Mapping the surgery exact sequence for topological manifolds to analysis

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Abstract

In this paper we prove the existence of a natural mapping from the surgery exact sequence for topological manifolds to the analytic surgery exact sequence of N. Higson and J. Roe. This generalizes the fundamental result of Higson and Roe, but in the treatment given by Piazza and Schick, from smooth manifolds to topological manifolds. Crucial to our treatment is the Lipschitz signature operator of Teleman.

We also give a generalization to the equivariant setting of the product defined by Siegel in his Ph.D. thesis. Geometric applications are given to stability results for rho classes. We also obtain a proof of the APS delocalised index theorem on odd dimensional manifolds, both for the spin Dirac operator and the signature operator, thus extending to odd dimensions the results of Piazza and Schick. Consequently, we are able to discuss the mapping of the surgery sequence in all dimensions.

Keywords. K-theory, Surgery Theory, Coarse Geometry, Lipschitz Manifolds

Contents

1 Introduction

Let M be a n-dimensional topological manifold, with $\Gamma = \pi_1(M)$ and let $\widetilde{M} \to M$ be its universal covering. We assume n greater than 5 and, initially, odd.

In [\[25\]](#page-24-0) Sullivan proves that there always exists a Lipschitz manifold structure on M and that it is unique up to a bi-Lipschitz homeomorphism isotopic to the identity. In [\[26,](#page-24-1) [27\]](#page-25-0) Teleman studies index theory in the Lipschitz context and in [\[8\]](#page-24-2) Hilsum develops it in the framework of unbounded Kasparov theory. In particular there is a signature operator arising from the Lipschitz structure and this operator determines a well defined class in the K-homology of M.

Thanks to these results it is possible to extend the work by Piazza and Schick [\[17\]](#page-24-3) (that follows the one by Higson and Roe [\[5,](#page-23-2) [6,](#page-23-3) [7\]](#page-24-4)) from the smooth to the topological category. Let us recall that in [\[17\]](#page-24-3) Piazza and Schick built a natural transformation

$$
L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow \mathcal{S}(M) \longrightarrow \mathcal{N}(M) \longrightarrow L_n(\mathbb{Z}\Gamma)
$$

\n
$$
\downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\rho} \qquad \qquad \downarrow_{\beta} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}}
$$

\n
$$
K_{n+1}(C_r^*(\Gamma)) \longrightarrow K_{n+1}(D^*(\widetilde{M})^{\Gamma}) \longrightarrow K_n(M) \longrightarrow K_n(C_r^*(\Gamma))
$$

from the surgery exact sequence for smooth manifolds to the analytic surgery exact sequence of Higson and Roe, using tools and methods in coarse index theory.

In this paper we check that this mapping also exists for the surgery sequence for topological manifolds. To this aim we will use as key tool the Lipschitz structure given by Sullivan theorem [\[25\]](#page-24-0). In particular we prove that the key results by Wahl, Piazza and Schick, have a true abstract and K-theoretical meaning, that does not depend on the smooth structure and the pseudodifferential calculus.

One significant difference between the smooth SES and the topological SES is that the second one is an exact sequence of groups, whereas the first one is not. In this paper we deal with the mapping at the set level: to prove that the diagram is commutative as a diagram of groups, the main difficulty is that the group structure of the topological structure set is rather quite hard to handle. The following question is wide open:

• is the map $\varrho: \mathcal{S}(M) \to K_{n+1}(D^*(\widetilde{M})^{\Gamma})$ a homomorphism of groups?

A positive answer to this question would have direct consequences to Conjecture 3.8 in [\[33\]](#page-25-1), using the methods in developed in [\[34\]](#page-25-2).

In the second part of the paper we give another realization of the group $K_*(D^*(\overline{M})^{\Gamma})$ in terms of the mapping cone of Kasparov bimodules, in order to generalize to the equivariant setting a product formula proved by Siegel in his Ph.D. thesis [\[21\]](#page-24-5). This product allows us to give stability results for ρ -invariants and to prove the delocalized APS index theorem of Piazza and Schick([\[18\]](#page-24-6)) in the odd dimensional case. This last result leads to define a natural mapping from the SES to the analytic SES of Higson and Roe when $\dim(M) = n$ is even, in both the smooth and the topological setting.

With the same method one can extendd the construction in [\[18\]](#page-24-6), about the Stolz exact sequence, to the case of even dimensional manifolds, but this was already proven in [\[35\]](#page-25-3).

We refer the reader to [\[17\]](#page-24-3) for a more detailed overview of the problem in the smooth setting.

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2 Signature operator on Lipschitz manifolds

We start recalling fundamental results on Lipschitz manifolds. For further details we refer to [\[26,](#page-24-1) [27,](#page-25-0) [8,](#page-24-2) [25,](#page-24-0) [30\]](#page-25-4).

Definition 2.1. A Lipschitz atlas on a topological manifold M is an atlas such that the map $\varphi \circ \psi^{-1}$ is a Lipschitz homeomorphism for any two charts $\varphi: U \to \mathbb{R}^n$ and $\psi: V \to \mathbb{R}^n$. By definition a Lipschitz manifold structure on M is a maximal Lipschitz atlas.

Theorem 2.2 ([\[25\]](#page-24-0)). Any topological manifold of dimension $n \neq 4$ has a Lipschitz atlas of coordinates. For any two such structures L_1 and L_2 , there exists a Lipschitz homeomorphism $h: L_1 \to L_2$ isotopic to the identity.

Theorem 2.3 ([\[26,](#page-24-1) [8\]](#page-24-2)). Let M be a closed oriented Lipschitz manifold of even dimension. Then from the complex of L^2 -differential forms on M (with respect to some choice of a Lipschitz Riemannian metric g) one obtains a signature operator D_q which is closed and self-adjoint. Therefore D_q determines a class $[D]$ in $K_0(M) \simeq KK(C(M), \mathbb{C})$ which is independent of the choice of the metric g. The image of $[D]$ in $K_0(pt) \simeq KK(\mathbb{C}, \mathbb{C})$ (i.e., the index of D_q) is the usual signature of the manifold.

In [\[9\]](#page-24-7) Hilsum proves that the signature operator gives a KK-class as above in the case of non compact manifolds too, provided the manifold M is endowed with a metric g such that it is metrically complete with respect to the induced structure of metric space. Moreover he showed a result on the finite propagation speed for solutions of the wave equation.

Theorem 2.4 (Hilsum). Let M be an oriented Lipschitz manifold with a Riemannian structure, such that the manifold is complete as metric space. Let d be the associated distance function and let D be the associated signature operator. For all $t \in \mathbb{R}$, we have that:

$$
supp(e^{itD}) \subset \{(x, y) \in M \times M \mid d(x, y) \leq t\}.
$$

For $f \in \mathcal{S}(\mathbb{R})$ such that supp $(\hat{f}) \subset [-a, a]$, with $a > 0$, we have that:

 $supp(f(D)) \subset \{(x,y) \in M \times M \mid d(x,y) \leq a\}.$

This theorem will be key in the coarse geometrical setting.

3 The ρ classes

We refer the reader to [\[7,](#page-24-4) sect.1] and [\[18,](#page-24-6) sect.1] for notations about coarse geometry and coarse algebras.

Let N be an oriented topological manifold of dimension $n \geq 5$; an element of the topological structure set of N is given by an orientation preserving homotopy equivalence $f: M \to N$. Two different homotopy equivalences $f: M \to N$ and $f': M' \to N$ are equivalent if there is a hcobordism W between them and a homotopy equivalence $F: W \to N \times [0,1]$, such that $F_{|M} = f$ and $F_{|M'} = f'$.

Definition 3.1. The topological structure set $\mathcal{S}^{TOP}(N)$ of N is defined as the set of the h-cobordism classes of oriented homotopy equivalences.

Given a class $[f: M \to N]$, we set $Z = M \cup -N$. Let Γ be the fundamental group of N. The universal covering $\tilde{N} \to N$ is induced by a map $u \colon N \to B\Gamma$, namely $\tilde{N} = u^*(E\Gamma)$, where BΓ is the classifying space of Γ and EΓ is its universal covering. Let \widetilde{M} be the Γ-Galois covering induced by $u_M := u \circ f$, then we get a Γ-Galois covering $\tilde{Z} = \tilde{M} \cup -\tilde{N}$ on Z. Let $\mathcal{F} = \widetilde{Z} \times_{\Gamma} C_r^*(\Gamma)$ be the associated Mischenko bundle.

Now, starting from a Lipschitz structure on Z given by Theorem [2.2,](#page-1-1) consider the L^2 -forms complex $L^2(Z, \Lambda_{\mathbb{C}}(Z))$, see [\[8,](#page-24-2) Section 2].

We get a differential d_Z and an involution τ_Z ; τ_Z is the operator $\omega \mapsto i^{p(p-1)+\frac{n}{2}} * \omega$ on forms of degree p . Like in the classical Hodge theory we can define the Lipschitz signature operator (with coefficients) as

$$
D_Z = (d_Z - \tau_Z d_Z \tau_Z)
$$

if n is even and

$$
D_Z = (\tau_Z d_Z + d_Z \tau_Z)
$$

if n is odd.

Like in [\[8\]](#page-24-2), we have that $(L^2(Z, \Lambda_{\mathbb{C}}(Z)), \mu, D_Z)$ defines an unbounded class $[D_Z] \in KK(C(Z), \mathbb{C}),$ where μ is the representation that associates the multiplication operator μ_f to a function f.

3.1 The perturbed signature operator

We want to associate a class in the K-group $K_*(D^*(\tilde{M})^\Gamma)$ to a homotopy equivalence $f\colon\thinspace M\to\thinspace N$ and show that this mapping is well defined on the h-cobordism classes.

The key result for what follows is the homotopy invariance of the index class of the signature operator for compact oriented smooth manifolds, proved by M. Hilsum and G. Skandalis in [\[11\]](#page-24-8). Remember that, in the equivariant setting, this class is given by

$$
\operatorname{Ind}_{\Gamma}(D_Z)=[\mathcal{F}]\otimes_{C(X)\otimes C^*_r(\Gamma)}[D_Z]\in KK(\mathbb{C},C^*_r(\Gamma)),
$$

where $[\mathcal{F}]$ is the class of Mishchenko bundle in $KK(\mathbb{C}, C(Z) \otimes C_r^*(\Gamma)).$

Theorem 3.2 (Hilsum-Skandalis). Let $f: M \to N$ be a homotopy equivalence. Then the class $\text{Ind}_{\Gamma}(D_Z) \in KK(\mathbb{C}, C_r^*(\Gamma))$ vanishes.

In remark [\[11,](#page-24-8) p.95] the authors observe that all arguments can be applied to the Lipschitz case: we can easily check that the objects do not need be smooth.

Remark 3.3. Of particular interest to us is a byproduct of the proof of Theorem 3.1, namely the construction of a homotopy \mathcal{D}_{α} between the signature operator $\mathcal{D}_{Z} = \mathcal{D}_{0}$ and an invertible operator \mathcal{D}_1 . This is the reason why the index class Ind_Γ(D_Z) vanishes. Here \mathcal{D}_Z is the signature operator twisted by the Mishchenko bundle. Moreover we point out that the perturbation creates a gap in its spectrum near 0.

Proposition 3.4. The difference $\mathcal{D}_0 - \mathcal{D}_1$ is a C*-module compact operator on $L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})$ both in the smooth and in the Lipschitz case.

Proof.

The proof of [\[11,](#page-24-8) Theorem 3.3] is based on the construction of an operator $T_{p,v}$, that plays the role of the pull-back of forms.

Let us take the following data:

- a submersion $p: M \times B^k \to N$, where B^k is the unit open disk of \mathbb{R}^k ;
- a smooth k-form v with compact support on B^k , such that $\int_{B^k} v = 1$. Put then $\omega = p_{B^k}^*(v).$

Then $p^*: L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N) \to L^2(M \times B^k, \Lambda_{\mathbb{C}}(M \times B^k) \otimes p^* \mathcal{F}_N)$ is a bounded operator and $T_{p,\nu}$ is defined as the operator $\xi \mapsto q_*(\omega \wedge p^*(\xi))$. Consider the following commutative diagram

$$
M \times B^{k}
$$
\n
$$
M \leftarrow \text{max } P
$$
\n
$$
M \leftarrow \text{max } M \times N \xrightarrow{p_N} N
$$
\n
$$
(3.1)
$$

where $t = id_M \times p$. We get that, for $\xi \in L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_{\mathcal{N}})$

$$
T_{p,\nu}(\xi) = q_*(\omega \wedge p^*(\xi)) = = (p_M)_* t_* (\omega \wedge (t)^* p_N^*(\xi)) = = (p_M)_* (t_* \omega \wedge p_N^*(\xi)).
$$

Notice that $(p_M)_*$ is nothing but the integration over N. Assume that k and p are chosen so that t is a submersion. If we denote the form $t_*\omega$ on $M \times N$ with $k(y, x)$, it turns out that $T_{p,\nu}(\xi) = \int_N k(x,y)\xi(x)$ is an integral operator with smooth kernel and consequently a smoothing operator.

The operator Y in [\[11,](#page-24-8) Lemma 2.1(c)], such that $1+T'_{p,v} \circ T_{p,v} = d_N \circ Y + Y \circ d_N$, is bounded of order -1 (see the proof of [\[32,](#page-25-5) Lemma 2.2] for an explicit expression of Y).

Now we can follow word by word the proof of [\[16,](#page-24-9) Lemma 9.14], using the conventions in [\[32,](#page-25-5) Section 3]. For simplicity let us consider the odd case. The perturbed signature operator is then given by

$$
\mathcal{D}_t = -iU_t(d_t \circ S_t + S_t \circ d_t) \circ U_t^{-1}
$$

where

$$
d_t = \begin{pmatrix} d_M & tT'_{p,v} \\ 0 & -d_N \end{pmatrix}, \quad S_t = \text{sign}(\tau_Z \circ L_t) \text{ and } U_t = |\tau_Z \circ L_t|^{\frac{1}{2}},
$$

with

$$
L_t = \begin{pmatrix} 1 + T'_{p,v} \circ T_{p,v} & (1 - it\gamma \circ Y) \circ T'_{p,v} \\ T_{p,v} \circ (1 + it\gamma \circ Y) & 1 \end{pmatrix}
$$

 $\sqrt{L_t^* L_t} = \sqrt{1 + R_t}$, with R_t smoothing. Observe that $0 < L_t^* L_t = 1 + R_t$ implies that $R_t > -1$. One can easily see that $L_t = 1 + H_t$, with H_t smoothing. Moreover one has that $|\tau_Z \circ L_t| =$ $\sqrt{L_t^* L_t} - \sqrt{1 + R_t}$, when R_t smoothing. Observe that $0 \leq L_t L_t = 1 + R_t$ implies that $R_t > -1$.
It follows that $\sqrt{L_t^* L_t} - 1 = f(R_t)$, where $f(x) = \sqrt{1 + x} - 1$ is holomorphic on the spectrum of R_t (-1 is a branch point for f). Since $f(0) = 0$, we have that $f(z) = az + zh(z)z$, where h is a holomorphic function.

Let us point out that if S_0 and S_1 are smoothing operators and T is a bounded operator, then $S_0 \circ T \circ S_1$ is smoothing. Then we immediately get that $F_t := |\tau_Z \circ L_t| - 1 = f(R_t)$ is smoothing. With the same argument one can prove that $U_t = 1 + H_t$ with H_t smoothing.

By [\[16,](#page-24-9) Lemma A.12], $|\tau_Z \circ L_t|^{-1} = 1 + F'_t$ and $U_t = 1 + H'_t$ with F'_t and H'_t smoothing. Then one obtains that, $S_t = \tau_Z + G_t$ and $d_t = d_Z + E_t$, where G_t and E_t are smoothing operators.

Consequently one has that

$$
\mathcal{D}_t = -i(1 + H_t) \left((d + E_t) \circ (\tau_Z + G_t) + (\tau_Z + G_t) \circ (d + E_t) \right) \circ (1 + H_t')
$$

is equal to $\mathcal{D} + C_f$ with C_f a compact operator.

Now we have to prove that the Lipschitz Hilsum-Skandalis perturbation is bounded. In the smooth case we tackled the problem geometrically, here we try with a more analytical approach. An operator of order $-n$ is a bounded operator between $H^s(Z, E)$ and $H^{s+n}(Z, E)$, the Sobolev sections of E of order s and $s + n$, for any s. An operator is regularizing if it is of order $-\infty$. Equivalently one can say that an operator T is regularizing (of order $-\infty$) if $D^n \circ T \circ D^m$ is a bounded operator on L^2 -section for any $m, n \in \mathbb{Z}$.

By [\[8,](#page-24-2) Proposition 5.6] we know that the signature operator has compact resolvent, therefore its spectrum is a countable and discrete subset $\{\lambda_n\}_{n\in\mathbb{N}}$ of R such that $\lim_{n\to\infty}\lambda_n^2=+\infty$.

Now let $\psi \in C_0^{\infty}(\mathbb{R})$ be a rapidly decreasing even function such that $\psi(0) = 1$. Since ψ is even, it turns out that $\psi(d_N + d_N^*)$ maps even/odd degree forms to even/odd degree forms and it is a Hilbert-Schmidt operator: the proof of the first statement of [\[19,](#page-24-10) Prop. 5.31] works putting 'Hilbert-Schmidt' instead of 'smoothing'. Let us denote its kernel by $k(x, y) \in L^2(N \times N, \text{End}(\Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N)).$

Define the compact operator $T_f: L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N) \to L^2(M, \Lambda_{\mathbb{C}}(M) \otimes f^* \mathcal{F}_N)$ as the integral operator with kernel $\bar{q}_*(\omega \wedge \bar{p}^*k) \in L^2(M \times N, \text{Hom}(\Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N, \Lambda_{\mathbb{C}}(M) \otimes f^* \mathcal{F}_N)),$ where $\bar{f} = f \times id_N$ for f equal to p and q as in diagram [\(3.1\)](#page-3-1).

This operator satisfies the hypothesis of [\[11,](#page-24-8) Lemma 2.1]. Indeed, because of our choice of ψ , we have that $1 - \psi(x) = x \cdot \varphi(x)$, where φ is a rapidly decreasing odd function. Moreover $d_N^* \circ \varphi(d_N + d_N^*) = \varphi(d_N + d_N^*) \circ d_N$, since φ is odd. Then we get the following formula

$$
1-\psi(d_N+d_N^*)=d_N\circ\varphi(d_N+d_N^*)+\varphi(d_N+d_N^*)\circ d_N
$$

and by construction $T_f' T_f = \psi (d_N + d_N^*)' \psi (d_N + d_N^*)$. Now it is easy to check that there exists an operator $Y \in \mathbb{B}(L^2(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N))$ such that $Y(\text{dom}(d_N)) \subset \text{dom}(d_N)$ and that $1 - T'_f \circ T_f = d_N \circ Y + Y \circ d_N$:

$$
1 - T'_f \circ T_f = 1 - \psi' \circ \psi =
$$

\n
$$
= 1 - (1 - d \circ \varphi - \varphi \circ d)' \circ (1 - d \circ \varphi - \varphi \circ d) =
$$

\n
$$
= 1 - (1 - d \circ \varphi - \varphi \circ d - \varphi'd' - d' \circ \varphi' +
$$

\n
$$
+ d \circ \varphi \circ \varphi' \circ d + d \circ \varphi \circ d' \circ \varphi' + \varphi \circ d \circ \varphi' \circ d' + \varphi \circ d \circ d' \circ \varphi') =
$$

\n
$$
= d \circ \varphi + \varphi \circ d - \varphi'd - d \circ \varphi' =
$$

\n
$$
= d \circ Y + Y \circ d,
$$

with $Y = \varphi(d_N + d_N^*) - \varphi(d_N + d_N^*)'$. We simplified some notations and we denoted d_N by d, $\varphi(d_N + d_N^*)$ by φ and the same for ψ . Moreover we used the following facts: $d'_N = -d_N$, $\varphi \circ d = d^* \circ \varphi$ and $\varphi' \circ d' = (d')^* \circ \varphi'.$

It is not difficult to check that the operator T_f is a regularizing operator (and hence a compact operator), therefore the image of T_f is in the domain of the Lipschitz signature operator.

Then the boundedness of the Lipschitz Hilsum-Skandalis perturbation follows as in the smooth case. The only thing we have to care about is the dependence of this construction on the choice of the metric on N. In particular we have to check that $\psi(d_N + d_N^*)$ is Hilbert-Schmidt no matter which metric we use to take the adjoint.

If we have two different metrics g_0 and g_1 on N, then by [\[8,](#page-24-2) Lemma 5.1] we can complete the $Lip(N)$ -module $Lip(N, \Lambda_{\mathbb{C}}(N) \otimes \mathcal{F}_N)$ with respect to the two metrics and we obtain two isomorphic $C(N)$ -Hilbert modules with compatible metrics:

$$
K^{-1}||\cdot||_1 \le ||\cdot||_0 \le K||\cdot||_1 \ \exists K \in \mathbb{R}^+ \setminus \{0\}.
$$

Then by the Minmax Theorem $|\lambda_n^0| \leq K^2 |\lambda_n^1|$, where for any $n \in \mathbb{N}$, λ_n^i is the *n*-th eigenvalue of $d + d_i^*$.

So it is easy to check that if ψ is a rapidly decreasing function on the spectrum of $d + d_0^*$, it is rapidly decreasing on the spectrum of $d + d_1^*$ too. Therefore $\psi(d + d^*)$ is Hilbert-Schmidt independently of the metric we choice.

 \Box

Definition 3.5. Let $f: M \to N$ be a homotopy equivalence and $Z = M \cup -N$. Denote by C_f the perturbation of \mathcal{D}_Z arising in the previous remark and call it a trivializing perturbation. Note that it depends on the homotopy equivalence f .

We recall from [\[17\]](#page-24-3) that there is an isomorphism of C^* -algebras

$$
\mathbb{K}(L^2(Z,\Lambda_{\mathbb{C}}(Z)\otimes\mathcal{F}))\simeq C^*(\widetilde{Z})^{\Gamma}.
$$

By [\[15,](#page-24-11) Proposition 2.1], the above isomorphism gives an isomorphism at the level of multiplier algebras

$$
\mathbb{B}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F})) \simeq \mathcal{M}(C^*(\widetilde{Z})^{\Gamma}).
$$
\n(3.2)

This isomorphism is given by the map L_{π} defined in [\[17,](#page-24-3) Section 2.2.1]. Hence we can go from the Mishchenko bundle setting to the covering one. From now on C_f will be the element in $\mathcal{M}(C^*(\widetilde{Z})^{\Gamma})$ associated to $\mathcal{C}_f \in \mathbb{B}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F}))$ through the map L_{π} . Moreover \widetilde{D}_Z will indicate the operator on the covering induced by the signature without coefficients in the Mishchenko bundle.

Remark 3.6. Consider a chopping function $\psi \in C_b(\mathbb{R})$ with compactly supported Fourier transform. Thanks to Theorem [2.4](#page-2-1) we can prove that the functional calculus through ψ of the operator D_Z is an operator of finite propagation. The pseudolocality of D_Z comes from [\[8,](#page-24-2) 6.1]. Hence $\psi(\widetilde{D}_Z) \in D^*(\widetilde{Z})^{\Gamma}$.

Proposition 3.7. The difference between $\psi(\tilde{D}_Z)$ and $\psi(\tilde{D}_Z + C_f)$ belongs to $C^*(\tilde{Z})^{\Gamma}$.

Proof. Moving to the Mishchenko bundle setting through (3.2) , we should prove that the difference $\psi(\mathcal{D}_Z) - \psi(\mathcal{D}_Z + \mathcal{C}_f)$ belongs to $\mathbb{K}(L^2(Z, \Lambda_{\mathbb{C}}(Z) \otimes \mathcal{F}))$. If $\psi_1(t) = t(1+t^2)^{-\frac{1}{2}}$, by [\[2,](#page-23-4) Proposition 2.2] we have that $[\psi_1(\mathcal{D}_Z), a]$ belongs to the algebra of compact C^{*}-module operators. Therefore if we consider the matrices $\begin{bmatrix} \mathcal{D}_Z & 0 \\ 0 & \mathcal{D}_Z + C_f \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, their bracket consists $\lim_{\mathcal{C}\downarrow} \left[\begin{array}{cc} 0 & -\mathcal{C}_f \\ \mathcal{C}_f & 0 \end{array} \right]$ $\begin{bmatrix} 0 & -C_f \\ C_f & 0 \end{bmatrix}$, that is known to be bounded. Then, after applying the functional calculus through ψ_1 , we deduce that the matrix components in the bracket

$$
\pm(\psi_1(\mathcal{D}_Z)-\psi_1(\mathcal{D}_Z+\mathcal{C}_f))
$$

are compact.

Now notice that two different chopping functions differ by a function in $C_0(\mathbb{R})$. Taking into account the correspondence stated in [\(3.2\)](#page-5-0), we have that the resolvent of \mathcal{D}_Z , given by $(i + \mathcal{D}_Z)^{-1}$, is compact (see [\[8,](#page-24-2) Proposition 5.6]) and since $\phi(t) = (i + t)^{-1}$ generates $C_0(\mathbb{R})$, the functional calculus of D_Z through a function in $C₀(\mathbb{R})$ gives a compact operator. Then if ψ' is any chopping function, it turns out that

$$
\psi(\mathcal{D}_Z) - \psi(\mathcal{D}_Z + \mathcal{C}_f) = \psi_1(\mathcal{D}_Z) - \psi_1(\mathcal{D}_Z + \mathcal{C}_f) + \text{compacts operators}
$$

and the right-hand side term is compact.

Corollary 3.8. The operator $\chi(\tilde{D}_Z+C_f)$, with $\chi(x)=\frac{x}{|x|}$, is a bounded involution in $D^*(\tilde{Z})^{\Gamma}$.

Thanks to Corollary [3.8](#page-6-1) we can define a class by setting

$$
\varrho(\widetilde{D}_Z+C_f) = \left[\frac{1}{2}(1+\chi(\widetilde{D}_Z+C_f))\right] \in K_0(D^*(\widetilde{Z})^{\Gamma}).
$$

Now consider the map $\varphi: Z \to M$ such that $\varphi_{|N} = f$ and $\varphi_{|-M} = -\mathrm{Id}_M$; we can clearly see that φ is covered by a Γ-equivariant map $\widetilde{\varphi} : \widetilde{Z} \to \widetilde{M}$.

Definition 3.9. Let $f: M \to N$ be a homotopy equivalence between two compact oriented Lipschitz manifolds. Consider $Z = M \cup -N$ and its covering \tilde{Z} associated, as above, to a classifying map $u: Z \to B\Gamma$. Let \widetilde{D}_Z be the Lipschitz signature operator and let C_f be the trivializing perturbation associated to f . We define

$$
\varrho(f\colon M\to N):=\widetilde{\varphi}_*\left[\frac{1}{2}(1+\chi(\widetilde{D}_Z+C_f))\right]\in K_0(D^*(\widetilde{M})^{\Gamma})
$$

and

$$
\varrho_{\Gamma}(f \colon M \to N) = u_* \varrho(f \colon M \to N) \in K_0(D_{\Gamma}^*).
$$

Proposition 3.10. The ρ -class does not depend on the choice of the Lipschitz structure.

Proof. The second part of Theorem [2.2](#page-1-1) can be formulated as follows: let \mathcal{L}_1 and \mathcal{L}_2 be two different Lipschitz structures on Z, then there exists a bi-Lipschitz homeomorphism $\phi: Z \to Z$, isotopic to the identity through a path ϕ^t and such that $\phi^*(\mathcal{L}_2) = \mathcal{L}_1$, where $\phi^*: C(Z) \to C(Z)$ is the induced *-homomorphism. Because of the functoriality of Teleman's construction we know that $\phi_*(\varrho_1) = \varrho_2$, where ϱ_1 and ϱ_2 are the invariants associated to two different Lipschitz structures. The isotopy ϕ^t induces a paths of *-isomorphisms $\phi_*^t: D^*(\tilde{Z})^{\Gamma} \to D^*(\tilde{Z})^{\Gamma}$. Then $\phi^t_*(\varrho_1)$ gives a homotopy between ϱ_2 and ϱ_1 .

3.2 Perturbed signature operator on manifolds with cylindrical ends

In this section we are going to check that the construction we made for ϱ and ϱ_{Γ} are well defined on the structure set $\mathcal{S}^{TOP}(N)$.

For this purpose we will use the results presented in [\[32,](#page-25-5) [18,](#page-24-6) [17\]](#page-24-3), where the authors have developed the theory in the smooth setting. Their methods are rather abstract and they also hold in the Lipschitz context.

In order to develop the theory for manifolds with cylindrical ends, we are going to use the same notations as [\[17,](#page-24-3) 2.19].

The geometrical setting is the following: let $f: M \to N$ and $f': M' \to N$ be two topological structures for N; let W be a cobordism between $\partial_0 W = M$ and $\partial_1 W = M'$ and let W_{∞} be the manifold with the infinite semi-cylinder $\partial W \times \mathbb{R}_{\leq 0}$ attached to the boundary; let $V = N \times [0, 1]$ and let V_{∞} be the complete cylinders with base $\partial V = N$; there is a homotopy equivalence $F: W_\infty \to V_\infty$ which has the product form $F_\partial \times id_{\mathbb{R}_{\leq 0}}$ on the cylindrical ends, where $F_{\partial_0} = f : M \to N$ and $F_{\partial_1} = f' : M' \to N$, both of them being homotopy equivalences.

Thanks to the results presented in [\[9\]](#page-24-7) we have a well defined Lipschitz signature complex on the manifold $X = W_{\infty} \cup -V_{\infty}$. Notice that $\partial_0 X = Z$ and $\partial_1 X = Z'$.

First of all we need a generalization of Theorem [3.2](#page-3-2) for manifolds with cylindrical ends. This result is given by [\[32,](#page-25-5) Proposition 8.1], where a perturbation of the signature operator is associated to the homotopy equivalence F . Such a perturbation makes the operator invertible. as in the usual case.

Remark 3.11. As well as in the case presented in Theorem [3.2,](#page-3-2) the generalization developed in [\[32,](#page-25-5) Proposition 8.1] is still valid in the Lipschitz setting.

The goal of this section is to check that the ρ -class is well defined on the h-cobordism classes: as pointed out in [\[17,](#page-24-3) Proposition 4.7], this is obtained by the combination of [\[32,](#page-25-5) Theorem 8.4] and [\[17,](#page-24-3) Corollary 3.3].

In [\[32,](#page-25-5) Theorem 8.4] all constructions work in the Lipschitz framework, where we do not consider the parameter ε . Wahl builds a perturbation $\mathcal{C}_F^{\text{cyl}}$ of the signature operator, supported on the cylindrical ends, from the perturbations on Z and Z' ; hence she constructs a homotopy of operators between $\mathcal{D}_X + C_F^{\text{cyl}}$ and an other operator that, thanks to the Bunke's relative index theorem, has vanishing index.

For the proof of the equality we just mentioned, the only not obvious point in the Lipschitz case is the one concerning the relative index theorem proved in [\[3\]](#page-23-5), since what remains of the proof uses abstract theory of unbounded operators and spectral flow methods.

It is worth formulating Bunke's Theorem in the Lipschitz case and giving a sketch of its proof.

3.2.1 Bunke's relative index theorem for Lipschitz manifolds

The idea of the theorem is the following: let X be a manifold, let $E \to X$ be a bundle and D a Fredholm operator on the sections of this bundle; if there exists a hypersurface Y in X such that the operator is invertible near Y , we can cut the manifold (and the bundle) along Y and we can paste a semicylinder to the boundary of both parts obtained after cutting, extending the bundle and the operator along the semicylinder. Then we obtain an operator whose index equals the index of the original operator.

More precisely we are considering the following data: the Lipschitz manifold X we have defined in the previous subsection, the Hilbert module $L^2(X, \Lambda_{\mathbb{C}}(X)\otimes \mathcal{F})$ of L^2 -forms on X twisted by the Mishchenko bundle, that we are going to denote by \mathcal{H}^0 ; a regular operator G that is the twisted Lipschitz signature operator, possibly perturbed by a bounded operator; we suppose that there is a Lipschitz function with compact support $f \geq 0$ and $(G^2 + f)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$ (here \mathcal{H}^2 is the maximal domain of the square of the signature operator).

Definition 3.12. Let $\text{Lip}_K(X)$ be the set of bounded Lipschitz functions h such that for all $\varepsilon > 0$ there exists a compact $C \subset X$, with $||dh_{|X\setminus C}||_{L^{\infty}} < \varepsilon$. Let us call $C_K(X)$ the closure of $\mathrm{Lip}_K(X)$ in the sup-norm.

For the comfort of the reader, we recall the theorem stated in the Lipschitz setting. Let $E_i \to X_i$, $i = 1, 2$, be the two $C^*(\Gamma)$ -C* bundles $\Lambda_{\mathbb{C}}(X_i) \otimes \mathcal{F}_i$, with operator G_i , associated to them as above. Let $W_i \cup_{Y_i} V_i$ be a partition of X_i where Y_i is a compact hypersurface. Assume that there is a commutative diagram of isomorphisms of all structures

where $U(Y_i)$ is a tubular neighbourhood of Y_i , for $i = 1, 2$. We cut X_i at Y_i , glue the pieces together interchanging the boundary components and obtain $X_3 := W_1 \cup_Y V_2$ and $X_4 :=$ $W_2 \cup_Y V_1$. Moreover, we glue the bundles using Ψ , which yields A-C* bundles $E_3 \to X_3$ and

 $E_4 \to X_4$ and we assume that G_i , $i = 3, 4$ are again invertible at infinity. We define $[X_i]$ as the class $[\mathcal{H}_i^0, \frac{G_i}{G_i^2+f}] \in KK(C_K(X_i), C^*(\Gamma)).$ The algebra $C_K(X)$ is unital. Hence, there is an embedding $i: \tilde{\mathbb{C}} \to C_K(X)$ and an induced map

$$
i^* \colon KK(C_K(X), C_r^*(\Gamma)) \to KK(\mathbb{C}), C_r^*(\Gamma)).
$$

Set $\{X_i\} := i^*[X_i] \in KK(\mathbb{C}, C_r^*(\Gamma))$ for $i = 1, ..., 4$.

Theorem 3.13 ([\[3\]](#page-23-5)).

$$
\{X_l\} + \{X_2\} - \{X_3\} - \{X_4\} = 0.
$$

Here are two facts:

- thanks to [\[26,](#page-24-1) Theorem 7.1] we have the following Rellich-type result: the inclusion $\mathcal{H}^2 \hookrightarrow \mathcal{H}^0$ is compact;
- for any f Lipschitz function compactly supported on X , the multiplication operator $f: \mathcal{H}^2 \to \mathcal{H}^0$ is compact. And this also holds for the Clifford multiplication by grad (f) , the gradient of f.

Let $R(\lambda)$ be the bounded operator $(G^2 + f + \lambda)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$, for $\lambda \geq 0$; because of the Rellich-type result, we know that $R(\lambda)$ defines a compact operator in $\mathbb{B}(\mathcal{H}^0)$ and that there is a positive constant C such that $||R(\lambda)|| \leq (C + \lambda)^{-1}$.

Lemma 3.14. The integral

$$
F = \frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda
$$

is convergent and defines an operator in $\mathbb{B}(\mathcal{H}^0)$.

Lemma 3.15. The operator $[D, R(\lambda)]$ extends to a bounded operator that coincides with

$$
-R(\lambda)\text{grad}(f)R(\lambda).
$$

Moreover such an operator is compact.

Proof. See [\[3,](#page-23-5) Lemma 1.7 and Lemma 1.8].

Lemma 3.16. For any $h \in C_K(X)$, $h(F^2 - I) \in \mathbb{K}(\mathcal{H}^0)$.

Proof. We have

$$
\begin{split}\n&\left(\frac{G}{\pi}\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda\right)\left(\frac{G}{\pi}\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda\right)=\\
&\frac{G^{2}}{\pi^{2}}\left(\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda\right)^{2}+\frac{G}{\pi}\left[\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda,\frac{G}{\pi}\right]\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda=\\
&\frac{G^{2}}{\pi^{2}}\left(G^{2}+f\right)^{-1}-\frac{G}{\pi}\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)\text{grad}(f)R(\lambda)d\lambda\int_{0}^{\infty}\lambda^{-\frac{1}{2}}R(\lambda)d\lambda\sim\\
&\frac{G^{2}}{\pi^{2}}\left(G^{2}+f\right)^{-1},\n\end{split}
$$

where in the third step we have used Lemma [3.15.](#page-8-0) Here ∼ means "equal modulo compacts". Hence

$$
h(F^2 - I) \sim h \frac{f}{G^2 + f}
$$

that is compact, since the multiplication by f is compact.

Lemma 3.17. For any $h \in C_K(X)$, $[F, h] \in \mathbb{K}(\mathcal{H}^0)$.

 \Box

Proof. Since we chose G as a perturbation of the signature operator D and since the perturbation becomes compact under bounded transform, we have that

$$
[F, h] \sim \left[\frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, h\right] =
$$

$$
\frac{D}{\pi} \left[\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, h\right] + \left[\frac{D}{\pi}, h\right] \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda =
$$

$$
\frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [R(\lambda), h] d\lambda + \text{grad}(h) \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \sim
$$

$$
\frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [R(\lambda), h] d\lambda.
$$

The term in the last line is compact as in the proof of [\[3,](#page-23-5) Lemma 1.12].

Lemma 3.18. Let f and f_1 be two positive and compactly supported Lipschitz functions such that $(G^2 + f)^{-1}$, $(G^2 + f_1)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$. Then the two associated operators F, F₁ differ each other by a compact operator.

Proof. See [\[3,](#page-23-5) Lemma 1.10].

The lemmas we presented yield to the following result.

Proposition 3.19. The pair (\mathcal{H}^0, F) defines a Kasparov $(C_K(X), C_r^*(\Gamma))$ -module and its class in $KK(C_K(X), C_r^*(\Gamma))$ does not depend on the choice of f.

After checking this technical part, the proof of Theorem [3.13](#page-8-1) is completely abstract and it follows in the Lipschitz case as in the smooth one.

Now we treat another fundamental result proved by Piazza ans Schick: the delocalized Atiyah-Patodi-Singer index theorem. As noticed in [\[17,](#page-24-3) Section 5.2], the proof of the delocalized APS index theorem is based on abstract functional analysis for unbounded operators on Hilbert spaces. The reader can check that it almost completely works in the Lipschitz context as well as in the smooth one and we will not give all the details again.

The only proof to be modified is [\[17,](#page-24-3) Prop 5.33]. Assume the context and the notation understood, then we state the following Proposition.

Proposition 3.20. Given a Dirac type operator D, the operator $(1 + D^2)^{-1}$: $L^2 \rightarrow H^2$ is a norm limit of finite propagation operators $G_n: L^2 \to H^2$ with the property that $[\varphi, G_n]: L^2 \to$ $H²$ is compact for any compactly supported continuous function on M.

Proof. It is an easy computation showing that

$$
\frac{1}{1+x^2} = \int_{-\infty}^{+\infty} \frac{e^{-|t|}}{2} e^{-itx} dt.
$$

Let $f: \mathbb{R} \to \mathbb{R}$ be a C^{∞} function such that

- $\bullet \ \ 0 \leq f \leq 1,$
- $f = 1$ on a neighbourhood of 0,
- f has compact support.

Define $G_n = \int_{-\infty}^{+\infty} f\left(\frac{t}{n}\right) \frac{e^{-|t|}}{2}$ $\frac{e^{-itD}}{2}e^{-itD}dt$.

Finite propagation: since $f\left(\frac{t}{n}\right) \frac{e^{-|t|}}{2}$ $\frac{1}{2}$ has compact support, G_n has finite propagation speed. Pseudolocality: thanks to the above formula, $(1+D^2)^{-1}-G_n = \int_{-\infty}^{+\infty} (1-f\left(\frac{t}{n}\right))\frac{e^{-|t|}}{2}$ $\frac{e^{-itD}}{2}e^{-itD}dt$. Notice that $(1-f(\frac{t}{n}))\frac{e^{-|t|}}{2}$ $\frac{1}{2}$ is C^{∞} and moreover it is a rapidly decreasing function on the

 \Box

spectrum of D. By [\[19,](#page-24-10) Prop 5.31], $(1+D^2)^{-1}-G_n$ is a bounded operator from H^m to H^k for any $m, k \in \mathbb{N}$, hence G_n is pseudolocal because so is $(1+D^2)^{-1}$. Indeed, using Jacobi identities for commutators and the fact that $[\varphi, D] = c(d\varphi), [\varphi, (1 + D^2)^{-1}] = (1 + D^2)^{-1}c(d\phi)D(1 + D^2)$ $(D^2)^{-1} + (1+D^2)^{-1}Dc(d\phi)(1+D^2)^{-1}$ is compact, because the Clifford multiplication $c(d\phi)$ is compact.

In fact we need less: it is sufficient to show that $(1 + D^2)^{-1} - G_n$ is a bounded operator from L^2 to H^3 and then, by Rellich Theorem, the commutator $[\varphi,(1+D^2)^{-1}-G_n]$ turns out to be a compact operator from L^2 to H^2 . To prove this, we only need that the third derivative of $(1-f(\frac{t}{n}))\frac{e^{-|t|}}{2}$ $\frac{1}{2}$ has a bounded supremum norm (less than being rapidly decreasing).

In fact, under these hypotheses and by the properties of the Fourier transform, we get that

$$
||(1+D^2)^{-1} - G_n||_{L^2 \to H^3} = \left\| \left(\left(1 - f\left(\frac{t}{n}\right)\right) \frac{e^{-|t|}}{2} \right)^m \right\|_{\infty} \tag{3.3}
$$

is bounded. Moreover $\left(\left(1-f\left(\frac{t}{n}\right)\right)\frac{e^{-|t|}}{2}\right)$ $\frac{(-|t|)}{2}$ is equal to

$$
-\frac{1}{n^3}f'''\left(\frac{t}{n}\right)e^{-|t|}+\frac{3}{n^2}f''\left(\frac{t}{n}\right)|t|e^{-|t|}-\frac{3}{n}f'\left(\frac{t}{n}\right)e^{-|t|}-\left(1-f\left(\frac{t}{n}\right)\right)|t|^3e^{-|t|}
$$

that clearly goes to zero as n goes to infinity. This also holds in the Lipschitz case.

Now we can state the delocalized Atiyah-Patodi-Singer index theorem, that also holds in the Lipschitz context.

Theorem 3.21 ([\[17\]](#page-24-3)). If $i: C^*(\tilde{X})^{\Gamma} \hookrightarrow D^*(\tilde{X})^{\Gamma}$ is the inclusion and $j_*: D^*(\partial \tilde{X})^{\Gamma} \to D^*(\tilde{X})^{\Gamma}$ is the map induced by the inclusion $j: \partial \widetilde{X} \hookrightarrow \widetilde{X}$, we have

$$
i_*(\text{Ind}_{\Gamma}(D_X+C_F^{\text{cyl}}))=j_*(\varrho(D_{\partial X}+C_{F_{\partial}}))\in K_0(D^*(\widetilde{X})^{\Gamma}).
$$

Using the functoriality of the classifying map $u \circ F \cup u: \widetilde{X} \to E\Gamma$ and the map $\Phi :=$ $\pi_1 \circ (F \cup -\mathrm{id}_{V \times [0,1]})$ we obtain

$$
i_*\widetilde{\Phi}_*(\text{Ind}_{\Gamma}(D_X+C_F^{\text{cyl}}))=\varrho(F_{\partial})\in K_0(D^*(\widetilde{V})^{\Gamma})
$$

$$
i_* u_* \widetilde{\Phi}_*(\text{Ind}_{\Gamma}(D_X + C_F^{\text{cyl}})) = \varrho_{\Gamma}(F_{\partial}) \in K_0(D_{\Gamma}^*).
$$

Observe that ϱ_{Γ} is additive on disjoint unions as $\partial X = Z \cup -Z'$ and in particular that

$$
\varrho(F_{\partial}) = \varrho(f) - \varrho(f').
$$

Combining this with [\[32,](#page-25-5) Theorem 8.4], we finally have that

$$
\varrho(f) = \varrho(f'),
$$

and similarly for ϱ_{Γ} , hence they are well defined on $\mathcal{S}^{TOP}(N)$.

4 Mapping surgery to analysis: the odd dimensional case

We refer the reader to [\[17,](#page-24-3) Section 4] for the definitions that we are going to recall:

• $\mathcal{N}(N)$ is the set of normal maps. Its elements are degree 1 normal maps $[f: M \to N]$ where M is an oriented manifold. Two such maps are equivalent if there is a normal cobordism between them. There is a map $\beta \colon \mathcal{N}(N) \to K_n(N)$ such that $[f: M \to N]$ goes to the class $f_*[\mathcal{D}_M] - [\mathcal{D}_N] \in K_*(N)$, where $[\mathcal{D}_M]$ and $[\mathcal{D}_V]$ are the K-homology classes of the signature operators. The map β already appears in the work of Higson and Roe where it is proved to be well defined.

• The map Ind_Γ: $L_{n+1}(\mathbb{Z}\Gamma) \to K_{n+1}(C_r^*(\Gamma))$ has been defined by Wahl, following the results of Hilsum-Skandalis [\[11\]](#page-24-8) and Piazza-Schick [\[18\]](#page-24-6). Recall that an element $x \in$ $L_{n+1}(\mathbb{Z}\Gamma)$ is represented by a quadruple $(F: W \to X \times [0,1], u: X \to B\Gamma)$ with W a cobordism between two orientable manifolds $\partial_1 W$ and $\partial_2 W$, X an orientable manifold, $F: (W, \partial W) \to (X \times [0, 1], \partial(X \times [0, 1]))$ a degree one normal map of pairs, $f_1 := F_{|\partial \cdot W}$ and $f_2 := F_{\partial_2 W}$ oriented homotopy equivalences and $u: X \to B\Gamma$ a classifying map. Let $f = f_1 \sqcup f_2$ denote the restriction of F to ∂W . Consider $Z := W \sqcup X \times [0,1]$ and Z_{∞} the manifold obtained attaching an infinite cylinder to the boundary ∂Z . Since f is a homotopy equivalence, one can perturb the signature operator only on the cylindrical ends and the obtained operator has a well defined index in $K_{n+1}(C_r^*(\Gamma))$, that is the image of $(F: W \to X \times [0,1], u: X \to B\Gamma)$ through the map Ind_Γ.

Now we can state the main theorem.

Theorem 4.1. Let N be an n-dimensional closed oriented topological manifold with fundamental group Γ. Assume that $n \geq 5$ is odd. Then there is a commutative diagram with exact rows

$$
L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow S^{TOP}(N) \longrightarrow \mathcal{N}^{TOP}(N) \longrightarrow L_n(\mathbb{Z}\Gamma)
$$

\n
$$
\downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow \varnothing \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}}
$$

\n
$$
K_{n+1}(C_r^*(\Gamma)) \longrightarrow K_{n+1}(D^*(\widetilde{N})^{\Gamma}) \longrightarrow K_n(N) \longrightarrow K_n(C_r^*(\Gamma))
$$

and through the classifying map u: $N \to B\Gamma$ of the universal cover \tilde{N} of N, we have the analogous commutative diagram that involves the universal Higson-Roe exact sequence

$$
L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow S^{TOP}(N) \longrightarrow \mathcal{N}^{TOP}(N) \longrightarrow L_n(\mathbb{Z}\Gamma)
$$

\n
$$
\downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{er}} \qquad \qquad \downarrow_{\text{Gr}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}}
$$

\n
$$
K_{n+1}(C_r^*(\Gamma)) \longrightarrow K_{n+1}(D_{\Gamma}^*) \longrightarrow K_n(B\Gamma) \longrightarrow K_n(C_r^*(\Gamma))
$$

Remark 4.2. Let us recall the fact that, despite $\mathcal{S}^{TOP}(N)$ has a group structure, we do not deal with it and the top row is considered just as a sequence of sets, as in the smooth case.

Thanks to the results of the previous section we can check that the results in [\[17,](#page-24-3) Sections 4.2 and 4.3] hold in terms of the category of topological manifolds instead of the one of smooth manifolds: all proofs are still valid in the Lipschitz context. Thanks to the work of C.Whal [\[32,](#page-25-5) Theorem 9.1], that can be combined with Theorem [2.4,](#page-2-1) the first vertical arrow is well defined in the Lipschitz setting. The second one is also well defined for the previous section. Concerning the third one there are no significant problems.

The same method used in Propositio[n3.10](#page-6-2) applies to the class of the signature and its index class, then all vertical arrows do not depend on the chosen Lipschitz structure.

One has to check the commutativity of the three squares.

- The third square is obviously commutative: let $(f: M \rightarrow N)$ be a normal map in $\mathcal{N}^{TOP}(N)$, it is sent horizontally to the same map forgetting that it is normal and then through Ind_Γ to the difference Ind_Γ(D_M) – Ind_Γ(D_N); on the other hand $\beta(f: M \to$ $N = f_*[D_M] - [D_N]$, that gives, through the analytic assembly map, the index class just founded.
- Let us study the second square: let $(f: M \to N)$ be a structure in $S^{TOP}(N)$, it goes to the same map forgetting that f is a homotopy equivalence; the ρ -class $\rho(f)$, as in Definition [3.9,](#page-6-3) is the push-forward through $\tilde{\varphi}$ of the class

$$
\left[\frac{1}{2}(1+\chi(\widetilde{D}_Z+C_f))\right] \in K_0(D^*(\widetilde{Z})^{\Gamma});
$$

this goes horizontally to the class in $K_0(D^*(\tilde{Z})^{\Gamma}/C^*(\tilde{Z})^{\Gamma})$ that represents, by Paschke duality, the K-homology class of the signature operator of Z; then by functoriality of $\tilde{\varphi}_*$ and the fact that $\beta(f: M \to N) = f_*[D_M] - [D_N]$, we obtain the commutativity of the second square.

• For the commutativity of first square, the proof is exactly the same as in $[17, 4.10]$. Let $a \in L_{n+1}(\mathbb{Z}\Gamma)$ and let $(f: M \to N)$ be a structure in $\mathcal{S}^{TOP}(N)$. The commutativity of the first square means that the following equation holds:

 $i_*(\text{Ind}_{\Gamma}(a)) = \varrho(a[f: M \to N]) - \varrho([f: M \to N]) \in K_0(D^*(\widetilde{Z})^{\Gamma});$

this is proved identifying the right hand side with the class predicted by the APS delocalized index theorem, that, as we know, holds in the Lipschitz case too. The proof is based on an addition formula, as in [\[32,](#page-25-5) 7.1], and algebraic identifications of ρ -classes, that the reader can check still holding, word-for-word, in the Lipschitz case.

5 Products

Let M and N be two Cartesian products with a common factor, namely $M = M_1 \times M_2$ and $N = N_1 \times M_2$, and let $f_1: M_1 \to N_1$ be a homotopy equivalence. Therefore $f = f_1 \times id$: $M \to N$ is a homotopy equivalence.

Observe that the signature operator on $Z = M \cup (-N)$ has this form: $D_Z = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$, i.e. the graded tensor product of the signature operator D_1 on $M_1 \cup (-N_1)$ and the signature operator D_2 on M_2 .

As before we construct from f a bounded operator C_f that produces an invertible perturbation $D_Z + C_f$. Notice that, from the construction in [\[11\]](#page-24-8) and as it has been pointed out in [\[32,](#page-25-5) (6.1)], the operator C_f has the form $C_{f_1}\hat{\otimes}1$, where all grading operators are understood in the graded tensor product. We have

$$
D_Z + C_f = (D_1 + C_{f_1})\hat{\otimes}1 + 1\hat{\otimes}D_2
$$

so we can associate an invertible perturbation of D_Z to an invertible perturbation of $D₁$.

We would like to state a product formula involving the ρ -class invariant of the first factor and the K-homology class of the second one. For this aim it will be useful to give another realization of the group $K_*(D^*(\tilde{X})^{\Gamma}).$

In [\[13,](#page-24-12) II.2] the Grothendieck group of a functor $\varphi: \mathcal{C} \to \mathcal{C}'$ is defined as the set of triples (E, F, α) , where E and F are objects in the category C and α is an isomorphism $\varphi(E) \to \varphi(F)$ in the category C', modulo the following equivalence relation: two triples (E, F, α) and (E', F', α') are equivalent if there exist two isomorphisms $f: E \to E'$ and $g: F \to F'$ such that the following diagram

$$
\varphi(E) \xrightarrow{\alpha} \varphi(F)
$$

\n
$$
\downarrow \varphi(f)
$$

\n
$$
\varphi(E') \xrightarrow{\alpha'} \varphi(F')
$$

commutes.

In [\[13,](#page-24-12) II.3.28] it is shown that, when φ is the restriction of vector bundles over a space X to a closed subspace Y, one obtains the relative K-group $K(X, Y)$ as the K-theory of the mapping cone of the inclusion $i: Y \hookrightarrow X$.

In [\[22\]](#page-24-13), G. Skandalis used the same idea: considering an element x in $KK(A, B)$ as a functor from $K(A)$ to $K(B)$ through the Kasparov product, one can construct a relative K-group $K(x)$ and one can also prove that it is isomorphic to the K-theory of a mapping cone C*-algebra. Moreover this relative K-group fits in a long exact sequence

$$
\ldots \longrightarrow K(B \otimes C_0(0,1)) \longrightarrow K(x) \longrightarrow K(A) \longrightarrow \ldots
$$

such that the boundary map is given by the Kasparov product with x .

More generally, if we fix a separable C^* -algebra D, we can consider an element x in $KK(A, B)$ as a functor from $KK(D, A)$ to $KK(D, B)$, through the Kasparov product with x on the right and we can still obtain a relative KK-group $K(D, x)$ that turns out to be isomorphic to the group $KK(D, C_{\psi})$, where C_{ψ} is the mapping cone C^{*}-algebra of a suitable *-homomorphism ψ .

So we have seen that constructing relative KK-groups corresponds, in a philosophical way, to taking the Grothendieck group of a functor or, in a more concrete way, to taking the Grothendieck group of a mapping cone. We want to construct a long exact sequence of groups such that the boundary map is the assembly map. Notice that the difficulty resides in the fact that the assembly map is not induced by a morphism nor by a Kasparov product on the right. But it is still possible to construct a group.

5.1 The analytic structure set and products

Let X be a proper and cocompact Γ -space, we would like to give an explicit construction of the cycles of $K_*(D^*(X)^{\Gamma})$ in terms of Kasparov bimodules, so that one can define a product by means of Kasparov products.

In [\[20\]](#page-24-14) J. Roe shows that the following diagram

$$
K_{*+1}(D^*(X)^{\Gamma}/C^*(X)^{\Gamma}) \xrightarrow{\text{Ind}} K_*(C^*(X)^{\Gamma})
$$

\n
$$
\downarrow P \qquad \qquad \downarrow \cong
$$

\n
$$
KK_{\Gamma}^*(C_0(X), \mathbb{C}) \xrightarrow{\mu_X^{\Gamma}} K_*(C_r^*(\Gamma))
$$

\n(5.1)

is commutative. Here P is given by the Paschke duality and μ_X^{Γ} is the assembly map defined by Kasparov in [\[14\]](#page-24-15). Let us recall some notation and definitions. For any Γ-C*-algebras A and B, there exists a descent homomorphism

$$
j^{\Gamma} \colon KK_{\Gamma}(A, B) \to KK(A \rtimes \Gamma, B \rtimes \Gamma)
$$

which is functorial and compatible with respect to Kasparov products. It associates to an equivariant KK-cycle $[H, \phi, F]$ the Kasparov bimodule $[H \rtimes \Gamma, \phi, F]$, where

- $H \rtimes \Gamma$ is the $A \rtimes \Gamma \cdot B \rtimes \Gamma$ -bimodule given by the completion of $C_c(\Gamma, H)$, with the usual $C_c(\Gamma, B)$ -valued inner product and left $C_c(\Gamma, A)$ -action;
- $\widetilde{\phi}$ is the extension to $A \rtimes \Gamma$ of the representation of $C_c(\Gamma, A)$ induced by ϕ ;
- \widetilde{F} is the extension to $H \rtimes \Gamma$ of the operator F that associates to $\gamma \mapsto \alpha(\gamma)$ the element $\gamma \mapsto F(\alpha(\gamma))$ on $C_c(\Gamma, H)$.

Moreover we know that for any proper and cocompact Γ -space X one can construct an imprimitivity $C(X/\Gamma)$ -C₀ (X) × Γ -bimodules E_X . Since $C(X/\Gamma)$ is unital E_X defines an element in $KK(\mathbb{C}, C_0(X) \rtimes \Gamma)$.

Definition 5.1. The assembly map μ_X^{Γ} is defined as the composition

$$
KK^*_{\Gamma}(C_0(X), \mathbb{C}) \xrightarrow{j^{\Gamma}} KK^*(C_0(X) \rtimes \Gamma, C_r^*(\Gamma)) \xrightarrow{[E_X] \otimes -} K_*(C_r^*(\Gamma)) .
$$

Let us sketch how the commutativity of the diagram [\(5.1\)](#page-13-1) is proved: if $F \in B(L^2(X))$ is an element of $D^*(X)$ ^{Γ} such that $F^2 = 1$ and $F^* = F$ modulo $C^*(X)$ ^{Γ}, a representative for

$$
\mu_X^{\Gamma} [L^2(X), \varphi \colon C_0(X) \to \mathbf{B}(L^2(X)), F]
$$

is given in the following way. Since we can assume that F is exactly of finite propagation, it defines an operator F_c on the pre-Hilbert space $L_c^2(X)$ of the compactly supported L^2 -functions of X. One can endow $L_c^2(X)$ with the following C_Γ-valued inner product

$$
\langle f, g \rangle_{\mathbb{C}\Gamma}(\gamma) = \langle f^{\gamma}, g \rangle_{\mathbb{C}}
$$

where $\langle f^{\gamma}, g \rangle_{\mathbb{C}}$ is the standard inner product between g and the function f translated by γ . With a standard double completion of the pair $(L_c^2(X), \mathbb{C}\Gamma)$ we obtain an Hilbert module over $C_r^*(\Gamma)$ that we denote by $L^2_{\Gamma}(X)$. Now F_c extends to an adjointable operator \widetilde{F} on $L^2_{\Gamma}(X)$ and the class $[L_{\Gamma}^2(X), 1 \otimes \tilde{\varphi}, \tilde{F}] \in KK(\mathbb{C}, C_r^*(\Gamma))$ is equal to $\mu_X^{\Gamma} [L^2(X), \varphi, F]$.

Remark 5.2. Notice that if F is invertible, then \widetilde{F} is also invertible.

Moreover one can prove that $L^2_{\Gamma}(X)$ is a complemented sub-Hilbert module of $L^2(X) \rtimes \Gamma$. In fact if $\phi: X \to [0, 1]$ is a compactly supported function such that

$$
\sum_{\gamma\in\Gamma}(\phi^2)^{\gamma}=1,
$$

then the projection

$$
p = \sum_{\gamma \in \Gamma} \phi \cdot \phi^{\gamma^{-1}}[\gamma] \in C_0(X) \rtimes \Gamma
$$

has as range the $C_r^*(\Gamma)$ -module $L^2_{\Gamma}(X)$. Actually the projection p gives the class $[E_X] \in$ $KK(\mathbb{C}, C_0(X) \rtimes \Gamma)$ used in the Definition [5.1](#page-13-2) of the assembly map.

Definition 5.3. Let X be as above. A Γ -equivariant analytic structure cycle on X consists of the following data:

- an equivariant selfadjoint Kasparov bimodule $(H, \phi, T) \in \mathbb{E}^{\Gamma}(C_0(X), \mathbb{C})$;
- a Kasparov bimodule $(\mathcal{E}(t), \psi(t), S(t)) \in (\mathbb{C}, C^*_r(\Gamma)[0, 1)),$ such that $\mathcal{E}(0) = E_X \otimes_{C_0(X) \rtimes \Gamma}$ $H \rtimes \Gamma$, $\psi(0) = \mathrm{id} \otimes_{C_0(X) \rtimes \Gamma} \phi$ (that from now on we will denote in short by $\mathrm{id} \otimes \phi$), $S(0)$ is a \widetilde{T} -connection and $S(1)$ is invertible. Here $\widetilde{\phi}$ and \widetilde{T} are as in the definition of the descent homomorphism. That is the class of $(\mathcal{E}(0), \psi(0), S(0))$ is equal to $\mu_X^{\Gamma}(H, \phi, T)$.

Such a cycle is said to be degenerate if both (H, ϕ, T) and $(\mathcal{E}(t), \psi(t), S(t))$ are degenerate Kasparov bimodules.

Definition 5.4. Let $(H_i, \phi_i, T_i, \mathcal{E}(t)_i, \psi(t)_i, S_i(t)), i = 0, 1$, be two Γ -equivariant analytic structure cycles.

We will say that they are homotopic if there exists a path $(H_s, \phi_s, T_s, \mathcal{E}_s(t), \psi(t)_s, S_s(t))$ of Γ-equivariant analytic structure cycles that joins them. Then we denote by $S_j^{\Gamma}(X)$ the Grothendieck group generated by all homotopy classes of Γ-equivariant analytic structure cycles on X.

We can define in an analogous way a group $S^{\Gamma}_*(X, A)$, where A is any Γ -C*-algebra, using the assembly map with coefficient $\mu_{X,A}^{\Gamma}: KK^{\Gamma}(C_0(X), A) \to KK(\mathbb{C}, A \rtimes \Gamma)$.

Remark 5.5. Note that one can give the definition of $S^{\Gamma}_*(X, A)$ also when Γ is a groupoid instead of a group.

Proposition 5.6. There is a commutative diagram

$$
\cdots \longrightarrow K_j(C^*(X)^{\Gamma}) \longrightarrow K_j(D^*(X)^{\Gamma}) \longrightarrow K_j(D^*(X)^{\Gamma}/C^*(X)^{\Gamma}) \longrightarrow \cdots
$$

\n
$$
\downarrow \beta \qquad \qquad \downarrow \alpha \qquad \qquad \downarrow \rho
$$

\n
$$
\cdots \longrightarrow KK^{j-1}(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0, 1)) \longrightarrow S_{j-1}^{\Gamma}(X) \longrightarrow KK^{j-1}_{\Gamma}(C_0(X), \mathbb{C}) \longrightarrow \cdots
$$

whose vertical arrows are isomorphisms.

Proof. Let $(L^2(X), \phi: C_0(X) \to \mathbf{B}(L^2(X)))$ be the *Γ*-equivariant $C_0(X)$ -module used to construct the algebra $D^*(X)^{\Gamma}$. The map P is given by the Paschke duality. The homomorphism β is given by the composition of the isomorphism between $C^*(X)^{\Gamma}$ and $C^*_r(\Gamma)$, and the Bott periodicity. Let us describe the homomorphism $\alpha: K_0(D^*(X)^{\Gamma}) \to K_1(\mu_X^{\Gamma})$. It associates to a projection p over $D^*(X)$ ^{Γ} the cycle $(H, \varphi, F, \mathcal{E}(t), \psi(t), S(t))$, where

- $(H, \varphi, F) = (L^2(X), \varphi, 2p 1);$
- $(\mathcal{E}(t), \psi(t), S(t))$ is given by the path constantly equal to $(L^2(\Gamma), id \otimes \tilde{\varphi}, F)$, that is the triple built in the discussion at the beginning of the present section triple built in the discussion at the beginning of the present section.

Observe that $L^2(\mathcal{X})$ is nothing but $E_X \otimes_{C_0(X)\rtimes\Gamma} L^2(X) \rtimes \Gamma$ and that \widetilde{F} is invertible by construction (see Remark [5.2\)](#page-14-0).

The homomorphism β associates to a projection p over $C^*(X)^{\Gamma}$ the Kasparov bimodule $[L_{\Gamma}²(X), \mathrm{id} \otimes \phi, G(t)],$ where $G(t)$ is the loop of invertible elements $\widetilde{F}(1-e^{2i\pi t})-1$ over $C^*(X)^{\Gamma}$, given by the Bott periodicity.

The second square is obviously commutative. Concerning the first one, since $(L^2(X), \phi, F)$ is degenerate as Kasparov bimodule, it is easy to produce a homotopy of cycles between $(0, 0, 0, L₁²(X), id \otimes \tilde{\varphi}, G(t))$ and $\alpha(i_*[p]) = [H, \varphi, F, L₁²(X), id \otimes \tilde{\varphi}, S(t)],$ where $S(t)$ is the constant path equal to \widetilde{F} . To do that, observe that $G\left(\frac{1}{2}\right) = \widetilde{F}$ and that $G_s(t) = G\left((1-s)t + \frac{1}{2}s\right)$ does the job.

Remark 5.7. Let $\varphi: A \to B$ a C^{*}-algebras morphism and C_{φ} its mapping cone. Then we obtain naturally the long exact sequence of K-groups

$$
\cdots \longrightarrow K_*(SB) \longrightarrow K_*(C_{\varphi}) \longrightarrow K_*(A) \xrightarrow{\varphi_*} K_*(B) \longrightarrow \cdots
$$

whose boundary morphism is induced by φ . Conversely, if we start from a homomorphism $K_*(A) \to K_*(B)$ induced by a morphism $\varphi: A \to B$, then this homomorphism fits into a sequence as above.

As explained before, the idea behind the construction of $S_j^{\Gamma}(X)$ is considering the assembly map as a functor. But instead of a Kasparov product on the right, we have the assembly map, that is the composition of the descent morphism and a Kasparov product on the left, and instead of the K-theory of a mapping cone of C*-algebras we obtain the Grothendieck group of a "mapping cone" of Kasparov bimodules.

Definition 5.8. Let Y, Z be two spaces and assume that a group Γ acts on Y and Z in a proper and cocompact way. Let $f: Y \to X$ be a Γ-equivariant continuous map. We can define a homomorphism

$$
f_* \colon S_j^{\Gamma}(Y) \to S_j^{\Gamma}(X)
$$

such that $f_*[H, \phi, T, \mathcal{E}(t), \psi(t), S(t)] = [H, \phi \circ f, T, \mathcal{E}'(t), \psi'(t), S'(t))].$

Here $(\mathcal{E}'(t), \psi'(t), S'(t))$ is the concatenation of the path we are going to describe and the path $(\mathcal{E}(t), \psi(t), S(t))$. The first one is the path connecting the Kasparov bimodules $(E_X \otimes_{C_0(X)\rtimes\Gamma} H\rtimes\Gamma, \text{id} \otimes \widetilde{(\phi\circ f)}, S')$ and $(E_Y \otimes_{C_0(Y)\rtimes\Gamma} H\rtimes\Gamma, \text{id} \otimes \widetilde{\phi}, S)$, where S' is a T-connection on $E_Y \otimes_{C_0(Y)\rtimes\Gamma} H \rtimes \Gamma$. This path always exists thanks to the functoriality of the assembly map: since $\mu_Y^{\Gamma} = \mu_X^{\Gamma} \circ f_* \colon KK_{\Gamma}(C_0(Y)) \to K_0(C_r^*(\Gamma))$, it turns out that $(E_X \otimes_{C_0(X)\rtimes\Gamma} H\rtimes \Gamma, \mathrm{id}\otimes \widetilde{(\phi\circ f)}, S')$ and $(E_Y \otimes_{C_0(Y)\rtimes\Gamma} H\rtimes \Gamma, \mathrm{id}\otimes \widetilde{\phi}, S)$ define the same class. Moreover the class obtained does not depend on the choice of this path.

Lemma 5.9. Let Y, X be two Riemannian manifolds and assume that a group Γ acts on Y and X freely, isometrically and such that Y/Γ is compact. Let $f: Y \to X$ be a Γ -equivariant continuous coarse map and let $V: H_Y \to H_X$ be an isometry that covers f in the D^{*}-sense ($\left(18, Definition\ 1.7\right)$). Then the following diagram

16

commutes.

Proof. Let $(H_Y, \phi_Y : C_0(Y) : \mathbf{B}(H_Y))$ and $(H_X, \phi_X : C_0(X) : \mathbf{B}(H_X))$ the representations used to construct the algebras $D^*(Y)^{\Gamma}$ and $D^*(X)^{\Gamma}$. Let p be a projection over $D^*(Y)^{\Gamma}$. Remember that $f_*[p] = [\text{Ad}_V(p)] \in K_0(D^*(X)^{\Gamma})$. Then we get two elements of $S_1^{\Gamma}(X)$:

- the first one is $\alpha([Ad_V(p)]) = [H_X, \phi_X, T, \mathcal{E}_X, id \otimes \widetilde{\phi}_X, S].$ Here $T \cong 2\text{Ad}_V(p) 1$, $\mathcal{E}_X = E_X \otimes_{C_0(X) \rtimes \Gamma} H_X \rtimes \Gamma$ and S is the path constantly equal to a \widetilde{T} -connection;
- the second one is $f_*(\alpha[p]) = [H_Y, \phi_Y \circ f^*, U, \mathcal{E}'(t), \psi'(t), S'(t)].$ Here $U = 2p 1$ and $(\mathcal{E}'(t), \psi'(t), S'(t))$ is the path connecting $(E_X \otimes_{C_0(X)\rtimes\Gamma} H \rtimes \Gamma, \text{id} \otimes (\widetilde{\phi \circ f}), S')$ and $(E_Y \otimes_{C_0(Y)\rtimes \Gamma} H \rtimes \Gamma, \text{id} \otimes \widetilde{\phi}, S)$, where S' is a \widetilde{U} -connection on $E_Y \otimes_{C_0(Y)\rtimes \Gamma} H \rtimes \Gamma$.

We have to prove that these two classes are the same.

Consider the projection $Q = V V^*$, then we can decompose $\alpha([{\rm Ad}_V(p)])$ in two direct summands:

$$
\alpha([Ad_V(p)]) = [QH_X, \phi_X, T_1, R\mathcal{E}_X, \text{id} \otimes \widetilde{\phi}_X, S_1] \oplus [(1-Q)H_X, \phi_X, T_2, (1-R)\mathcal{E}_X, \text{id} \otimes \widetilde{\phi}_X, S_2],
$$

where $T_1 = QTQ, T_2 = (1 - Q)T(1 - Q), R$ is a \tilde{Q} -connection and S_1 and S_2 are defined similarly.

The second summand is clearly degenerate and, since to the following diagram

$$
C_0(X) \xrightarrow{f^*} C_0(Y) \xrightarrow{\varphi_Y} \mathbb{B}(H_Y)
$$

$$
\varphi_X \longrightarrow \mathbb{B}(QH_X)
$$

commutes, modulo compacts operators, the first one is equal to $f_*(\alpha[p])$.

Remark 5.10. If we define the group $\hat{K}_j^{\Gamma}(X)$ as in definition [5.3,](#page-14-1) but dropping the condition of $S(1)$ being invertible, we get that the map

$$
\hat{K}_j^{\Gamma}(X) \ni [H, \phi, T, \mathcal{E}(t), \psi(t), S(t)] \mapsto [H, \phi, T] \in KK_{\Gamma}^j(C_0(X), \mathbb{C})
$$

is a group isomorphism. Indeed one can easily check that the kernel of this map is isomorphic to $KK_j(\mathbb{C}, C^*_r(\Gamma) \otimes C_0(0,1])$, that is trivial since $C^*_r(\Gamma) \otimes C_0(0,1]$ is a cone. The inverse map is obviously given by

 $[H, \phi, T] \mapsto [H, \phi, T, \mathcal{E}, \mathrm{id} \otimes \widetilde{\phi}, S]$

where S is the path constantly equal to any \widetilde{T} -connection.

Definition 5.11. Let Γ be a discrete group, we can define the following exact sequence of groups

$$
\ldots \longrightarrow KK_*(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0,1)) \longrightarrow S_*^{\Gamma} \longrightarrow \hat{K}_*^{\Gamma} \longrightarrow \ldots
$$

as the direct limit of

$$
\ldots \longrightarrow KK_*(\mathbb{C}, C_r^*(\Gamma) \otimes C_0(0,1)) \longrightarrow S_*^{\Gamma}(X) \longrightarrow \hat{K}_*^{\Gamma}(X) \longrightarrow \ldots
$$

over all cocompact Γ-subspaces X of $E\Gamma$.

Thus we obtain the same groups defined in [\[18,](#page-24-6) Definition 1.11]. This follows easily from Proposition [5.6](#page-14-2) and Lemma [5.9.](#page-15-0)

Let

$$
\xi = [H_1, \phi_1, T_1, \mathcal{E}_1(t), \psi_1(t), S_1(t)] \in S_j^{\Gamma_1}(X_1)
$$

and let

$$
\lambda = [H_2, \phi_2, T_2, \mathcal{E}_2(t), \psi_2(t), S_2(t)] \in \hat{K}_i^{\Gamma_2}(X_2),
$$

where X_1 and X_2 are two proper and cocompact spaces with respect to Γ_1 and Γ_2 respectively. Let $(H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, T)$ be an exterior Kasparov product of (H_1, ϕ_1, T_1) and (H_2, ϕ_2, T_2) . Let $(\mathcal{E}(t), \psi(t), S(t))$ be the restriction to the diagonal of the Kasparov product of $(\mathcal{E}_1(t), \psi_1(t), S_1(t))$ and $(\mathcal{E}_2(t), \psi_2(t), S_2(t))$ (that is a Kasparov C-A-bimodule, where A is equal to the algebra $C_r^*(\Gamma_1) \otimes C_r^*(\Gamma_2) \otimes C_0([0,1]^2 \setminus \{1\} \times [0,1])).$

Definition 5.12. We define a product

$$
S_j^{\Gamma_1}(X_1) \times \hat{K}_i^{\Gamma_2}(X_2) \to S_{j+i}^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2)
$$

that associates to $\xi \times \lambda$ the class

$$
\xi \boxtimes \lambda := [H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, T, \mathcal{E}(t), \psi(t), S(t)]
$$

where the entries are as described above. The product is compatible with homotopies in both factors and so it is well defined.

Remark 5.13. A similar product is defined in an obvious way on $KK^{j-1}(\mathbb{C}, C_r^*(\Gamma_1) \otimes C_0(0,1))$ and $\hat{K}_j^{\Gamma_1}(C(X_1), \mathbb{C})$. It is natural in the sense that the following diagram

$$
\cdots \longrightarrow KK^j(\mathbb{C}, A) \times \hat{K}_i^{\Gamma_2}(X_2) \longrightarrow S_j^{\Gamma_1}(X_1) \times \hat{K}_i^{\Gamma_2}(X_2) \longrightarrow \hat{K}_j^{\Gamma}(X_1) \times \hat{K}_i^{\Gamma_2}(X_2) \longrightarrow \cdots
$$

\n
$$
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow
$$

\n
$$
\cdots \longrightarrow KK^{j+i}(\mathbb{C}, B) \longrightarrow S_{j+i}^{\Gamma_1 \times \Gamma_2}(X_1 \times \tilde{X}_2) \longrightarrow \hat{K}_{j+i}^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2) \longrightarrow \cdots
$$

is commutative. Here $A = C^*(\widetilde{X}_1)^{\Gamma_1} \otimes C_0(0,1)$ and $B = C^*(\widetilde{X}_1 \times \widetilde{X}_2)^{\Gamma_1 \times \Gamma_2} \otimes C_0(0,1)$.

Lemma 5.14. Let Y, X, Z be three spaces and assume that a group Γ_1 acts properly and cocompactly on Y and X and Γ_2 acts properly and cocompactly on Z. Let $f: Y \to X$ be a Γ-equivariant continuous map. Then the following diagram

$$
S_i^{\Gamma_1}(Y) \times \hat{K}_j^{\Gamma_2}(Z) \xrightarrow{f_* \times id} S_i^{\Gamma_1}(X) \times \hat{K}_j^{\Gamma_2}(Z)
$$

\n
$$
\downarrow \qquad \qquad \downarrow
$$

\n
$$
S_{j+i}^{\Gamma_1 \times \Gamma_2}(Y \times Z) \xrightarrow{(f \times id_Z)_*} S_{j+i}^{\Gamma_1 \times \Gamma_2}(X \times Z)
$$

where the vertical arrows are given by [5.12,](#page-16-0) is commutative.

Proof. This is straightforward since $(\phi_1 \otimes \phi_2) \circ (f \times id_Z)^* = (\phi_1 \circ f^*) \otimes \phi_2$. \Box

5.2 Stability of ρ classes

5.2.1 The signature operator

Let $f: M \to N$ be a structure in $\mathcal{S}^{TOP}(N)$ and $\varrho(f)$ be the associated ϱ -class in $K_*(D^*(\tilde{Z})^{\Gamma})$. Let us see the different realisations of this class with respect to the different models of the analytical structure set.

- In $K_0(D^*(\widetilde{Z})^{\Gamma})$ we have the element $\left[\frac{1}{2}(1+\chi(\widetilde{D}_Z+C_f))\right]$.
- In $S_1^{\Gamma}(\widetilde{Z})$ this element turns into

$$
\left[H, \phi, F, \mathcal{E}, \mathrm{id} \otimes \widetilde{\phi}, G\right],
$$

where $F = (\chi(\widetilde{D}_Z + C_f)), \mathcal{E} = E_Z \otimes_{C_0(Z) \rtimes \Gamma} H \rtimes \Gamma$ and G is the path constantly equal to the \widetilde{F} -connection used in the proof of Proposition [5.6.](#page-14-2)

• Finally observe that the image of the last element through the natural map $S_1^{\Gamma}(\tilde{Z}) \to S_1^{\Gamma}$ is the image of $\varrho_{\Gamma} \in K_0(D_{\Gamma}^*)$ by means of the obvious isomorphism.

$$
\varrho(f_1 \times \mathrm{id}_{M_2}) = \varrho(f_1) \boxtimes [D_2] \in S_1^{\Gamma_1 \times \Gamma_2}(\widetilde{M}_1 \times \widetilde{M}_2)
$$

and the same holds for ρ_{Γ} .

Proof. Let $Z_1 = M_1 \cup N_1$ and $Z_2 = M_1 \times M_2 \cup N_1 \times M_2$. The class $\varrho(f_1)$ is represented in $S_1^{\Gamma_1}(Z_1)$ by the cycle

$$
\left[H_1,\phi_1,F_1,\mathcal{E}_1,\mathrm{id}\otimes\widetilde{\phi}_1,G_1\right],
$$

where $F_1 = \chi(D_{Z_1} + C_{f_1}).$

The class $[D_2] \in \hat{K}_1^{\Gamma_2}(M_2)$ is represented by

$$
\Big[H_2,\phi_2,F_2,\mathcal{E}_2,\mathrm{id}\otimes \widetilde{\phi}_2,G_2\Big],
$$

where $F_2 = \psi(D_{M_2}).$

Finally the class $\rho(f_1 \times \mathrm{id}_{M_2}) \in S_1^{\Gamma_1 \times \Gamma_2}(Z_1 \times M_2)$ is represented by

$$
\left[H_1\otimes H_2,\phi_1\otimes \phi_2,F,\mathcal{E}_1\otimes \mathcal{E}_2,\mathrm{id}\otimes \widetilde{\phi}_1\otimes \widetilde{\phi}_2,G\right],
$$

where $F = \chi(D_{Z_1} \otimes 1 + 1 \otimes D_{M_1} + C_{f_1 \times \mathrm{id}_{M_2}}).$

We have to prove the identity of the last class mentioned with the product $\varrho(f) \boxtimes [D_2] \in$ $S_1^{\Gamma_1 \times \Gamma_2} (Z_1 \times M_2)$ given by

$$
\left[H_1\otimes H_2,\phi_1\otimes \phi_2,F',\mathcal{E}_1\otimes \mathcal{E}_2,1\otimes \widetilde{\phi}_1\otimes \widetilde{\phi}_2,G'\right],
$$

where $F' = \chi(\widetilde{D}_{Z_1} + C_f) \otimes 1 + 1 \otimes \psi(\widetilde{D}_{M_2}).$

Since $D_{Z_1} \otimes 1 + 1 \otimes D_{M_1} + C_{f_1 \times \mathrm{id}_{M_2}} = (D_{Z_1} + C_f) \otimes 1 + 1 \otimes D_{M_2}$ and that χ and ψ differ by a function in $C_0(\mathbb{R})$, the identity follows from [\[2\]](#page-23-4).

Trivially this holds for ρ_{Γ} too.

We would like that, after fixing a non zero K-homology class λ , under suitable assumptions the product with this element is an injective map.

To prove that, we need to define a new group that we will denote by $\mathcal{T}_{*}^{\Gamma_1,\Gamma_2}(X_1,X_2)$ (notice that the order of X_1 and X_2 is not irrelevant).

Definition 5.16. A cycle of $\mathcal{T}_j^{\Gamma_1,\Gamma_2}(X_1,X_2)$ consists of the following data:

- a Kasparov bimodule $(H, \phi, T) \in \mathbb{E}^{\Gamma_1 \times \Gamma_2} (C_0(X_1) \otimes C_0(X_2), \mathbb{C});$
- a Kasparov bimodule $(\mathcal{E}_s, \psi_s, S_s) \in \mathbb{E}^{\Gamma_1} (C_0(X_1), C_r^*(\Gamma_2) \otimes C[0,1]),$ where \mathcal{E}_0 is equal to $E_{X_2} \otimes_{C_0(X_2) \rtimes \Gamma_2} H \rtimes \Gamma_2$, $\psi_0 = \text{id} \otimes \widetilde{\phi}$ and S_0 is any \widetilde{T} -connection;
- a Kasparov bimodule $(\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s}) \in \mathbb{E}(\mathbb{C}, C^*_r(\Gamma_1) \otimes C^*_r(\Gamma_2) \otimes C_0(\mathcal{T}))$, where $\mathcal T$ is the triangle $\{(t, s) \in [0, 1]^2 \setminus \{1, 1\} | t \leq s\}, \mathcal{E}'_{0,s} = E_{X_1} \otimes_{C_0(X_1) \rtimes \Gamma_1} \mathcal{E}_s \rtimes \Gamma_1, \psi'_{0,s} = id \otimes \widetilde{\psi}$ and $S'_{0,s}$ is any \widetilde{S}_s -connection;

modulo homotopies of cycles, defined in a obvious way.

Remark 5.17. To have an intuition of what this group is, accordingly with the idea in Remark [5.7,](#page-15-1) one can think of it as the restriction to the triangle $\mathcal{T} = \{(t, s) \in [0, 1]^2 \setminus \{1, 1\} | t \leq s\}$ of the product of the "mapping cone" $\mu_{X_1}^{\Gamma_1}$ and the "mapping cylinder" of $\mu_{X_2}^{\Gamma_2}$. This idea was used in Definition [5.12](#page-16-0) too.

Lemma 5.18. The group $\mathcal{T}_{*}^{\Gamma_1,\Gamma_2}(X_1,X_2)$ is isomorphic to $S^{\Gamma_1\times \Gamma_2}(X_1\times X_2)$.

Proof. Define the homomorphism $\Phi \colon \mathcal{T}_{*}^{\Gamma_1,\Gamma_2}(X_1,X_2) \to S^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2)$ given by

 $((H, \phi, T), (\mathcal{E}_s, \psi_s, S_s), (\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s})) \mapsto (H, \phi, T, \mathcal{E}'_{t,t}, \psi'_{t,t}, S'_{t,t}).$

Define the following homomorphism

$$
\Psi\colon (H,\phi,T,\mathcal{H}_t,\alpha_t,U_t)\mapsto \big((H,\phi,T),(\mathcal{E}_s,\psi_s,S_s),(\mathcal{E}'_{t,s},\psi'_{t,s},S'_{t,s})\big)
$$

where

- $(\mathcal{E}_s, \psi_s, S_s)$ is the path constantly equal to $(E_{X_2} \otimes_{C_0(X_2) \rtimes \Gamma_2} H \rtimes \Gamma_2, \mathrm{id} \otimes \widetilde{\phi}, S)$, with S any \widetilde{T} -connection:
- for all fixed $t \in [0,1)$, $(\mathcal{E}'_{t,s}, \psi'_{t,s}, S'_{t,s})$ is the paths constantly equal to $(\mathcal{H}_t, \alpha_t, U_t)$.

It is easy to check that the third condition in Definition [5.16](#page-18-0) is satisfied and that Φ and Ψ are inverse to each other.

 \Box

Proposition 5.19. Let λ be a class in $\hat{K}_i^{\Gamma_2}(X_2)$. If there exists a class $\zeta \in KK^{-i}(\mathcal{C}_r^*(\Gamma_2), \mathbb{C})$ such that $\mu_{X_2}^{\Gamma_2}(\lambda) \otimes_{C_r^*(\Gamma_2)} \zeta = n$ with $n \neq 0$, then

$$
\boxtimes \lambda \colon S_i^{\Gamma_1}(X_1) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to S_{i+j}^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2) \otimes \mathbb{Z}\left[\frac{1}{n}\right]
$$

is injective. In particular if $\mu_{X_2}^{\Gamma_2}(\lambda) \otimes_{C_r^*(\Gamma_2)} \zeta = 1$, then the product with λ is honestly injective.

Proof. To prove the Lemma we are going to build a left inverse for $\boxtimes \lambda$. Define the map c_{ζ} as the composition of the following ones:

- the isomorphism $\Psi: S_*^{\Gamma_1 \times \Gamma_2}(X_1 \times X_2) \to \mathcal{T}_*^{\Gamma_1, \Gamma_2}(X_1, X_2),$
- the evaluation at $s = 1$, $ev_{s=1}$: $\mathcal{T}_{*}^{\Gamma_{1},\Gamma_{2}}(X_{1},X_{2}) \to S_{*}^{\Gamma_{1}}(X_{1},C_{r}^{*}(\Gamma_{2}))$, given by

$$
\big((H,\phi,T),(\mathcal{E}_s,\psi_s,S_s),(\mathcal{E}_{t,s}',\psi_{t,s}',S_{t,s}')\big)\mapsto \big((\mathcal{E}_1,\psi_1,S_1),(\mathcal{E}_{t,1}',\psi_{t,1}',S_{t,1}')\big)
$$

• the morphism $S^{\Gamma_1}_*(X_1, C^*_r(\Gamma_2)) \to S^{\Gamma_1}_{*-i}(X_1)$ given by

$$
(H, \phi, T, \mathcal{E}(t), \psi(t), S(t)) \mapsto (H', \phi', T', \mathcal{E}'(t), \psi'(t), S'(t)),
$$

where (H', ϕ', T') is any Kasparov product of $(H, \phi, T, \mathcal{E}(t))$ and ζ , and $(\mathcal{E}'(t), \psi'(t), S'(t))$ is any Kasparov product of $(\mathcal{E}(t), \psi(t), S(t))$ and ζ .

It is easy to check that $ev_{s=1} \circ \Psi \circ \boxtimes \lambda \colon S_i^{\Gamma_1}(X_1) \to S_{i+j}^{\Gamma_1}(X_1, C_r^*(\Gamma_2))$ is just the exterior product with $\mu_{X_2}^{\Gamma_2}(\lambda)$. Then, by hypothesis, $c_{\zeta}(x \boxtimes \lambda) = n \cdot x$ for any $x \in S_i^{\Gamma_1}(X_1)$. After inverting n, we get an inverse for $\boxtimes \lambda$.

Remark 5.20. The same argument fits to prove that if we fix an element $x \in \hat{K}_i^{\Gamma_2}(X_2)$ satisfying the above condition, then the vertical arrows of the following diagram

are rationally injective. Here $A = C^*(X_1)^{\Gamma_1} \otimes C_0(0,1)$ and $B = C^*(X_1 \times X_2)^{\Gamma_1 \times \Gamma_2} \otimes C_0(0,1)$.

We can obtain the condition of Lemma [5.19](#page-19-0) under certain hypotheses on Γ_2 : we impose that the group has a γ element, this means that there exists a C^{*}-algebra on which Γ acts properly and elements

$$
\eta \in KK_{\Gamma}(\mathbb{C}, A)
$$
 and $d \in KK_{\Gamma}(A, \mathbb{C}),$

such that $\gamma = \eta \otimes_A d \in KK_\Gamma(\mathbb{C}, \mathbb{C})$ satisfies $p^*\gamma = 1 \in KK_{E\Gamma \rtimes \Gamma}(C_0(\underline{E}\Gamma), C_0(\underline{E}\Gamma)),$ where EΓ is the classifying space for proper actions of Γ and $p: E\overline{\Gamma} \rtimes \overline{\Gamma} \to \Gamma$ is the homomorphism defined by $p(z, q) = q$. We refer the reader to [\[28,](#page-25-6) [29\]](#page-25-7).

The existence of the γ element implies that the Baum-Connes assembly map (with coefficients) is split injective and that the group is K-amenable: this last property gives the existence of a non trivial element $\zeta \in KK(C_r^*(\Gamma_2), \mathbb{C})$ such that, if $\xi = [L^2(\widetilde{X}_2), D] \in KK_{\Gamma_2}(\widetilde{X}_2, \mathbb{C})$ is
the class given by an equivariant elliptic operator D, then $\mu_{\widetilde{Y}}^{\Gamma_2}(D) \otimes_{C^*(\Gamma_2)} \zeta$ is equal to th $\frac{\Gamma_2}{\tilde{X}_2}(D) \otimes_{C_r^*(\Gamma_2)} \zeta$ is equal to the Fredholm index of the induced operator on \widetilde{X}_2/Γ .

Corollary 5.21. Let M_2 be an even dimensional Lipschitz manifold with fundamental group Γ_2 such that it has a γ element and $[D_2] \in K_*(M_2)$ has non zero index. If $f_1: N_1 \to M_1$ and $f_1' : N_1' \rightarrow M_1$ are homotopy equivalences between odd dimensional Lipschitz manifolds, with $differential$ ρ -class invariants, then

$$
[f_1 \times \mathrm{id}_{M_2}] \neq [f'_1 \times \mathrm{id}_{M_2}] \in \mathcal{S}^{TOP}(M_1 \times M_2).
$$

5.2.2 Dirac operators and positive scalar curvature

We would like to apply the methods of the previous sections to get similar results about the secondary invariants described in [\[18\]](#page-24-6).

Let us recall [\[18,](#page-24-6) Definition 1.6]: let (M, g) be a Riemannian spin manifold of dimension $n > 0$, with fundamental group Γ. If g has uniformly positive scalar curvature then the Dirac operator \mathcal{D}_M is invertible and $\chi(\mathcal{D}_M)$, the bounded transform of the lift of \mathcal{D}_M to the universal covering of M, defines a class $\varrho_g \in D^*(M)^\Gamma$.

Thanks to that and the APS-delocalized Theorem, for n odd, one obtains the following commutative diagram

$$
\Omega_{n+1}^{\text{spin}}(M) \longrightarrow \mathrm{R}_{n+1}^{\text{spin}}(M) \longrightarrow \mathrm{Pos}_{n}^{\text{spin}}(M) \longrightarrow \Omega_{n}^{\text{spin}}(M)
$$

\n
$$
\downarrow \beta \qquad \qquad \downarrow \mathrm{Ind}_{\Gamma} \qquad \qquad \downarrow \varrho \qquad \qquad \downarrow \beta
$$

\n
$$
K_{n+1}(M) \longrightarrow K_{n+1}(C_{r}^{*}(\Gamma)) \longrightarrow K_{n+1}(D^{*}(\widetilde{M})^{\Gamma}) \longrightarrow K_{n}(M)
$$

where M is a compact space with fundamental group Γ and universal covering \widetilde{M} . The first row in the diagram is the Stolz exact sequence, see for instance [\[18,](#page-24-6) Definition 1.39].

In the $S^{\Gamma}_{*}(M)$ picture of the analytic structure set, the class ρ_{g} is given by the quadruple

$$
[L^2(M, \mathcal{G}), C(M), \mathcal{D}_M, \widetilde{\chi(\mathcal{D}_M)}].
$$

Here the last term is the constant path $\chi(\vec{D}_M)$ because the operator is invertible and there is no need to perturb it.

Remark 5.22. If (M, g) has positive scalar curvature and (N, h) is another Riemannian manifold, then for $\varepsilon > 0$ small enough, $(M \times N, g \times \varepsilon h)$ has positive scalar curvature. Hence if M admits a metric with positive scalar curvature, so does $M \times N$.

Proposition 5.23. Let M be a spin manifold of dimension n and let q be a Riemannian metric with positive scalar curvature on M . Let N be a spin manifold of dimension m and h a Riemannian metric such that $(M \times N, g \times h)$ has positive scalar curvature. Then

$$
\varrho_g \boxtimes [\not{D}_h] = \varrho_{g \times h} \in S_{n+m}^{\Gamma_1 \times \Gamma_2} (M \times N),
$$

where Γ_1 and Γ_2 are the fundamental groups of M and N respectively and $[\phi_h]$ is the class of the Dirac operator on N in $K_m(N)$.

Proof. We can prove the result as in [5.15.](#page-17-2) Moreover since the class ρ_q is represented by a quadruple whose last term is the constant path $\chi(\vec{p}_h)$, it turns out that we can prove it in an easier way (see for instance [\[21,](#page-24-5) Proposition 6.2.13]). \Box

Corollary 5.24. Let M be a spin manifold of odd dimension n with fundamental group Γ_1 and let g_1 and g_2 be two Riemannian metrics with positive scalar curvature on M such that $\varrho_{g_1} \neq \varrho_{g_2} \in S_n^{\Gamma}(M)$. Let (N, h) be a Riemannian spin manifold of even dimension m with fundamental group Γ_2 , such that the index of $[\mathcal{D}_N] \in K_m(N)$ is $k \neq 0$, Γ_2 has a γ element and $g_i \times h$ has positive scalar curvature on $M \times N$, for $i = 1, 2$.

Then

$$
[g_1 \times h] \neq [g_2 \times h] \in \text{Pos}_{n+m}^{\text{spin}}(M \times N) \otimes \mathbb{Z}\left[\frac{1}{k}\right].
$$

Proof. We can use the arguments we used for Lemma [5.19](#page-19-0) to obtain immediately the result. \Box

5.3 The delocalized APS index Theorem in the odd-dimensional case

Another application of the product formula is the proof of the delocalized APS index theorem for odd dimensional cobordisms.

We will do it for the perturbed signature operator, the theorem for the Dirac operator on a spin manifold with positive scalar curvature is completely analogous.

Because of motivations well explained in [\[17,](#page-24-3) Remark 4.6], we will prove the theorem at the cost of inverting 2. We recall that here and in [\[17\]](#page-24-3) the signature operator on an odd dimensional manifold is not the odd signature operator of Atiyah, Patodi and Singer, but the direct sum of two (unitarily equivalent) versions of this operator.

Since in the statement of the delocalized APS index theorem in the odd dimensional case we will compare the ρ invariant of the boundary with the index of the APS odd signature operator on the cobordism, it is worth to specify the notation we shall follow: on an odd dimensional manifold we denote by D^{APS} the odd signature operator of Atiyah, Patodi and Singer and we denote by D the odd signature operator that we used so far.

The strategy of the proof is to reduce the odd dimensional case to the even dimensional one through the product by the K-homology class of the signature operator on the circle. Then it is useful to review the behavior of the signature operator with respect to cartesian products of manifolds. For a detailed treatment we refer the reader to sections 5 and 6 of [\[31\]](#page-25-8).

Let W be an n-dimensional manifold with boundary ∂W endowed with a cocompact free Γ-action. We assume that *n* is odd and that the boundary of *W* is composed by a pair of homotopy equivalent manifolds. Let $j: \partial W \hookrightarrow W$ and $j': \partial W \times \mathbb{R} \hookrightarrow W \times \mathbb{R}$ be the obvious inclusions. Let us recall some useful facts:

- the even signature operator $D_{W\times S^1}$ is equivalent to the direct sum of two copies of the exterior product $D_W^{APS} \hat{\otimes} 1 + 1 \hat{\otimes} D_{S^1}^{APS}$, see [\[31,](#page-25-8) Section 6.3]. Since D_{S^1} is the sum of two equivalent versions of $D_{S^1}^{APS}$, one has that $D_{W\times S^1}$ is equivalent to $D_W^{APS} \hat{\otimes} 1$ + $1\hat{\otimes}D_{S^1}$. Consequently the higher index of $(D_{W\times S^1}+C_{F\times \text{id}}^{\text{cyl}})^+$ is equal to the class given by the product $\frac{1}{2}[\text{Ind}_{\Gamma}(D_{W}^{APS} + C_{F}^{\text{cyl}})] \boxtimes [D_{S}^{\{1\}}]$, where here $\boxtimes: K_i(C_r^*(\Gamma)) \times K_j(S^1) \rightarrow$ $K_{i+j}(C_r^*(\Gamma\times\mathbb{Z}))$;
- the operator $D_{\partial W\times S^1}^{APS}$ is equivalent to the exterior product of the even dimensional signature operator $D_{\partial W}$ and the odd dimensional signature operator $D_{S_1}^{APS}$, see [\[31,](#page-25-8) Section 6.1]. Thus we obtain that the odd dimensional operator $D_{\partial W \times S^1}$ is equivalent to the exterior product of the even dimensional signature operator $D_{\partial W}$ and the odd dimensional signature operator D_{S^1} . In particular this means that $\varrho(D_{\partial W} + C_{F_{\partial}}) \boxtimes [D_{S^1}]$ is equal to $\varrho(D_{\partial W \times S^1} + C_{F_{\partial} \times \text{id}})$, where here \boxtimes : $S_i^{\Gamma}(\widetilde{W}) \times K_j(S^1) \to S_{i+j}^{\Gamma \times \mathbb{Z}}(\widetilde{W} \times \mathbb{R})$.

Remark 5.25. Notice that, since $D_{S^1}^{APS}$ is nothing else than the Dirac operator on the circle and since D_{S^1} is unitarily equivalent to two copies of $D_{S^1}^{APS}$, its index is two times the generator of $C_r^*(\mathbb{Z})$. Now $KK(C_r^*(\mathbb{Z}), \mathbb{C}) \cong KK(C(S^1), \mathbb{C})$ by Fourier transform and $KK(C(S^1), \mathbb{C}) \cong$ $Hom(K_0(C(S^1)), \mathbb{Z})$, by [\[4,](#page-23-6) Theorem 7.5.5] for instance. So choosing any homomorphism from

 $K_0(C(S^1))$ to Z that sends the index of $D_{S^1}^{APS}$ to 1, we obtain a class $\zeta \in KK(C_r^*(\mathbb{Z}), \mathbb{C})$ that satisfies the assumptions of Lemma [5.19,](#page-19-0) with $n = 2$.

Theorem 5.26. If $i: C^*(\overline{W})^{\Gamma} \hookrightarrow D^*(\overline{W})^{\Gamma}$ is the inclusion and $j_*: D^*(\partial \overline{W})^{\Gamma} \to D^*(\overline{W})^{\Gamma}$ is the map induced by the inclusion j: $\partial \widetilde{W} \hookrightarrow \widetilde{W}$, we have

$$
i_*\left(\frac{1}{2}\mathrm{Ind}_{\Gamma}(D_W^{\mathrm{APS}} + C_F^{\mathrm{cyl}})\right) = j_*\left(\varrho(D_{\partial W} + C_{F_{\partial}})\right) \in K_0(D^*(\widetilde{W})^{\Gamma}) \otimes \mathbb{Z}\left[\frac{1}{2}\right],
$$

where $\frac{1}{2} \text{Ind}_{\Gamma} (D_W^{APS} + C_F^{\text{cyl}}) \in K_0 (C^*(\widetilde{W})^{\Gamma}) \otimes \mathbb{Z} \left[\frac{1}{2} \right]$.

Proof. Let W be as above. Because of Proposition [5.6](#page-14-2) and Lemma [5.9](#page-15-0) we will prove the theorem in the $S^{\Gamma}_{*}(\cdot)$ setting.

 $\mathrm{Let} \ \prod_D : \ S_0^{\Gamma}(\widetilde{W}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \ \rightarrow \ S_0^{\Gamma \times \mathbb{Z}}(\widetilde{W} \times \mathbb{R}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \ \text{and} \ \Pi_C : \ K_1(C^*(\widetilde{W})^{\Gamma}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \ \rightarrow$ $K_1(C^*(\widetilde{W} \times \mathbb{R})^{\mathbb{Z} \times \Gamma}) \otimes \mathbb{Z} \left[\frac{1}{2}\right]$ be the morphism induced by the product with the of class of the signature operator D_{S^1} in $K_1(S^1)$. By Lemma [5.19,](#page-19-0) we have that

$$
i_* \left(\frac{1}{2} \operatorname{Ind}_{\Gamma} (D_W^{APS} + C_F^{\text{cyl}})\right) = j_S(\varrho(D_{\partial W} + C_{F_{\partial}}))
$$
\n(5.2)

holds if and only if

$$
\Pi_D\left(i_*\left(\frac{1}{2}\mathrm{Ind}_{\Gamma}(D_W^{APS} + C_F^{\mathrm{cyl}})\right)\right) = \Pi_D(j_S(\varrho(D_{\partial W} + C_{F_{\partial}})))
$$

holds.

But by Remark [5.13](#page-17-3) it turns out that

$$
\Pi_D\left(i_*\left(\frac{1}{2}\mathrm{Ind}_{\Gamma}(D_W^{APS} + C_F^{\mathrm{cyl}})\right)\right) = i_*\left(\Pi_C\left(\frac{1}{2}\mathrm{Ind}_{\Gamma}(D_W^{APS} + C_F^{\mathrm{cyl}})\right)\right)
$$

and, and by [\[31,](#page-25-8) Section 6.3], that

$$
\Pi_C \left(\frac{1}{2} \text{Ind}_{\Gamma} (D_W^{APS} + C_F^{\text{cyl}}) \right) = \text{Ind}_{\Gamma} (D_{W \times S^1} + C_{F \times \text{id}}^{\text{cyl}}).
$$

Moreover by Lemma [5.14](#page-17-4) it follows that

$$
\Pi_D(j_S(\varrho(D_{\partial W}+C_{F_{\partial}})))=j_S'(\Pi_D(\varrho(D_{\partial W}+C_{F_{\partial}})))
$$

and, by Proposition [5.23,](#page-20-1) that

$$
\Pi_D \left(\varrho(D_{\partial W} + C_{F_{\partial}}) \right) = \varrho(D_{\partial W \times S^1} + C_{F_{\partial} \times \text{id}}).
$$

Thus we have that [\(5.2\)](#page-22-0) holds if and only if

$$
i_* \left(\operatorname{Ind}_{\Gamma} (D^{APS}_{W \times S^1} + C^{cyl}_{F \times \text{id}}) \right) = j'_S \left(\varrho (D_{\partial W \times S^1} + C_{F_{\partial} \times \text{id}}) \right)
$$

holds. But, since $W \times S^1$ is even dimensional, the equality on the right-hand side holds by [3.21](#page-10-1) and the Theorem is proved. \Box

If W is a Spin Riemannian manifold with boundary, such that the metric on the boundary has positive scalar curvature, then we can state the analogous theorem for the ρ invariants associated to Dirac operators.

Theorem 5.27. If $i: C^*(\overline{W})^{\Gamma} \hookrightarrow D^*(\overline{W})^{\Gamma}$ is the inclusion and $j_*: D^*(\partial \overline{W})^{\Gamma} \to D^*(\overline{W})^{\Gamma}$ is the map induced by the inclusion $j : \partial W \hookrightarrow W$, we have

$$
i_*(\text{Ind}_{\Gamma}(\mathcal{D}_W)) = j_*(\varrho(\mathcal{D}_{\partial W})) \in K_0(D^*(\widetilde{W})^{\Gamma}).
$$

Notice that in this case it is not necessary to invert 2. Moreover the proof of the theorem is very similar to the case of the signature operator, but easier because we do not have to perturb the Dirac operator to obtain an invertible operator.

6 Mapping surgery to analysis: the even dimensional case

The extension to the odd dimensional case of the delocalized APS index theorem allows us to state the following result (with proof almost identical to the the one given in the odd dimensional case).

Theorem 6.1. Let N be an n-dimensional closed oriented topological manifold with fundamental group Γ. Assume that $n \geq 5$ is even. Then there is a commutative diagram with exact rows

$$
L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow S^{TOP}(N) \longrightarrow \mathcal{N}^{TOP}(N) \longrightarrow L_n(\mathbb{Z}\Gamma)
$$

\n
$$
\downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad
$$

and through the classifying map u: $N \to B\Gamma$ of the universal cover \tilde{N} of N, we have the analogous commutative diagram that involves the universal Higson-Roe exact sequence

$$
L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow S^{TOP}(N) \longrightarrow \mathcal{N}^{TOP}(N) \longrightarrow L_n(\mathbb{Z}\Gamma)
$$

\n
$$
\downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow_{\text{Pr}} \qquad \qquad \downarrow_{\text{Ind}_{\Gamma}} \qquad \qquad \downarrow
$$

The same is true if we consider the surgery exact sequence for smooth manifolds.

Remark 6.2. Thanks to Theorem [5.27,](#page-22-1) we can enunciate the analogous statement for the Stolz sequence. With the same notations as in subsection [5.2.2,](#page-20-0) we obtain the following commutative diagram

$$
\Omega_{n+1}^{\text{spin}}(M) \longrightarrow R_{n+1}^{\text{spin}}(M) \longrightarrow \text{Pos}_n^{\text{spin}}(M) \longrightarrow \Omega_n^{\text{spin}}(M)
$$

\n
$$
\downarrow \beta \qquad \qquad \downarrow \text{Ind}_{\Gamma} \qquad \qquad \downarrow \varrho \qquad \qquad \downarrow \beta
$$

\n
$$
K_{n+1}(M) \longrightarrow K_{n+1}(C_r^*(\Gamma)) \longrightarrow K_{n+1}(D^*(\widetilde{M})^{\Gamma}) \longrightarrow K_n(M)
$$

with $n \geq 5$ even.

Corollary 6.3. Corollaries [5.21](#page-20-2) and [5.24](#page-21-1) are true irrespective of the dimesions of M and N.

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