NONEXISTENCE OF COUNTABLE EXTREMALLY DISCONNECTED GROUPS WITH MANY OPEN SUBGROUPS

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ABSTRACT. It is proved that the existence of a countable extremally disconnected Boolean topological group containing a family of open subgroups whose intersection has empty interior implies the existence of a rapid ultrafilter.

1. Preliminaries

The problem of the existence in ZFC of a nondiscrete Hausdorff extremally disconnected topological group was posed by Arhangel'skii in 1967 and has been extensively studied since then. Several consistent examples have been constructed [2–7], but Arhangel'skii's problem remains unsolved.

All (consistent) examples of extremally disconnected groups known to the author have a base of neighborhoods of the identity element consisting of subgroups. The main result of this paper is that a countable ZFC example cannot be constructed in this way; to be more precise, we prove that the existence of a countable extremally disconnected Boolean topological group containing a family of open subgroups whose intersection has empty interior (or, equivalently, admitting a continuous isomorphism onto the direct sum $\bigoplus_{\omega} \mathbb{Z}_2$ of countably many copies of \mathbb{Z}_2 with the product topology) implies the existence of a rapid ultrafilter. Thus, if there exists in ZFC a countable nondiscrete extremally disconnected group, then there must exist such a group without open subgroups.

Recall that a topological space is said to be *extremally disconnected* if the closure of any open set in this space is open (or, equivalently, the closures of two disjoint open sets are disjoint). Malykhin [4] showed that any extremally disconnected topological group must contain an open (and therefore closed) Boolean subgroup (i.e., a subgroup consisting of elements of order 2). Thus, in studying the existence of extremally disconnected groups, it suffices to consider Boolean groups.

Any countable Boolean group is a countable-dimensional vector space over the field \mathbb{Z}_2 and, therefore, can be represented as the set of finitely supported maps $g: \omega \to \mathbb{Z}_2$ (i.e., is isomorphic to the direct sum $\bigoplus_{\omega} \mathbb{Z}_2$ of countably many copies of \mathbb{Z}_2). Any such isomorphic representation $\varphi: G \to \bigoplus_{\omega} \mathbb{Z}_2$ is determined by the choice of a basis $E = \{e_n : n \in \omega\}$ in G for which

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 $\varphi(e_i) = (\underbrace{0, \ldots, 0}_{i \text{ times}}, 1, 0, 0 \ldots).$ We assume $\bigoplus_{\omega} \mathbb{Z}_2$ to be endowed with the prod-

uct topology. Thus, each isomorphism $\varphi \colon G \to \bigoplus_{\omega} \mathbb{Z}_2$ induces a topology on G, which we refer to as the *product topology induced* by φ (or *associated* with the corresponding basis E).

Given an isomorphism $\varphi \colon G \to \bigoplus_{\omega} \mathbb{Z}_2$, we can treat elements of $G \cong \bigoplus_{\omega} \mathbb{Z}_2$ as maps to \mathbb{Z}_2 finitely supported on ω , so that the maps

$$\min: G \to \omega, \qquad \min g = \min \operatorname{supp} g,$$

and

$$\max: G \to \omega, \qquad \max g = \max \operatorname{supp} g$$

are defined.

We denote the zero element of G by **0**.

All topological groups under consideration are assumed to be Hausdorff.

Given $k, n \in \omega$, we write [k, n] for $\{i \in \omega : k \leq i \leq n\}$. The sign \bigsqcup denotes disjoint union. Thus, $S = \bigsqcup_{n \in \omega} S_n$ means that the S_i form a disjoint partition of S.

An ultrafilter \mathcal{U} on ω is a *P*-point ultrafilter if, given any partition $\omega = \bigcup_{n \in \omega} A_n$ with $A_n \notin \mathcal{U}$, $n \in \omega$, there exists an $A \in \mathcal{U}$ such that $|A \cap A_n| < \aleph_0$ for any *n*. The nonexistence of *P*-point ultrafilters is consistent with ZFC [8].

An ultrafilter \mathcal{U} on ω is *selective*, or *Ramsey*, if, given any partition $\omega = \bigcup_{n \in \omega} A_n$ with $A_n \notin \mathcal{U}$, $n \in \omega$, there exists an $A \in \mathcal{U}$ such that $|A \cap A_n| \leq 1$ for any n.

An ultrafilter \mathcal{U} on ω is said to be *rapid* if every function $\omega \to \omega$ is majorized by the increasing enumeration of some element of \mathcal{U} .

Any Ramsey ultrafilter is rapid (and a P-point). The nonexistence of rapid ultrafilters is consistent with ZFC [9].

Note that if X and Y are sets, \mathcal{U} is an ultrafilter on X, and $f: X \to Y$ is a map, then the family $\mathcal{V} = \{A \subset Y : f^{-1}(A) \in \mathcal{U}\}$ is an ultrafilter on f(X). We denote it by $f(\mathcal{U})$.

2. Statements

The main result of this paper is the following theorem.

Theorem. The existence of a countable extremally disconnected Boolean topological group containing a family of open subgroups whose intersection has empty interior implies the existence of a rapid ultrafilter.

Corollary 1. The nonexistence of a countable extremally disconnected Boolean topological group containing a family of open subgroups whose intersection has empty interior is consistent with ZFC.

The proof of the theorem uses the following two lemmas.

Lemma 1. Suppose that G is a countable Boolean topological group which can be represented as $G \cong \bigoplus_{\omega} \mathbb{Z}_2$ in such a way that

for any function
$$f: \omega \to \omega$$
, the set
 $U_f = \{x \neq \mathbf{0} : \max x > f(\min x)\} \cup \{\mathbf{0}\}$ (1)
contains an open neighborhood of $\mathbf{0}$.

Let $f: \omega \to \omega$ be any increasing function, and let V be a neighborhood of zero such that $2V \subset U_g$ for $g: \omega \to \omega$ defined by $g(n) = f(2^{n+1})$. Then the increasing enumeration $\{m_0, m_1, \ldots\}$ of $\max(V \setminus \{\mathbf{0}\})$ majorizes f, i.e., $m_i > f(i)$ for all $i \in \omega$.

Lemma 2. Let G be a countable nondiscrete Boolean topological group which contains a (countable) decreasing family of open subgroups $G_i \subset G$, $i \in \omega$, with $\bigcap G_i = \{\mathbf{0}\}$. Then there exists a continuous isomorphism $\varphi \colon G \to \bigoplus_{\omega} \mathbb{Z}_2$, *i.e.*, an isomorphism φ such that the product topology on G induced by φ is contained in the given topology.

These lemmas also imply the following assertions.

Corollary 2. The existence of a countable nondiscrete group G satisfying condition (1) for some representation $G \cong \bigoplus_{\omega} \mathbb{Z}_2$ implies the existence of a rapid ultrafilter.

Proof. Any nonprincipal ultrafilter \mathcal{U} on G converging to zero contains all neighborhoods of zero. Hence $\max \mathcal{U}$ contains $\max V$ for any neighborhood V of zero and is therefore rapid.

It is easy to see that, for any countable Boolean group G with a fixed isomorphism $\varphi: G \to \bigoplus_{\omega} \mathbb{Z}_2$ and any map $f: \omega \to \omega$, the set

 $C_f = G \setminus U_f = \{x \in G \setminus \{\mathbf{0}\} : \max x \le f(\min x)\}$

is discrete in the product topology induced by φ , and $C_f \cup \{\mathbf{0}\}$ is closed in this topology. This, together with Lemma 2, implies the following corollary, which gives a partial answer to Protasov's question on the existence in ZFC of a countable nondiscrete topological group in which all discrete subsets are closed (see [10, Chapter 13, Question 16].

Corollary 3. Let (G, τ) be a countable nondiscrete Boolean topological group. Suppose that G has no discrete subsets with a unique limit point and contains a family of open subgroups $G_i \subset G$, $i \in \omega$, such that $\bigcap G_i = \{\mathbf{0}\}$. Then there exists a rapid ultrafilter.

In the next corollary, by a maximal nondiscrete group topology we mean a nondiscrete group topology which is maximal among all nondiscrete group topologies.

Corollary 4. The following statement is consistent with ZFC: Any countable Boolean group with a maximal nondiscrete group topology contains a discrete subset with a unique limit point. *Proof.* This statement is true in any model of ZFC containing no rapid ultrafilters. Indeed, let G be a countable Boolean group with a maximal nondiscrete group topology τ . Choose any basis E in the vector space G. If τ contains the product topology \mathbb{T} associated with E, then we can apply Corollary 3. If $\tau \not\supseteq \mathbb{T}$, then the maximality of τ implies the existence of neighborhoods of zero U_0 in τ and V_0 in \mathbb{T} with $U_0 \cap V_0 = \{\mathbf{0}\}$. For each $f: \omega \to \omega$, all sets $\{x \in C_f : \min x \leq n\}$ are finite and, hence,

$$\{U \cap C_f : \mathbf{0} \in U \in \tau\} \supset \{V \cap C_f : \mathbf{0} \in V \in \mathbb{T}\}.$$

Consequently, G satisfies condition (1) with $V = U_0$, and Corollary 2 applies.

3. Proofs

The short proof of Lemma 1 presented below was found and kindly communicated to the author by E. A. Reznichenko.

Proof of Lemma 1. Note that if $X \subset G \setminus \{0\}$ and $|X| > 2^k$, then there are $x, y \in X$ for which $\min(x + y) > k$: it suffices to take any x and y whose first k coordinates coincide. This observation was essentially made in [10, Proof of Theorem 5.19, Case 2].

Let $\{m_0, m_1, \ldots\}$ be the increasing enumeration of $\max(V \setminus \{\mathbf{0}\})$. For each $i \in \omega$, choose $x_i \in V$ with $\max x_i = m_i$. We have $m_0 > g(\min x_0) > f(\min x_0) \ge f(0)$. Let us show that $m_n > f(n)$ for any n > 0. Take k for which $2^k < n \le 2^{k+1}$. It follows from the above observation that $\min(x_i + x_j) > k$ for some $i < j \le n$. Clearly, $\max x_j = m_j > m_i = \max x_i$ implies $\max(x_i + x_j) = \max x_j$. We have

$$f(n) \le f(2^{k+1}) = g(k) \le g(\min(x_i + x_j)) < \max(x_i + x_j) = m_j \le m_n.$$

Proof of Lemma 2. We treat G as a vector space over the field \mathbb{Z}_2 and the G_i as its subspaces. Obviously, to prove the lemma, it suffices to construct a basis $E = \{e_n : n \in \omega\}$ such that, for every $i \in \omega$, there exists a $J_i \subset \omega$ for which $G_i = \langle e_n : n \in J_i \rangle$. Indeed, if E is such a basis, then the assumption $\bigcap G_i = \{\mathbf{0}\}$ implies $\bigcap J_i = \emptyset$, and all linear spans $\langle e_k : k \geq n \rangle$ (which form a base of neighborhoods of zero in the product topology associated with the basis E) are open as subgroups with nonempty interior.

In each (nontrivial) quotient space G_i/G_{i+1} , we take a basis $\{\varepsilon_{\alpha} : \gamma \in I_i\}$, where $|I_i| = \dim G_i/G_{i+1}$, and let e_{α} be representatives of ε_{γ} in G_i . We assume the (at most countable) index sets I_i to be well ordered and disjoint, let $I = \bigcup_{i \in \omega} I_i$, and endow I with the lexicographic order (for $\alpha, \beta \in I$, we say that $\alpha < \beta$ if $\alpha \in I_i, \beta \in I_j$, and either i < j or i = j and $\alpha < \beta$ in I_i). For any $i \in \omega$ and $\alpha \in I_i$, we define H_{α} to be the subspace of G spanned by $\{e_{\beta} : \alpha \in I_i, \beta \geq \alpha\}$ and G_{i+1} . Thus, H_{β} is defined for each $\beta \in I$; moreover, if $\beta, \gamma \in I$ and $\beta < \gamma$ in I, then $H_{\beta} \supset H_{\gamma}$, and if α is the least element of I_i , then $H_{\alpha} = G_i$. This means that the subspaces H_{α} form a decreasing (with respect to the order induced by I) chain of subspaces refining the chain $G_0 \supset G_1 \supset \ldots$. Note that the lexicographic order on I is a well-order, so for each $\alpha \in I$, its immediate successor $\alpha + 1$ is defined; by construction, we have $\dim H_{\alpha}/H_{\alpha+1} = 1$ for every $\alpha \in I$.

Clearly, it suffices to construct a basis $E' = \{e'_{\alpha} : \alpha \in I\}$ with the property

$$e'_{\alpha} \in H_{\alpha} \setminus H_{\alpha+1}$$
 for every $\alpha \in I;$ (*)

the required basis E is then obtained by reordering E'. Moreover, property (\star) ensures the linear independence of E', so it is sufficient to construct a set of vectors spanning G with this property.

Take any basis $E'' = \{e''_n : n \in \omega\}$ in G. We construct E' by induction on n. Let α_0 be the (unique) element of I for which $e''_0 \in H_{\alpha_0} \setminus H_{\alpha_0+1}$. We set $e'_{\alpha_0} = e''_0$.

Suppose that k is a positive integer and we have already defined elements $\alpha_i \in I$ and vectors e'_{α_i} for i < k so that $e'_{\alpha_i} \in H_{\alpha_i} \setminus H_{\alpha_i+1}$ and $\langle e'_{\alpha_0}, \ldots, e'_{\alpha_{k-1}} \rangle = \langle e''_0, \ldots, e''_{k-1} \rangle$. Take the (unique) $\alpha \in I$ for which $e''_k \in H_\alpha \setminus H_{\alpha+1}$. If α is unoccupied, that is, $\alpha \neq \alpha_i$ for any i < k, then we set $\alpha_k = \alpha$ and $e'_{\alpha_k} = e''_k$. If α is already occupied, i.e., $\alpha = \alpha_m$ for some m < k, then we take the (unique) $\beta \in I$ for which $e''_k + e'_{\alpha_m} \in H_\beta \setminus H_{\beta+1}$. (Clearly, $\beta > \alpha$, because $\dim H_\alpha/H_{\alpha+1} = 1$ and $e''_k, e'_{\alpha_m} \in H_\alpha \setminus H_{\alpha+1}$). If β is unoccupied, then we set $\alpha_k = \beta$ and $e'_{\alpha_k} = e''_k + e'_{\alpha_m}$; if $\beta = \alpha_l$ for l < k, then we take the (unique) $\gamma \in I$ for which $e''_k + e'_{\alpha_m} \in H_\gamma \setminus H_{\gamma+1}$ (clearly, $\gamma > \beta > \alpha$), and so on. Only finitely many (k-1) indices from I are occupied; therefore, after finitely many steps, we obtain $e''_k + e'_{\alpha_m} + e'_{\alpha_l} + \cdots + e'_{\alpha_s} \in H_\delta \setminus H_{\delta+1}$ for an unoccupied index δ . We set $\alpha_k = \delta$ and $e'_{\alpha_k} = e''_k + e'_{\alpha_m} + e'_{\alpha_l} + \cdots + e'_{\alpha_s} + e'_{\alpha_l} + \cdots + e'_{\alpha_s}$.

many steps, we obtain $e_k'' + e_{\alpha_m}' + e_{\alpha_l}' + \dots + e_{\alpha_s}' \in H_{\delta} \setminus H_{\delta+1}$ for an unoccupied index δ . We set $\alpha_k = \delta$ and $e_{\alpha_k}' = e_k'' + e_{\alpha_m}' + e_{\alpha_l}' + \dots + e_{\alpha_s}'$. As a result, we obtain a set of vectors $E' = \{e_{\alpha_n}' : n \in \omega\}$ such that $e_{\alpha_n}' \in H_{\alpha_n} \setminus H_{\alpha_n+1}$ and $\langle e_{\alpha_0}', \dots, e_{\alpha_n}' \rangle = \langle e_0'', \dots, e_n'' \rangle$ for every $n \in \omega$. The latter means that E' spans G, because so does the basis E''.

Formally, it may happen that not all of the indices $\alpha \in I$ are occupied, that is, $\{\alpha_n : n \in \omega\} = J \subsetneq I$. In this case, we take arbitrary vectors $e'_{\alpha} \in H_{\alpha} \setminus H_{\alpha+1}$ for $\alpha \in I \setminus J$ and put them to E'. The set E' thus enlarged satisfies condition (\star) and is therefore linearly independent; thus, it cannot differ from the initial E', because the latter spans G, and J in fact coincides with I, i.e., $E' = \{e'_{\alpha_n} : n \in \omega\} = \{e'_{\alpha} : \alpha \in I\}$. This completes the proof of the lemma. \Box

Proof of the theorem. Let G be a countable extremally disconnected Boolean group, and let $G_i \subset G$, $i \in \omega$, be its open (and hence closed) subgroups such that the intersection $\bigcap_{i\in\omega} G_i$ has empty interior. Then the quotient of G by $\bigcap_{i\in\omega} G_i$ is Hausdorff, nondiscrete, and extremally disconnected (being an open image of an extremally disconnected group). Thus, we can assume without loss of generality that $\bigcap_{i\in\omega} G_i = \{\mathbf{0}\}$. We can also assume that $G_{i+1} \subset G_i$. According to Lemma 2, there exists a continuous isomorphism $\varphi: G \to \bigoplus_{\omega} \mathbb{Z}_2$.

If G satisfies condition (1) for min and max associated with this isomorphism, then the existence of a rapid ultrafilter follows by Lemma 1. Suppose that G does not satisfy (1), i.e., there exists a function $f: \omega \to \omega$ such that **0**

is a limit point of C_f , where

$$C_f = U_f \setminus \{\mathbf{0}\} = \{x \in G \setminus \{\mathbf{0}\} : \max x \le f(\min x)\}.$$

Let us enumerate C_f as $\{c_0, c_1, \dots\}$ in such a way that

$$\min c_k < \min c_n \implies k < n. \tag{2}$$

This can be done, because all sets of the form $\{x \in C_f : \min x = n\}$ are finite. For each $n \in \omega$, we set

$$U_{c_n} = \{ x \in G : \min x > f(\min c_n) \} \cup \{ \mathbf{0} \}.$$

All U_{c_n} are open neighborhoods of zero in the product topology on G induced by φ (and, therefore, in the topology of G). Clearly,

$$(c_k + U_{c_k}) \cap (c_n + U_{c_n}) = \emptyset$$
 for $k \neq n$.

Thus, in Zelenyuk's terminology [11], C_f is a strongly discrete set in the extremally disconnected group G, and according to [11, Lemma 2], the subsets of C_f containing **0** in their closures form an ultrafilter on C_f . Let us denote it by \mathcal{U} . In the proof of Theorem 3 of [11] Zelenyuk showed that this ultrafilter is partially selective. This means that there exists a map $u: C_f \to \mathcal{U}$ such that, given any partition $C_f = \bigsqcup_{i \in \omega} A_i$ with $A_i \notin \mathcal{U}$, there exists an $A \in \mathcal{U}$ for which $A \cap A_i \cap u(x) = \emptyset$ whenever $x \in A \cap A_i$. (Formally, Zelenyuk's definition of a partially selective ultrafilter differs from that given above, but it is easy to see that these definitions are equivalent.) Substituting $D = C_f$ in Zelenyuk's proof of his Theorem 3 in [11] (no other substitutions or changes are needed), we see that, for any partition $C_f = \bigsqcup_{i \in \omega} A_i$ with $A_i \notin \mathcal{U}$, there exists an $A \in \mathcal{U}$ such that, for any $i \in \omega$, we have $A \cap A_i \cap U_c = \emptyset$ whenever $c \in A \cap A_i$ (in our terminology, this means that the partial selectivity of \mathcal{U} is witnessed by the map $c \mapsto U_c$.)

Note that, by virtue of the condition (2) on the enumeration of C_f , we have

$$U_{c_n} \cap C_f = \{c_i : i \ge \min\{j : \min c_j > f(\min c_n)\}\}$$

for each n. Bearing this in mind and using the natural bijection $C_f \rightleftharpoons \omega$ defined by $c_n \mapsto n$, we can summarize the preceding paragraph as follows. To each $n \in \omega$ we can assign an $m_n \in \omega$ so that there exists an ultrafilter \mathcal{V} on ω with the following property:

for any partition
$$\omega = \bigsqcup_{i \in \omega} A_i$$
 with $A_i \notin \mathcal{V}$,
there exists an $A \in \mathcal{V}$ such that, for any $i \in \omega$,
 $n \in A \cap A_i$ implies $A \cap A_i \cap \{j : j \ge m_n\} = \emptyset$. (3)

This means, in particular, that \mathcal{V} is a *P*-point ultrafilter. It remains to apply the following lemma.

Lemma 3. For any ultrafilter \mathcal{U} on ω satisfying condition (3) for some correspondence $n \mapsto m_n$, there exists a map $\Phi: \omega \to \omega$ such that $\Phi(\mathcal{U})$ is Ramsey.

Proof. We set $M_0 = m_0$, $M_1 = m_{m_0}$, $M_2 = m_{m_m}$, and so on; for positive $n \in \omega$, we set

$$M_n = m_{M_{n-1}},$$
 i.e., $M_n = m_{\underbrace{m_{m \dots m_0}}_{n \text{ times}}}.$

Without loss of generality, we can assume that the m_n strictly increase. In this case, the M_n strictly increase as well. The required map Φ can be defined as follows: given an integer $i \in \omega$, we find the interval $[M_n, M_{n+1} - 1]$ to which it belongs and set $\Phi(i) = n$. The map Φ thus defined contracts each interval of the form $[M_n, M_{n+1} - 1]$ to the point n and is surjective.

Let us show that the ultrafilter $\Phi(\mathcal{U})$ is Ramsey. Consider a partition $\omega = \bigsqcup_{i \in \omega} B_i$ with $B_i \notin \Phi(\mathcal{U})$. We must choose $B \in \Phi(\mathcal{U})$ so that $|B \cap B_i| \leq 1$ for all $i \in \omega$. Let $A_i = \Phi^{-1}(B_i)$. We have $A_i \notin \mathcal{U}$ and $\omega = \bigsqcup_i A_i$. Take $A \in \mathcal{U}$ as in (3), i.e., such that, for any $i \in \omega$, $n \in A \cap A_i$ implies $A \cap A_i \cap \{j : j \geq m_n\} = \emptyset$. Since \mathcal{U} is an ultrafilter, we can choose A so that either

$$A \cap \bigcup \{ [M_n, M_{n+1} - 1] : n \text{ is odd} \} = \emptyset$$

or

$$A \cap \bigcup \{ [M_n, M_{n+1} - 1] : n \text{ is even} \} = \emptyset.$$

Suppose for definiteness that A satisfies the former condition. Let us show that $|\Phi(A) \cap B_i| \leq 1$ for all $i \in \omega$. Consider any nonempty intersection $\Phi(A) \cap B_i$. Let n be its least element. We have $n = \Phi(k)$ for some $k \in A \cap A_i$. By assumption, $A \cap A_i \cap \{j : j \geq m_k\} = \emptyset$. We have: $k \in [M_n, M_{n+1} - 1]$ (because $\Phi(k) = n$), n is even (by the choice of A), $A \cap [M_{n+1}, M_{n+2} - 1] = \emptyset$, and $m_k \in [M_{n+1}, M_{n+2} - 1]$ (because $k < M_{n+1}$ and $k \geq M_n$). Hence $A \cap A_i \cap \{j : j \geq M_{n+1}\} = \emptyset$, and

 $\Phi(A) \cap \Phi(A_i) \cap \Phi(\{j : j \ge M_{n+1}\}) = \Phi(A) \cap B_i \cap \{j : j \ge n+1\} = \emptyset.$ This means that $\Phi(A) \cap B_i = \{n\}.$

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