Cayley-Bacharach Formulas

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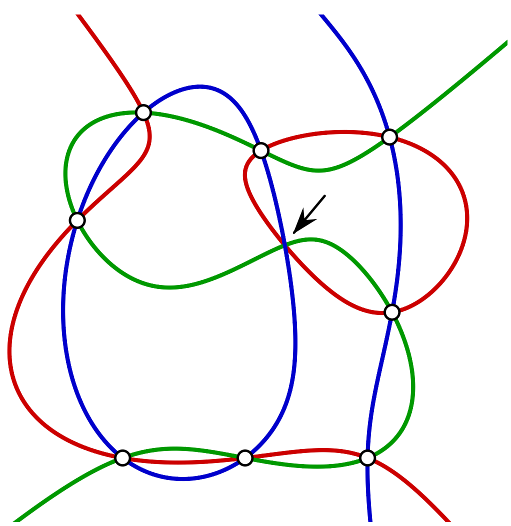
Abstract

The Cayley-Bacharach Theorem states that all cubic curves through eight given points in the plane also pass through a unique ninth point. We write that point as an explicit rational function in the other eight.

1 Introduction

This note concerns the following result from classical algebraic geometry.

Theorem 1 (Cayley-Bacharach). *Let $P_1, \ldots, P_8$ be eight distinct points in the plane, no three on a line, and no six on a conic. There exists a unique ninth point P_9 such that every cubic curve through $P_1, \ldots, P_8$ also contains P_9.*



All cubics passing through the eight white points meet in a unique ninth point

This result refers to the projective plane $\mathbb{P}^2$. It appears in most textbooks on plane algebraic curves. For instance, Kirwan asks for a proof in [9, Exercise 3.13]. Theorem 1 dates back to classical 19th century work of Hart [6], Weddle [18], Chasles [3], Cayley [2] and others. While the 1851 articles of Hart and Weddle are mainly focused on geometric constructions for the ninth point, Cayley's 1862 article is more algebraic and gives a complete proof.

In this paper we present explicit formulas for the Cayley-Bacharach point in terms of algebraic invariants of the other eight points. Our motivation arose from computational projective geometry [13]. The aim was to devise numerically stable schemes for plotting P_9 when eight points $P_1, \ldots, P_8$ move in animations of the Cayley-Bacharach Theorem created with `Cinderella` [14]. The formulas displayed in (2), (11) and (13) are useful for that purpose.

In what follows we present our first formula. In Section 2 we offer two proofs. The first exposits Cayley's arguments in [2], while the second is a verification using modern computer algebra. In Section 3 we present our second formula. That one is optimal with respect to degree and symmetry. In Section 4 we close with a discussion on related issues and further reading.

We write the *Cayley-Bacharach point* P_9 as a rational expression in terms of

$$P_1 = (x_1 : y_1 : z_1),\ P_2 = (x_2 : y_2 : z_2),\ \ldots,\ P_8 = (x_8 : y_8 : z_8).$$

Such a formula exists because of the following argument. Consider the linear system of cubic curves through $P_1, P_2, \ldots, P_8$. Its dimension is at least #degrees of freedom − #constraints = $10-8 = 2$. Choose two distinct cubics C_1 and C_2 in that system. Let $P_9 = (x_9 : y_9 : z_9)$ be their 9th intersection point. In light of Theorem 1, the point P_9 depends only on $P_1, P_2, \ldots, P_8$. From this one finds that the Cayley-Bacharach Theorem holds over any field.

Remark 2. *The quotients y_9/x_9 and z_9/x_9 can be written as rational functions in the 24 unknowns $x_1, y_1, z_1, \ldots, y_8, z_8$. The numerators and denominators of these rational functions are polynomials with integer coefficients.*

We now define some polynomials that serve as ingredients in our formulas. The condition for three points to lie on a line is the cubic polynomial

$$[123] \;=\; \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}.$$

The condition for six points to lie on a conic is given by the polynomial

$$\begin{aligned} C(P_1, P_2, \ldots, P_6) \quad &= \quad \det \begin{pmatrix} x_1^2 & x_1y_1 & x_1z_1 & y_1^2 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & x_2z_2 & y_2^2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & x_3z_3 & y_3^2 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & x_4z_4 & y_4^2 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & x_5z_5 & y_5^2 & y_5z_5 & z_5^2 \\ x_6^2 & x_6y_6 & x_6z_6 & y_6^2 & y_6z_6 & z_6^2 \end{pmatrix} \\ &= \quad [123][145][246][356] - [124][135][236][456]. \end{aligned} \tag{1}$$

Here is one more geometric condition of interest to us: eight points lie on a cubic curve that is singular at the first point. This condition is expressed by a polynomial of degree $7 \cdot 3 + 3 \cdot 2 = 27$, namely $D(P_1; P_2, \ldots, P_8) =$

$$\det \begin{pmatrix} x_2^3 & x_2^2y_2 & x_2^2z_2 & x_2y_2^2 & x_2y_2z_2 & x_2z_2^2 & y_2^3 & y_2^2z_2 & y_2z_2^2 & z_2^3 \\ x_3^3 & x_3^2y_3 & x_3^2z_3 & x_3y_3^2 & x_3y_3z_3 & x_3z_3^2 & y_3^3 & y_3^2z_3 & y_3z_3^2 & z_3^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_8^3 & x_8^2y_8 & x_8^2z_8 & x_8y_8^2 & x_8y_8z_8 & x_8z_8^2 & y_8^3 & y_8^2z_8 & y_8z_8^2 & z_8^3 \\ 3x_1^2 & 2x_1y_1 & 2x_1z_1 & y_1^2 & y_1z_1 & z_1^2 & 0 & 0 & 0 & 0 \\ 0 & x_1^2 & 0 & 2x_1y_1 & x_1z_1 & 0 & 3y_1^2 & 2y_1z_1 & z_1^2 & 0 \\ 0 & 0 & x_1^2 & 0 & x_1y_1 & 2x_1z_1 & 0 & y_2^2 & 2y_1z_1 & 3z_1^2 \end{pmatrix}.$$

In these formulas we can change the indices. For any $i, j, \ldots, k \in \{1, 2, \ldots, 8\}$, the expressions $[ijk]$, $C(P_i, P_j, \ldots, P_k)$ and $D(P_i; P_j, \ldots, P_k)$ are well-defined homogeneous polynomials with integer coefficients in 24 unknowns x_i, y_j, z_k.

To state the first main result of this note, we abbreviate

$$\begin{aligned} C_x &= C(P_1, P_4, P_5, P_6, P_7, P_8), & C_y &= C(P_2, P_4, P_5, P_6, P_7, P_8), \\ C_z &= C(P_3, P_4, P_5, P_6, P_7, P_8), & D_x &= D(P_1; P_2, P_3, P_4, P_5, P_6, P_7, P_8), \\ D_y &= D(P_2; P_3, P_1, P_4, P_5, P_6, P_7, P_8), & D_z &= D(P_3; P_1, P_2, P_4, P_5, P_6, P_7, P_8). \end{aligned}$$

Theorem 3. *The Cayley-Bacharach point is given by the formula*

$$P_9 \;=\; C_xD_yD_z \cdot P_1 \;+\; D_xC_yD_z \cdot P_2 \;+\; D_xD_yC_z \cdot P_3. \tag{2}$$

Equivalently, the coordinates of P_9 are the rational functions

$$\begin{aligned} x_9 &= C_xD_yD_zx_1 + D_xC_yD_zx_2 + D_xD_yC_zx_3, \\ y_9 &= C_xD_yD_zy_1 + D_xC_yD_zy_2 + D_xD_yC_zy_3, \\ z_9 &= C_xD_yD_zz_1 + D_xC_yD_zz_2 + D_xD_yC_zz_3. \end{aligned} \tag{3}$$

The following identity allows us to write the coefficients in (2) in terms of the brackets $[ijk]$. This can be verified using a computer algebra system.

$$\begin{aligned} D(P_7; P_1, P_2, P_3, P_4, P_5, P_6, P_8) \ = \\ 3 \cdot (& [647][857][478][128][173][423][573][526][176] \\ & -[647][857][473][428][178][123][573][526][176] \\ & +[647][857][473][428][178][576][126][173][523] \\ & +[657][847][573][528][178][123][473][426][176] \\ & -[657][847][578][128][173][523][473][426][176] \\ & -[657][847][573][528][178][476][126][173][423]). \end{aligned} \tag{4}$$

The following lemma is implied by the bracket expansions in (1) and (4).

Lemma 4. *Let T be a projective transformation on $\mathbb{P}^2$, expressed as a 3×3 matrix that acts on the homogeneous coordinates of the points P_i. Then*

$$\begin{aligned} C(T(P_1), T(P_2), \ldots, T(P_6)) &= \det(T)^4 \cdot C(P_1, P_2, \ldots, P_6), \\ D(T(P_1); T(P_2), \ldots, T(P_8)) &= \det(T)^9 \cdot D(P_1; P_2, \ldots, P_8). \end{aligned}$$

In the next section we shall present two proofs of Theorem 3.

2 From Cayley to Computer Algebra

In his 1862 paper [2], Cayley describes a geometric construction for expressing P_9 rationally in $P_1, \ldots, P_8$. The key step is an implicit characterization of P_9 in terms of certain cross ratios. We set $[\![123456]\!] = C(P_1, P_2, \ldots, P_6)$ and

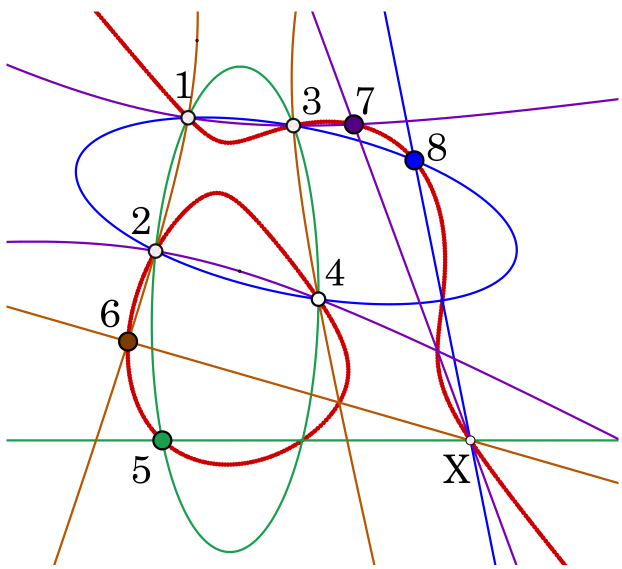
$$(1,2,3,4)_5 := \frac{[513][524]}{[514][523]} \quad \text{and} \quad (1,2,3,4)_{5678} := \frac{[\![567813]\!][\![567824]\!]}{[\![567814]\!][\![567823]\!]}.$$

The first expression is the cross ratio of the four lines spanned by P_5 and one of P_1, P_2, P_3 or P_4. The second expression is the *cross ratio of four conics* passing through P_5, P_6, P_7, P_8 and one of the points P_1, P_2, P_3, P_4. It is also the cross ratio of the four tangents at any of the intersection points.

First Proof of Theorem 3. Cayley characterizes the point P_9 by the identity

$$(5,6,7,8)_9 \ = \ (5,6,7,8)_{1234}. \tag{5}$$

We shall prove this identify and then derive Theorem 3 from it. Let $C_{\lambda,\mu} = \lambda C_1 + \mu C_2$ denote the pencil of conics through the points P_1, P_2, P_3, P_4, and let $L_{\lambda,\mu} = \lambda L_1 + \mu L_2$ be the pencil of lines through an auxiliary point X. The intersection of these two pencils, as $(\lambda : \mu)$ runs through $\mathbb{P}^1$, is the cubic curve defined by $C_1 L_2 = C_2 L_1$. This cubic contains the points P_1, P_2, P_3, P_4, X. If we identify the pencil $C_{\lambda,\mu}$ with $\mathbb{P}^1$ via coordinates $(\lambda : \mu)$ then one can verify that the cross ratio of the four conics $C_{\lambda,\mu}$ through P_5, P_6, P_7, P_8 equals $(5,6,7,8)_{1234}$. Similarly the cross ratio of the four lines in $L_{\lambda,\mu}$ through these points is $(5,6,7,8)_X$. Hence if P_5, P_6, P_7, P_8 are chosen on the cubic then $(5,6,7,8)_{1234} = (5,6,7,8)_X$. Expanding this equation reveals that X lies on a certain conic that passes through P_5, P_6, P_7, P_8. This conic is specified by the condition $(5,6,7,8)_X = l$, for some constant l. For each X on this conic in general position, with proper choice of C_1, C_2, L_1, L_2, we recover a cubic that passes through $P_1, \ldots, P_8$. In fact, it is the set of intersections of conics and lines that have identical cross ratios with respect to P_5, P_6, P_7 in the above sense. Every cubic that passes through $P_1, \ldots, P_8$ arises this way. Note that this point-cubic correspondence depends only on $P_1, \ldots, P_7$.



Construction of cubic curves. Corresponding lines and conics are drawn in the same color.

Consider the unique cubic through nine points $P_1, \ldots, P_9$ in general position. It arises by applying the previous construction to any eight of them. The corresponding point X is in the intersection of the two conics $\mathcal{A} = \{X : (5,6,7,8)_{1234} = (5,6,7,8)_X\}$ and $\mathcal{B} = \{X : (5,6,7,9)_{1234} = (5,6,7,9)_X\}$. The other intersections are P_5, P_6, P_7, so the point X is uniquely specified.

Now assume that P_9 is the Cayley-Bacharach point of the other eight. Then there is no unique cubic through $P_1, \ldots, P_9$. The cubics passing through $P_1, \ldots, P_8$ are exactly the same as the cubics passing through $P_1, \ldots, P_7, P_9$. In the sense of the above point-cubic correspondence, that means the two conics $\mathcal{A}$ and $\mathcal{B}$ must coincide. Hence P_9 lies on the conic $\mathcal{A} = \{X : (5,6,7,8)_{1234} = (5,6,7,8)_X\}$. We conclude that Cayley's condition (5) holds.

We next derive Theorem 3 from (5). Suppose that $P_1, \ldots, P_8$ are given. By symmetry, the Cayley-Bacharach point P_9 satisfies the two equations

$$(5,6,7,8)_9 = (5,6,7,8)_{1234} =: l \;\text{ and }\; (4,6,7,8)_9 = (4,6,7,8)_{1235} =: m. \quad (6)$$

Under the non-degeneracy assumption $[678] \neq 0$, we can write $P_9 = aP_6 + bP_7 + cP_8$. We regard $(a : b : c)$ as homogeneous coordinates on $\mathbb{P}^2$. Inserting this expression for P_9 into $l = (5,6,7,8)_9$ creates the formula

$$l \;=\; \frac{[957][968]}{[958][967]} \;=\; \frac{(a[657] + c[857])b[768]}{(a[658] + b[758])c[867]}.$$

This can be simplified to

$$[657] \cdot ab \,+\, l[658] \cdot ac \,+\, (1-l)[857] \cdot bc \;=\; 0.$$

Similarly, inserting $P_9 = aP_6 + bP_7 + cP_8$ into $m = (4,6,7,8)_9$ leads to

$$[647] \cdot ab \,+\, m[648] \cdot ac \,+\, (1-m)[847] \cdot bc \;=\; 0.$$

These two quadratic equations have four solutions in $\mathbb{P}^2$. Three of them are (1:0:0), (0:1:0) and (0:0:1), corresponding to our basis points P_6, P_7 and P_8. The fourth solution $(a : b : c)$ gives the Cayley-Bacharach point P_9. It equals

$$\Big((-[647][857](l-1) + [657][847|(m-1))([658][847]l(m-1) - [648][857](l-1)m) :$$
$$-([647][658]l - [648][657]m)([658][847]l(m-1) - [648][857](l-1)m) :$$
$$-([647][658]l - [648][657]m)([647][857](l-1) + [657][847] - [847]m))\Big).$$

We now replace l and m in this expression by the right hand sides in (6). After clearing denominators, expanding, dividing by common factors, and rewriting bracket monomials, we arrive at the formula (2) for P_9. □

Theorem 3 can also be proved directly, by clever use of computer algebra.

Second Proof of Theorem 3. The ring $\mathbb{Z}[x.,y.,z.]$ is $\mathbb{Z}^8$-graded with $\deg(x_i) = \deg(y_i) = \deg(z_i) = \mathbf{e}_i$. The right hand side of (2) is a vector of homogeneous polynomials of multidegree $(9,9,9,8,8,8,8,8)$. By Lemma 4, it is equivariant under projective transformations on $\mathbb{P}^2$, up to a constant factor. We may fix

$$\begin{array}{l} P_1 = (1:0:0),\ P_2 = (0:1:0),\ P_3 = (0:0:1),\ P_4 = (1:1:1), \\ P_5 = (1:a:b),\ P_6 = (1:c:d),\ P_7 = (1:e:f),\ P_8 = (1:g:h). \end{array} \tag{7}$$

If (2) holds for such configurations of eight points then it holds in general.

Let $u = y_9/x_9$ and $v = z_9/x_9$. Since $P_1, P_2, \ldots, P_8, P_9$ lie on two linearly independent cubics C_1 and C_2, the following matrix has rank at most 8:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 \\ 1 & c & d & c^2 & cd & d^2 & c^3 & c^2d & cd^2 & d^3 \\ 1 & e & f & e^2 & ef & f^2 & e^3 & e^2f & ef^2 & f^3 \\ 1 & g & h & g^2 & gh & h^2 & g^3 & g^2h & gh^2 & h^3 \\ 1 & u & v & u^2 & uv & v^2 & u^3 & u^2v & uv^2 & v^3 \end{pmatrix} \tag{8}$$

Hence the 9×9-minors of (8) are zero. This gives 10 equations in u and v whose coefficients are polynomials in $a, b, \ldots, h$. Each equation is of the form

$$A_1u^2v + A_2uv^2 + A_3u^2 + A_4uv + A_5v^2 + A_6u + A_7v \;=\; 0, \tag{9}$$

where $A_1, A_2, \ldots, A_7 \in \mathbb{Z}[a,b,c,d,e,f,g,h]$ are the cofactors in (8).

For our special choices of $P_1, \ldots, P_8, P_9$, the formula in Theorem 3 states

$$u \;=\; \frac{D_xC_y}{C_xD_y} \quad \text{and} \quad v \;=\; \frac{D_xC_z}{C_xD_z}. \tag{10}$$

To show this, we must argue that (9) holds after the substitution (10). Equivalently, to prove Theorem 3, we need to verify the 10 identities of the form

$$\begin{array}{r} A_1C_y^2C_zD_x^2D_z + A_2C_yC_z^2D_x^2D_y + A_3C_xC_y^2D_xD_z^2 + A_4C_xC_yC_zD_xD_yD_z \\ +A_5C_xC_z^2D_xD_y^2 + A_6C_x^2C_yD_yD_z^2 + A_7C_x^2C_zD_y^2D_z \;=\; 0. \end{array}$$

The left hand side lies in $\mathbb{Z}[a,b,c,d,e,f,g,h]$. We will show that it is zero.

The computation needed to multiply out each term on the left hand side is still too large for a standard computer. A symbolic proof using the computer algebra system `sage` [16] involves some tricks to control intermediate expression growth as follows. Namely, we evaluate it in the following form:

$$\frac{C_zD_x\frac{A_1C_yD_z+A_2C_zD_y}{C_x}+A_3C_yD_z^2}{D_y}+A_4C_zD_z+\frac{C_zD_y\frac{A_5C_zD_x+A_7C_xD_z}{C_y}+A_6C_xD_z^2}{D_x}.$$

After computing $A_1C_yD_z+A_2C_zD_y$, one verifies that the result is divisible by C_x. Similarly, all other fractions in the above expression leave polynomial quotients. Therefore, the sizes of the intermediate results are limited. □

3 Formula of Minimal Degree

A natural question is whether the formula (2) is optimal in the sense that it has the lowest degree possible. The answer is "no". We can do better.

The three polynomials in (3) have multidegree $(9,9,9,8,8,8,8,8)$, and they are all divisible by [123]. Removing that common factor, we obtain three polynomials in 24 unknowns with greatest common divisor 1. The following statement can be verified with symbolic computations. A theoretical proof was given in the PhD dissertation of the first author in [12, Chapter 5].

Corollary 5. *The following formula for the Cayley-Bacharach point is invariant under the symmetric group S_8 and contains no extraneous factor:*

$$P_9 \;=\; \frac{1}{[123]}\cdot\big(C_xD_yD_z\cdot P_1\;+\;D_xC_yD_z\cdot P_2\;+\;D_xD_yC_z\cdot P_3\big) \qquad (11)$$

Its coordinates are homogeneous polynomials of degree $(8,8,8,8,8,8,8,8)$.

The expression (11) is still not satisfactory because it involves division. Our second main result is a highly symmetric formula of optimal degree for P_9. We shall use the following bracket monomial of degree $(8,8,8,8,8,8,8,7)$:

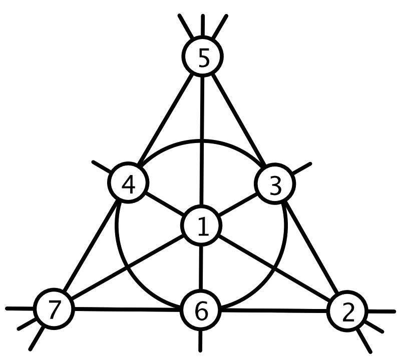
$$F(1,2,3,4,5,6,7;8) \;:=\; \begin{array}{l}[128][238][348][458][568][678][718]\\ \cdot[124][235][346][457][561][672][713]\\ \cdot[126][237][341][452][563][674][715].\end{array} \qquad (12)$$

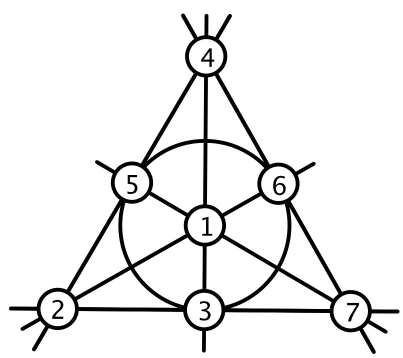
Theorem 6. *The Cayley-Bacharach point P_9 is given by the formula*

$$\sum_{\pi \in S_8} sign(\pi) \cdot F(\pi(1), \ldots \pi(7); \pi(8)) \cdot P_{\pi(8)}. \qquad (13)$$

Here π runs over all 40320 *permutations in the symmetric group S_8.*

Before addressing the validity of this formula, we discuss how it was found. We looked for a bracket expression of degree $(8, 8, 8, 8, 8, 8, 8, 8)$ that calculates P_9 in terms of the other eight points. The points $P_1, \ldots, P_8$ play a symmetric role in the calculation of P_9. Switching any two of them should give the same result. Furthermore if two points coincide then the formula should create the zero vector as an indication for degeneracy. Thus we searched for a formula that was antisymmetric in $1, 2, 3, 4, 5, 6, 7, 8$. This is ensured by the signed summation over S_8. Now let us focus on the structure of one summand. We needed a product of 21 brackets that is multiplied with the homogeneous coordinates of one of the points, say P_8. In that bracket monomial each other point must occur 8 times while P_8 occurs 7 times. Our $F(1, 2, 3, 4, 5, 6, 7; 8)$ reflects a particularly nice choice. The first row involves point P_8 seven times, while the pair P_1P_2 is cyclically shifted. A reasonable assumption is that the remaining 14 brackets are separated into two 7_3 *configurations*, i.e. seven brackets with each point occurring in exactly three triplets. Up to isomorphism there is only one 7_3 configuration: *the Fano plane*. There are $7!/168 = 30$ different ways to label a Fano plane. Among these precisely two are invariant under cyclically shifting the indices $1, \ldots, 7$. These are precisely the Fano planes in the second and third row of (12):





The two Fano planes appearing in $F(1, 2, 3, 4, 5, 6, 7; 8)$.

Proof of Theorem 6. The formula was verified using `Mathematica` by comparing (13) with the point $(x_9 \colon y_9 \colon z_9)$ created by the formula in Theorem 3. Let $(X_9 : Y_9 : Z_9)$ be the point calculated in (13). To prove Theorem 6, it is sufficient to show that $x_9Y_9 = X_9y_9$ and $x_9Z_9 = X_9z_9$ for arbitrary choices of the points $P_1, \ldots, P_8$. It suffices to verify this for the coordinates in (7).

Strong confidence in our identities can be created by checking random specializations in exact arithmetic. A brute force approach by fully expanding (13) ends up in combinatorial explosion because summing over S_8 creates 40320 terms. However, one can apply the symmetries of the expression (12) to significantly reduce the number of summands. By cyclic shifting, we have

$$F(1,2,3,4,5,6,7;8) \;=\; F(7,1,2,3,4,5,6;8).$$

Replacing the cycle $1, 2, \ldots, 7$ by its mirror image negates the expression:

$$F(1,2,3,4,5,6,7;8) \;=\; -F(7,6,5,4,3,2,1;8).$$

These two symmetries allow us to perform the summation only over $2880 = 8!/14$ summands. With this simplification, we derived a computer algebra proof of Theorem 6 using `Mathematica`. As in the proof of Theorem 3, we may assume that (7) holds. A straightforward simplification still ends in a combinatorial explosion. However the test can be carried out in approximately six hours if one variable is set to a fixed integer value. Since the degree of each variable is just 8, it suffices to perform this test of 9 different choices of this variable. This leads to a computer algebra proof, via `Mathematica`, that runs for approximately two days on current standard hardware. □

4 Discussion

Our contribution in this paper are two explicit formulas, in Theorems 3 and 6, for the Cayley-Bacharach point P_9 in terms of eight given points in $\mathbb{P}^2$. This adds to the geometric constructions known from the 19th century literature.

A natural analogue to the Cayley-Bacharach Theorem exists for eight points in 3-space. It states: *all quadric surfaces through seven given points in* $\mathbb{P}^3$ *also pass through a unique eighth point.* The formula for that eighth is easier to derive than the one in Theorem 3. It can be found in [11, §7]. Both versions of the Cayley-Bacharach Theorem play a prominent role in work of Blekherman [1] on sum of squares polynomials. These are motivated by recent advances in polynomial optimization. Work of Iliman and De Wolff [8, §3] suggests that our formulas will be useful in such domains of application.

Computing the Cayley-Bacharach point also makes sense in *tropical geometry* [10]. In that setting, all expressions in our formulas should be evaluated using arithmetic in the min-plus semiring, with the determinant in the definitions of $C_x, C_y, C_z, D_x, D_y, D_z$ replaced by the *tropical determinant*. To assess the combinatorial structure and complexity of the tropicalization of (11), one examines the Newton polytopes of the numerators and denominators.

For example, suppose $P_1 = (1:0:0), P_2 = (0:1:0), P_3 = (0:0:1)$, and $P_4 = (1:1:1)$. Then P_9 is given by the formula in (10). The factors $C_x, C_y, C_z, D_x, D_y, D_z$ are polynomials in 12 variables x_i, y_i, z_i for $5 \leq i \leq 8$. It can be verified with the software `polymake` [7] that the six Newton polytopes are isomorphic. The f-vector for that common Newton polytope is

$$(120, 1980, 7430, 11470, 8720, 3460, 700, 60).$$

That is, the polytope has 120 vertices, 1980 edges, and 60 facets. The tropical polynomials $\mathrm{trop}(C_x), \ldots, \mathrm{trop}(D_z)$ are piecewise linear functions, each given as the minimum of 120 linear functions on $\mathbb{R}^{12}$. From this, we obtain an explicit piecewise linear formula for $\mathrm{trop}(P_9)$ in terms of $\mathrm{trop}(P_1), \ldots, \mathrm{trop}(P_8)$. That formula is valid for scalars x_i, y_i, z_i in a field with valuation, such as the p-adic numbers, provided there is no cancelation of lowest terms when evaluating (11). Unfortunately, cancellations do occur in many situations, and this topic deserves to be studied further. We note that a *tropical Cayley-Bacharach Theorem* with weaker hypotheses was given by Tabera in [17].

The Cayley-Bacharach Theorem offers students a friendly point of entry into classical algebraic geometry [4, 15]. Those who use computer algebra systems will appreciate our explicit formulas for P_9 in terms of $P_1, \ldots, P_8$. While the expressions (2), (11) and (13) seem to be new, they rest on geometric constructions that are very old and well known, notably from [2, 3, 6, 18, 19].

Here is one especially nice construction, related to *del Pezzo surfaces*. Let $\mathcal{S}$ be the cubic surface in $\mathbb{P}^3$ that is obtained by blowing up the plane $\mathbb{P}^2$ at the first six points $P_1, \ldots, P_6$. Write $\tilde{P}_7$ and $\tilde{P}_8$ for the images on $\mathcal{S}$ of P_7 and P_8. The line in $\mathbb{P}^3$ through $\tilde{P}_7$ and $\tilde{P}_8$ meets the cubic surface $\mathcal{S}$ in one other point $\tilde{P}_9$, namely the image in $\mathcal{S}$ of the desired Cayley-Bacharach point P_9.

A referee kindly explained to us how Theorem 3 can be derived from the *Geiser involution*; see [4, Section 8.7.2] or [15, Section 8.1]. This is a Cremona transformation $G : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by seven fixed points $P_1, \ldots, P_7$.

Algebraic geometers should think of fixing a marked del Pezzo surface of degree 2. The corresponding Geiser involution is the map G that takes P_8 to the Cayley-Bacharach point P_9. In coordinates, one can write $G : (x{:}y{:}z) \mapsto (G_0(x,y,z) : G_1(x,y,z) : G_2(x,y,z))$ where G_i are ternary forms of degree 8 with triple points at $P_1, \ldots, P_7$. The punchline is that our $C_x D_y D_z$, $D_x C_y D_z$ and $D_x D_y C_z$ are such polynomials of degree 8 in the unknown $P_8 = (x : y : z)$.

In the literature, one can find numerous generalizations of Theorem 1 that also carry the name "Cayley-Bacharach". To learn more about these, our readers might start with the 1949 book of Semple and Roth [15, Section V.1.1], and then proceed to the 1996 article of Eisenbud, Green and Harris [5].

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