_H}, \qquad (66)$$

where $I_{j\ell m}$ is defined in equation (58). The last integral we need to perform is

$$J_m^{\mathrm{low}} = \int_{-\infty}^{\infty} \frac{h(r)^2 - (\omega - m\varpi(r))^2}{2h(r)} \, \mathrm{d}r_*. \qquad (67)$$

Note that with our choice of $h(r)$, the integrand in above integral is zero over much of the relevant range. To be more precise, we are interested only in

$$J_m^{\mathrm{low}} = \int_{r_H}^{r_0} \frac{(\omega - m\Omega_H)^2 - (\omega - m\varpi(r))^2}{2(m\Omega_H - \omega)} \frac{r^2 + a^2}{\Delta} \, \mathrm{d}r. \qquad (68)$$

The upper limit of integration r_0 is defined by the condition

$$m\left[\Omega_H + \varpi(r_0)\right] = 2\omega, \qquad (69)$$

or we can write down r_0 explicitly as

$$r_0 = \sqrt{r_H^2 + \frac{2(m - m_*)}{2m_* - m}(r_H^2 + a^2)}. \qquad (70)$$

Notice that the upper limit $r_0 > r_H$ for $m \in [m_*, 2m_*)$. Then

$$J_m^{\mathrm{low}} = \frac{m}{2(m\Omega_H - \omega)} \int_{r_H}^{r_0} (\Omega_H - \varpi(r))(m\varpi(r) + m\Omega_H - 2\omega) \frac{r^2 + a^2}{\Delta} \, \mathrm{d}r. \qquad (71)$$

However, for the relevant domain of integration we have

$$0 \leq (m\varpi(r) + m\Omega_H - 2\omega) \leq 2(m\Omega_H - \omega). \qquad (72)$$

Then we can conclude that

$$J_m^{\mathrm{low}} \leq m \int_{r_H}^{r_0} (\Omega_H - \varpi(r)) \frac{r^2 + a^2}{\Delta} \, \mathrm{d}r = m\Omega_H \int_{r_H}^{r_0} \frac{r^{n-1}(r - r_H)(r + r_H)}{r^{n-1}(r^2 + a^2) - r_H^{n-1}(r_H^2 + a^2)} \, \mathrm{d}r. \qquad (73)$$

This integral is finite, and one can evaluate it exactly for each value of n. (The *integrand* is in fact finite as $r \to r_H$ by the l'Hôpital rule.) By now combining all these results, we have

$$T_{j\ell, m\in[m_*, 2m_*)} \geq \mathrm{sech}^2 \left\{ -\frac{1}{2}\ln(m/m_* - 1) + \frac{I_{j\ell, m\in[m_*, 2m_*)}}{2r_H\omega(m/m_* - 1)} + J_m^{\mathrm{low}} \right\}. \qquad (74)$$

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B. Highly super-radiant modes $(m \geq 2m_*)$.

In this region we have $(\omega - m\Omega_H)^2 > \omega^2$, so we can choose

$$h(r) = \max\{m\varpi(r) - \omega, \omega\}. \tag{75}$$

It is not difficult to see that $h(r)$ is both positive and monotone decreasing as we move from the horizon to spatial infinity. Note also that $h(r) \geq \omega$ for the relevant domain. By using equation (75), we have

$$\int_{-\infty}^{\infty} \frac{|h'(r)|}{h(r)}\,\mathrm{d}r_* = |\ln h(r)|_{r_H}^{\infty} = \ln\left(\frac{m\Omega_H - \omega}{\omega}\right) = \ln(m/m_* - 1). \tag{76}$$

We also obtain

$$\int_{-\infty}^{\infty} \frac{V_{j\ell m}}{2h(r)}\,\mathrm{d}r_* \leq \int_{-\infty}^{\infty} \frac{V_{j\ell m}}{2\omega}\mathrm{d}r_* = \frac{I_{j\ell m}}{2\omega r_H}, \tag{77}$$

where $I_{j\ell m}$ is defined in equation (58) as for the previous cases. Finally, we are left with the integral

$$J_m^{\mathrm{high}} = \int_{-\infty}^{\infty} \frac{h(r)^2 - (\omega - m\varpi(r))^2}{2h(r)}\,\mathrm{d}r_*. \tag{78}$$

Again the integrand is zero over much of the domain of integration. That is, we are only interested in

$$J_m^{\mathrm{high}} = \int_{r_0}^{\infty} \frac{\omega^2 - (\omega - m\varpi(r))^2}{2\omega}\frac{r^2 + a^2}{\Delta}\,\mathrm{d}r. \tag{79}$$

Here the lower bound of integration, r_0, is now defined by

$$m\varpi(r_0) = 2\omega, \tag{80}$$

implying

$$r_0 = a\sqrt{\frac{m}{2\omega a} - 1}. \tag{81}$$

Recall that $m \geq 2m_*$ in this region, we have

$$r_0 \geq a\sqrt{\frac{m_*}{\omega a} - 1} = a\sqrt{\frac{r_H^2 + a^2}{a^2} - 1} = r_H. \tag{82}$$

The integral J_m^{high} is finite. (In fact, the *integrand* is finite as $r \to r_0$, and falls of as $1/r^2$ as $r \to \infty$.) After assembling all results we have, we finally obtain

$$T_{j\ell,m \geq 2m_*} \geq \operatorname{sech}^2\left\{\frac{1}{2}\ln(m/m_* - 1) + \frac{I_{j\ell, m \geq 2m_*}}{2r_H\omega} + J_m^{\mathrm{high}}\right\}. \tag{83}$$

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VI. SUMMARY OF THE GENERAL CASE.

Collecting the results for the low-lying and highly super-radiant modes, together with the non-super-radiant modes, we have the following bounds for the transmission probabilities:

$$
T_{j\ell m} \geq
\begin{cases}
\operatorname{sech}^2\left\{\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2r_H\omega}I_{j\ell m}\right\} & \text{for } m < 0; \\[2ex]
\operatorname{sech}^2\left\{\frac{1}{2r_H\omega}I_{j\ell m}\right\} & \text{for } m = 0; \\[2ex]
\operatorname{sech}^2\left\{-\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2r_H\omega(1-m/m_*)}I_{j\ell m}\right\} & \text{for } 0 < m < m_*; \\[2ex]
\operatorname{sech}^2\left\{-\frac{1}{2}\ln(m/m_* - 1) + \frac{1}{2r_H\omega(m/m_*-1)}I_{j\ell m} + J_m^{\text{low}}\right\} & \text{for } m_* \leq m < 2m_*; \\[2ex]
\operatorname{sech}^2\left\{\frac{1}{2}\ln(m/m_* - 1) + \frac{1}{2r_H\omega}I_{j\ell m} + J_m^{\text{high}}\right\} & \text{for } m \geq 2m_*.
\end{cases}
\tag{84}
$$

Here m_* is the "critical" azimuthal angular momentum defined by $m_* = \omega/\Omega_H$, while the quantity $I_{j\ell m}$ is defined in equation (58).

VII. FOUR-DIMENSIONAL CASE $n = 0$.

When $n = 0$ the Myers–Perry spacetime reduces to the usual Kerr spacetime. Furthermore, the separation constant and effective potential reduce to those discussed in reference 11. Ultimately the bounds on the greybody factors reduce (as they should) to those of reference 11.

VIII. FIVE-DIMENSIONAL CASE $n = 1$.

Let us now take a look at a special case with only *one* extra dimension $n = 1$. These are the (3+1+1)-dimensional [five-dimensional] Myers–Perry black holes. In this case we have the simplification

$$
\Delta \to r^2 + a^2 - \mu.
\tag{85}
$$

A brief computation, starting from equation (58), now yields

$$
I_{j\ell m}^{n=1} = \left(\frac{3}{8a\Omega_H} - \frac{1}{8} + j^2 - \frac{a\Omega_H}{4}\right) + \left(\frac{3}{2} - j^2 - \frac{3}{8a\Omega_H} + \lambda_{j\ell m}\right)\frac{r_H}{a}\arctan\left(\frac{a}{r_H}\right).
\tag{86}
$$

Interestingly, J_m^{low} has a very simple bound in five-dimensional space-time. For $n = 1$, we have

$$
\left. J_m^{\text{low}}\right|_{n=1} \leq m\Omega_H(r_0 - r_H) = \omega\frac{m}{m_*}(r_0 - r_H).
\tag{87}
$$

Let us now consider J_m^{high}; this also takes a simpler form in five-dimensional space-time

$$
\left. J_m^{\text{high}}\right|_{n=1} = \int_{r_0}^{\infty} \frac{ma}{2\omega}\left[\frac{2\omega - m\varpi(r)}{(r - r_H)(r + r_H)}\right]\,\mathrm{d}r.
\tag{88}
$$

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For the relevant domain of integration, $2\omega > m\varpi(r)$, then we can conclude that

$$J_m^{\text{high}}\big|_{n=1} \le ma \int_{r_0}^{\infty} \frac{1}{(r - r_H)(r + r_H)}\, \mathrm{d}r = \frac{ma}{r_H} \ln \sqrt{\frac{r_0 + r_H}{r_0 - r_H}}. \tag{89}$$

Collecting results, we finally deduce a quite explicit bound for scalar emission from five-dimensional simply rotating Myers–Perry black holes. The bound is given by:

$$T_{j\ell m}^{(n=1)} \ge \begin{cases} \operatorname{sech}^2\left\{\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2r_H\omega}I_{j\ell m}^{n=1}\right\} & \text{for } m < 0; \\[2ex] \operatorname{sech}^2\left\{\frac{1}{2r_H\omega}I_{j\ell m}^{n=1}\right\} & \text{for } m = 0; \\[2ex] \operatorname{sech}^2\left\{-\frac{1}{2}\ln(1 - m/m_*) + \frac{1}{2r_H\omega(1-m/m_*)}I_{j\ell m}^{n=1}\right\} & \text{for } 0 < m < m_*; \\[2ex] \operatorname{sech}^2\left\{-\frac{1}{2}\ln(m/m_* - 1) + \frac{1}{2r_H\omega(m/m_*-1)}I_{j\ell m}^{n=1} + \omega\frac{m}{m_*}(r_0 - r_H)\right\} & \text{for } m_* \le m < 2m_*; \\[2ex] \operatorname{sech}^2\left\{\frac{1}{2}\ln(m/m_* - 1) + \frac{1}{2r_H\omega}I_{j\ell m}^{n=1} + \frac{ma}{r_H}\ln\sqrt{\frac{r_0+r_H}{r_0-r_H}}\right\} & \text{for } m \ge 2m_*. \end{cases} \tag{90}$$

Here $I_{j\ell m}^{n=1}$ is as given in equation (86).

IX. DISCUSSION.

In this article we have established certain rigorous bounds on the greybody factors (mode dependent transmission probabilities) for the Myers–Perry black holes. We have also obtained (mutatis mutandis) certain rigorous bounds on the emission rates for the super-radiant modes. In the absence of exact results, (the relevant differential equations seem highly resistant to explicit analytic solution), quantitative bounds along these lines seem to be the best one can do.

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REFERENCES

[1] S. W. Hawking,
"Black hole explosions",
Nature **248** (1974) 30.

[2] S. W. Hawking,
"Particle Creation by Black Holes",
Commun. Math. Phys. **43** (1975) 199 [Erratum-ibid. **46** (1976) 206].

[3] D. N. Page,
"Particle Emission Rates from a Black Hole:
Massless Particles from an Uncharged, Nonrotating Hole",
Phys. Rev. D **13** (1976) 198.
doi: 10.1103/PhysRevD.13.198

[4] D. N. Page,
"Particle Emission Rates from a Black Hole. 2.
Massless Particles from a Rotating Hole",
Phys. Rev. D **14** (1976) 3260.
doi: 10.1103/PhysRevD.14.3260

[5] J. D. Bekenstein and A. Meisels,
"Einstein A and B coefficients for a black hole",
Phys. Rev. D **15** (1977) 2775.
doi: 10.1103/PhysRevD.15.2775

[6] Jorge Escobedo,
"Greybody Factors: Hawking Radiation in Disguise",
M.Sc. Thesis, U Amsterdam, 2008.

[7] Petarpa Boonserm and Matt Visser,
"Bounding the greybody factors for Schwarzschild black holes",
Phys. Rev. D **78** (2008) 101502 [arXiv:0806.2209 [gr-qc]].

[8] T. Ngampitipan and P. Boonserm,
"Bounding the Greybody Factors for Non-rotating Black Holes",
Int. J. Mod. Phys. **D22** (2013) 1350058,
DOI: 10.1142/S0218271813500582 arXiv:1211.4070 [math-ph].

[9] T. Ngampitipan and P. Boonserm,
"Bounding the greybody factors for the Reissner–Nordström black holes",
J. Phys. Conf. Ser. **435** (2013) 012027 [arXiv:1301.7527 [math-ph]], ICAST2012.

[10] P. Boonserm, T. Ngampitipan and M. Visser,
"Regge-Wheeler equation, linear stability, and greybody factors for dirty black holes",
Phys. Rev. D **88** (2013) 041502 [arXiv:1305.1416 [gr-qc]].

[11] P. Boonserm, T. Ngampitipan and M. Visser,
"Bounding the greybody factors for scalar field excitations of the
Kerr-Newman spacetime",
Journal of High Energy Physics 1403 (2014) 113; arXiv:1401.0568 [gr-qc].
doi:10.1007/JHEP03(2014)113

[12] D. Ida, Y. Uchida and Y. Morisawa,
"The scalar perturbation of higher-dimensional rotating black holes",
Phys. Rev. D **67** (2003) 084019; arXiv: gr-qc/0212035.
doi: 10.1103/PhysRevD.67.084019

[13] S. Creek, O. Efthimiou, P. Kanti, and K. Tamvakis,
"Scalar emission in the bulk in a rotating black hole background",
Phys. Lett. B **656** (2007) 102111; arXiv: 0709.0241 [hep-th].
doi: 10.1016/j.physletb.2007.09.050

[14] S. Creek, O. Efthimiou, P. Kanti and K. Tamvakis,
"Greybody factors for brane scalar fields in a rotating black-hole background",
Phys. Rev. D **75** (2007) 084043; arXiv: hep-th/0701288.
doi: 10.1103/PhysRevD.75.084043

[15] M. Visser,
"Some general bounds for 1-D scattering",
Phys. Rev. A **59** (1999) 42738;arXiv: quant-ph/9901030.
doi: 10.1103/PhysRevA.59.427

[16] P. Boonserm and M. Visser,
"Transmission probabilities and the Miller–Good transformation",
J. Phys. A **42**, 045301 (2009) [arXiv:0808.2516 [math-ph]].

[17] P. Boonserm and M. Visser,
"Bounding the Bogoliubov coefficients",
Annals Phys. **323** (2008) 2779 [arXiv:0801.0610 [quant-ph]].

[18] P. Boonserm and M. Visser,
"Analytic bounds on transmission probabilities",
Annals Phys. **325** (2010) 1328 [arXiv:0901.0944 [math-ph]].

[19] P. Boonserm and M. Visser,
"Reformulating the Schrödinger equation as a Shabat–Zakharov system",
J. Math. Phys. **51** (2010) 022105 [arXiv:0910.2600 [math-ph]].

[20] P. Boonserm and M. Visser,
"Quasi-normal frequencies: Key analytic results",
JHEP **1103** (2011) 073 [arXiv:1005.4483 [math-ph]].

[21] P. Boonserm and M. Visser,
"One Dimensional Scattering Problems: A Pedagogical Presentation of the Relationship between Reflection and Transmission Amplitudes",
Thai Journal of Mathematics **8** (2010) 83–97.

[22] P. Boonserm and M. Visser,
"Compound transfer matrices: Constructive and destructive interference",
J. Math. Phys. **53** (2012) 012104 [arXiv:1101.4014 [math-ph]].

[23] P. Boonserm and M. Visser,
"Bounds on variable-length compound jumps",
J. Math. Phys. **54** (2013) 092105 [arXiv:1301.7524 [math-ph]].

[24] R. C. Myers and M. J. Perry,
"Black Holes in Higher Dimensional Space-Times",

Annals Phys. **172** (1986) 304.

[25] R. Emparan and H. S. Reall,
"Black Holes in Higher Dimensions",
Living Rev. Rel. **11** (2008) 6; [arXiv:0801.3471 [hep-th]].

[26] W. H. Press and S. A. Teukolsky,
"Perturbations of a Rotating Black Hole. II.   Dynamical Stability of the Kerr Metric",
Astrophys. J. **185** (1973) 649–673.

[27] Shibata, M., Sasaki, M., Tagoshi, H. and Tanaka, T.,
"Gravitational waves from a particle orbiting around a rotating black hole: Post-Newtonian expansion",
Phys. Rev. D **51** (1995) 1646–1663.

[28] Tagoshi, H., Shibata, M., Tanaka, T. and Sasaki, M.,
"Post-Newtonian expansion of gravitational waves from a particle in circular orbits around a rotating black hole: Up to $O(v^8)$ beyond the quadrupole formula",
Phys. Rev. D **54** (1996) 1439–1459.

[29] K. D. Kokkotas and B. G. Schmidt,
"Quasinormal modes of stars and black holes",
Living Rev. Rel. **2** (1999) 2 [gr-qc/9909058].

[30] Misao Sasaki and Hideyuki Tagoshi,
"Analytic black hole perturbation approach to gravitational radiation"
Living Reviews in Relativity **6** (2003) 6,
e-Print: gr-qc/0306120

[31] V. Ferrari,
"Stellar Perturbations",
Lecture Notes in Physics **617** (2003) 89–112.

[32] L. Samuelsson and N. Andersson,
"Neutron Star Asteroseismology. Axial Crust Oscillations in the Cowling Approximation",
Mon. Not. Roy. Astron. Soc. **374** (2007) 256 [astro-ph/0609265].

[33] R. A. Konoplya and A. Zhidenko,
"Quasinormal modes of black holes: From astrophysics to string theory",
Rev. Mod. Phys. **83** (2011) 793 [arXiv:1102.4014 [gr-qc]].

[34] V. Ferrari,
"Gravitational waves from perturbed stars",
Bulletin of the Astronomical Society of India **39** (2011) 203–224.
arXiv:1105.1678 [gr-qc].

[35] V. P. Frolov and D. Kubiznak,
"Hidden Symmetries of Higher Dimensional Rotating Black Holes",
Phys. Rev. Lett. **98** (2007) 011101; arXiv:gr-qc/0605058 [gr-qc].
doi: 10.1103/PhysRevLett.98.011101

[36] B. Carter,
"Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations",
Commun. Math. Phys. **10** (1968) 280.

[37] C. Muller,
Lecture Notes in Mathematics: Spherical Harmonics
(Springer-Verlag, Berlin-Heidelberg, 1966).

[38] Robert S. Maier,
"The 192 solutions of the Heun equation",
Mathematics of Computation **76** (2007): 811–843, arXiv:math/0408317,
doi:10.1090/S0025-5718-06-01939-9, MR 2291838

[39] A. Ronveaux, ed.,
Heun's differential equations,
(Clarendon Press, Oxford University Press, England, 1995).
ISBN 978-0-19-859695-0, MR 1392976

[40] B. D. Sleeman and V. B. Kuznetzov, "Heun functions",
in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W.,
NIST Handbook of Mathematical Functions, (Cambridge University Press, England, 2010).
ISBN 978-0521192255, MR2723248

[41] Galliano Valent,
"Heun functions versus elliptic functions",
in *Difference equations, special functions and orthogonal polynomials*,
(World Scientific, Singapore, 2007), pp. 664–686, arXiv:math-ph/0512006,
doi:10.1142/9789812770752_0057, MR 2451210

[42] E. D. Fackerell and R. G. Crossman,
"Spin-weighted angular spheroidal functions",
J. Math. Phys. **18** (1997) 1849–1854.

[43] M. Richartz, S. Weinfurtner, A. J. Penner and W. G. Unruh,
"General universal superradiant scattering",
Phys. Rev. D **80** (2009) 124016 [arXiv:0909.2317 [gr-qc]].

[44] Ya. B. Zel'dovich,
"Amplification of cylindrical electromagnetic waves reflected from a rotating body",
JETP **35** (1972) 1085.