YOKONUMA-SCHUR ALGEBRAS

WEIDENG CUI

ABSTRACT. In this paper, we define the Yokonuma-Schur algebra $YS_q(r, n)$ as the endomorphism algebra of a permutation module for the Yokonuma-Hecke algebra $Y_{r,n}(q)$. We prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis following the approach in [DJM], and we further show that it is a quasi-hereditary cover of $Y_{r,n}(q)$ in the sense of Rouquier following [HM2]. We also introduce the tilting modules for $YS_q(r, n)$. In the appendix, we define and study the cyclotomic Yokonuma-Schur algebra in a similar way.

1. INTRODUCTION

1.1. The Yokonuma-Hecke algebra was first introduced by Yokonuma [Yo] as a centralizer algebra associated to the permutation representation of a Chevalley group G with respect to a maximal unipotent subgroup of G. Juyumaya [Ju1] gave a new presentation of the Yokonuma-Hecke algebra, which is commonly used for studying this algebra.

The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is a quotient of the group algebra of the modular framed braid group $(\mathbb{Z}/r\mathbb{Z}) \wr B_n$, where B_n is the braid group of type A on n strands. It can also be regraded as a deformation of the group algebra of the complex reflection group G(r, 1, n), which is isomorphic to the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, where \mathfrak{S}_n is the symmetric group on n letters. It is well-known that there exists another deformation of the group algebra of G(r, 1, n), the Ariki-Koike algebra $H_{r,n}$ [AK]. The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is quite different from $H_{r,n}$. For example, the Iwahori-Hecke algebra of type A is canonically a subalgebra of $H_{r,n}$, whereas it is an obvious quotient of $Y_{r,n}(q)$, but not an obvious subalgebra of it.

In the past few years, many people are largely motivated to study $Y_{r,n}(q)$ in order to construct its associated knot invariant; see the papers [Ju2], [JuL] and [ChL]. In particular, Juyumaya and Kannan [Ju2, JuK] found a basis of $Y_{r,n}(q)$, and then defined a Markov trace on it.

Some other people are particularly interested in the representation theory of $Y_{r,n}(q)$, and also its application to knot theory. Chlouveraki and Poulain d'Andecy [ChPA1] gave explicit formulas for all irreducible representations of $Y_{r,n}(q)$ over $\mathbb{C}(q)$, and obtained a semisimplicity criterion for it. In their subsequent paper [ChPA2], they defined and studied the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$, and constructed several bases for them, and then showed how to define Markov traces on these algebras. Moreover, they gave the classification of irreducible representations of $Y_{r,n}^d(q)$ in the generic semisimple case, defined the canonical symmetrizing form on it and computed the associated Schur elements directly.

1.2. Recently, Jacon and Poulain d'Andecy [JaPA] constructed an explicit algebraic isomorphism between the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and a direct sum of matrix algebras over tensor products of Iwahori-Hecke algebras of type A, which is in fact a special case of the results by G. Lusztig [Lu, Section 34]. This allows them to give a description of the modular representation theory of $Y_{r,n}(q)$ and a complete classification of all Markov traces for it. Chlouveraki and Sécherre [ChS, Theorem 4.3] proved that the affine Yokonuma-Hecke algebra is a particular case of the pro-*p*-Iwahori-Hecke algebra defined by Vignéras in [Vi].

Espinoza and Ryom-Hansen [ER] gave a new proof of Jacon and Poulain d'Andecy's isomorphism theorem by giving a concrete isomorphism between $Y_{r,n}(q)$ and Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. Moreover, they showed that $Y_{r,n}(q)$ is a cellular algebra by giving an explicit cellular basis. Combining the results of [DJM] with those of [ER], we [C1] proved that the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$ is cellular by constructing an explicit cellular basis, and showed that the Jucys-Murphy elements for $Y_{r,n}^d(q)$ are JM-elements in the abstract sense introduced by Mathas [Ma3].

We [CW] have established an equivalence between a module category of the affine (resp. cyclotomic) Yokonuma-Hecke algebra $\hat{Y}_{r,n}(q)$ (resp. $Y_{r,n}^d(q)$) and its suitable counterpart for a direct sum of tensor products of affine Hecke algebras of type A (resp. cyclotomic Hecke algebras), which allows us to give the classification of simple modules of affine Yokonuma-Hecke algebras and of the associated cyclotomic Yokonuma-Hecke algebras over an algebraically closed field of characteristic p when p does not divide r, and also describe the classification of blocks for these algebras. In addition, the modular branching rules for cyclotomic (resp. affine) Yokonuma-Hecke algebras are obtained, and they are further identified with crystal graphs of integrable modules for affine lie algebras of type A. In a subsequent paper, we [C2] have established an explicit algebra isomorphism between the affine Yokonuma-Hecke algebras of type A. As an application, we proved that $\hat{Y}_{r,n}(q)$ is affine cellular in the sense of Koenig and Xi, and studied its homological properties.

1.3. In [DJM], they constructed a cellular basis for the cyclotomic q-Schur algebra $S(\Lambda)$ by firstly constructing a cellular basis for the Ariki-Koike algebra $H_{r,n}$. They further obtained a complete set of non-isomorphic irreducible $S(\Lambda)$ -modules and showed that it is quasi-hereditary. Now, there exists a cellular basis on $Y_{r,n}(q)$ by [ER], it is natural to try to define and study the corresponding Schur algebra for the Yokonuma-Hecke algebra $Y_{r,n}(q)$ by using this cellular basis.

In this paper, we will define the Yokonuma-Schur algebra $YS_q(r, n)$ as the endomorphism algebra of a permutation module associated to the Yokonuma-Hecke algebra $Y_{r,n}(q)$. Combining the results of [DJM] with those of [SS], we prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis, and further prove that it is quasi-hereditary. We also investigate the indecomposable tilting modules for $YS_q(r, n)$ and prove that they are self-dual.

This paper is organized as follows. In Section 2, we recall the definition of the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and the construction of a cellular basis of $Y_{r,n}(q)$ following [ER]. In Section 3, we will define the Yokonuma-Schur algebra $YS_q(r,n)$ as the endomorphism algebra of a permutation module associated to the Yokonuma-Hecke algebra $Y_{r,n}(q)$. We prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis, and further prove that it is quasi-hereditary by combining the results of [DJM] with those of [SS]. In Section 4, following the approach in [HM2], we will construct an exact functor from the category of $YS_q(r, n)$ -modules to the category of $Y_{r,n}(q)$ -modules. In Section 5, we introduce the tilting modules for $YS_q(r, n)$ and the closely related Young modules for $Y_{r,n}(q)$ following [Ma2]. In the appendix, we will generalize these results to define and study the cyclotomic Yokonuma-Schur algebra by using the cellular basis of $Y_{r,n}^d(q)$ constructed in [C1]. Since this approach is very similar, we only mention the main results and skip all the details.

Many ideas of this paper originate from the references [DJM, Ma2, SS], although it should be noted that the basic set-up here is different from theirs; anyhow, we expect that the Yokonuma-Schur algebra and its cyclotomic analog defined here deserve further study.

2. Cellular Bases for Yokonuma-Hecke Algebras

In this section, we recall the definition of the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and the construction of a cellular basis of $Y_{r,n}(q)$ presented in [ER, Section 4].

Let $r, n \in \mathbb{N}$, $r \geq 1$, and let $\zeta = e^{2\pi i/r}$. Let q be an indeterminate. Let \mathfrak{S}_n be the symmetric group on n letters, which acts on the set $\{1, 2, \ldots, n\}$ on the right by convention.

Let $\mathcal{R} = \mathbb{Z}[\frac{1}{r}][q, q^{-1}, \zeta]$. The Yokonuma-Hecke algebra $Y_{r,n} = Y_{r,n}(q)$ is an \mathcal{R} -associative algebra generated by the elements $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}$ satisfying the following relations:

$$\begin{array}{ll}
g_{i}g_{j} = g_{j}g_{i} & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \geq 2; \\
g_{i}g_{i+1}g_{i} = g_{i+1}g_{i}g_{i+1} & \text{for all } i = 1, \dots, n-2; \\
t_{i}t_{j} = t_{j}t_{i} & \text{for all } i, j = 1, \dots, n; \\
g_{i}t_{j} = t_{js_{i}}g_{i} & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n; \\
t_{i}^{r} = 1 & \text{for all } i = 1, \dots, n; \\
g_{i}^{2} = 1 + (q - q^{-1})e_{i}g_{i} & \text{for all } i = 1, \dots, n-1, \\
\end{array}$$
(2.1)

where s_i is the transposition (i, i + 1), and for each $1 \le i \le n - 1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

Note that the elements e_i are idempotents in $Y_{r,n}$. The elements g_i are invertible, with the inverse given by

$$g_i^{-1} = g_i - (q - q^{-1})e_i$$
 for all $i = 1, \dots, n - 1.$ (2.2)

Let $w \in \mathfrak{S}_n$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w. By Matsumoto's lemma, the element $g_w := g_{i_1}g_{i_2}\cdots g_{i_r}$ does not depend on the choice of the reduced expression of w, that is, it is well-defined. Let l denote the length function on \mathfrak{S}_n . Then we have

$$g_{i}g_{w} = \begin{cases} g_{s_{i}w} & \text{if } l(s_{i}w) > l(w); \\ g_{s_{i}w} + (q - q^{-1})e_{i}g_{w} & \text{if } l(s_{i}w) < l(w). \end{cases}$$
(2.3)

Using the multiplication formulae in (2.3), Juyumaya [Ju2] has proved that the following set is an \mathcal{R} -basis of $Y_{r,n}$:

$$\mathcal{B}_{r,n} = \{ t_1^{k_1} \cdots t_n^{k_n} g_w \mid 0 \le k_1, \dots, k_n \le r - 1 \text{ and } w \in \mathfrak{S}_n \}.$$
(2.4)

Thus, $Y_{r,n}$ is a free \mathcal{R} -module of rank $r^n n!$.

Set $\mathbf{s} := \{1, 2, \dots, n\}$. Let $i, k \in \mathbf{s}$ and set

$$e_{i,k} := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_k^{-s}.$$
(2.5)

Note that $e_{i,i} = 1$, $e_{i,k} = e_{k,i}$, and that $e_{i,i+1} = e_i$. It can be easily checked that

$$\begin{array}{ll}
e_{i,k}^{2} = e_{i,k} & \text{for all } i, k = 1, \dots, n, \\
t_{i}e_{j,k} = e_{j,k}t_{i} & \text{for all } i, j, k = 1, \dots, n, \\
e_{i,j}e_{k,l} = e_{k,l}e_{i,j} & \text{for all } i, j, k, l = 1, \dots, n, \\
e_{i}e_{k,l} = e_{s_{i}(k),s_{i}(l)}e_{i} & \text{for all } i = 1, \dots, n - 1 \text{ and } k, l = 1, \dots, n, \\
e_{j,k}g_{i} = g_{i}e_{js_{i},ks_{i}} & \text{for all } i = 1, \dots, n - 1 \text{ and } j, k = 1, \dots, n. \\
\end{array}$$
(2.6)

In particular, we have $e_i g_i = g_i e_i$ for all $i = 1, 2, \ldots, n-1$.

For any nonempty subset $I \subseteq \mathbf{s}$ we define the following element E_I by

$$E_I := \prod_{i,j \in I; i < j} e_{i,j},$$

where by convention $E_I = 1$ if |I| = 1.

We also need a further generalization of this. We say that the set $A = \{I_1, I_2, \ldots, I_k\}$ is a set partition of **s** if the I_j 's are nonempty and disjoint subsets of **s**, and their union is **s**. We refer to them as the blocks of A. We denote by $S\mathcal{P}_n$ the set of all set partitions of **s**. For $A = \{I_1, I_2, \ldots, I_k\} \in S\mathcal{P}_n$ we then define $E_A := \prod_i E_{I_i}$.

We extend the right action of \mathfrak{S}_n on **s** to a right action on \mathfrak{SP}_n by defining $Aw := \{I_1w, \ldots, I_kw\} \in \mathfrak{SP}_n$ for $w \in \mathfrak{S}_n$. Then we can easily get the following lemma.

Lemma 2.1. For $A \in SP_n$ and $w \in \mathfrak{S}_n$, we have

$$g_w E_A = E_{Aw^{-1}} g_w.$$

In particular, if w leaves invariant every block of A, or more generally permutes some of the blocks of A, then g_w commutes with E_A .

 $\mu = (\mu_1, \ldots, \mu_k)$ is called a composition of n if it is a finite sequence of nonnegative integers whose sum is n. A composition μ is a partition of n if its parts are non-increasing. We write $\mu \models n$ (resp. $\lambda \vdash n$) if μ is a composition (resp. partition) of n, and we define $|\mu| := n$ (resp. $|\lambda| := n$).

We associate a Young diagram to a composition μ , which is the set

$$[\mu] := \{(i, j) \mid i \ge 1 \text{ and } 1 \le j \le \mu_i\}.$$

We will regard $[\mu]$ as an array of boxes, or nodes, in the plane. For $\mu \models n$, we define a μ -tableau by replacing each node of $[\mu]$ by one of the integers $1, 2, \ldots, n$, allowing no repeats.

For $\mu \models n$, we say that a μ -tableau t is row standard if the entries in each row of t increase from left to right. A μ -tableau t is standard if μ is a partition, t is row standard and the entries in each column increase from top to bottom. For a composition μ of n, we denote by t^{μ} the μ -tableau in which $1, 2, \ldots, n$ appear in increasing order from left to right along the rows of $[\mu]$.

The symmetric group \mathfrak{S}_n acts from the right on the set of μ -tableaux by permuting the entries in each tableau. For any composition $\mu = (\mu_1, \ldots, \mu_k)$ of n we define the Young subgroup $\mathfrak{S}_{\mu} := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k}$, which is the row stabilizer of \mathfrak{t}^{μ} .

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$ be two compositions of n. We say that $\lambda \ge \mu$ if

$$\sum_{i=1}^{j} \lambda_i \ge \sum_{i=1}^{j} \mu_i \text{ for all } j \ge 1.$$

If $\lambda \geq \mu$ and $\lambda \neq \mu$, we write $\lambda > \mu$.

We extend the partial order above to tableaux as follows. If \mathfrak{v} is a row standard λ -tableau and $1 \leq k \leq n$, then the entries $1, 2, \ldots, k$ in \mathfrak{v} occupy the diagram of a composition; let $\mathfrak{v}_{\downarrow k}$ denote this composition. Let λ and μ be two compositions of n. Suppose that \mathfrak{s} is a row standard λ -tableau and that \mathfrak{t} is a row standard μ -tableau. We say that \mathfrak{s} dominates \mathfrak{t} , and we write $\mathfrak{s} \supseteq \mathfrak{t}$ if $\mathfrak{s}_{\downarrow k} \supseteq \mathfrak{t}_{\downarrow k}$ for all k. If $\mathfrak{s} \supseteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$, then we write $\mathfrak{s} \supset \mathfrak{t}$.

Following [ChPA1, Section 4.3], the combinatorial objects appearing in the representation theory of the Yokonuma-Hecke algebra $Y_{r,n}$ will be *r*-compositions (resp. *r*partitions). By definition, an *r*-composition (resp. *r*-partition) of *n* is an ordered *r*-tuple $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(r)})$ of compositions (resp. partitions) $\mu^{(k)}$ such that $\sum_{k=1}^{r} |\mu^{(k)}| = n$. We denote by $\mathcal{C}_{r,n}$ (resp. $\mathcal{P}_{r,n}$) the set of *r*-compositions (resp. *r*-partitions) of *n*. The Young diagram [$\boldsymbol{\mu}$] of an *r*-composition $\boldsymbol{\mu}$ is the ordered *r*-tuple of the Young diagram of its components.

Let $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(r)})$ be an *r*-composition of *n*. A $\boldsymbol{\mu}$ -tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ is obtained by placing each node of $[\boldsymbol{\mu}]$ by one of the integers $1, 2, \dots, n$, allowing no repeats. We will call the number *n* the size of \mathfrak{t} and the $\mathfrak{t}^{(k)}$'s the components of \mathfrak{t} .

For each $\mu \in \mathcal{C}_{r,n}$, a μ -tableau is called row standard if the numbers increase along any row (from left to right) of each diagram in $[\mu]$. For each $\lambda \in \mathcal{P}_{r,n}$, a λ -tableau is called standard if the numbers increase along any row (from left to right) and down any column (from top to bottom) of each diagram in $[\lambda]$. For $\mu \in \mathcal{C}_{r,n}$, we denote by r-Std (μ) the set of row standard μ -tableaux of size n, which is endowed with an action of \mathfrak{S}_n from the right by permuting the entries in each μ -tableau. For $\lambda \in \mathcal{P}_{r,n}$, let Std (λ) denote the set of standard λ -tableaux of size n.

For each $\boldsymbol{\mu} \in \mathbb{C}_{r,n}$, we denote by $\mathfrak{t}^{\boldsymbol{\mu}}$ the standard $\boldsymbol{\mu}$ -tableau in which $1, 2, \ldots, n$ appear in increasing order from left to right along the rows of the first diagram, and then along the rows of the second diagram, and so on. For each $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}) \in \mathbb{C}_{r,n}$, we have a Young subgroup

$$\mathfrak{S}_{\boldsymbol{\mu}} := \mathfrak{S}_{\boldsymbol{\mu}^{(1)}} \times \mathfrak{S}_{\boldsymbol{\mu}^{(2)}} \cdots \times \mathfrak{S}_{\boldsymbol{\mu}^{(r)}},$$

which is exactly the row stabilizer of \mathfrak{t}^{μ} .

For each $\mu \in \mathcal{C}_{r,n}$ and a row standard μ -tableau \mathfrak{s} , let $d(\mathfrak{s})$ be the unique element of \mathfrak{S}_n such that $\mathfrak{s} = \mathfrak{t}^{\mu} d(\mathfrak{s})$. Then $d(\mathfrak{s})$ is a distinguished right coset representative of \mathfrak{S}_{μ} in

 \mathfrak{S}_n , that is, $l(wd(\mathfrak{s})) = l(w) + l(d(\mathfrak{s}))$ for any $w \in \mathfrak{S}_{\mu}$. In this way, we obtain a bijection between the set $r\text{-Std}(\mu)$ of row standard μ -tableaux and the set \mathcal{D}_{μ} of distinguished right coset representatives of \mathfrak{S}_{μ} in \mathfrak{S}_n .

Let $\boldsymbol{\mu} \in \mathcal{C}_{r,n}$ and \mathfrak{t} be a $\boldsymbol{\mu}$ -tableau. For $j = 1, \ldots, n$, we define $p_{\mathfrak{t}}(j) = k$ if j appears in the k-th component $\mathfrak{t}^{(k)}$ of \mathfrak{t} . When $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\mu}}$, we write $p_{\boldsymbol{\mu}}(j)$ instead of $p_{\mathfrak{t}^{\boldsymbol{\mu}}}(j)$.

We now define a partial order on the set of r-compositions, which is similar to the case of compositions.

Definition 2.2. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(r)})$ be two *r*-compositions of *n*. We say that λ dominates μ , and we write $\lambda \geq \mu$ if and only if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{j=1}^{l} \lambda_l^{(k)} \ge \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{j=1}^{l} \mu_l^{(k)}$$

for all k and l with $1 \le k \le r$ and $l \ge 0$. If $\lambda \ge \mu$ and $\lambda \ne \mu$, we write $\lambda > \mu$.

We now fix once and for all a total order on the set of r-th roots of unity via setting $\zeta_k := \zeta^{k-1}$ for $1 \leq k \leq r$. Set $S := \{\zeta_1, \zeta_2, \ldots, \zeta_r\}$. Then we define a set partition $A_{\lambda} \in S\mathcal{P}_n$ for any r-composition λ .

Definition 2.3. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{C}_{r,n}$. Suppose that we choose all $1 \leq i_1 < i_2 < \cdots < i_p \leq r$ such that $\lambda^{(i_1)}, \lambda^{(i_2)}, \ldots, \lambda^{(i_p)}$ are the nonempty components of $[\lambda]$. Define $a_k := \sum_{j=1}^k |\lambda^{(i_j)}|$ for $1 \leq k \leq p$. Then the set partition A_{λ} associated with λ is defined as

$$A_{\boldsymbol{\lambda}} := \{\{1, \dots, a_1\}, \{a_1 + 1, \dots, a_2\}, \dots, \{a_{p-1} + 1, \dots, n\}\},\$$

which may be written as $A_{\lambda} = \{I_1, I_2, \dots, I_p\}$, and is referred to the blocks of A_{λ} in the order given above.

Definition 2.4. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{C}_{r,n}$, and let $a_k := \sum_{j=1}^k |\lambda^{(i_j)}| \ (1 \le k \le p)$ be defined as above. Then we define

$$u_{\boldsymbol{\lambda}} := u_{a_1,i_1} u_{a_2,i_2} \cdots u_{a_p,i_p},$$

where $u_{i,k} = \prod_{l=1; l \neq k}^{r} (t_i - \zeta_l)$ for $1 \le i \le n$ and $1 \le k \le r$.

Definition 2.5. Let $\lambda \in \mathbb{C}_{r,n}$. We set $U_{\lambda} := u_{\lambda} E_{A_{\lambda}}$, and define $x_{\lambda} = \sum_{w \in \mathfrak{S}_{\lambda}} q^{l(w)} g_{w}$. Then we define the element m_{λ} of $Y_{r,n}$ as follows:

$$m_{\lambda} := U_{\lambda} x_{\lambda} = u_{\lambda} E_{A_{\lambda}} x_{\lambda}.$$

Let * denote the \mathcal{R} -linear anti-automorphism of $Y_{r,n}$, which is determined by $g_i^* = g_i$ and $t_i^* = t_j$ for $1 \le i \le n-1$ and $1 \le j \le n$.

Definition 2.6. Let $\lambda \in C_{r,n}$, and let \mathfrak{s} and \mathfrak{t} be two row standard λ -tableaux. We then define $m_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} m_{\lambda} g_{d(\mathfrak{t})}$.

For each $\lambda \in \mathcal{P}_{r,n}$, let $Y_{r,n}^{\triangleright \lambda}$ be the \mathcal{R} -submodule of $Y_{r,n}$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in$ Std (μ) for various $\mu \in \mathcal{P}_{r,n}$ such that $\mu \triangleright \lambda$.

Theorem 2.7. (See [ER, Theorem 20].) The algebra $Y_{r,n}$ is a free \mathcal{R} -module with a cellular basis

$$\mathcal{B}_{r,n} = \{ m_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } r - partition \ \lambda \text{ of } n \},\$$

that is, the following properties hold:

(i) The \mathcal{R} -linear map determined by $m_{\mathfrak{st}} \mapsto m_{\mathfrak{ts}} (m_{\mathfrak{st}} \in \mathcal{B}_{r,n})$ is an anti-automorphism on $Y_{r,n}$.

(ii) For a given $h \in Y_{r,n}$, $\mu \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \mathrm{Std}(\mu)$, there exist $r_{\mathfrak{vt}}(h) \in \mathcal{R}$ such that for all $\mathfrak{s} \in \mathrm{Std}(\mu)$, we have

$$m_{\mathfrak{st}}h \equiv \sum_{\mathfrak{v}\in \operatorname{Std}(\boldsymbol{\mu})} r_{\mathfrak{vt}}(h)m_{\mathfrak{sv}} \, \operatorname{mod} \, \operatorname{Y}_{r,n}^{\rhd \boldsymbol{\mu}},$$

where $r_{\mathfrak{vt}}(h)$ may depend on $\mathfrak{v}, \mathfrak{t}$ and h, but not on \mathfrak{s} .

For each $\lambda \in \mathcal{P}_{r,n}$, let \overline{m}_{λ} be the image of m_{λ} under the algebra homomorphism $Y_{r,n} \to Y_{r,n}/Y_{r,n}^{\triangleright \lambda}$. We denote by S^{λ} the right $Y_{r,n}$ -submodule of $Y_{r,n}/Y_{r,n}^{\triangleright \lambda}$ generated by \overline{m}_{λ} , which is called the Specht module associated to λ . By Theorem 2.7, S^{λ} is a free \mathcal{R} -module with basis $\{\overline{m}_{\lambda}g_{d(\mathfrak{t})} | \mathfrak{t} \in \mathrm{Std}(\lambda)\}$. We can define an associative symmetric bilinear form on S^{λ} by

$$m_{\lambda}g_{d(\mathfrak{s})}g_{d(\mathfrak{t})}^{*}m_{\lambda} \equiv \langle \overline{m}_{\lambda}g_{d(\mathfrak{s})}, \overline{m}_{\lambda}g_{d(\mathfrak{t})} \rangle m_{\lambda} \mod Y_{r,n}^{\triangleright \lambda}.$$

Let rad $S^{\boldsymbol{\lambda}} = \{u \in S^{\boldsymbol{\lambda}} \mid \langle u, v \rangle = 0 \text{ for all } v \in S^{\boldsymbol{\lambda}}\}$. Consequently, rad $S^{\boldsymbol{\lambda}}$ is a $Y_{r,n}$ -submodule of $S^{\boldsymbol{\lambda}}$. Let $D^{\boldsymbol{\lambda}} = S^{\boldsymbol{\lambda}}/\text{rad} S^{\boldsymbol{\lambda}}$ for each $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$. By a general theory of cellular basis, if $\mathcal{R} = \mathbb{K}$ is an algebraically closed field of characteristic $p \geq 0$ such that p does not divide r, the set $\{D^{\boldsymbol{\lambda}} \neq 0 \mid \boldsymbol{\lambda} \in \mathcal{P}_{r,n}\}$ gives a complete set of non-isomorphic irreducible $Y_{r,n}$ -modules. In fact, by [ER, Theorem 7 and (46)] (see also [JaPA, §4.1] and [CW, Theorem 6.3]), $\{\boldsymbol{\lambda} \in \mathcal{P}_{r,n} \mid D^{\boldsymbol{\lambda}} \neq 0\}$ is just the set $\mathcal{K}_{r,n}$, where

$$\mathcal{K}_{r,n} = \big\{ \boldsymbol{\lambda} \in \mathcal{P}_{r,n} \mid \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \text{ with each } \lambda^{(i)} \text{ being an } e - \text{restricted partition} \big\}.$$

3. Yokonuma-Schur Algebra and its cellular basis

For an r-composition λ of n, a λ -tableau S = $(S^{(1)}, \ldots, S^{(r)})$ is a map S : $[\lambda] \rightarrow \{1, \ldots, n\} \times \{1, \ldots, r\}$, which can be regarded as the diagram $[\lambda]$, together with an ordered pair (i, k) $(1 \leq i \leq n, 1 \leq k \leq r)$ attached to each node. Given $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$, a λ -tableau S is said to be of type μ if the number of (i, k) in the entry of S is equal to $\mu_i^{(k)}$. Given $\mathfrak{s} \in \operatorname{Std}(\lambda)$, $\mu(\mathfrak{s})$, a λ -tableau of type μ , is defined by replacing each entry m in \mathfrak{s} by (i, k) if m is in the *i*-th row of the k-th component of \mathfrak{t}^{μ} .

We define a total order on the set of pairs (i, k) by $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$. Let $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$. Suppose that $S = (S^{(1)}, \ldots, S^{(r)})$ is a λ -tableau of type μ . S is said to be semistandard if each component $S^{(k)}$ is non-decreasing in rows, strictly increasing in columns, and all entries of $S^{(k)}$ are of the form (i, l) with $l \geq k$. We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard λ -tableau of type μ .

Let us consider the special case when r = 1. Suppose that $\lambda, \mu \in \mathcal{C}_{1,n}$. A λ -tableau S of type μ is said to be row semistandard if the entries in each row of S are non-decreasing. S is said to be semistandard if $\lambda \in \mathcal{P}_{1,n}$, S is row semistandard and the entries in each

column are strictly increasing. Assume that $\lambda, \mu \in \mathcal{C}_{1,n}$ and put $\mathcal{D}_{\lambda\mu} = \mathcal{D}_{\lambda} \cap \mathcal{D}_{\mu}^{-1}$. Then $\mathcal{D}_{\lambda\mu}$ is the set of minimal length elements in the double cosets $\mathfrak{S}_{\lambda} \setminus \mathfrak{S}_n / \mathfrak{S}_{\mu}$, and the map $d \mapsto \mu(\mathfrak{t}^{\lambda}d)$ gives a bijection between the set $\mathcal{D}_{\lambda\mu}$ and the set of row semistandard λ -tableaux of type μ .

Let $d \in \mathcal{D}_{\lambda\mu}$, and put $S = \mu(\mathfrak{t}^{\lambda}d)$, $T = \lambda(\mathfrak{t}^{\mu}d^{-1})$. Then S and T are both row semistandard, and we have

$$\sum_{\substack{y\in\mathfrak{D}_{\boldsymbol{\mu}}\\\boldsymbol{\lambda}(\mathfrak{t}^{\boldsymbol{\mu}}y)=\mathrm{T}}} q^{l(y)}g_{y}^{*}x_{\boldsymbol{\mu}} = \sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}d\mathfrak{S}_{\boldsymbol{\mu}}} q^{l(w)}g_{w} = \sum_{\substack{x\in\mathfrak{D}_{\boldsymbol{\lambda}}\\\boldsymbol{\mu}(\mathfrak{t}^{\boldsymbol{\lambda}}x)=\mathrm{S}}} q^{l(x)}x_{\boldsymbol{\lambda}}g_{x}.$$
(3.1)

For any $\boldsymbol{\kappa} \in \mathcal{C}_{r,n}$, we define its type $\alpha(\boldsymbol{\kappa})$ by $\alpha(\boldsymbol{\kappa}) = (n_1, \ldots, n_r)$ with $n_i = |\boldsymbol{\kappa}^{(i)}|$. Assume that $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ and $\boldsymbol{\mu} \in \mathcal{C}_{r,n}$. We define a subset $\mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of $\mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by

$$\mathfrak{T}_0^+(\boldsymbol{\lambda},\boldsymbol{\mu}) = \{ S \in \mathfrak{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu}) \mid \alpha(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\mu}) \}.$$

Take $S \in \mathcal{T}_0(\boldsymbol{\lambda}, \mu)$. One can check that $S \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \mu)$ if and only if each entry of $S^{(k)}$ is of the form (i, k) for some *i*. Moreover, if $\mathfrak{s} \in \text{Std}(\boldsymbol{\lambda})$ is such that $\mu(\mathfrak{s}) = S$ with $S \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \mu)$, then the entries of the *i*-th component of \mathfrak{s} consist of numbers $a_i + 1, \ldots, a_{i+1}$, where $a_i = \sum_{k=1}^{i-1} n_k$. In particular, $d(\mathfrak{s}) \in \mathfrak{S}_{\alpha}$ for $\alpha = \alpha(\boldsymbol{\lambda})$.

Take $S \in \mathcal{T}_0^+(\lambda, \mu)$. Let $\mathfrak{s}_1 = \text{first}(S)$, which is the unique element of $\text{Std}(\lambda)$ satisfying the property that $\mu(\mathfrak{s}_1) = S$ and that $\mathfrak{s}_1 \succeq \mathfrak{s}$ for any $\mathfrak{s} \in \text{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$. Let $\alpha = \alpha(\lambda) = \alpha(\mu)$. Then $d = d(\mathfrak{s}_1) \in \mathfrak{S}_{\alpha}$, which is given as $d = (d_1, \ldots, d_r)$ with d_k a distinguished double coset representative in $\mathfrak{S}_{\lambda^{(k)}} \setminus \mathfrak{S}_{n_k} / \mathfrak{S}_{\mu^{(k)}}$. From (3.1) we have

$$\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\\boldsymbol{\mu}(\mathfrak{s})=\mathrm{S}}} q^{l(d(\mathfrak{s}))} x_{\lambda} g_{d(\mathfrak{s})} = \sum_{w\in\mathfrak{S}_{\lambda}d\mathfrak{S}_{\mu}} q^{l(w)} g_{w} = hg_{d} x_{\mu}, \tag{3.2}$$

where $h = \sum g_v$, the sum running over certain elements $v \in \mathfrak{S}_{\lambda}$.

For each $\mu \in \mathcal{C}_{r,n}$, let $M^{\mu} = m_{\mu} Y_{r,n}$. The following lemma gives a basis of M^{μ} as an \mathcal{R} -module.

Lemma 3.1. For each $\mu \in \mathcal{C}_{r,n}$, $\{m_{\mu}g_d \mid d \in \mathcal{D}_{\mu}\}$ is an \mathcal{R} -basis of M^{μ} .

Proof. Since $m_{\mu}g_d = \sum_{w \in \mathfrak{S}_{\mu}} U_{\mu}g_{wd}$ for each $d \in \mathfrak{D}_{\mu}$, and hence $\{m_{\mu}g_d\}$ is linearly independent. Since $m_{\mu}t_i = \zeta_{\mathfrak{P}_{\mu}(i)}m_{\mu}$ by [ER, Lemma 11(4)] and $m_{\mu}g_w = q^{l(w)}m_{\mu}$ for $w \in \mathfrak{S}_{\mu}$, then the set $\{m_{\mu}g_d \mid d \in \mathfrak{D}_{\mu}\}$ spans M^{μ} by (2.4). Thus, $\{m_{\mu}g_d \mid d \in \mathfrak{D}_{\mu}\}$, or equivalently, $\{m_{\mu}g_{d(\mathfrak{t})} \mid \mathfrak{t} \in \mathrm{r-Std}(\mu)\}$ is an \mathfrak{R} -basis of M^{μ} .

We now construct a basis of M^{μ} related to the cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$. For $S \in \mathfrak{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we define

$$m_{\mathrm{St}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \boldsymbol{\mu}(\mathfrak{s}) = \mathrm{S}}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{st}}.$$

We have

Lemma 3.2. Let $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in Std(\lambda)$. Then $m_{S\mathfrak{t}} \in M^{\mu}$.

Proof. By (3.2) we have

$$m_{\mathrm{St}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = \mathrm{S}}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} g^*_{d(\mathfrak{s})} x_{\lambda} U_{\lambda} g_{d(\mathfrak{t})}$$
$$= q^{l(d(\mathfrak{t}))} x_{\mu} g^*_{d} h^* U_{\lambda} g_{d(\mathfrak{t})}.$$

Since $h = \sum g_v$, where $v \in \mathfrak{S}_{\lambda}$, hence U_{λ} commutes with h^* by [ER, Lemma 11(3)]. Since $d \in \mathfrak{S}_{\alpha}$ with $\alpha = \alpha(\lambda)$, U_{λ} commutes with g_d^* by the same reason. Noting that $\alpha = \alpha(\lambda) = \alpha(\mu)$, we have $U_{\lambda} = U_{\mu}$. Thus, we see that

$$x_{\boldsymbol{\mu}}g_d^*h^*U_{\boldsymbol{\lambda}} = x_{\boldsymbol{\mu}}U_{\boldsymbol{\mu}}g_d^*h^* \in m_{\boldsymbol{\mu}}Y_{r,n} = M^{\boldsymbol{\mu}},$$

and $m_{\mathrm{St}} \in M^{\mu}$ as required.

Proposition 3.3. For each $\mu \in \mathcal{C}_{r,n}$, M^{μ} is free with an \mathcal{R} -basis

$$\{m_{\mathrm{St}} \mid \mathrm{S} \in \mathfrak{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu}) \text{ and } \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \}.$$

Proof. The basis elements $m_{\mathfrak{st}}$ contained in the expression of $m_{\mathfrak{St}}$ are disjoint for different $m_{\mathfrak{St}}$. It follows that $m_{\mathfrak{St}}$ are linearly independent. By Lemma 3.1, M^{μ} is a free \mathcal{R} -module, and its rank is equal to the number of $\{m_{\mathfrak{St}}\}$ given in the proposition by [SS, Lemma 2.5(ii) and Corollary 4.5(ii)]. Hence, any element in M^{μ} can be written as a linear combination of various $m_{\mathfrak{St}}$ with coefficients in the quotient field of \mathcal{R} . But since the set $\{q^{l(d(\mathfrak{s}))+l(d(\mathfrak{t}))}m_{\mathfrak{st}}\}$ is an \mathcal{R} -basis of $Y_{r,n}$, these coefficients are actually in \mathcal{R} . This proves the proposition.

Let $\mu, \nu \in \mathcal{C}_{r,n}$ be such that $\alpha(\mu) = \alpha(\nu) = \alpha$. Put $\alpha = (n_1, \ldots, n_r)$. Let us take $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_{\alpha}$. We have $d = (d_1, \ldots, d_r)$ with $d_k \in \mathcal{D}_{\mu^{(k)}\nu^{(k)}}$ with respect to \mathfrak{S}_{n_k} . Then we can define a map $\varphi^d_{\mu\nu} : M^{\nu} \to M^{\mu}$ by

$$\varphi^d_{\mu\nu}(m_{\nu}h) = \sum_{w \in \mathfrak{S}_{\mu}d\mathfrak{S}_{\nu}} q^{l(w)} U_{\mu}g_w h$$

for all $h \in Y_{r,n}$. In fact, by (3.1), we have

$$\sum_{\substack{y\in\mathcal{D}_{\boldsymbol{\nu}}\cap\mathfrak{S}_{\alpha}\\\boldsymbol{\mu}(\mathfrak{t}^{\boldsymbol{\nu}}y)=\mathrm{T}}} q^{l(y)} U_{\boldsymbol{\mu}} g_{y}^{*} x_{\boldsymbol{\nu}} = \sum_{w\in\mathfrak{S}_{\boldsymbol{\mu}}d\mathfrak{S}_{\boldsymbol{\nu}}} q^{l(w)} U_{\boldsymbol{\mu}} g_{w} = \sum_{\substack{x\in\mathcal{D}_{\boldsymbol{\mu}}\cap\mathfrak{S}_{\alpha}\\\boldsymbol{\nu}(\mathfrak{t}^{\mu}x)=\mathrm{S}}} q^{l(x)} m_{\boldsymbol{\mu}} g_{x}, \tag{3.3}$$

where $S = \mu(\mathfrak{t}^{\nu}d)$ and $T = \nu(\mathfrak{t}^{\mu}d^{-1})$ are row semistandard tableaux. Noting that $U_{\mu} = U_{\nu}$ and $y \in \mathfrak{S}_{\alpha}$, we have $U_{\mu}g_{y}^{*}x_{\nu} = g_{y}^{*}U_{\nu}x_{\nu} = g_{y}^{*}m_{\nu}$, and $\varphi_{\mu\nu}^{d}$ is well-defined.

The proof of the next proposition is inspired by that of [SS, Proposition 5.2], although it should be noted that the basic set-up there is different from ours. It allows us to restrict ourselves to considering the subset $\mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Proposition 3.4. Let $\mu, \nu \in \mathcal{C}_{r,n}$. Then

(i) Assume that $\alpha(\boldsymbol{\mu}) \neq \alpha(\boldsymbol{\nu})$. Then $\operatorname{Hom}_{Y_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}}) = 0$.

(ii) Assume that $\alpha(\boldsymbol{\mu}) = \alpha(\boldsymbol{\nu})$. Then $\operatorname{Hom}_{Y_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}})$ is a free \mathbb{R} -module with basis $\{\varphi_{\boldsymbol{\mu}\boldsymbol{\nu}}^d \mid d \in \mathcal{D}_{\boldsymbol{\mu}\boldsymbol{\nu}} \cap \mathfrak{S}_{\alpha}\}.$

Proof. Suppose that $\varphi \in \operatorname{Hom}_{Y_{r,n}}(M^{\nu}, M^{\mu})$. Then, for all $h \in Y_{r,n}$, we have $\varphi(m_{\nu}h) = \varphi(m_{\nu})h$; hence φ is completely determined by $\varphi(m_{\nu})$. Since $\varphi(m_{\nu}) \in M^{\mu}$, by Lemma 3.1, there exist some $c_x \in \mathcal{R}$ such that $\varphi(m_{\nu}) = \sum_{x \in \mathcal{D}_{\mu}} c_x m_{\mu} g_x$. By [ER, Lemma 10(49)], for each k, we have

$$\varphi(m_{\boldsymbol{\nu}}t_k) = \varphi(\zeta_{\mathbf{p}_{\boldsymbol{\nu}}(k)}m_{\boldsymbol{\nu}}) = \sum_{x \in \mathfrak{D}_{\boldsymbol{\mu}}} \zeta_{\mathbf{p}_{\boldsymbol{\nu}}(k)}c_x m_{\boldsymbol{\mu}}g_x.$$
(3.4)

Now assume that $c_y \neq 0$ for some $y \in \mathcal{D}_{\mu}$, which is equal to some $d(\mathfrak{s})$ for some row standard μ -tableau \mathfrak{s} . Then we have

$$(c_y m_{\boldsymbol{\mu}} g_y) t_k = c_y m_{\boldsymbol{\mu}} t_{kd(\mathfrak{s})^{-1}} g_y = \zeta_{\mathrm{P}_{\boldsymbol{\mu}}(kd(\mathfrak{s})^{-1})} c_y m_{\boldsymbol{\mu}} g_y.$$

Since $\mathfrak{s} = \mathfrak{t}^{\mu} d(\mathfrak{s})$, we have that $p_{\mu}(kd(\mathfrak{s})^{-1}) = p_{\mathfrak{s}}(k)$, and hence

$$(c_y m_\mu g_y) t_k = \zeta_{\mathbf{p}_s(k)} c_y m_\mu g_y. \tag{3.5}$$

By comparing (3.4) and (3.5), we have $p_{t\nu}(k) = p_{\mathfrak{s}}(k)$ for all $k = 1, \ldots, n$. This implies that $\alpha(\boldsymbol{\mu}) = \alpha(\boldsymbol{\nu})$. Thus, (i) is proved.

Now assume that $\alpha(\boldsymbol{\mu}) = \alpha(\boldsymbol{\nu}) = \alpha$. Since $p_{t\nu}(k) = p_{t\mu}(kd(\mathfrak{s})^{-1})$ for all $k = 1, \ldots, n$, we must have $y = d(\mathfrak{s}) \in \mathfrak{S}_{\alpha}$. Let d be the unique minimal length element in $\mathfrak{S}_{\boldsymbol{\mu}} y \mathfrak{S}_{\boldsymbol{\nu}}$. Then $d \in \mathcal{D}_{\boldsymbol{\mu}\boldsymbol{\nu}} \cap \mathfrak{S}_{\alpha}$, and a similar argument as in the proof of [Ma1, Theorem 4.8] implies that $c_d \neq 0$. Set $\varphi' = \varphi - c_d \varphi^d_{\boldsymbol{\mu}\boldsymbol{\nu}}$. Then $\varphi' \in \operatorname{Hom}_{Y_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}})$, and $\varphi'(m_{\boldsymbol{\nu}})$ can be written as $\varphi'(m_{\boldsymbol{\nu}}) = \sum_{x \in \mathcal{D}_{\boldsymbol{\mu}}} a_x m_{\boldsymbol{\mu}} g_x$, where $a_x = c_x$ if $\mathfrak{S}_{\boldsymbol{\mu}} x \mathfrak{S}_{\boldsymbol{\nu}} \neq \mathfrak{S}_{\boldsymbol{\mu}} d\mathfrak{S}_{\boldsymbol{\nu}}$, and $a_x = 0$ for $x \in \mathfrak{S}_{\boldsymbol{\mu}} d\mathfrak{S}_{\boldsymbol{\nu}}$ by the argument as in the proof of [Ma1, Theorem 4.8]. Hence, by induction we can write φ as a linear combination of $\varphi^d_{\boldsymbol{\mu}\boldsymbol{\nu}}$ with $d \in \mathcal{D}_{\boldsymbol{\mu}\boldsymbol{\nu}} \cap \mathfrak{S}_{\alpha}$ as required.

Finally, we have to show that $\{\varphi_{\mu\nu}^d \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_{\alpha}\}$ is linearly independent. This follows from the fact that $\varphi_{\mu\nu}^d(m_{\nu})$ is a linearly independent subset of M^{μ} , since the set $\{U_{\mu}g_w\}$ is linearly independent by the basis theorem for $Y_{r,n}$.

We write $M^{\nu*} = (M^{\nu})^* = Y_{r,n}m_{\nu}$. As a corollary to Proposition 3.4, we have the next result.

Corollary 3.5. Let $\mu, \nu \in \mathcal{C}_{r,n}$. Then $\operatorname{Hom}_{Y_{r,n}}(M^{\nu}, M^{\mu})$ and $M^{\nu*} \cap M^{\mu}$ are canonically isomorphic as \mathcal{R} -modules.

Proof. Every homomorphism φ in $\operatorname{Hom}_{Y_{r,n}}(M^{\nu}, M^{\mu})$ is determined by $\varphi(m_{\nu})$, and moreover, $\varphi(m_{\nu}) \in M^{\nu*} \cap M^{\mu}$ by Proposition 3.4. As a result, the map $\operatorname{Hom}_{Y_{r,n}}(M^{\nu}, M^{\mu}) \to M^{\nu*} \cap M^{\mu}$ given by $\varphi \mapsto \varphi(m_{\nu})$ is an isomorphism of \mathcal{R} -modules.

Remark 3.6. It is shown in [CR, 61.2] that whenever A is a quasi-hereditary algebra, $a \in A$ and J is an ideal of A then $\text{Hom}_A(aA, J) \cong Aa \cap J$. By [ChPA1, Proposition 10] (see also [C1, Corollary 4.5]), $Y_{r,n}$ is quasi-Frobenius, so this gives another proof of Corollary 3.5.

Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{C}_{r,n}$ and $\boldsymbol{\lambda} \in \mathbb{P}_{r,n}$. We assume that $\alpha(\boldsymbol{\mu}) = \alpha(\boldsymbol{\nu}) = \alpha(\boldsymbol{\lambda})$. For $S \in \mathbb{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $T \in \mathbb{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$, put

$$m_{\rm ST} = \sum_{\mathfrak{s},\mathfrak{t}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}},$$

where the sum is taken over all $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = \mathrm{S}$ and $\nu(\mathfrak{t}) = \mathrm{T}$.

Proposition 3.7. Suppose that $\mu, \nu \in \mathcal{C}_{r,n}$ with $\alpha(\mu) = \alpha(\nu)$. Then the set

$$\{m_{\mathrm{ST}} \mid \mathrm{S} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}, \boldsymbol{\mu}) \text{ and } \mathrm{T} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n}\}$$

is an \mathfrak{R} -basis of $M^{\boldsymbol{\nu}*} \cap M^{\boldsymbol{\mu}}$.

Proof. Since

$$m_{\mathrm{ST}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \boldsymbol{\mu}(\mathfrak{s}) = \mathrm{S}}} m_{\mathrm{T}\mathfrak{s}}^* = \sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \boldsymbol{\nu}(\mathfrak{t}) = \mathrm{T}}} m_{\mathrm{S}\mathfrak{t}}$$

we see that $m_{\mathrm{ST}} \in M^{\nu*} \cap M^{\mu}$ by Lemma 3.2. Moreover, the elements m_{ST} are linearly independent since the basis elements $m_{\mathfrak{st}}$ involved in the m_{ST} are distinct. Now suppose that $h \in M^{\nu*} \cap M^{\mu}$. Since $h \in Y_{r,n}$, we may express h with respect to the standard basis, that is, we may write $h = \sum r_{\mathfrak{st}} m_{\mathfrak{st}}$ for some $r_{\mathfrak{st}} \in \mathcal{R}$. Since $h \in M^{\mu}$, by Proposition 3.3 if $r_{\mathfrak{st}} \neq 0$ then $\mu(\mathfrak{s}) \in \mathcal{T}_0^+(\lambda, \mu)$ for some $\lambda \in \mathcal{P}_{r,n}$ and $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}}$ whenever $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$. Similarly, since $h \in M^{\nu*}$, if $r_{\mathfrak{st}} \neq 0$ then $\nu(\mathfrak{t}) \in \mathcal{T}_0^+(\lambda, \nu)$ for some $\lambda \in \mathcal{P}_{r,n}$ and $r_{\mathfrak{st}} = r_{\mathfrak{s}\mathfrak{t}'}$ whenever $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$. Consequently, if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}') \in \mathcal{T}_0^+(\lambda, \mu)$ and $\nu(\mathfrak{t}) = \nu(\mathfrak{t}') \in \mathcal{T}_0^+(\lambda, \nu)$, then $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}\mathfrak{t}'}$. This proves the proposition.

Definition 3.8. Suppose that $M_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} M^{\mu}$. We define the Yokonuma-Schur algebra $YS_n^r = YS_q(r, n)$ as the endomorphism algebra

$$YS_n^r = End_{Y_{r,n}}(M_n^r),$$

which is isomorphic to $\bigoplus_{\mu,\nu\in\mathcal{C}_{r,n}} \operatorname{Hom}_{Y_{r,n}}(M^{\nu}, M^{\mu}).$

Let $S \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $T \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$. In view of Proposition 3.7, we can define $\varphi_{ST} \in Hom_{Y_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}})$ by

$$\varphi_{\rm ST}(m_{\nu}h) = m_{\rm ST}h \tag{3.6}$$

for all $h \in Y_{r,n}$. We extend φ_{ST} to an element of YS_n^r by defining φ_{ST} to be zero on M^{κ} for $\nu \neq \kappa \in \mathcal{C}_{r,n}$. For any $\lambda \in \mathcal{P}_{r,n}$, let $\mathcal{T}_0^+(\lambda) = \bigcup_{\mu \in \mathcal{C}_{r,n}} \mathcal{T}_0^+(\lambda,\mu)$. We denote by $YS_{r,n}^{\triangleright \lambda}$ the \mathcal{R} -submodule of YS_n^r spanned by φ_{ST} such that $S, T \in \mathcal{T}_0^+(\nu)$ with $\nu \triangleright \lambda$. Then we have the next theorem.

Theorem 3.9. The Yokonuma-Schur algebra YS_n^r is free as an \mathcal{R} -module with a basis

$$\{\varphi_{\mathrm{ST}} \mid \mathrm{S}, \mathrm{T} \in \mathfrak{T}_0^+(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \}.$$

Moreover, this basis satisfies the following properties:

(i) The \Re -linear map $*: YS_n^r \to YS_n^r$ determined by $\varphi_{ST}^* = \varphi_{TS}$, for all $S, T \in \mathcal{T}_0^+(\lambda)$ and all $\lambda \in \mathcal{P}_{r,n}$, is an anti-automorphism of YS_n^r .

(ii) Let $T \in \mathcal{T}_0^+(\boldsymbol{\lambda})$ and $\varphi \in YS_n^r$. Then for each $V \in \mathcal{T}_0^+(\boldsymbol{\lambda})$, there exists $r_V = r_{V,T,\varphi} \in \mathcal{R}$ such that for all $S \in \mathcal{T}_0^+(\boldsymbol{\lambda})$, we have

$$\varphi_{\mathrm{ST}}\varphi \equiv \sum_{\mathrm{V}\in\mathfrak{T}_0^+(\boldsymbol{\lambda})} r_{\mathrm{V}}\varphi_{\mathrm{SV}} \mod \mathrm{YS}_{r,n}^{\rhd\boldsymbol{\lambda}}.$$

In particular, this basis $\{\varphi_{\rm ST}\}$ is a cellular basis of YS_n^r .

Proof. The proof is similar to that of [Ma1, Theorem 4.14] and [DJM, Theorem 6.6]. By Corollary 3.5 and Proposition 3.7, the set $\{\varphi_{ST}\}$ is an \mathcal{R} -basis of YS_n^r . Next we need to verify (i) and (ii).

(i) Let $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{C}_{r,n}$, and take $\mathbf{S} \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathbf{T} \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Then $\varphi_{\mathrm{ST}}^*(m_{\boldsymbol{\mu}}) = m_{\mathrm{TS}} = (m_{\mathrm{ST}})^* = (\varphi_{\mathrm{ST}}(m_{\boldsymbol{\nu}}))^*$. By the \mathcal{R} -linearity, we have $\varphi^*(m_{\boldsymbol{\mu}}) = (\varphi(m_{\boldsymbol{\nu}}))^*$ for any $\varphi \in \mathrm{Hom}_{\mathbf{Y}_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}})$. Given $\varphi \in \mathrm{Hom}_{\mathbf{Y}_{r,n}}(M^{\boldsymbol{\nu}}, M^{\boldsymbol{\mu}})$ and $\psi \in \mathrm{Hom}_{\mathbf{Y}_{r,n}}(M^{\boldsymbol{\kappa}}, M^{\boldsymbol{\lambda}})$, we may assume that $\boldsymbol{\mu} = \boldsymbol{\kappa}$ since otherwise $\psi \varphi = 0$. Write $\varphi(m_{\boldsymbol{\nu}}) = x_1 m_{\boldsymbol{\nu}}$ and $\psi(m_{\boldsymbol{\mu}}) = x_2 m_{\boldsymbol{\mu}}$ for some $x_1, x_2 \in \mathbf{Y}_{r,n}$. We have

$$(\psi\varphi)^{*}(m_{\lambda}) = (\psi\varphi(m_{\nu}))^{*} = (x_{2}x_{1}m_{\nu})^{*} = m_{\nu}x_{1}^{*}x_{2}^{*}$$
$$= \varphi^{*}(m_{\mu})x_{2}^{*} = \varphi^{*}(m_{\mu}x_{2}^{*}) = \varphi^{*}\psi^{*}(m_{\lambda}).$$

Hence, $(\psi \varphi)^* = \varphi^* \psi^*$ and * is an anti-automorphism.

(ii) Take $S \in \mathfrak{T}_0^+(\lambda, \mu)$, $T \in \mathfrak{T}_0^+(\lambda, \nu)$. We may assume that $\varphi \in \operatorname{Hom}_{Y_{r,n}}(M^{\kappa}, M^{\nu})$ for some $\kappa \in \mathcal{C}_{r,n}$ with $\alpha(\kappa) = \alpha(\nu)$. We have $\varphi(m_{\kappa}) = m_{\nu}h$ for some $h \in Y_{r,n}$. Then $\varphi_{ST}\varphi(m_{\kappa}) = m_{ST}h$. By Corollary 3.5, we see that $m_{ST}h \in M^{\kappa*} \cap M^{\mu}$. Hence by Proposition 3.7, we may write $m_{ST}h = \sum_{U,V} r_{UV}m_{UV}$, where $r_{UV} \in \mathcal{R}$, and the sum is over $U \in \mathfrak{T}_0^+(\alpha, \mu)$ and $V \in \mathfrak{T}_0^+(\alpha, \kappa)$ for some $\alpha \in \mathcal{P}_{r,n}$. By applying Theorem 2.7(ii), we can write $m_{ST}h$ as

$$m_{\rm ST}h = \sum_{\mathbf{V}\in\mathcal{T}_0^+(\boldsymbol{\lambda},\boldsymbol{\kappa})} r_{\mathbf{V}}m_{\rm SV} + \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{P}_{r,n}\\\boldsymbol{\alpha}\succ\boldsymbol{\lambda}}} \sum_{\substack{\mathbf{U}'\in\mathcal{T}_0^+(\boldsymbol{\alpha},\boldsymbol{\mu})\\\mathbf{V}'\in\mathcal{T}_0^+(\boldsymbol{\alpha},\boldsymbol{\kappa})}} r_{\mathbf{U}'\mathbf{V}'}m_{\mathbf{U}'\mathbf{V}'},$$

where $r_{\rm V}, r_{{\rm U}'{\rm V}'} \in \mathcal{R}$. Therefore, we have

$$\varphi_{\mathrm{ST}}\varphi \equiv \sum_{\mathrm{V}\in\mathcal{T}^+_0(\boldsymbol{\lambda},\boldsymbol{\kappa})} r_{\mathrm{V}}\varphi_{\mathrm{SV}} \mod \mathrm{YS}_{r,n}^{\boldsymbol{\triangleright}\boldsymbol{\lambda}}.$$

We are done.

For each $\lambda \in \mathcal{P}_{r,n}$, let $T^{\lambda} = \lambda(t^{\lambda})$. Then $T^{\lambda} \in \mathcal{T}_{0}^{+}(\lambda, \lambda)$ and T^{λ} is the unique semistandard λ -tableau of type λ . Moreover $\mathfrak{t} = \mathfrak{t}^{\lambda}$ is the unique element in $\mathrm{Std}(\lambda)$ such that $\lambda(\mathfrak{t}) = T^{\lambda}$. Thus, $m_{T\lambda T\lambda} = m_{\mathfrak{t}\lambda\mathfrak{t}\lambda} = m_{\lambda}$, and $\varphi_{\lambda} = \varphi_{T\lambda T\lambda}$ is the identity map on M^{λ} .

The Weyl module W^{λ} is defined as the right YS_n^r -submodule of $YS_n^r/YS_{r,n}^{r,\lambda}$ spanned by the image of φ_{λ} . For each $S \in \mathcal{T}_0^+(\lambda, \mu)$, we denote by φ_S the image of $\varphi_{T^{\lambda}S}$ in $YS_n^r/YS_{r,n}^{r,\lambda}$. Then by Theorem 3.9, we see that W^{λ} , as an \mathcal{R} -module, is free with basis $\{\varphi_S \mid S \in \mathcal{T}_0^+(\lambda)\}.$

The Weyl module W^{λ} possesses an associative symmetric bilinear form, which is completely determined by the equation

$$\varphi_{\mathrm{T}^{\lambda}\mathrm{S}}\varphi_{\mathrm{T}\mathrm{T}^{\lambda}} \equiv \langle \varphi_{\mathrm{S}}, \varphi_{\mathrm{T}} \rangle \varphi_{\lambda} \mod \mathrm{YS}_{r,n}^{\rhd_{\lambda}}$$

for all S, T $\in \mathcal{T}_0^+(\lambda)$. Note that $\langle \varphi_S, \varphi_T \rangle = 0$ unless S and T are semistandard tableaux of the same type. Let $L^{\lambda} = W^{\lambda}/\operatorname{rad} W^{\lambda}$, where $\operatorname{rad} W^{\lambda} = \{x \in W^{\lambda} \mid \langle x, y \rangle = 0 \text{ for all } y \in W^{\lambda}\}.$

12

Proposition 3.10. Suppose that $\mathcal{R} = \mathbb{K}$ is a field. Then for each $\lambda \in \mathcal{P}_{r,n}$, L^{λ} is an absolutely irreducible YS_n^r -module. Moreover, $\{L^{\lambda} \mid \lambda \in \mathcal{P}_{r,n}\}$ is a complete set of non-isomorphic irreducible YS_n^r -modules.

Proof. For each $\lambda \in \mathcal{P}_{r,n}$, we have

$$\varphi_{\mathrm{T}^{\lambda}\mathrm{T}^{\lambda}}\varphi_{\mathrm{T}^{\lambda}\mathrm{T}^{\lambda}} \equiv \langle \varphi_{\mathrm{T}^{\lambda}}, \varphi_{\mathrm{T}^{\lambda}} \rangle \varphi_{\lambda} \mod \mathrm{YS}_{r.n}^{\triangleright_{\lambda}}.$$

But since $\varphi_{T^{\lambda}T^{\lambda}}\varphi_{T^{\lambda}T^{\lambda}} = \varphi_{\lambda}$ is the identity map on M^{λ} , we see that $\langle \varphi_{T^{\lambda}}, \varphi_{T^{\lambda}} \rangle = 1$, and so L^{λ} is nonzero. Then the assertions follow from [GL, (3.4)].

If $\lambda, \mu \in \mathcal{P}_{r,n}$, let $d_{\lambda\mu}$ denote the composition multiplicity of L^{μ} as a composition factor of W^{λ} . Then $(d_{\lambda\mu})_{\lambda,\mu\in\mathcal{P}_{r,n}}$ is the decomposition matrix of YS_n^r . The theory of cellular algebras [GL, (3.6)] yields the following result.

Corollary 3.11. Suppose that $\mathcal{R} = \mathbb{K}$ is a field. $(d_{\lambda\mu})_{\lambda,\mu\in\mathcal{P}_{r,n}}$ is unitriangular. That is, for $\lambda, \mu \in \mathcal{P}_{r,n}$, we have $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.

Combining Proposition 3.10 with [GL, (3.10)], we have the next result.

Corollary 3.12. Suppose that $\mathcal{R} = \mathbb{K}$ is a field. The Yokonuma-Schur algebra YS_n^r is quasi-hereditary.

Remark 3.13. For each $\lambda \in \mathcal{P}_{r,n}$ and for each $\mathfrak{t} \in \operatorname{Std}(\lambda)$, let $m_{\mathfrak{t}} \in S^{\lambda}$ be the image of $m_{\mathfrak{t}\lambda\mathfrak{t}}$ under the map $Y_{r,n}/Y_{r,n}^{\rhd\lambda}$. Then $\{m_{\mathfrak{t}}\} = \{\overline{m}_{\lambda}g_{d(\mathfrak{t})}\}$ gives an \mathcal{R} -basis of S^{λ} . For $T \in \mathcal{T}_{0}^{+}(\lambda,\mu)$, put $m_{T} = \sum_{\mathfrak{t}} q^{l(d(\mathfrak{t}))+l(d(\mathfrak{t}^{\lambda}))}m_{\mathfrak{t}} \in S^{\lambda}$, where the sum is taken over all \mathfrak{t} such that $\mu(\mathfrak{t}) = T$. Since m_{T} is the image of $m_{T\lambda T}$, one obtains a well-defined map $\varphi_{T} \in$ $\operatorname{Hom}_{Y_{r,n}}(M^{\mu}, S^{\lambda})$ by $\varphi_{T}(m_{\mu}) = m_{T}$, which is regarded as an element of $\operatorname{Hom}_{Y_{r,n}}(M_{n}^{r}, S^{\lambda})$ by extending by 0 outside. In a similar way as in [Ma1, Proposition 4.15], we see that W^{λ} is isomorphic to the $\operatorname{YS}_{n}^{r}$ -submodule of $\operatorname{Hom}_{Y_{r,n}}(M_{n}^{r}, S^{\lambda})$ with basis $\{\varphi_{T} \mid T \in \mathcal{T}_{0}^{+}(\lambda)\}$.

4. Schur functors

In this section, we will follow the approach in [HM2, §4.3] to define an exact functor from the category of YS_n^r -modules to the category of $Y_{r,n}$ -modules. For an algebra A, let A-mod be the category of finite dimensional right A-modules.

Let $\dot{\mathbb{C}}_{r,n} = \mathbb{C}_{r,n} \cup \{\omega\}$, where ω is a dummy symbol. Set $M^{\omega} = Y_{r,n}$ and $\dot{M}_n^r = M_n^r \oplus M^{\omega}$. The extended Yokonuma-Schur algebra is the algebra

$$\operatorname{YS}_n^r = \operatorname{End}_{\operatorname{Y}_{r,n}}(M_n^r).$$

Suppose that $\lambda \in \mathcal{P}_{r,n}$, and set $\mathcal{T}_0^+(\lambda, \omega) := \operatorname{Std}(\lambda)$. Let $m_\omega = 1$ so that $M^\omega = m_\omega Y_{r,n}$. Let $\mathfrak{t}^\omega = 1$ and $m_{\mathfrak{t}^\omega \mathfrak{t}^\omega} = 1$. We regard YS_n^r as a subalgebra of YS_n^r in the obvious way.

Extending (3.6), if $\lambda \in \mathcal{P}_{r,n}$, $\mu, \nu \in \dot{\mathcal{C}}_{r,n}$, and $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$, we define

$$\varphi_{\rm ST}(m_{\boldsymbol{\nu}}h) = m_{\rm ST}h$$

for all $h \in Y_{r,n}$. Then $\varphi_{ST} \in Y\dot{S}_n^r$. For each $\lambda \in \mathcal{P}_{r,n}$, set $\dot{\mathcal{T}}_0^+(\lambda) = \mathcal{T}_0^+(\lambda) \cup \mathcal{T}_0^+(\lambda, \omega) = \mathcal{T}_0^+(\lambda) \cup Std(\lambda)$.

Proposition 4.1. The algebra YS_n^r is a cellular algebra with a cellular basis

 $\{\varphi_{\mathrm{ST}} \mid S, T \in \dot{\mathfrak{T}}_0^+(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n}\}.$

Moreover, if $\mathfrak{R} = \mathbb{K}$ is a field, then \dot{YS}_n^r is a quasi-hereditary algebra with Weyl modules $\{\dot{W}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_{r,n}\}$ and simple modules $\{\dot{L}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_{r,n}\}$.

Proof. By definition, YS_n^r is a subalgebra of YS_n^r and, as an \mathcal{R} -module,

 $\dot{\mathrm{YS}}_{n}^{r} = \mathrm{YS}_{n}^{r} \oplus \mathrm{Hom}_{\mathrm{Y}_{r,n}}(M^{\omega}, M_{n}^{r}) \oplus \mathrm{Hom}_{\mathrm{Y}_{r,n}}(M_{n}^{r}, M^{\omega}) \oplus \mathrm{End}_{\mathrm{Y}_{r,n}}(M^{\omega}, M^{\omega}).$

For $\boldsymbol{\mu} \in \dot{\mathbb{C}}_{r,n}$, there are isomorphisms of \mathcal{R} -modules $M^{\boldsymbol{\mu}} \cong \operatorname{Hom}_{Y_{r,n}}(M^{\omega}, M^{\boldsymbol{\mu}})$ given by $m_{\operatorname{St}} \mapsto \varphi_{\operatorname{St}}$, for $\mathrm{S} \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with some $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$. For $\boldsymbol{\nu} \in \dot{\mathbb{C}}_{r,n}$, there are isomorphisms of \mathcal{R} -modules $M^{\boldsymbol{\nu}*} \cong \operatorname{Hom}_{Y_{r,n}}(M^{\boldsymbol{\nu}}, M^{\omega})$ given by $m_{\mathfrak{sT}} \mapsto \varphi_{\mathfrak{sT}}$, for $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\mathrm{T} \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$ with some $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, where $m_{\mathfrak{sT}} = m_{\mathrm{Ts}}^*$. Therefore, the elements in the statement of this proposition give a basis of YS_n^r by Proposition 3.3 and Theorem 3.9.

Now suppose that $\mathcal{R} = \mathbb{K}$ is a field. Repeating the arguments from Theorem 3.9 and Proposition 3.10 shows that \dot{YS}_n^r is a quasi-hereditary cellular algebra.

By Proposition 4.1, there exist Weyl modules \dot{W}^{λ} and simple modules $\dot{L}^{\lambda} = \dot{W}^{\lambda}/\operatorname{rad} \dot{W}^{\lambda}$ for \dot{YS}_{n}^{r} , for each $\lambda \in \mathcal{P}_{r,n}$. Let $\{\varphi_{S} \mid S \in \dot{\mathcal{T}}_{0}^{+}(\lambda)\}$ be the basis of \dot{W}^{λ} . For each $\mu \in \mathcal{C}_{r,n}$, let φ_{μ} be the identity map on M^{μ} . We extend φ_{μ} to an element of YS_{n}^{r} by defining φ_{μ} to be zero on M^{κ} for $\mu \neq \kappa \in \mathcal{C}_{r,n}$. In particular, $\varphi_{\mu} = \varphi_{T^{\mu}T^{\mu}}$ if $\mu \in \mathcal{P}_{r,n}$. As an \mathcal{R} -module, every YS_{n}^{r} -module M has a weight space decomposition

$$M = \bigoplus_{\boldsymbol{\mu} \in \mathcal{C}_{r,n}} M_{\boldsymbol{\mu}}, \quad \text{where } M_{\boldsymbol{\mu}} = M\varphi_{\boldsymbol{\mu}}.$$

$$\tag{4.1}$$

Set $\varphi_n^r = \sum_{\boldsymbol{\mu} \in \mathcal{C}_{r,n}} \varphi_{\boldsymbol{\mu}}$ and let φ_{ω} be the identity map on $M^{\omega} = Y_{r,n}$. Then φ_n^r is the identity element of YS_n^r and $\varphi_n^r + \varphi_{\omega}$ is the identity element of $Y\dot{S}_n^r$. By definition, φ_n^r and φ_{ω} are both idempotents in $Y\dot{S}_n^r$ and $\varphi_n^r Y\dot{S}_n^r \varphi_n^r \cong YS_n^r$. Therefore, by [HM2, (2.10)], there are exact functors

$$\dot{\mathbf{F}}_n^{\omega}: \mathbf{Y} \dot{\mathbf{S}}_n^r - \mathbf{mod} \to \mathbf{Y} \mathbf{S}_n^r - \mathbf{mod}, \quad \dot{\mathbf{G}}_n^{\omega}: \mathbf{Y} \mathbf{S}_n^r - \mathbf{mod} \to \mathbf{Y} \dot{\mathbf{S}}_n^r - \mathbf{mod}$$

given by $\dot{\mathbf{F}}_n^{\omega}(M) = M\varphi_n^r$ and $\dot{\mathbf{G}}_n^{\omega}(N) = N \otimes_{\mathrm{YS}_n^r} \varphi_n^r \mathrm{Y\dot{S}}_n^r$. By [HM, §2.4], there are functors $\mathrm{H}_n^{\omega} := \mathrm{H}_{\varphi_n^r}, \ \mathrm{O}_n^{\omega} := \mathrm{O}_{\varphi_n^r}, \ \mathrm{O}_{\omega}^n := \mathrm{O}_{\varphi_n^r}$ from $\mathrm{Y\dot{S}}_n^r - \mathrm{mod}$ to $\mathrm{YS}_n^r - \mathrm{mod}$ such that $\mathrm{H}_n^{\omega}(M) = M/\mathrm{O}_n^{\omega}(M)$.

Lemma 4.2. Suppose that $\mathcal{R} = \mathbb{K}$ is a field. Then the functors \dot{F}_n^{ω} and \dot{G}_n^{ω} induce mutually inverse equivalences of categories between \dot{YS}_n^r -mod and \dot{YS}_n^r -mod. Moreover, $\dot{F}_n^{\omega}(\dot{W}^{\lambda}) \cong W^{\lambda}$ and $\dot{F}_n^{\omega}(\dot{L}^{\lambda}) \cong L^{\lambda}$ for all $\lambda \in \mathcal{P}_{r,n}$.

Proof. Let M be a YS_n^r -module. Then, extending (4.1), M has a weight space decomposition

$$M = \bigoplus_{\mu \in \dot{\mathcal{C}}_{r,n}} M_{\mu}$$
, where $M_{\mu} = M \varphi_{\mu}$.

Then, essentially by definition, $\dot{\mathbf{F}}_{n}^{\omega}(M) = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_{r,n}} M_{\boldsymbol{\lambda}}$. That is, $\dot{\mathbf{F}}_{n}^{\omega}$ removes the ω -weight space of M. In particular, $\dot{\mathbf{F}}_{n}^{\omega}(\dot{W}^{\boldsymbol{\lambda}}) = W^{\boldsymbol{\lambda}}$ and $\dot{\mathbf{F}}_{n}^{\omega}(\dot{L}^{\boldsymbol{\lambda}}) = L^{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$. The fact that $\dot{\mathbf{F}}_{n}^{\omega}(\dot{L}^{\boldsymbol{\mu}}) = L^{\boldsymbol{\mu}}$ for all $\boldsymbol{\mu} \in \mathcal{P}_{r,n}$ implies that $O_{\omega}^{n}(M) = M$, $O_{n}^{\omega}(M) = 0$ for all $M \in Y\dot{\mathbf{S}}_{n}^{r}$ -mod. Therefore, \mathbf{H}_{n}^{ω} is the identity functor and $\dot{\mathbf{G}}_{n}^{\omega} \cong \mathbf{H}_{n}^{\omega} \circ \dot{\mathbf{G}}_{n}^{\omega}$. Hence, this lemma is an application of the theory of quotient functors given in [HM2, Theorem 2.11].

The identity map φ_{ω} on $Y_{r,n} = M^{\omega}$ is idempotent in YS_n^r and there is an isomorphism of \mathcal{R} -algebras $\varphi_{\omega} YS_n^r \varphi_{\omega} \cong Y_{r,n}$. Therefore, by [HM2, (2.10)], there are functors

$$\dot{\mathbf{F}}_n^r : \dot{\mathbf{YS}}_n^r - \mathrm{mod} \to \mathbf{Y}_{r,n} - \mathrm{mod}, \quad \dot{\mathbf{G}}_n^r : \mathbf{Y}_{r,n} - \mathrm{mod} \to \dot{\mathbf{YS}}_n^r - \mathrm{mod}$$

given by $\dot{\mathbf{F}}_n^r(M) = M\varphi_\omega = M_\omega$ and $\dot{\mathbf{G}}_n^r(N) = N \otimes_{\mathbf{Y}_{r,n}} \varphi_\omega \mathbf{Y} \dot{\mathbf{S}}_n^r$.

Proposition 4.3. Suppose that $\mathfrak{R} = \mathbb{K}$ is a field. Then there is an exact functor F_n^r : $YS_n^r - \text{mod} \to Y_{r,n} - \text{mod}$ given by $F_n^r(M) = (M \otimes_{YS_n^r} \varphi_n^r Y\dot{S}_n^r)\varphi_\omega$, for $M \in YS_n^r - \text{mod}$, such that if $\lambda, \mu \in \mathfrak{P}_{r,n}$, then $F_n^r(W^{\lambda}) \cong S^{\lambda}$, and

$$\mathbf{F}_{n}^{r}(L^{\boldsymbol{\mu}}) \cong \begin{cases} D^{\boldsymbol{\mu}} & \text{if } \boldsymbol{\mu} \in \mathcal{K}_{r,n}; \\ 0 & \text{if } \boldsymbol{\mu} \notin \mathcal{K}_{r,n}. \end{cases}$$
(4.2)

Proof. By definition, $\mathbf{F}_n^r = \dot{\mathbf{F}}_n^r \circ \dot{\mathbf{G}}_n^\omega$, so \mathbf{F}_n^r is an exact functor from \mathbf{YS}_n^r -mod to $\mathbf{Y}_{r,n}$ -mod. The functor $\dot{\mathbf{F}}_n^r$ is nothing more than projection onto the ω -weight space. Hence, if $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, then $\dot{\mathbf{F}}_n^r(\dot{W}^{\boldsymbol{\lambda}})$ is spanned by the maps $\{\varphi_t \mid t \in \mathrm{Std}(\boldsymbol{\lambda})\}$, since $\mathcal{T}_0^+(\boldsymbol{\lambda},\omega) = \mathrm{Std}(\boldsymbol{\lambda})$. The map $\varphi_t \mapsto m_t$, for $\mathbf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, defines an isomorphism $\dot{\mathbf{F}}_n^r(\dot{W}^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$ of $\mathbf{Y}_{r,n}$ -modules. Therefore, $\mathbf{F}_n^r(W^{\boldsymbol{\lambda}}) \cong S^{\boldsymbol{\lambda}}$ by Lemma 4.2.

By [HM2, Theorem 2.11], $F_n^r(L^{\mu})$ is an irreducible $Y_{r,n}$ -module whenever it is nonzero. Using the fact that $F_n^r(W^{\lambda}) \cong S^{\lambda}$ and Corollary 3.11, a straightforward argument by induction on the dominance ordering shows that $F_n^r(L^{\mu}) \cong D^{\mu}$ if $\mu \in \mathcal{K}_{r,n}$ and that $F_n^r(L^{\mu}) = 0$ otherwise.

Since \mathbf{F}_n^r is exact, we obtain the promised relationship between the decomposition numbers of \mathbf{YS}_n^r and $\mathbf{Y}_{r,n}$.

Corollary 4.4. Suppose that $\mathcal{R} = \mathbb{K}$ is a field and that $\lambda \in \mathcal{P}_{r,n}$, $\mu \in \mathcal{K}_{r,n}$. Then we have $[S^{\lambda} : D^{\mu}] = [W^{\lambda} : L^{\mu}]$.

Lemma 4.5. (A double centralizer property) There are canonical isomorphisms of algebras such that $Y\dot{S}_n^r = End_{Y_{r,n}}(\dot{M}_n^r)$ and $Y_{r,n} = End_{Y\dot{S}_n^r}(\dot{M}_n^r)$. In particular, the functor \dot{F}_n^r is fully faithful on projectives.

Proof. The first isomorphism is the definition of $Y\dot{S}_n^r$, whereas the second follows directly from the definition of $Y\dot{S}_n^r$ because

$$\mathbf{Y}_{r,n} \cong \operatorname{Hom}_{\mathbf{Y}_{r,n}}(\mathbf{Y}_{r,n},\mathbf{Y}_{r,n}) \cong \varphi_{\omega} \mathbf{Y} \mathbf{S}_{n}^{r} \varphi_{\omega} \cong \operatorname{End}_{\mathbf{Y} \mathbf{S}_{n}^{r}}(\varphi_{\omega} \mathbf{Y} \mathbf{S}_{n}^{r}),$$

and $\varphi_{\omega} \dot{\mathrm{YS}}_n^r \cong \dot{M}_n^r$ as a right $\dot{\mathrm{YS}}_n^r$ -module.

15

Corollary 4.6. YS_n^r is a quasi-hereditary cover of $Y_{r,n}$ in the sense of Rouquier [Ro, Definition 4.34].

Proof. Recall that $\dot{M}_n^r \cong \varphi_\omega Y\dot{S}_n^r$ is a projective $Y\dot{S}_n^r$ -module. Using the Morita equivalence between $Y\dot{S}_n^r$ and YS_n^r , we see that M_n^r is a projective YS_n^r -module. Because \dot{F}_n^r is fully faithful on projective modules by Lemma 4.5 and F_n^r is the composition of \dot{F}_n^r with an equivalence of categories, so is F_n^r . This implies that YS_n^r is a quasi-hereditary cover of $Y_{r,n}$ in the sense of Rouquier [Ro, Definition 4.34].

5. TILTING MODULES

In this section, we introduce the tilting modules for YS_n^r and the closely related Young modules for $Y_{r,n}$ following [Ma2]. Throughout this section we assume that \mathbb{K} is a field, which can be regarded as an \mathcal{R} -algebra.

5.1. Young modules. Recall from (4.1) that every YS_n^r -module has a weight space decomposition. Analogously, as a right YS_n^r -module, the regular representation of YS_n^r has a decomposition into a direct sum of left weight spaces

$$YS_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} Z^{\mu}, \quad \text{where } Z^{\mu} = \varphi_{\mu} YS_n^r \text{ for } \mu \in \mathcal{C}_{r,n}.$$

The next lemma gives some properties of the right YS_n^r -module Z^{μ} .

Lemma 5.1. Assume that $\mu \in \mathcal{C}_{r,n}$. Then the following hold: (i) Z^{μ} is free as an \mathcal{R} -module with a basis

$$\{\varphi_{\mathrm{ST}} \mid \mathrm{S} \in \mathfrak{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathrm{T} \in \mathfrak{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ for some } \boldsymbol{\nu} \in \mathfrak{C}_{r,n} \text{ and } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \}.$$

- (ii) Let $\mathfrak{M}^{\mu} = \operatorname{Hom}_{Y_{r,n}}(M_n^r, M^{\mu})$. As right YS_n^r -modules, we have $Z^{\mu} \cong \mathfrak{M}^{\mu}$.
- (iii) As $Y_{r,n}$ -modules, we have $F_n^r(Z^{\mu}) \cong M^{\mu}$.

Proof. (i) It follows from Theorem 3.9.

(ii) It follows from (i).

(iii) We have

$$\mathbf{F}_{n}^{r}(Z^{\boldsymbol{\mu}}) = \dot{\mathbf{F}}_{n}^{r} \circ \dot{\mathbf{G}}_{n}^{\omega}(\varphi_{\boldsymbol{\mu}}\mathbf{Y}\mathbf{S}_{n}^{r}) = \dot{\mathbf{F}}_{n}^{r}(\varphi_{\boldsymbol{\mu}}\mathbf{Y}\mathbf{S}_{n}^{r}\otimes_{\mathbf{Y}\mathbf{S}_{n}^{r}}\varphi_{n}^{r}\mathbf{Y}\dot{\mathbf{S}}_{n}^{r})$$
$$\cong \dot{\mathbf{F}}_{n}^{r}(\varphi_{\boldsymbol{\mu}}\mathbf{Y}\dot{\mathbf{S}}_{n}^{r}) \cong \mathrm{Hom}_{\mathbf{Y}_{r,n}}(\mathbf{Y}_{r,n}, M^{\boldsymbol{\mu}}) \cong M^{\boldsymbol{\mu}}.$$

We are done.

The next lemma claims that each Z^{μ} has a Weyl filtration.

Lemma 5.2. Assume that $\mu \in \mathcal{C}_{r,n}$. Then Z^{μ} has a Weyl filtration

$$Z^{\boldsymbol{\mu}} = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

such that for each $1 \leq i \leq k$ there exists some $\lambda_i \in \mathcal{P}_{r,n}$ with $\alpha(\lambda_i) = \alpha(\mu)$ satisfying $M_i/M_{i+1} \cong W^{\lambda_i}$. Moreover, $\sharp\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \sharp T_0^+(\lambda, \mu)$ for each $\lambda \in \mathcal{P}_{r,n}$ with $\alpha(\lambda) = \alpha(\mu)$.

Proof. Choose a total ordering $\{S_1, \ldots, S_k\}$ on the set $\cup_{\boldsymbol{\lambda} \in \mathcal{P}_{r,n}} \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that i > jwhenever $\boldsymbol{\lambda}_i \triangleright \boldsymbol{\lambda}_j$, where $S_i \in \mathcal{T}_0^+(\boldsymbol{\lambda}_i, \boldsymbol{\mu}), S_j \in \mathcal{T}_0^+(\boldsymbol{\lambda}_j, \boldsymbol{\mu})$. For each $1 \le i \le k$, let M_i be the \mathcal{R} -submodule of $Z^{\boldsymbol{\mu}}$ with basis $\{\varphi_{S_jT} \mid j \ge i \text{ and } T \in \mathcal{T}_0^+(\boldsymbol{\lambda}_j)\}$. Then M_i is a YS_n^r -module by Theorem 3.9. Further, there is an isomorphism of YS_n^r -modules $W^{\boldsymbol{\lambda}_i} \cong M_i/M_{i+1}$ given by $\varphi_T \mapsto \varphi_{S_iT} + M_{i+1}$ for $T \in \mathcal{T}_0^+(\boldsymbol{\lambda}_i)$, because $M_i \cap YS_{r,n}^{\triangleright \boldsymbol{\lambda}_i} \subseteq M_{i+1}$. Since YS_n^r is quasi-hereditary, $[Z^{\boldsymbol{\mu}} : W^{\boldsymbol{\lambda}}]$ is independent of the choice of Weyl filtration. \Box

Applying the Schur functor F_n^r , by Proposition 4.3 and Lemma 5.1(iii), we also have the following result.

Corollary 5.3. Assume that $\mu \in \mathcal{C}_{r,n}$. Then M^{μ} has a Specht filtration

 $M^{\boldsymbol{\mu}} = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$

such that for each $1 \leq i \leq k$ there exists some $\lambda_i \in \mathcal{P}_{r,n}$ with $\alpha(\lambda_i) = \alpha(\mu)$ satisfying $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, $\sharp\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \sharp \mathcal{T}_0^+(\lambda, \mu)$ for each $\lambda \in \mathcal{P}_{r,n}$ with $\alpha(\lambda) = \alpha(\mu)$.

Since φ_{μ} is an idempotent in YS_n^r , Z^{μ} is a projective YS_n^r -module. Notice that if $\mathcal{T}_0(\lambda,\mu) \neq \emptyset$, then $\lambda \geq \mu$. Thus, W^{λ} appears in Z^{μ} only if $\lambda \geq \mu$. For each $\mu \in \mathcal{P}_{r,n}$, let P^{μ} be the projective cover of L^{μ} . Then by [Ma1, Lemma 2.16], P^{μ} has a filtration by Weyl modules in which W^{λ} appears with multiplicity $[P^{\mu}:W^{\lambda}] = [W^{\lambda}:L^{\mu}]$. From these facts, we can easily get the following lemma.

Lemma 5.4. Assume that $\mu \in \mathcal{P}_{r,n}$. Then

$$Z^{\boldsymbol{\mu}} \cong P^{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} c_{\boldsymbol{\lambda} \boldsymbol{\mu}} P^{\boldsymbol{\lambda}}$$

for some non-negative integer $c_{\lambda\mu}$.

Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$. Recall that \mathcal{M}^{μ} is defined as $\operatorname{Hom}_{Y_{r,n}}(M_n^r, M^{\mu})$. Thus, we can define a YS_n^r -module homomorphism $\Phi_{ST} : \mathcal{M}^{\nu} \to \mathcal{M}^{\mu}$ by $\Phi_{ST}(f) = \varphi_{ST}f$ for all $f \in \mathcal{M}^{\nu}$. In fact, these maps give a basis of $\operatorname{Hom}_{YS_n^r}(\mathcal{M}^{\nu}, \mathcal{M}^{\mu})$.

Lemma 5.5. Suppose that $\mu, \nu \in \mathbb{C}_{r,n}$. Then $\operatorname{Hom}_{\operatorname{YS}_n^r}(\mathcal{M}^{\nu}, \mathcal{M}^{\mu})$ is free as an \mathbb{R} -module with basis $\{\Phi_{\operatorname{ST}} \mid \mathrm{S} \in \mathcal{T}_0^+(\lambda, \mu), \mathrm{T} \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$

For each $\lambda \in \mathcal{P}_{r,n}$, let $Y^{\lambda} = \mathcal{F}_n^r(P^{\lambda})$, which we call a Young module of $Y_{r,n}$.

Proposition 5.6. Suppose that $\mathcal{R} = \mathbb{K}$ is a field and that $\lambda \in \mathcal{P}_{r,n}$. Then the following hold:

(i) Each Y^{λ} is an indecomposable $Y_{r,n}$ -module.

(ii) If μ is another r-partition of n, then $Y^{\lambda} \cong Y^{\mu}$ if and only if $\lambda = \mu$.

(iii) We have

$$M^{\lambda} \cong Y^{\lambda} \oplus \bigoplus_{\nu \succ \lambda} c_{\nu\lambda} Y^{\nu}.$$

(iv) The Young module Y^{λ} has a Specht filtration in which S^{μ} appears with multiplicity $[Y^{\lambda}: S^{\mu}] = [W^{\mu}: L^{\lambda}].$

Proof. (i) By Corollary 4.6, the functor F_n^r is fully faithful on projective modules, so $\operatorname{End}_{Y_{r,n}}(Y^{\lambda}) \cong \operatorname{End}_{YS_n^r}(P^{\lambda})$ is a local ring since P^{λ} is indecomposable.

(ii) If $Y^{\lambda} \cong Y^{\mu}$, then $\operatorname{Hom}_{Y_{r,n}}(Y^{\lambda}, Y^{\mu})$ contains an isomorphism and this lifts to give an isomorphism $P^{\mu} \cong P^{\lambda}$, so $\lambda = \mu$.

(iii) Applying the Schur functor F_n^r , it follows from Lemma 5.1(iii) and Lemma 5.4.

(iv) Recall that P^{λ} has a Weyl filtration $P^{\lambda} = P_1 \supset P_2 \supset \cdots P_k \supset P_{k+1} = 0$. Moreover, for each $\mu \in \mathcal{P}_{r,n}$, $\sharp \{1 \leq i \leq k \mid P_i/P_{i+1} \cong W^{\mu}\} = [P^{\lambda} : W^{\mu}] = [W^{\mu} : L^{\lambda}]$. Setting $Y_i = F_n^r(P_i)$, and using Proposition 4.3, gives a filtration of Y^{λ} with the required properties.

The next proposition identify the projective Young modules, and its proof is similar to that of [HM2, Proposition 5.9].

Proposition 5.7. Suppose that $\mu \in \mathcal{K}_{r,n}$. Then Y^{μ} is the projective cover of D^{μ} .

Proof. Recall that P^{μ} is the projective cover of L^{μ} and $F_n^r(P^{\mu}) = Y^{\mu}$, $F_n^r(L^{\mu}) = D^{\mu}$ if $\mu \in \mathcal{K}_{r,n}$. Since F_n^r is exact, there is a surjective map $Y^{\mu} \mapsto D^{\mu}$. Therefore, it suffices to show that Y^{μ} is projective since it is indecomposable by Proposition 5.6(i).

Recall that $\dot{YS}_n^r = End_{Y_{r,n}}(\dot{M}_n^r)$, where $\dot{M}_n^r = M_n^r \oplus Y_{r,n}$. There is also a Schur functor \dot{F}_n^r from \dot{YS}_n^r -mod to $Y_{r,n}$ -mod given by $\dot{F}_n^r(M) = M\varphi_{\omega}$. In particular, $\dot{F}_n^r(\dot{M}_n^r) \cong Y_{r,n}$ as right $Y_{r,n}$ -modules.

As a \dot{YS}_n^r -module, $\dot{M}_n^r \cong \varphi_{\omega} \dot{YS}_n^r$. In particular, \dot{M}_n^r is a projective \dot{YS}_n^r -module. If $\lambda \in \mathcal{P}_{r,n}$, let \dot{P}^{λ} be the projective cover of the irreducible \dot{YS}_n^r -module \dot{L}^{λ} . The multiplicity of \dot{P}^{λ} as a summand of \dot{M}_n^r is equal to

$$\dim \operatorname{Hom}_{\mathrm{Y}\dot{\mathrm{S}}_n^r}(\dot{M}_n^r, \dot{L}^{\boldsymbol{\lambda}}) = \dim \operatorname{Hom}_{\mathrm{Y}\dot{\mathrm{S}}_n^r}(\varphi_{\omega} \mathrm{Y}\dot{\mathrm{S}}_n^r, \dot{L}^{\boldsymbol{\lambda}}) = \dim \dot{L}^{\boldsymbol{\lambda}}\varphi_{\omega} = \dim D^{\boldsymbol{\lambda}}.$$

Consequently, $\dot{M}_n^r \cong \bigoplus_{\lambda \in \mathcal{K}_{r,n}} (\dim D^{\lambda}) \dot{P}^{\lambda}$ as a $Y\dot{S}_n^r$ -module. By definition, $Y^{\lambda} = F_n^r(P^{\lambda}) = \dot{F}_n^r(\dot{P}^{\lambda})$ for all $\lambda \in \mathcal{P}_{r,n}$. Therefore,

$$\mathbf{Y}_{r,n} \cong \dot{\mathbf{F}}_n^r(\dot{M}_n^r) \cong \bigoplus_{\boldsymbol{\lambda} \in \mathcal{K}_{r,n}} (\dim D^{\boldsymbol{\lambda}}) Y^{\boldsymbol{\lambda}}$$

as a right $Y_{r,n}$ -module. The result follows.

5.2. Twisted Young modules. Let $\mathcal{Z} = \mathbb{Z}[\frac{1}{r}][\dot{q}, \dot{q}^{-1}, \zeta]$, where \dot{q} is an indeterminate, and let $Y_{r,n}^{\mathcal{Z}}$ be the Yokonuma-Hecke algebra over \mathcal{Z} . It is easy to see that $Y_{r,n}^{\mathcal{Z}}$ has a \mathbb{Z} -algebra involution ' which is determined by

$$g'_i = g_i, \quad \dot{q}' = -\dot{q}^{-1}, \quad \text{and } t'_j = t_j, \quad \zeta'_j = \zeta_{r-j+1}$$

for $1 \le i \le n-1$ and $1 \le j \le n$.

For each $\mu \in \mathcal{C}_{r,n}$, let $y_{\mu} = (x_{\mu})' = \sum_{w \in \mathfrak{S}_{\mu}} (-\dot{q})^{-l(w)} g_w$, and set $n_{\mu} := (U_{\mu} x_{\mu})' = U'_{\mu} y_{\mu}$. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda)$. We define $n_{\mathfrak{s}\mathfrak{t}} := g^*_{d(\mathfrak{s})} n_{\lambda} g_{d(\mathfrak{t})}$. Then by definition we have $n_{\mathfrak{s}\mathfrak{t}} = (m_{\mathfrak{s}\mathfrak{t}})'$. Because ' is a \mathbb{Z} -algebra involution, $\{n_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis of $Y^{\mathbb{Z}}_{r,n}$ by Theorem 2.7. The ring \mathcal{R} is naturally a \mathcal{Z} -module under specialization; that is, \dot{q} acts on \mathcal{R} as multiplication by q. Because $Y_{r,n}$ is \mathcal{R} -free, this induces an isomorphism of \mathcal{R} -algebras $Y_{r,n} \cong Y^{\mathbb{Z}}_{r,n} \otimes_{\mathbb{Z}} \mathcal{R}$ via $g_i \mapsto g_i \otimes \mathbb{1}_{\mathcal{R}}$ $(1 \leq i \leq n-1)$ and $t_j \mapsto t_j \otimes \mathbb{1}_{\mathcal{R}}$ $(1 \leq j \leq n)$.

18

Hereafter, we will identify the algebra $Y_{r,n}$ and $Y_{r,n}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ via the isomorphism above. Thus, we have the following result.

The Yokonuma-Hecke algebra $Y_{r,n}$ is free as an \mathbb{R} -module with a cellular basis $\{n_{\mathfrak{st}}|\mathfrak{s},\mathfrak{t}\in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$

Now we can apply the general theory of cellular algebras. For each $\lambda \in \mathcal{P}_{r,n}$, we define the dual Specht module S_{λ} to be the right $Y_{r,n}$ -module $(n_{\lambda} + Y_{\triangleright\lambda}^{r,n})Y_{r,n}$, where $Y_{\triangleright\lambda}^{r,n} = (Y_{r,n}^{\triangleright\lambda})'$ is the two-sided ideal of $Y_{r,n}$ with basis n_{uv} with $u, v \in Std(\nu)$ for various $\nu \in \mathcal{P}_{r,n}$ such that $\nu \triangleright \lambda$. Then S_{λ} is \mathcal{R} -free with basis $\{n_t | t \in Std(\lambda)\}$, where $n_t = n_{t\lambda t} + Y_{\triangleright\lambda}^{r,n}$. Let $D_{\lambda} = S_{\lambda}/radS_{\lambda}$, where $radS_{\lambda}$ is the radical of the bilinear form on S_{λ} which is defined with respect to the cellular basis $\{n_{st}\}$.

For each $\mu \in \mathcal{C}_{r,n}$, let $N^{\mu} = n_{\mu} Y_{r,n}$. If $S \in \mathcal{T}_{0}^{+}(\lambda, \mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we define

$$n_{\mathrm{St}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = \mathrm{S}}} (-q)^{-l(d(\mathfrak{s})) - l(d(\mathfrak{t}))} n_{\mathfrak{st}}.$$

From the definition, we have $n_{\text{St}} = (m_{\text{St}})'$. Therefore, Proposition 3.3 and the usual specialization argument show that the following holds.

Corollary 5.8. Suppose that $\mu \in \mathbb{C}_{r,n}$. Then N^{μ} is free as an \mathbb{R} -module with basis $\{n_{\mathrm{St}} \mid \mathrm{S} \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$

Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{C}_{r,n}$ and $\boldsymbol{\lambda} \in \mathbb{P}_{r,n}$. Suppose that $\alpha(\boldsymbol{\mu}) = \alpha(\boldsymbol{\nu}) = \alpha(\boldsymbol{\lambda})$. For $S \in \mathbb{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $T \in \mathbb{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$, let

$$n_{\mathrm{ST}} = \sum_{\substack{\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\\ \boldsymbol{\mu}(\mathfrak{s}) = \mathrm{S}, \, \boldsymbol{\nu}(\mathfrak{t}) = \mathrm{T}}} (-q)^{-l(d(\mathfrak{s})) - l(d(\mathfrak{t}))} n_{\mathfrak{st}}.$$

We can now define the twisted Yokonuma-Schur algebra as

$$YS_r^n = End_{Y_{r,n}}(N_n^r)$$

where $N_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} N^{\mu}$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$, we can also define the homomorphism φ'_{ST} by $\varphi'_{ST}(n_{\alpha}h) = \delta_{\alpha\nu}n_{ST}h$ for all $h \in Y_{r,n}$ and $\alpha \in \mathcal{C}_{r,n}$. Then $\varphi'_{ST} \in YS_r^n$. The proof of the next proposition is in exactly the same way as that of [Ma2, Proposition 4.3], and we skip the details.

Proposition 5.9. (i) The twisted Yokonuma-Schur algebra YS_r^n is free as an \mathcal{R} -module with a cellular basis

$$\{\varphi'_{\mathrm{ST}} \mid \mathrm{S}, \mathrm{T} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathfrak{C}_{r,n} \text{ and } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \}.$$

- (ii) The twisted Yokonuma-Schur algebra YS_r^n is quasi-hereditary.
- (iii) The \mathcal{R} -algebras YS_n^r and YS_n^r are canonically isomorphic.

Let W_{λ} and L_{λ} ($\lambda \in \mathcal{P}_{r,n}$) be the Weyl modules and simple modules of YS_r^n , respectively; they are defined in exactly the same way as the corresponding modules for YS_n^r . As in Section 4, we can define an exact Schur functor F_r^n from YS_r^n -mod to $Y_{r,n}$ -mod.

Moreover, we have $F_r^n(W_{\lambda}) \cong S_{\lambda}$, $F_r^n(L_{\lambda}) \cong D_{\lambda}$ and $[W_{\lambda} : L_{\mu}] = [S_{\lambda} : D_{\mu}]$ whenever $D_{\mu} \neq 0$.

For each $\lambda \in \mathcal{P}_{r,n}$, let P_{λ} be the projective cover of L_{λ} . Define $Y_{\lambda} = F_r^n(P_{\lambda})$, which is called a twisted Young module. The next proposition can be proved in exactly the same way as in Proposition 5.6.

Proposition 5.10. Suppose that $\mathcal{R} = \mathbb{K}$ is a field and that $\mu \in \mathcal{P}_{r,n}$. Then we have

(i) Each Y_{μ} is an indecomposable $Y_{r,n}$ -module.

(ii) If λ is another r-partition of n, then $Y_{\lambda} \cong Y_{\mu}$ if and only if $\lambda = \mu$. (iii)

$$N^{\boldsymbol{\mu}} \cong Y_{\boldsymbol{\mu}} \oplus \bigoplus_{\boldsymbol{\lambda} \rhd \boldsymbol{\mu}} c_{\boldsymbol{\lambda} \boldsymbol{\mu}} Y_{\boldsymbol{\lambda}},$$

where the integers $c_{\lambda\mu}$ are the same as those appearing in Lemma 5.4.

(iv) The twisted Young module Y_{μ} has a dual Specht filtration in which the number of subquotients equal to S_{λ} is $[W_{\lambda} : L_{\mu}]$.

5.3. Non-degenerate bilinear forms. If σ is a composition, its conjugate is the partition $\sigma' = (\sigma'_1, \sigma'_2, \ldots)$, where σ'_i is the number of nodes in column *i* of the diagram of σ . If $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathbb{C}_{r,n}$, its conjugate $\boldsymbol{\lambda}'$ is the *r*-partition $\boldsymbol{\lambda}' = ((\lambda^{(r)})', \ldots, (\lambda^{(1)})')$. Similarly, the conjugate of a standard $\boldsymbol{\lambda}$ -tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(r)})$ is the standard $\boldsymbol{\lambda}'$ -tableau $\mathfrak{t}' = ((\mathfrak{t}^{(r)})', \ldots, (\mathfrak{t}^{(1)})')$, where $(\mathfrak{t}^{(k)})'$ is the tableau obtained by interchanging the rows and columns of $\mathfrak{t}^{(k)}$.

If \mathfrak{v} is a standard tableau, let $\mathfrak{v}_{\downarrow k}$ be the subtableau of \mathfrak{v} which contains $1, 2, \ldots, k$, and let shape $(\mathfrak{v}_{\downarrow k})$ be the associated *r*-composition. Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Suppose that \mathfrak{s} is a standard λ -tableau and that \mathfrak{t} is a standard μ -tableau, we say that \mathfrak{s} dominates \mathfrak{t} , and write $\mathfrak{s} \succeq \mathfrak{t}$ if shape $(\mathfrak{s}_{\downarrow k}) \succeq$ shape $(\mathfrak{t}_{\downarrow k})$ for all $1 \leq k \leq n$. If $\mathfrak{s} \succeq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$, then we write $\mathfrak{s} \succ \mathfrak{t}$. Note that $\mathfrak{s} \succeq \mathfrak{t}$ if and only if $\mathfrak{t}' \succeq \mathfrak{s}'$. We extend the dominance order to pairs of standard tableaux by defining $(\mathfrak{s}, \mathfrak{t}) \succeq (\mathfrak{u}, \mathfrak{v})$ if $\mathfrak{s} \succeq \mathfrak{u}$ and $\mathfrak{t} \succeq \mathfrak{v}$. We write $(\mathfrak{s}, \mathfrak{t}) \triangleright (\mathfrak{u}, \mathfrak{v})$ if $(\mathfrak{s}, \mathfrak{t}) \succeq (\mathfrak{u}, \mathfrak{v})$ and $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$.

For each $\lambda \in \mathcal{P}_{r,n}$, let $\mathfrak{t}_{\lambda} = (\mathfrak{t}^{\lambda'})'$; that is, \mathfrak{t}_{λ} is the standard λ -tableau with the numbers $1, 2, \ldots, n$ entered in order first down the columns of $\mathfrak{t}_{\lambda}^{(r)}$, and then the columns of $\mathfrak{t}_{\lambda}^{(r-1)}$ and so on. If $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we define two elements $d(\mathfrak{t})$ and $d'(\mathfrak{t})$ in \mathfrak{S}_n by $\mathfrak{t} = \mathfrak{t}^{\lambda} d(\mathfrak{t})$ and $\mathfrak{t} = \mathfrak{t}_{\lambda} d'(\mathfrak{t})$. Conjugating either of the two equations shows that $d'(\mathfrak{t}) = d(\mathfrak{t}')$. Let $w_{\lambda} = d(\mathfrak{t}_{\lambda})$. In particular, we have $w_{\lambda} = w_{\lambda'}^{-1}$. Moreover, it is easy to see that $w_{\lambda} = d(\mathfrak{t})d'(\mathfrak{t})^{-1}$ and $l(w_{\lambda}) = l(d(\mathfrak{t})) + l(d'(\mathfrak{t}))$ for all $\mathfrak{t} \in \mathrm{Std}(\lambda)$.

Recall that there is a unique anti-automorphism * on $Y_{r,n}$ such that $g_i^* = g_i$ for $1 \leq i \leq n-1$ and $t_j^* = t_j$ for $1 \leq j \leq n$. Given a right $Y_{r,n}$ -module M, we define its contragredient dual M^{\circledast} to be the dual module $\operatorname{Hom}_{\mathfrak{R}}(M, \mathfrak{R})$ equipped with the right $Y_{r,n}$ -action $(\varphi h)(m) = \varphi(mh^*)$ for all $\varphi \in M^{\circledast}$, $h \in Y_{r,n}$ and $m \in M$. A module M is self-dual if $M \cong M^{\circledast}$. Equivalently, M is self-dual if and only if M possesses a non-degenerate associative bilinear form $\langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is associative if $\langle xh, y \rangle = \langle x, yh^* \rangle$ for all $x, y \in M$ and $h \in Y_{r,n}$.

Recall that the following Jucys-Murphy elements J_i $(1 \le i \le n)$ in $Y_{r,n}$ have been introduced in [ChPA1] by induction

$$J_1 := 1 \text{ and } J_{i+1} := g_i J_i g_i \text{ for } i = 1, \dots, n-1.$$
 (5.1)

If $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, for $1 \leq k \leq n$, we define the content of k in \mathfrak{t} as the element $c_{\mathfrak{t}}(k) := q^{2(j-i)}$ if k appears in row i and column j of some component $\mathfrak{t}^{(s)}$ of \mathfrak{t} . The following proposition is proved in [ER].

Proposition 5.11. (See [ER, Proposition 3].) Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{s}, \mathfrak{t}$ are two standard λ -tableaux. For each $1 \leq k \leq n$, there exist $r_{\mathfrak{uv}} \in \mathcal{R}$ such that

$$m_{\mathfrak{st}}J_k = \operatorname{res}_{\mathfrak{t}}(k)m_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v})} r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}}, \qquad (5.2)$$

where the sum is over the pair $(\mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}^2(\mu) = \mathrm{Std}(\mu) \times \mathrm{Std}(\mu)$ such that $(\mathfrak{u}, \mathfrak{v}) \triangleright (\mathfrak{s}, \mathfrak{t})$ and $\alpha(\mu) = \alpha(\lambda)$ with $\mu \in \mathcal{P}_{r,n}$. Moreover, we have

$$m_{\mathfrak{s}\mathfrak{t}}t_k = \zeta_{\mathrm{p}_\mathfrak{t}(k)}m_{\mathfrak{s}\mathfrak{t}}.\tag{5.3}$$

Remark 5.12. There are two ways to define the dominance order on $\operatorname{Std}^2(\mathcal{P}_{r,n}) = \{(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$. If $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\lambda)$ and $(\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mu)$, then we define

 $(\mathfrak{s},\mathfrak{t}) \succeq (\mathfrak{u},\mathfrak{v}) \text{ if } \lambda \rhd \mu, \text{ or } \lambda = \mu \text{ and } \mathfrak{s} \succeq \mathfrak{u} \text{ and } \mathfrak{t} \succeq \mathfrak{v}.$

By definition, $(\mathfrak{s}, \mathfrak{t}) \geq (\mathfrak{u}, \mathfrak{v})$ implies that $(\mathfrak{s}, \mathfrak{t}) \succeq (\mathfrak{u}, \mathfrak{v})$, but the inverse is false in general. In fact, it is proved in [ER] that the equality (5.2) holds under the dominance order \succeq . But it is easy to see that the equality (5.2) still holds under the stronger dominance order \succeq . Besides, the proof of Proposition 5.11 essentially reduces to the case of r = 1, from which we can easily get the second condition $\alpha(\mu) = \alpha(\lambda)$ in the summation of (5.2). These facts are crucial to the following arguments.

Let $\mathbb{K} = \mathbb{Q}(q, \zeta)$. We shall first consider the split semisimple \mathbb{K} -algebra $Y_{r,n}^{\mathbb{K}} := Y_{r,n}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$. In particular, we can apply all the results in [Ma3, Section 3].

We shall follow the arguments of [Ma3, Section 3] to construct a "seminormal" basis of $Y_{r,n}^{\mathbb{K}}$. For $1 \leq k \leq n$, we define the following two sets:

$$\mathfrak{C}(k) := \{ \mathfrak{c}_{\mathfrak{t}}(k) \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \},$$

and

$$\overline{\mathcal{C}(k)} := \{ \zeta_{\mathrm{p}_{\mathfrak{t}}(k)} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathcal{P}_{r,n} \}.$$

Definition 5.13. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$.

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \left(\prod_{\substack{c \in \mathcal{C}(k) \\ c \neq \mathfrak{c}_{\mathfrak{t}}(k)}} \frac{J_{k} - c}{\mathfrak{c}_{\mathfrak{t}}(k) - c} \cdot \prod_{\substack{\bar{c} \in \overline{\mathcal{C}}(k) \\ \bar{c} \neq \zeta_{\mathrm{P}_{\mathfrak{t}}(k)}}} \frac{t_{k} - \bar{c}}{\zeta_{\mathrm{P}_{\mathfrak{t}}(k)} - \bar{c}} \right).$$
(5.4)

(ii) We set $f_{\mathfrak{st}} := F_{\mathfrak{s}} m_{\mathfrak{st}} F_{\mathfrak{t}}$ and $g_{\mathfrak{st}} := F_{\mathfrak{s}} n_{\mathfrak{st}} F_{\mathfrak{t}}$

By Proposition 5.11, we can now apply the general theory developed in [Ma3, Section 3] to get the following results.

Proposition 5.14. (i) Suppose that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ for some $\lambda \in \mathcal{P}_{r,n}$. In $Y_{r,n}^{\mathbb{K}}$ we have

$$m_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})} r_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}}, \tag{5.5}$$

where $r_{\mathfrak{u}\mathfrak{v}} \in \mathbb{K}$ and the sum is over the pair $(\mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}^2(\mu)$ $(\mu \in \mathcal{P}_{r,n})$ such that $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$ and $\alpha(\mu) = \alpha(\lambda)$;

$$n_{\mathfrak{s}\mathfrak{t}} = g_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})} s_{\mathfrak{u}\mathfrak{v}}g_{\mathfrak{u}\mathfrak{v}},\tag{5.6}$$

where $s_{\mathfrak{u}\mathfrak{v}} \in \mathbb{K}$ and the sum is over the pair $(\mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}^2(\boldsymbol{\nu})$ $(\boldsymbol{\nu} \in \mathfrak{P}_{r,n})$ such that $s_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \triangleright (\mathfrak{s}, \mathfrak{t})$ and $\alpha(\boldsymbol{\nu}) = \alpha(\boldsymbol{\lambda})$.

- (ii) The set $\{f_{\mathfrak{st}} | \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$ is a \mathbb{K} -basis of $Y_{r,n}^{\mathbb{K}}$.
- (iii) For $\lambda, \mu \in \mathcal{P}_{r,n}$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$, we have

$$f_{\mathfrak{st}}J_k = c_{\mathfrak{t}}(k)f_{\mathfrak{st}}, \qquad f_{\mathfrak{st}}t_k = \zeta_{p_{\mathfrak{t}}(k)}f_{\mathfrak{st}}, \qquad f_{\mathfrak{st}}F_{\mathfrak{u}} = \delta_{\mathfrak{t},\mathfrak{u}}f_{\mathfrak{su}}, \tag{5.7}$$

and moreover, there exists a scalar $0 \neq \gamma_t \in \mathbb{K}$ such that

$$f_{\mathfrak{s}\mathfrak{t}}f_{\mathfrak{u}\mathfrak{v}} = \begin{cases} \gamma_{\mathfrak{t}}f_{\mathfrak{s}\mathfrak{v}} & \text{if } \boldsymbol{\lambda} = \boldsymbol{\mu} \text{ and } \mathfrak{t} = \mathfrak{u}; \\ 0 & \text{otherwise.} \end{cases}$$
(5.8)

In particular, $\gamma_{\mathfrak{t}}$ depends only on \mathfrak{t} and the set $\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ and } \lambda \in \mathcal{P}_{r,n}\}$ is a cellular basis of $Y_{r,n}^{\mathbb{K}}$.

(iv) For $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$, we have $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}}$. Moreover, these elements $\{F_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}\}$ give a complete set of pairwise orthogonal primitive idempotents for $Y_{r,n}^{\mathbb{K}}$.

(v) For $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we have

$$F_{\mathfrak{t}}J_k = J_k F_{\mathfrak{t}} = c_{\mathfrak{t}}(k)F_{\mathfrak{t}}, \qquad F_{\mathfrak{t}}t_k = t_k F_{\mathfrak{t}} = \zeta_{\mathrm{p}_{\mathfrak{t}}(k)}F_{\mathfrak{t}}.$$
(5.9)

(vi) The Jucys-Murphy elements $J_1, \ldots, J_n, t_1, \ldots, t_n$ generate a maximal commutative subalgebra of $Y_{r,n}^{\mathbb{K}}$.

From the definitions, we see that for $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we have $(c_{\mathfrak{t}}(k))' = c_{\mathfrak{t}'}(k)$ and $\zeta'_{\mathrm{p}_{\mathfrak{t}}(k)} = \zeta_{\mathrm{p}_{\mathfrak{t}'}(k)}$. This implies the following lemma.

Lemma 5.15. For $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$, we have $F'_{\mathfrak{t}} = F_{\mathfrak{t}'}$, and hence $g_{\mathfrak{s}\mathfrak{t}} = f'_{\mathfrak{s}\mathfrak{t}}$ in $Y_{r,n}^{\mathbb{K}}$.

By Proposition 5.14(iv) and Lemma 5.15, we can easily get the following result.

Lemma 5.16. Suppose that $\lambda, \mu \in \mathcal{P}_{r,n}$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$. Then in $\mathrm{Y}_{r,n}^{\mathbb{K}}$, we have $f_{\mathfrak{s}\mathfrak{t}}g_{\mathfrak{u}\mathfrak{v}} = 0$ unless $\mathfrak{t} = \mathfrak{u}'$.

By Proposition 5.14(i) and Lemma 5.16, we can easily get the following lemma.

Lemma 5.17. Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Suppose that \mathfrak{s} and \mathfrak{t} are standard λ -tableaux and that \mathfrak{u} and \mathfrak{v} are standard μ -tableaux. If $m_{\mathfrak{st}}n_{\mathfrak{u}\mathfrak{v}} \neq 0$, then $\mathfrak{u}' \succeq \mathfrak{t}$ and $\alpha(\mu') = \alpha(\lambda)$.

Recall that $\{t_1^{k_1} \cdots t_n^{k_n} g_w \mid 0 \leq k_1, \ldots, k_n \leq r-1 \text{ and } w \in \mathfrak{S}_n\}$ is an \mathfrak{R} -basis of $Y_{r,n}$. We can define an \mathfrak{R} -linear map $\tau : Y_{r,n} \to \mathfrak{R}$ by

$$\tau(t_1^{k_1}\cdots t_n^{k_n}g_w) = \begin{cases} 1 & \text{if } k_1 \equiv k_2 \equiv \cdots \equiv 0 \pmod{r} \text{ and } w = 1; \\ 0 & \text{otherwise.} \end{cases}$$
(5.10)

This map τ was introduced in [ChPA1, Proposition 10] and was shown to be a trace form; that is, $\tau(ab) = \tau(ba)$ for all $a, b \in Y_{r,n}$. Moreover, we have

$$\tau(t_1^{k_1} \cdots t_n^{k_n} g_w g_{w'} t_1^{l_1} \cdots t_n^{l_n}) = \begin{cases} 1 & \text{if } w^{-1} = w' \text{ and } k_i + l_i \equiv 0 \pmod{r} \text{ for } 1 \le i \le n; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we get that $\tau(h^*) = \tau(h)$ for all $h \in Y_{r,n}$.

We now define a symmetric associative bilinear form $\langle \cdot, \cdot \rangle$ on $Y_{r,n}$ by $\langle h_1, h_2 \rangle = \tau(h_1 h_2^*)$. We then have the following crucial result.

Theorem 5.18. Suppose that $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{r,n}$ and we choose all $1 \leq i_1 < i_2 < \cdots < i_p \leq r$ such that $\lambda^{(i_1)}, \lambda^{(i_2)}, \ldots, \lambda^{(i_p)}$ are the nonempty components of $[\lambda]$. Define $c_k := |\lambda^{(i_k)}|$ for $1 \leq k \leq p$. Let $\mu \in \mathcal{P}_{r,n}$. Suppose that $\mathfrak{s}, \mathfrak{t}$ are two standard λ -tableaux and that $\mathfrak{u}, \mathfrak{v}$ are two standard μ -tableaux. Then we have

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{u}\mathfrak{v}} \rangle = \begin{cases} \frac{r^p (\zeta_{i_1} \cdots \zeta_{i_p})^{-2}}{r^{\sum_{k=1}^p \binom{c_k}{2}}} & \text{if } (\mathfrak{u}', \mathfrak{v}') = (\mathfrak{s}, \mathfrak{t}); \\ 0 & \text{if } (\mathfrak{u}', \mathfrak{v}') \not \geq (\mathfrak{s}, \mathfrak{t}). \end{cases}$$
(5.11)

Proof. Suppose first that $\langle m_{\mathfrak{st}}, n_{\mathfrak{uv}} \rangle \neq 0$. By definition, $\langle m_{\mathfrak{st}}, n_{\mathfrak{uv}} \rangle = \tau(m_{\mathfrak{st}}n_{\mathfrak{vu}})$, so $m_{\mathfrak{st}}n_{\mathfrak{vu}} \neq 0$; hence $\mathfrak{v}' \supseteq \mathfrak{t}$ by Lemma 5.17. Since τ is a trace form and $\tau(h^*) = \tau(h)$ for all $h \in \mathbf{Y}_{r,n}$, we also have $\tau(m_{\mathfrak{st}}n_{\mathfrak{vu}}) = \tau(n_{\mathfrak{vu}}m_{\mathfrak{st}}) = \tau(m_{\mathfrak{ts}}n_{\mathfrak{uv}})$; hence $m_{\mathfrak{ts}}n_{\mathfrak{uv}} \neq 0$ and $\mathfrak{u}' \supseteq \mathfrak{s}$ by Lemma 5.17. Therefore, if $\langle m_{\mathfrak{st}}, n_{\mathfrak{uv}} \rangle \neq 0$, then $(\mathfrak{u}', \mathfrak{v}') \supseteq (\mathfrak{s}, \mathfrak{t})$.

Now assume that $(\mathfrak{u}',\mathfrak{v}') = (\mathfrak{s},\mathfrak{t})$. Then $g_{w_{\lambda}} = g_{d(\mathfrak{t})}g^*_{d(\mathfrak{t}')} = g_{d(\mathfrak{s})}g^*_{d(\mathfrak{s}')}$. Therefore, we have

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{s}'\mathfrak{t}'} \rangle = \tau(m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{t}'\mathfrak{s}'})$$

$$= \tau(g_{d(\mathfrak{s})}m_{\lambda}g_{d(\mathfrak{t})}g_{d(\mathfrak{t}')}^{*}n_{\lambda'}g_{d(\mathfrak{s}')})$$

$$= \tau(g_{d(\mathfrak{s}')}g_{d(\mathfrak{s})}^{*}m_{\lambda}g_{w_{\lambda}}n_{\lambda'})$$

$$= \tau(g_{w_{\lambda}}^{*}m_{\lambda}g_{w_{\lambda}}n_{\lambda'})$$

$$= \tau(m_{\lambda}g_{w_{\lambda}}n_{\lambda'}g_{w_{\lambda}}^{-1})$$

$$= \tau(u_{\lambda}E_{A_{\lambda}}x_{\lambda}g_{w_{\lambda}}U_{\lambda'}'y_{\lambda'}g_{w_{\lambda'}})$$

$$= \tau(u_{\lambda}E_{A_{\lambda}}g_{w_{\lambda}}u_{\lambda'}'E_{A_{\lambda'}}y_{\lambda'}g_{w_{\lambda'}}x_{\lambda}).$$

$$(5.12)$$

By definition, we have

$$w_{\boldsymbol{\lambda}} = \begin{pmatrix} 1 & 2 & \cdots & |\lambda^{(i_1)}| \\ j_1 & j_2 & \cdots & j_{|\lambda^{(i_1)}|} \\ \end{pmatrix} \begin{pmatrix} \cdots & \cdots & n \\ \cdots & \cdots & k_1 \\ k_1 & \cdots & k_{|\lambda^{(i_p)}|} \end{pmatrix},$$
(5.13)

where $\{j_1, j_2, \dots, j_{|\lambda^{(i_1)}|}\} = \{n - |\lambda^{(i_1)}| + 1, \dots, n\}$ and $\{k_1, \dots, k_{|\lambda^{(i_p)}|}\} = \{1, \dots, |\lambda^{(i_p)}|\}.$

By (5.13), we have

$$w_{\lambda}^{-1} = \begin{pmatrix} 1 & 2 & \cdots & |\lambda^{(i_p)}| \\ k_1 & k_2 & \cdots & k_{|\lambda^{(i_p)}|} \\ \end{pmatrix} \begin{pmatrix} \cdots & \cdots & n \\ \cdots & \cdots & n \\ j_1 & \cdots & k_{|\lambda^{(i_1)}|} \end{pmatrix}, \quad (5.14)$$

where $\{k_1, k_2, \dots, k_{|\lambda^{(i_p)}|}\} = \{n - |\lambda^{(i_p)}| + 1, \dots, n\}$ and $\{j_1, \dots, j_{|\lambda^{(i_1)}|}\} = \{1, \dots, |\lambda^{(i_1)}|\}$. Define $A_{\lambda} = \{I_1, I_2, \dots, I_p\}$ as in Definition 2.3. By assumption, we have $\lambda' = \{I_1, I_2, \dots, I_p\}$ as (i) before (i) by a sumption (ii) by a sumption (iii) by a sumption).

Define $A_{\boldsymbol{\lambda}} = \{I_1, I_2, \ldots, I_p\}$ as in Definition 2.3. By assumption, we have $\boldsymbol{\lambda} = ((\lambda^{(r)})', \ldots, (\lambda^{(1)})') \in \mathcal{P}_{r,n}$ with $(\lambda^{(i_p)})', (\lambda^{(i_{p-1})})', \ldots, (\lambda^{(i_1)})'$ being all the nonempty components of $[\boldsymbol{\lambda}']$. Suppose that $A_{\boldsymbol{\lambda}'} = \{I'_p, \ldots, I'_1\}$, where $I'_p = \{1, 2, \ldots, |\lambda^{(i_p)}|\}, \ldots$, and $I'_1 = \{n - |\lambda^{(i_1)}| + 1, \ldots, n\}$. By definition 2.4, we have

$$u'_{\lambda'}E_{A_{\lambda'}} = \prod_{l \neq r+1 - (r+1-i_p)} (t_{|\lambda^{(i_p)}|} - \zeta_l) \cdots \prod_{l \neq r+1 - (r+1-i_1)} (t_n - \zeta_l) \cdot E_{I'_p}E_{I'_{p-1}} \cdots E_{I'_1}.$$
 (5.15)

By (5.15), Lemma 2.1, together with the fact that $g_w t_k = t_{(k)w^{-1}}g_w$, we get that

$$g_{w_{\lambda}}u'_{\lambda'}E_{A_{\lambda'}} = u_{\lambda}E_{A_{\lambda}}g_{w_{\lambda}}.$$
(5.16)

By (5.12) and (5.16), we have

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{s}'\mathfrak{t}'} \rangle = \tau(E_{A_{\lambda}} u_{\lambda} U_{\lambda} g_{w_{\lambda}} y_{\lambda'} g_{w_{\lambda'}} x_{\lambda}).$$
(5.17)

By [ER, Lemma 10(49)], we have

$$t_i U_{\lambda} = \zeta_{i_k} U_{\lambda}$$
 if $i \in I_k$ for some $1 \le k \le p$. (5.18)

Recall that $S = \{\zeta_1, \zeta_2, \dots, \zeta_r\}$. We also have, for any fixed *r*-th root of unity ξ ,

$$\prod_{\xi \neq \alpha \in S} (\xi - \alpha) = r\xi^{-1}.$$
(5.19)

By (5.18) and (5.19), we get that

$$u_{\lambda}U_{\lambda} = r^p(\zeta_{i_1}\cdots\zeta_{i_p})^{-1}U_{\lambda}.$$
(5.20)

Combining (5.17) and (5.20), we get that

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{s}'\mathfrak{t}'} \rangle = r^p (\zeta_{i_1} \cdots \zeta_{i_p})^{-1} \tau (E_{A_{\lambda}} u_{\lambda} g_{w_{\lambda}} y_{\lambda'} g_{w_{\lambda'}} x_{\lambda}).$$
(5.21)

By (5.10) and $\zeta_1 \cdots \zeta_r = (-1)^{r-1}$, we have

$$\tau(E_{A_{\lambda}}u_{\lambda}) = \prod_{k=1}^{p} \frac{1}{r^{(|\lambda^{(i_{k})}|)}} \cdot \prod_{k=1}^{p} \left(\prod_{l \neq i_{k}} (-1)^{r-1} \zeta_{l}\right)$$
$$= (\zeta_{i_{1}} \cdots \zeta_{i_{p}})^{-1} \frac{1}{r^{\sum_{k=1}^{p} \binom{c_{k}}{2}}}.$$
(5.22)

By (5.21) and (5.22), we have

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{s}'\mathfrak{t}'} \rangle = \frac{r^p (\zeta_{i_1} \cdots \zeta_{i_p})^{-2}}{r^{\sum_{k=1}^p \binom{c_k}{2}}} \tau(g_{w_{\lambda}} y_{\lambda'} g_{w_{\lambda'}} x_{\lambda}).$$
(5.23)

Since $\mathfrak{S}_{\lambda} \cap {}^{w_{\lambda}}\mathfrak{S}_{\lambda'} = \{1\}$ and $w_{\lambda} = w_{\lambda'}^{-1}$ is a distinguished $(\mathfrak{S}_{\lambda}, \mathfrak{S}_{\lambda'})$ -double coset representative, we have

$$g_{w_{\lambda}}y_{\lambda'}g_{w_{\lambda'}}x_{\lambda} = \sum_{\substack{u \in \mathfrak{S}_{\lambda'} \\ v \in \mathfrak{S}_{\lambda}}} g_{w_{\lambda}} \cdot (-q)^{-l(u)}g_{u} \cdot g_{w_{\lambda}^{-1}} \cdot q^{l(v)}g_{v}$$
$$= \sum_{\substack{u \in \mathfrak{S}_{\lambda'} \\ v \in \mathfrak{S}_{\lambda}}} (-1)^{l(u)}q^{l(v)-l(u)}g_{w_{\lambda}}g_{uw_{\lambda}^{-1}v}.$$

Thus, we get $\tau(g_{w_{\lambda}}y_{\lambda'}g_{w_{\lambda'}}x_{\lambda}) = 1$. Therefore, by (5.23), we have

$$\langle m_{\mathfrak{st}}, n_{\mathfrak{s}'\mathfrak{t}'}
angle = rac{r^p (\zeta_{i_1} \cdots \zeta_{i_p})^{-2}}{r^{\sum_{k=1}^p \binom{c_k}{2}}}.$$

We have proved the theorem.

Remark 5.19. Theorem 5.18 implies that τ is non-degenerate. Consequently, $Y_{r,n}$ is a symmetric algebra.

The next corollary can be proved in exactly the same way as in [Ma2, Corollary 5.7] using Theorem 5.18, which justify the term dual Specht module.

Corollary 5.20. Suppose that $\lambda \in \mathcal{P}_{r,n}$. Then $S^{\lambda'} \cong S^{\circledast}_{\lambda}$.

If $S = (S^{(1)}, \ldots, S^{(r)})$ is a λ -tableau of type μ , we define the conjugate of S by $S' = ((S^{(r)})', \ldots, (S^{(1)})')$ which is a λ' -tableau of type μ , where $(S^{(j)})'$ is the tableau obtained by interchanging the rows and columns of $S^{(j)}$ for each j. A λ -tableau S is called column semistandard if S' is semistandard. For $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$, let $\mathcal{T}^{cs}(\lambda, \mu) = \{S \mid S' \in \mathcal{T}^+_0(\lambda', \mu)\}.$

The proof of the next lemma is in exactly the same way as that of [Ma2, Lemma 5.8] by making use of Lemma 5.17. We skip the details.

Lemma 5.21. Suppose that $\boldsymbol{\mu} \in \mathcal{C}_{r,n}, \boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ and that $m_{\boldsymbol{\mu}} n_{\mathfrak{u}\mathfrak{v}} \neq 0$ or $n_{\boldsymbol{\mu}} m_{\mathfrak{u}\mathfrak{v}} \neq 0$ for some standard $\boldsymbol{\lambda}$ -tableaux \mathfrak{u} and \mathfrak{v} . Then $\boldsymbol{\mu}(\mathfrak{u})$ is column semistandard and $\alpha(\boldsymbol{\lambda}') = \alpha(\boldsymbol{\mu})$; that is, $\boldsymbol{\mu}(\mathfrak{u}) \in \mathbb{T}^{cs}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Remark 5.22. As mentioned in [HM1, p. 15], it is unfortunate that Mathas confused the two partial orders \geq and \geq on Std²($\mathcal{P}_{r,n}$) in [Ma2] and [Ma3]. Anyhow, we can adapt the approach in [Ma3, Section 3] to get Proposition 5.14 and then Lemmas 5.17 and 5.21. We leave the details to the reader; see also [HM1, Section 2] for details.

If $S \in \mathcal{T}^{cs}(\lambda, \mu)$, let \dot{S} be the unique standard λ -tableau such that $\mu(\dot{S}) = S$ and $d(\dot{S})$ is a distinguished $(\mathfrak{S}_{\lambda}, \mathfrak{S}_{\mu})$ -double coset representative; that is, $d(\dot{S})$ is the unique element of minimal length in its double coset.

Proposition 5.23. Suppose that $\mu \in C_{r,n}$. Then M^{μ} is free as an \mathbb{R} -module with basis $\{m_{\mu}n_{\mathrm{St}} | \mathrm{S} \in \mathbb{T}^{\mathrm{cs}}(\lambda,\mu) \text{ and } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathbb{P}_{r,n}\}$ and N^{μ} is free as an \mathbb{R} -module with basis $\{n_{\mu}m_{\mathrm{St}} | \mathrm{S} \in \mathbb{T}^{\mathrm{cs}}(\lambda,\mu) \text{ and } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathbb{P}_{r,n}\}.$

Proof. We only prove the claim for M^{μ} . Recall that $\{n_{\mathfrak{st}}\}$ is an \mathfrak{R} -basis of $Y_{r,n}$, so M^{μ} is spanned by the elements $m_{\mu}n_{\mathfrak{st}}$, where $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^{2}(\mathfrak{P}_{r,n})$. Furthermore, if $m_{\mu}n_{\mathfrak{st}} \neq 0$ then $\mu(\mathfrak{s})$ is column semistandard and $\alpha(\lambda') = \alpha(\mu)$ if $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^{2}(\lambda)$ by Lemma 5.21. Hence, M^{μ} is spanned by the elements $m_{\mu}n_{\mathfrak{st}}$, where $\mu(\mathfrak{s})$ is column semistandard and $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^{2}(\lambda)$ for various $\lambda \in \mathfrak{P}_{r,n}$ with $\alpha(\lambda') = \alpha(\mu)$.

For each such element $m_{\mu}n_{\mathfrak{s}\mathfrak{t}}$, where $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\nu)$ with $\alpha(\nu') = \alpha(\mu)$. Since $\mu(\mathfrak{s}) =$ S is column semistandard, we choose $\dot{S} \in \operatorname{Std}(\nu)$ as above and get that $\mu(\mathfrak{t}^{\nu}d(\mathfrak{s})) =$ $\mu(\mathfrak{t}^{\nu}d(\dot{S}))$. Thus, $d(\mathfrak{s})$ and $d(\dot{S})$ lie in the same $(\mathfrak{S}_{\nu}, \mathfrak{S}_{\mu})$ -double coset. By definition $d(\dot{S})$ is the unique element of minimal length in its double coset, therefore we get $m_{\mu}g^*_{d(\mathfrak{s})}n_{\nu} =$ $\pm q^a m_{\mu}g^*_{d(\dot{S})}n_{\nu}$ for some integer a. Because M^{μ} is \mathcal{R} -free and the number of elements in our spanning set is exactly the rank of M^{μ} , thus we have proved the first claim. The second statement can be proved in a similar way.

Combining Lemma 5.21 and Proposition 5.23, we get the next result.

Corollary 5.24. Suppose that $\boldsymbol{\mu} \in \mathcal{C}_{r,n}, \boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ and that \mathfrak{u} and \mathfrak{v} are two standard $\boldsymbol{\lambda}$ -tableaux. Then $m_{\boldsymbol{\mu}}n_{\mathfrak{u}\mathfrak{v}} \neq 0$ if and only if $\boldsymbol{\mu}(\mathfrak{u})$ is column semistandard and $\alpha(\boldsymbol{\lambda}') = \alpha(\boldsymbol{\mu})$. Similarly, $n_{\boldsymbol{\mu}}m_{\mathfrak{u}\mathfrak{v}} \neq 0$ if and only if $\boldsymbol{\mu}(\mathfrak{u})$ is column semistandard and $\alpha(\boldsymbol{\lambda}') = \alpha(\boldsymbol{\mu})$.

Using Proposition 5.23 we can get the following result by repeating the argument of Lemma 5.2.

Corollary 5.25. Suppose that $\mu \in \mathcal{C}_{r,n}$. Then there exist filtrations

 $M^{\mu} = H^{1} \supset H^{2} \supset \cdots \supset H^{k} \supset H^{k+1} = 0 \text{ and } N^{\mu} = H_{1} \supset H_{2} \supset \cdots \supset H_{k} \supset H_{k+1} = 0$ of M^{μ} and N^{μ} , respectively, and r-partitions $\lambda_{1}, \ldots, \lambda_{k}$ such that $\mu' \supseteq \lambda_{i}, H^{i}/H^{i+1} \cong S_{\lambda_{i}}$ and $H_{i}/H_{i+1} \cong S^{\lambda_{i}}$ for $1 \leq i \leq k$. Moreover, for any $\lambda \in \mathcal{P}_{r,n}$ we have $\sharp \{1 \leq i \leq k \mid \lambda_{i} = \lambda\} = \sharp \mathbb{T}^{cs}(\lambda, \mu)$.

Now we can define a bilinear form $\langle \cdot, \cdot \rangle_{\mu}$ on M^{μ} by

$$\langle m_{\mathrm{St}}, m_{\mu} n_{\dot{\mathrm{U}}\mathfrak{p}} \rangle_{\mu} = \langle m_{\mathrm{St}}, n_{\dot{\mathrm{U}}\mathfrak{p}} \rangle,$$

where $m_{\rm St}$ and $m_{\mu}n_{\dot{U}\mathfrak{p}}$ rum over the bases of Propositions 3.3 and 5.23, respectively.

The next proposition can be proved in exactly the same way as in [Ma2, Proposition 5.13]. We omit the details and leave them to the reader.

Proposition 5.26. Suppose that $\mu \in \mathcal{C}_{r,n}$. Then $\langle \cdot, \cdot \rangle_{\mu}$ is a non-degenerate associative bilinear form on M^{μ} . In particular, M^{μ} is self-dual. Similarly, N^{μ} is self-dual.

By induction and using Propositions 5.6 and 5.10 we can get the next result.

Corollary 5.27. Let $\lambda \in \mathcal{P}_{r,n}$. Then the Young module Y^{λ} and twisted Young module Y_{λ} are both self-dual.

5.4. Tilting modules. Recall that a YS^r_n-module T is a tilting module if it has both a filtration by Weyl modules W^{λ} ($\lambda \in \mathcal{P}_{r,n}$) and a filtration by dual Weyl modules. Since YS^r_n is quasi-hereditary, by [Ri], for each $\lambda \in \mathcal{P}_{r,n}$ there exists a unique indecomposable tilting module T^{λ} such that $[T^{\lambda} : W^{\lambda}] = 1$ and $[T^{\lambda} : W^{\mu}] \neq 0$ only if $\lambda \succeq \mu$. Moreover, any tilting module T can be written as a direct sum of these T^{λ} 's. The T^{λ} are the partial

tilting modules of YS_n^r . A full tilting module for YS_n^r is any tilting module which contain every T^{λ} ($\lambda \in \mathcal{P}_{r,n}$) as a direct summand.

For each $\boldsymbol{\nu} \in \mathcal{C}_{r,n}$, let $\theta_{\boldsymbol{\nu}} \in \operatorname{Hom}_{Y_{r,n}}(Y_{r,n}, N^{\boldsymbol{\nu}})$ be the map, which is defined by $\theta_{\boldsymbol{\nu}}(h) = n_{\boldsymbol{\nu}}h$ for all $h \in Y_{r,n}$. We define

$$E^{\boldsymbol{\nu}} := \dot{\mathbf{F}}_n^{\omega}(\theta_{\boldsymbol{\nu}} \mathbf{Y} \dot{\mathbf{S}}_n^r).$$

Since E^{ν} , by definition, is the set of maps from M_n^r to N^{ν} which factor through θ_{ν} , we get that E^{ν} is a right YS_n^r -module.

Definition 5.28. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mu, \nu \in \mathcal{C}_{r,n}$. For $S \in \mathcal{T}^{cs}(\lambda, \nu)$ and $T \in \mathcal{T}^+_0(\lambda, \mu)$ let θ_{ST} be the homomorphism in E^{ν} determined by $\theta_{ST}(m_{\alpha}h) = \delta_{\alpha\mu}n_{\nu}m_{ST}h$ for all $h \in Y_{r,n}$ and all $\alpha \in \mathcal{C}_{r,n}$.

Proposition 5.29. Let $\nu \in \mathcal{C}_{r,n}$. Then E^{ν} is free as an \mathcal{R} -module with basis

$$\{\theta_{\mathrm{ST}} \mid \mathrm{S} \in \mathfrak{T}^{\mathrm{cs}}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ and } \mathrm{T} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n}\}.$$

Proof. Let $\dot{E}^{\nu} = \theta_{\nu} Y \dot{S}_{n}^{r}$. Then \dot{E}^{ν} is a right $Y \dot{S}_{n}^{r}$ -module and $E^{\nu} = \dot{F}_{n}^{\omega}(\dot{E}^{\nu})$. By Proposition 4.1, \dot{E}^{ν} is spanned by the maps $\theta_{\nu}\varphi_{ST}$, where $S \in \mathcal{T}_{0}^{+}(\lambda, \sigma)$ and $T \in \mathcal{T}_{0}^{+}(\lambda, \mu)$ for various $\lambda \in \mathcal{P}_{r,n}$ and $\sigma, \mu \in \dot{\mathcal{C}}_{r,n}$. By definition, $\theta_{\nu}\varphi_{ST} = 0$ unless $\sigma = \omega$, that is, S is a standard λ -tableau; so \dot{E}^{ν} is spanned by the elements $\theta_{\nu}\varphi_{\mathfrak{s}T}$ with $\mathfrak{s} \in \mathrm{Std}(\lambda)$ and $T \in \mathcal{T}_{0}^{+}(\lambda, \mu)$ for $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \dot{\mathcal{C}}_{r,n}$. Furthermore, $\theta_{\nu}\varphi_{\mathfrak{s}T}(m_{\mu}h) = n_{\nu}m_{\mathfrak{s}T}h$ for all $h \in Y_{r,n}$. Thus, $\theta_{\nu}\varphi_{\mathfrak{s}T} \neq 0$ if and only if $\nu(\mathfrak{s})$ is column semistandard and $\alpha(\lambda') = \alpha(\nu)$ by Corollary 5.24, and in this case $\theta_{\nu}\varphi_{\mathfrak{s}T} = \pm q^{a}\theta_{ST}$ for some $a \in \mathbb{Z}$ and $S = \nu(\mathfrak{s})$. Hence these elements $\{\theta_{ST} \mid S \in \mathfrak{T}^{cs}(\lambda, \nu)$ and $T \in \dot{\mathfrak{T}}_{0}^{+}(\lambda)$ for some $\lambda \in \mathcal{P}_{r,n}\}$ span \dot{E}^{ν} .

On the other hand, the elements $\{\theta_{\mathrm{ST}}\}\$ are linearly independent by Proposition 5.23, so they are a basis of \dot{E}^{ν} . Since the functor \dot{F}_{n}^{ω} removes the ω -weight space of \dot{E}^{ν} , therefore \dot{F}_{n}^{ω} maps the basis $\{\theta_{\mathrm{ST}}\}\$ of \dot{E}^{ν} to the elements stated in the proposition, or to zero if $\mu = \omega$. Hence, $\{\theta_{\mathrm{ST}} \mid \mathrm{S} \in \mathbb{T}^{\mathrm{cs}}(\lambda, \nu) \text{ and } \mathrm{T} \in \mathbb{T}_{0}^{+}(\lambda) \text{ for some } \lambda \in \mathbb{P}_{r,n}\}$ is an \mathcal{R} -basis of E^{ν} .

The next proposition can be proved in exactly the same way as in [Ma2, Theorem 6.5] by using Proposition 5.29. We skip the details.

Proposition 5.30. Let $\boldsymbol{\nu} \in \mathbb{C}_{r,n}$. Then $E^{\boldsymbol{\nu}}$ admits a YS_n^r -module filtration $E^{\boldsymbol{\nu}} = E_1 \supset E_2 \supset \cdots \supset E_k \supset E_{k+1} = 0$ such that $E_i/E_{i+1} \cong W^{\boldsymbol{\lambda}_i}$ for some $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k \in \mathfrak{P}_{r,n}$ and $\boldsymbol{\nu}' \succeq \boldsymbol{\lambda}_i$ for all $1 \leq i \leq k$. Moreover, if $\boldsymbol{\lambda} \in \mathfrak{P}_{r,n}$, then $\sharp \{1 \leq i \leq k \mid \boldsymbol{\lambda}_i = \boldsymbol{\lambda}\} = \sharp \mathbb{T}^{\mathrm{cs}}(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

From Proposition 5.30 we can easily get the next corollary.

Corollary 5.31. Suppose that $\lambda, \mu \in \mathcal{P}_{r,n}$. Then $[E^{\lambda} : W^{\lambda'}] = 1$ and $[E^{\lambda} : W^{\mu}] \neq 0$ only if $\lambda' \succeq \mu$.

Definition 5.32. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mu, \nu \in \mathcal{C}_{r,n}$. For $A \in \mathcal{T}_0^+(\lambda, \nu)$ and $B \in \mathcal{T}^{cs}(\lambda, \mu)$ let θ'_{AB} be the homomorphism determined by $\theta'_{AB}(m_{\alpha}h) = \delta_{\alpha\mu}n_{A\dot{B}}m_{\mu}h$ for all $h \in Y_{r,n}$ and all $\alpha \in \mathcal{C}_{r,n}$.

Proposition 5.33. Let $\nu \in \mathcal{C}_{r.n}$. Then E^{ν} is free as an \mathbb{R} -module with basis

 $\{\theta'_{AB} \mid A \in \mathfrak{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ and } B \in \mathfrak{T}^{cs}(\boldsymbol{\lambda}, \boldsymbol{\mu}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n} \text{ and } \boldsymbol{\mu} \in \mathfrak{C}_{r,n} \}.$

Proof. We first show that $\theta'_{AB} \in E^{\nu}$. By Corollary 5.8, $n_{AB} = n_{\nu}x$ for some $x \in Y_{r,n}$. Therefore, we have

$$\theta_{\rm AB}'(m_{\mu}h) = n_{\rm AB}m_{\mu}h = n_{\nu}xm_{\mu}h = \theta_{\nu}(xm_{\mu}h).$$

That is, θ'_{AB} factors through θ_{ν} so that $\theta'_{AB} \in E^{\nu}$ as claimed. Moreover, the elements stated in the proposition are linearly independent by applying * to Proposition 5.23. Therefore, the elements $\{\theta'_{AB}\}$ give a basis of E^{ν} by counting dimensions using Proposition 5.29.

The contragredient dual E^{\circledast} of a YS_n^r -module E can be defined in exactly the same way as that of $Y_{r,n}$ -modules. Again, we say that E is self-dual if $E \cong E^{\circledast}$.

Using the two bases $\{\theta_{ST}\}$ and $\{\theta'_{AB}\}$ in Propositions 5.29 and 5.33, we now define a bilinear form $\{\cdot, \cdot\}_{\nu}$ on E^{ν} by

$$\{\theta_{\rm ST}, \theta_{\rm AB}'\}_{\nu} = \begin{cases} \langle m_{\rm \dot{S}T}, n_{\rm A\dot{B}} \rangle & \text{if Type}({\rm T}) = {\rm Type}({\rm B}); \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem can be proved in exactly the same way as in [Ma2, Theorem 6.17]. We skip the details.

Theorem 5.34. Suppose that $\nu \in \mathcal{C}_{r,n}$. Then $\{\cdot, \cdot\}_{\nu}$ defines a non-degenerate associative bilinear form on E^{ν} ; that is, E^{ν} is self-dual.

Using Corollary 5.31 and Theorem 5.34 we can easily get the next result.

Corollary 5.35. Let $\lambda \in \mathcal{P}_{r,n}$. Then we have

(i) E^{λ} is a tilting module. Moreover, $E^{\lambda} \cong T^{\lambda'} \oplus \bigoplus_{\lambda' \rhd \mu} e_{\lambda \mu} T^{\mu}$ for some non-negative integers $e_{\lambda \mu}$.

(ii) T^{λ} is self-dual. Moreover, the tilting modules of YS_n^r are the indecomposable direct summands of the modules $\{E^{\lambda} \mid \lambda \in \mathcal{P}_{r,n}\}$.

Recall that the Schur functor $F_n^r : YS_n^r - \text{mod} \to Y_{r,n} - \text{mod}$ defined in Proposition 4.3.

Lemma 5.36. Suppose that $\mu \in \mathcal{C}_{r,n}$. Then $F_n^r(E^{\mu}) \cong N^{\mu}$ as $Y_{r,n}$ -modules.

Proof. By Lemma 4.2 and Proposition 4.3 we have

$$\begin{aligned} \mathbf{F}_{n}^{r}(E^{\boldsymbol{\mu}}) &= \mathbf{F}_{n}^{r}(\dot{\mathbf{F}}_{n}^{\omega}(\theta_{\boldsymbol{\mu}}\mathbf{Y}\mathbf{S}_{n}^{r})) = \dot{\mathbf{F}}_{n}^{r}(\theta_{\boldsymbol{\mu}}\mathbf{Y}\mathbf{S}_{n}^{r}) \\ &= \theta_{\boldsymbol{\mu}}\mathbf{Y}\dot{\mathbf{S}}_{n}^{r}\varphi_{\omega} \cong \mathrm{Hom}_{\mathbf{Y}_{r,n}}(\mathbf{Y}_{r,n},N^{\boldsymbol{\mu}}) \\ &\cong N^{\boldsymbol{\mu}} \end{aligned}$$

as required.

Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{C}_{r,n}$. Recall that for each $S \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $T \in \mathcal{T}_0^+(\boldsymbol{\lambda}, \boldsymbol{\nu})$ there is a $Y_{r,n}$ module homomorphism $\varphi'_{ST} : N^{\boldsymbol{\nu}} \to N^{\boldsymbol{\mu}}$; this induces a YS_n^r -module homomorphism $\Phi'_{ST} : E^{\boldsymbol{\nu}} \to E^{\boldsymbol{\mu}}$ defined by $\Phi'_{ST}(\theta) = \varphi'_{ST}\theta$ for $\theta \in E^{\boldsymbol{\nu}}$. The next proposition, which can be proved in exactly the same way as in [Ma2, Proposition 7.1], shows that these maps $\{\Phi'_{ST}\}$ give a basis of all the YS_n^r -module homomorphisms from $E^{\boldsymbol{\nu}}$ to $E^{\boldsymbol{\mu}}$.

Proposition 5.37. Suppose that $\mu, \nu \in C_{r,n}$. Then $\operatorname{Hom}_{YS_n^r}(E^{\nu}, E^{\mu})$ is free as an \mathbb{R} -module with basis

$$\{\Phi'_{\mathrm{ST}} \mid \mathrm{S} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}, \boldsymbol{\mu}) \text{ and } \mathrm{T} \in \mathfrak{T}^+_0(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ for some } \boldsymbol{\lambda} \in \mathfrak{P}_{r,n}\}.$$

By definition $E_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} E^{\mu}$ is a full tilting module for YS_n^r . Define the Ringel dual of YS_n^r to be the algebra $\operatorname{End}_{YS_n^r}(E_n^r)$. If A is an algebra, let A^{op} be the opposite algebra in which the order of multiplication is reserved. The following corollary gives a description of the Ringel dual of YS_n^r .

Corollary 5.38. There exist canonical isomorphisms of \mathcal{R} -algebras

$$\operatorname{End}_{\operatorname{YS}_n^r}(E_n^r) \cong (\operatorname{YS}_n^n)^{\operatorname{op}}$$

Corollary 5.39. Suppose that $\lambda \in \mathcal{P}_{r,n}$. Then $F_n^r(T^{\lambda'}) \cong Y_{\lambda}$ as $Y_{r,n}$ -modules.

Proof. By Lemma 5.36 the natural map $\operatorname{Hom}_{Y_{r,n}}(N^{\nu}, N^{\mu}) \to \operatorname{Hom}_{YS_n^r}(E^{\nu}, E^{\mu})$ is injective; by Proposition 5.9(i) and Proposition 5.37 this is an isomorphism. Consequently, if an indecomposable tilting module T^{ν} is a direct summand of E^{λ} then $F_n^r(T^{\nu})$ is an indecomposable direct summand of N^{λ} . Now, $E^{\lambda} \cong T^{\lambda'} \oplus \bigoplus_{\lambda' \rhd \mu} e_{\lambda\mu} T^{\mu}$ by Corollary 5.35(i) and $F_n^r(E^{\lambda}) \cong N^{\lambda} \cong Y_{\lambda} \oplus \bigoplus_{\nu \rhd \lambda} c_{\nu\lambda} Y_{\nu}$ by Proposition 5.10(iii). Hence, the result follows by induction on the dominance order.

Corollary 5.40. Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Then $[T^{\lambda} : (W^{\mu})^{\circledast}] = [W^{\mu'} : L^{\lambda'}]$.

Proof. Recall that $F_n^r(W^{\mu}) \cong S^{\mu}$ by Proposition 4.3. By definition, it is easy to see that the functor F_n^r commutes with duality. Then we have $F_n^r((W^{\mu})^{\circledast}) \cong (F_n^r(W^{\mu}))^{\circledast} \cong S_{\mu'}$ by Corollary 5.20. Thus we have

$$[T^{\boldsymbol{\lambda}}: (W^{\boldsymbol{\mu}})^{\circledast}] = [\mathbf{F}_n^r(T^{\boldsymbol{\lambda}}): \mathbf{F}_n^r((W^{\boldsymbol{\mu}})^{\circledast})] = [Y_{\boldsymbol{\lambda}'}: S_{\boldsymbol{\mu}'}]$$
$$= [W_{\boldsymbol{\mu}'}: L_{\boldsymbol{\lambda}'}] = [W^{\boldsymbol{\mu}'}: L^{\boldsymbol{\lambda}'}],$$

where the last equality follows from Proposition 5.9(iii).

6. Appendix. Cyclotomic Yokonuma-Schur Algebras

In this appendix, we will generalize the results above to define and study the cyclotomic Yokonuma-Schur algebra by using the cellular basis of the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d$ constructed in [C1]. Since this approach is very similar, we only mention the main results and skip all the details of the proofs.

We first recall the definition of $Y_{r,n}^d$ and the construction of a cellular basis of it.

By definition, The affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n} = \widehat{Y}_{r,n}(q)$ is an \mathcal{R} -associative algebra generated by the elements $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}, X_1^{\pm 1}$, in which the generators $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}$ satisfy the following relations:

 $g_{i}g_{j} = g_{j}g_{i} \qquad \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \ge 2;$ $g_{i}g_{i+1}g_{i} = g_{i+1}g_{i}g_{i+1} \qquad \text{for all } i = 1, \dots, n-2;$ $t_{i}t_{j} = t_{j}t_{i} \qquad \text{for all } i, j = 1, \dots, n;$ $g_{i}t_{j} = t_{js_{i}}g_{i} \qquad \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n;$ $t_{i}^{r} = 1 \qquad \text{for all } i = 1, \dots, n;$ $g_{i}^{2} = 1 + (q - q^{-1})e_{i}g_{i} \qquad \text{for all } i = 1, \dots, n-1,$ (6.1)

where s_i is the transposition (i, i + 1), and for each $1 \le i \le n - 1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s},$$

together with the following relations concerning the generators $X_1^{\pm 1}$:

$$X_{1}X_{1}^{-1} = X_{1}^{-1}X_{1} = 1;$$

$$g_{1}X_{1}g_{1}X_{1} = X_{1}g_{1}X_{1}g_{1};$$

$$g_{i}X_{1} = X_{1}g_{i} for all i = 2, \dots, n-1;$$

$$t_{j}X_{1} = X_{1}t_{j} for all j = 1, \dots, n.$$
(6.2)

We define inductively the following elements in $\widehat{Y}_{r,n}$:

$$X_{i+1} := g_i X_i g_i \quad \text{for } i = 1, \dots, n-1.$$
 (6.3)

Let $d \ge 1$ and v_1, \ldots, v_d be some invertible indeterminates. Set $f_1 := (X_1 - v_1) \cdots (X_1 - v_d)$. Let \mathcal{J}_d denote the two-sided ideal of $\widehat{Y}_{r,n}$ generated by f_1 , and define the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d = Y_{r,n}^d(q)$ to be the quotient

$$\mathbf{Y}_{r,n}^d = \widehat{\mathbf{Y}}_{r,n} / \mathcal{J}_d.$$

It has been proved in [C1] (see also [ChPA2, Theorem 4.15]) that the set of the following elements

 $\{t_1^{\beta_1}\cdots t_n^{\beta_n}X_1^{\alpha_1}\cdots X_n^{\alpha_n}g_w \mid 0 \le \alpha_1,\ldots,\alpha_n \le d-1, \ 0 \le \beta_1,\ldots,\beta_n \le r-1, \ w \in \mathfrak{S}_n\}$ forms an \mathfrak{R} -basis of $\mathbf{Y}_{r,n}^d$.

Let $d \in \mathbb{Z}_{\geq 1}$. Following [ChPA2, Section 3.1], the combinatorial objects appearing in the representation theory of the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d$ will be *m*compositions (resp. *m*-partitions) with m = rd, which can also be identified with *r*-tuples of *d*-compositions (resp. *d*-partitions). We will call such an object an (r, d)-composition (resp. (r, d)-partition). By definition, an (r, d)-composition (resp. (r, d)-partition) of *n* is an ordered *r*-tuple $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)}) = ((\lambda_1^{(1)}, \ldots, \lambda_d^{(1)}), \ldots, (\lambda_1^{(r)}, \ldots, \lambda_d^{(r)}))$ of *d*compositions (resp. *d*-partitions) $(\lambda_1^{(k)}, \ldots, \lambda_d^{(k)})$ $(1 \le k \le r)$ such that $\sum_{k=1}^r \sum_{j=1}^d |\lambda_j^{(k)}| =$ *n*. We denote by $\mathbb{C}_{r,n}^d$ (resp. $\mathcal{P}_{r,n}^d$) the set of (r, d)-compositions (resp. (r, d)-partitions) of *n*. We will say that the *l*-th composition (resp. partition) of the *k*-th *r*-tuple has position (k, l).

A triplet $\boldsymbol{\theta} = (\theta, k, l)$ consisting of a node θ , an integer $k \in \{1, \ldots, r\}$, and an integer $l \in \{1, \ldots, d\}$ is called an (r, d)-node. We shall say that the (r, d)-node $\boldsymbol{\theta}$ has position (k, l). We shall denote by $[\boldsymbol{\lambda}]$ the set of (r, d)-nodes such that the subset consisting of the (r, d)-nodes having position (k, l) forms a usual composition (resp. partition) $\lambda_l^{(k)}$, for any $k \in \{1, \ldots, r\}$ and $l \in \{1, \ldots, d\}$.

Let $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)}))$ be an (r, d)-composition of n. An (r, d)-tableau $\mathbf{t} = ((\mathbf{t}_1^{(1)}, \dots, \mathbf{t}_d^{(1)}), \dots, (\mathbf{t}_1^{(r)}, \dots, \mathbf{t}_d^{(r)}))$ of shape $\underline{\lambda}$ is obtained by placing each (r, d)-node of $[\underline{\lambda}]$ by one of the integers $1, 2, \dots, n$, allowing no repeats. We will call the number n the size of \mathbf{t} and the $\mathbf{t}_l^{(k)}$'s the components of \mathbf{t} . Each (r, d)-node $\boldsymbol{\theta}$ of \mathbf{t} is labelled by ((a, b), k, l) if it lies in row a and column b of the component $\mathbf{t}_l^{(k)}$ of \mathbf{t} .

For each $\underline{\mu} \in \mathbb{C}_{r,n}^d$, an (r, d)-tableau of shape $\underline{\mu}$ is called row standard if the numbers increase along any row (from left to right) of each diagram in $[\underline{\mu}]$. For each $\underline{\lambda} \in \mathcal{P}_{r,n}^d$, an (r, d)-tableau of shape $\underline{\lambda}$ is called standard if the numbers increase along any row (from left to right) and down any column (from top to bottom) of each diagram in $[\underline{\lambda}]$. From now on, we denote by $\operatorname{Std}(\underline{\lambda})$ the set of all standard (r, d)-tableaux of size n and of shape $\underline{\lambda}$, which is endowed with an action of \mathfrak{S}_n from the right by permuting the entries in each (r, d)-tableau.

For each $\underline{\lambda} \in \mathbb{C}^{d}_{r,n}$, we denote by $\mathfrak{t}^{\underline{\lambda}}$ the standard (r, d)-tableau of shape $\underline{\lambda}$ in which $1, 2, \ldots, n$ appear in increasing order from left to right along the rows of the first diagram, and then along the rows of the second diagram, and so on.

For each $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathcal{C}_{r,n}^d$, we have a Young subgroup

$$\mathfrak{S}_{\underline{\lambda}} := \mathfrak{S}_{\lambda_1^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda_d^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda_1^{(r)}} \times \cdots \times \mathfrak{S}_{\lambda_d^{(r)}},$$

which is exactly the row stabilizer of $t^{\underline{\lambda}}$.

For each $\underline{\lambda} \in \mathbb{C}_{r,n}^d$ and a row standard (r, d)-tableau \mathfrak{s} of shape $\underline{\lambda}$, let $d(\mathfrak{s})$ be the element of \mathfrak{S}_n such that $\mathfrak{s} = \mathfrak{t}^{\underline{\lambda}} d(\mathfrak{s})$. Then $d(\mathfrak{s})$ is a distinguished right coset representative of $\mathfrak{S}_{\underline{\lambda}}$ in \mathfrak{S}_n , that is, $l(wd(\mathfrak{s})) = l(w) + l(d(\mathfrak{s}))$ for any $w \in \mathfrak{S}_{\underline{\lambda}}$. In this way, we obtain a correspondence between the set of row standard (r, d)-tableaux of shape $\underline{\lambda}$ and the set of distinguished right coset representatives of $\mathfrak{S}_{\underline{\lambda}}$ in \mathfrak{S}_n .

We now define a partial order on the set of (r, d)-compositions.

Definition 6.1. Let $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)}))$ and $\underline{\mu} = ((\mu_1^{(1)}, \dots, \mu_d^{(1)}), \dots, (\mu_1^{(r)}, \dots, \mu_d^{(r)}))$ be two (r, d)-compositions of n. We say that $\underline{\lambda}$ dominates $\underline{\mu}$, and we write $\underline{\lambda} \ge \underline{\mu}$ if and only if

$$\sum_{i=1}^{k-1} \sum_{j=1}^{d} |\lambda_j^{(i)}| + \sum_{j=1}^{l-1} |\lambda_j^{(k)}| + \sum_{i=1}^{p} \lambda_{l,i}^{(k)} \ge \sum_{i=1}^{k-1} \sum_{j=1}^{d} |\mu_j^{(i)}| + \sum_{j=1}^{l-1} |\mu_j^{(k)}| + \sum_{i=1}^{p} \mu_{l,i}^{(k)}$$

for all k, l and p with $1 \le k \le r$, $1 \le l \le d$ and $p \ge 0$. If $\underline{\lambda} \ge \underline{\mu}$ and $\underline{\lambda} \ne \underline{\mu}$, we write $\underline{\lambda} > \underline{\mu}$.

Definition 6.2. Let $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathbb{C}_{r,n}^d$. Suppose that we choose all $1 \leq i_1 < i_2 < \dots < i_p \leq r$ such that $(\lambda_1^{(i_1)}, \dots, \lambda_d^{(i_1)}), (\lambda_1^{(i_2)}, \dots, \lambda_d^{(i_2)}), \dots, (\lambda_1^{(i_p)}, \dots, \lambda_d^{(i_p)})$ are nonempty. Define $a_k := \sum_{j=1}^k |\boldsymbol{\lambda}^{(i_j)}|$ for $1 \leq k \leq p$, where $|\boldsymbol{\lambda}^{(i_j)}| = \sum_{l=1}^d |\lambda_l^{(i_j)}|$. Then the set partition $A_{\underline{\lambda}}$ associated with $\underline{\lambda}$ is defined as

 $A_{\underline{\lambda}} := \{\{1, \dots, a_1\}, \{a_1 + 1, \dots, a_2\}, \dots, \{a_{p-1} + 1, \dots, n\}\},\$

which may be written as $A_{\underline{\lambda}} = \{I_1, I_2, \dots, I_p\}$, and is referred to the blocks of $A_{\underline{\lambda}}$ in the order given above.

Definition 6.3. Let $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathbb{C}_{r,n}^d$, and let $a_k := \sum_{i=1}^k |\lambda^{(i_j)}| \ (1 \le k \le p)$ be defined as above. Then we define

$$u_{\underline{\lambda}} := u_{a_1,i_1} u_{a_2,i_2} \cdots u_{a_p,i_p}$$

Definition 6.4. Let $\underline{\lambda} = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathbb{C}_{r,n}^d$. Associated with $\underline{\lambda}$ we can define the following elements a_l^k and b_k :

$$a_l^k := \sum_{m=1}^{l-1} |\lambda_m^{(k)}|, \ b_k := \sum_{j=1}^{k-1} \sum_{i=1}^d |\lambda_i^{(j)}| \text{ for } 1 \le k \le r \text{ and } 1 \le l \le d.$$

Associated with these elements we can define an element $u_{\mathbf{a}}^+ := u_{\mathbf{a},1} u_{\mathbf{a},2} \cdots u_{\mathbf{a},r}$, where

$$u_{\mathbf{a},k} := \prod_{l=1}^{d} \prod_{j=1}^{a_l^k} (X_{b_k+j} - v_l).$$

We can now define the key ingredient of the cellular basis for $Y_{r,n}^d$.

Definition 6.5. Let $\underline{\lambda} \in \mathcal{C}_{r,n}^d$ and define $u_{\mathbf{a}}^+$ as above. Let $x_{\underline{\lambda}} = \sum_{w \in \mathfrak{S}_{\underline{\lambda}}} q^{l(w)} g_w$. Then we define the element $m_{\underline{\lambda}}$ of $Y_{r,n}^d$ as follows:

$$m_{\underline{\lambda}} := U_{\underline{\lambda}} u_{\mathbf{a}}^{+} x_{\underline{\lambda}} = u_{\underline{\lambda}} E_{A_{\underline{\lambda}}} u_{\mathbf{a}}^{+} x_{\underline{\lambda}}.$$
(6.4)

Let * denote the \mathcal{R} -linear anti-automorphism of $Y_{r,n}^d$, which is determined by

$$g_i^* = g_i, \quad t_j^* = t_j, \quad X_j^* = X_j \quad \text{for } 1 \le i \le n-1 \text{ and } 1 \le j \le n.$$

Definition 6.6. Let $\underline{\lambda} \in \mathbb{C}^d_{r,n}$, and let \mathfrak{s} and \mathfrak{t} be two row standard (r, d)-tableaux of shape $\underline{\lambda}$. We then define $m_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} m_{\underline{\lambda}} g_{d(\mathfrak{t})}$.

For each $\underline{\lambda} \in \mathcal{P}_{r,n}^d$, let $Y_{r,n}^{d, \triangleright \underline{\lambda}}$ be the \mathcal{R} -submodule of $Y_{r,n}^d$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in$ Std (μ) for various $\underline{\mu} \in \mathcal{P}_{r,n}^d$ such that $\underline{\mu} \triangleright \underline{\lambda}$.

Theorem 6.7. (See [C1, Theorem 6.18].) The algebra $Y_{r,n}^d$ is a free \mathcal{R} -module with a cellular basis

 $\mathcal{B}_{r,n}^d = \{ m_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\underline{\lambda}) \text{ for some } (r,d) - partition \ \underline{\lambda} \text{ of } n \},\$

that is, the following properties hold:

(i) The \mathcal{R} -linear map determined by $m_{\mathfrak{st}} \mapsto m_{\mathfrak{ts}} (m_{\mathfrak{st}} \in \mathcal{B}^d_{r,n})$ is an anti-automorphism on $Y^d_{r,n}$.

(ii) For a given $h \in Y_{r,n}^d$, $\underline{\mu} \in \mathcal{P}_{r,n}^d$ and $\mathfrak{t} \in \mathrm{Std}(\underline{\mu})$, there exist $r_{\mathfrak{vt}}(h) \in \mathfrak{R}$ such that for all $\mathfrak{s} \in \mathrm{Std}(\mu)$, we have

$$m_{\mathfrak{s}\mathfrak{t}}h\equiv\sum_{\mathfrak{v}\in\mathrm{Std}(\underline{\mu})}r_{\mathfrak{v}\mathfrak{t}}(h)m_{\mathfrak{s}\mathfrak{v}}\ \mathrm{mod}\ Y^{d,\rhd\underline{\mu}}_{r,n},$$

where $r_{\mathfrak{vt}}(h)$ may depend on $\mathfrak{v}, \mathfrak{t}$ and h, but not on \mathfrak{s} .

For $\underline{\lambda} \in \mathbb{C}_{r,n}^d$, a $\underline{\lambda}$ -tableau $\mathbf{S} = ((S_1^{(1)}, \dots, S_d^{(1)}), \dots, (S_1^{(r)}, \dots, S_d^{(r)}))$ is a map $\mathbf{S} : [\underline{\lambda}] \to \{1, \dots, n\} \times \{1, \dots, d\} \times \{1, \dots, r\}$, which can be regarded as the diagram $[\underline{\lambda}]$, together with an ordered triple (i, j, k) $(1 \le i \le n, 1 \le j \le d, 1 \le k \le r)$ attached to each node. Given $\underline{\lambda} \in \mathcal{P}_{r,n}^d$ and $\underline{\mu} \in \mathbb{C}_{r,n}^d$, a $\underline{\lambda}$ -tableau S is said to be of type $\underline{\mu}$ if the number of (i, j, k) in the entry of S is equal to $\mu_{j,i}^{(k)}$. Given $\mathfrak{s} \in \operatorname{Std}(\underline{\lambda}), \underline{\mu}(\mathfrak{s}),$ a $\underline{\lambda}$ -tableau of type $\underline{\mu}$,

is defined by replacing each entry m in \mathfrak{s} by (i, j, k) if m is in the *i*-th row of the (k, j)-th component of $\mathfrak{t}^{\underline{\mu}}$.

We define a total order on the set of triples (i, j, k) by $(i_1, j_1, k_1) < (i_2, j_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $j_1 < j_2$, or $k_1 = k_2$, $j_1 = j_2$, and $i_1 < i_2$, Let $\underline{\lambda} \in \mathcal{P}^d_{r,n}$ and $\underline{\mu} \in \mathcal{C}^d_{r,n}$. Suppose that $S = ((S_1^{(1)}, \dots, S_d^{(1)}), \dots, (S_1^{(r)}, \dots, S_d^{(r)}))$ is a $\underline{\lambda}$ -tableau of type $\underline{\mu}$. S is said to be semistandard if each component $S_j^{(k)}$ is non-decreasing in rows, strictly increasing in columns, and all entries of $S_j^{(k)}$ are of the form (i, h, l) with $h \ge j$ and $l \ge k$. We denote by $\mathcal{T}_0(\underline{\lambda}, \underline{\mu})$ the set of semistandard $\underline{\lambda}$ -tableaux of type $\underline{\mu}$.

For any $\underline{\kappa} \in \mathbb{C}^d_{r,n}$, we define its type $\alpha(\underline{\kappa})$ by $\alpha(\underline{\kappa}) = (n_1, \ldots, n_r)$ with $n_i = |\kappa^{(i)}|$. Assume that $\underline{\lambda} \in \mathbb{P}^d_{r,n}$ and $\underline{\mu} \in \mathbb{C}^d_{r,n}$. We define a subset $\mathbb{T}^+_0(\underline{\lambda}, \underline{\mu})$ of $\mathbb{T}_0(\underline{\lambda}, \underline{\mu})$ by

$$\mathfrak{T}_{0}^{+}(\underline{\lambda},\underline{\mu}) = \{ S \in \mathfrak{T}_{0}(\underline{\lambda},\underline{\mu}) \mid \alpha(\underline{\lambda}) = \alpha(\underline{\mu}) \}.$$

For each $\underline{\mu} \in \mathbb{C}^d_{r,n}$, let $M^{\underline{\mu}} = m_{\underline{\mu}} Y^d_{r,n}$. We now construct a basis of $M^{\underline{\mu}}$ related to the cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ in Theorem 6.7. For $S \in \mathcal{T}^+_0(\underline{\lambda}, \underline{\mu})$ and $\mathfrak{t} \in \mathrm{Std}(\underline{\lambda})$, we define

$$m_{\mathrm{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\underline{\lambda}) \\ \underline{\mu}(\mathfrak{s}) = \mathrm{S}}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}.$$

The following theorem can be proved in exactly the same way as in [DJM, Theorem 4.14] by combining [DJM] with [C1].

Theorem 6.8. Let $S \in \mathcal{T}_0^+(\underline{\lambda}, \underline{\mu})$ and $\mathfrak{t} \in \text{Std}(\underline{\lambda})$ for some $\underline{\lambda} \in \mathcal{P}_{r,n}^d$ and $\underline{\mu} \in \mathcal{C}_{r,n}^d$. Then $m_{\text{St}} \in M^{\underline{\mu}}$. Moreover, $M^{\underline{\mu}}$ is free with an \mathcal{R} -basis

$$\{m_{\mathrm{St}} \mid \mathrm{S} \in \mathfrak{T}_0^+(\underline{\lambda}, \underline{\mu}) \text{ and } \mathfrak{t} \in \mathrm{Std}(\underline{\lambda}) \text{ for some } \underline{\lambda} \in \mathcal{P}_{r,n}^d\}.$$

Let $\underline{\mu}, \underline{\nu} \in \mathcal{C}^d_{r,n}$ and $\underline{\lambda} \in \mathcal{P}^d_{r,n}$. We assume that $\alpha(\underline{\mu}) = \alpha(\underline{\nu}) = \alpha(\underline{\lambda})$. For $S \in \mathcal{T}^+_0(\underline{\lambda}, \underline{\mu})$, $T \in \mathcal{T}^+_0(\underline{\lambda}, \underline{\nu})$, put

$$m_{\rm ST} = \sum_{\mathfrak{s},\mathfrak{t}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}$$

where the sum is taken over all $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\underline{\lambda})$ such that $\underline{\mu}(\mathfrak{s}) = S$ and $\underline{\nu}(\mathfrak{t}) = T$. We then have the next proposition by making use of Theorem 6.8.

Proposition 6.9. Suppose that $\underline{\mu}, \underline{\nu} \in C^d_{r,n}$ with $\alpha(\underline{\mu}) = \alpha(\underline{\nu})$. Then the set

$$\{m_{\mathrm{ST}} \mid \mathrm{S} \in \mathfrak{T}_0^+(\underline{\lambda}, \underline{\mu}) \text{ and } \mathrm{T} \in \mathfrak{T}_0^+(\underline{\lambda}, \underline{\nu}) \text{ for some } \underline{\lambda} \in \mathfrak{P}_{r,n}^d\}$$

is an \mathcal{R} -basis of $M^{\underline{\nu}*} \cap M^{\underline{\mu}}$.

Definition 6.10. Suppose that $M_n^{r,d} = \bigoplus_{\underline{\mu} \in \mathbb{C}_{r,n}^d} M^{\underline{\mu}}$. We define the cyclotomic Yokonuma-Schur algebra $YS_n^{r,d}$ as the endomorphism algebra

$$YS_n^{r,d} = End_{Y_{r,n}^d}(M_n^{r,d}),$$

which is isomorphic to $\bigoplus_{\mu,\underline{\nu}\in\mathcal{C}_{r,n}^d} \operatorname{Hom}_{Y_{r,n}^d}(M^{\underline{\nu}}, M^{\underline{\mu}}).$

Let $S \in \mathcal{T}_0^+(\underline{\lambda}, \underline{\mu})$ and $T \in \mathcal{T}_0^+(\underline{\lambda}, \underline{\nu})$. In view of Proposition 6.9, we can define $\varphi_{ST} \in Hom_{Y_{d-1}^d}(M^{\underline{\nu}}, M^{\underline{\mu}})$ by

$$\varphi_{\rm ST}(m_{\nu}h) = m_{\rm ST}h$$

for all $h \in Y_{r,n}^d$. We extend φ_{ST} to an element of $YS_n^{r,d}$ by defining φ_{ST} to be zero on $M^{\underline{\kappa}}$ for any $\underline{\nu} \neq \underline{\kappa} \in \mathcal{C}_{r,n}^d$. For each $\underline{\lambda} \in \mathcal{P}_{r,n}^d$, let $\mathcal{T}_0^+(\underline{\lambda}) = \bigcup_{\underline{\mu} \in \mathcal{C}_{r,n}^d} \mathcal{T}_0^+(\underline{\lambda}, \underline{\mu})$. We denote by $YS_{r,n}^{d, \geq \underline{\lambda}}$ the \mathcal{R} -submodule of $YS_n^{r,d}$ spanned by φ_{ST} such that $S, T \in \mathcal{T}_0^+(\underline{\alpha})$ with $\underline{\alpha} \geq \underline{\lambda}$. Then we can prove the following theorem by a similar argument as in [DJM, Theorem 6.6].

Theorem 6.11. The Yokonuma-Schur algebra $YS_n^{r,d}$ is free as an \mathcal{R} -module with a basis

 $\{\varphi_{\mathrm{ST}} \mid \mathrm{S}, \mathrm{T} \in \mathfrak{T}_0^+(\underline{\lambda}) \text{ for some } \underline{\lambda} \in \mathfrak{P}_{r,n}^d\}.$

Moreover, this basis satisfies the following properties.

(i) The \mathcal{R} -linear map $*: \mathrm{YS}_n^{r,d} \to \mathrm{YS}_n^{r,d}$ determined by $\varphi_{\mathrm{ST}}^* = \varphi_{\mathrm{TS}}$, for all $\mathrm{S}, \mathrm{T} \in \mathcal{T}_0^+(\underline{\lambda})$ and all $\underline{\lambda} \in \mathcal{P}_{r,n}^d$, is an anti-automorphism of $\mathrm{YS}_n^{r,d}$.

(ii) Let $T \in \mathcal{T}_0^+(\underline{\lambda})$ and $\varphi \in YS_n^{r,d}$. Then for each $V \in \mathcal{T}_0^+(\underline{\lambda})$, there exists $r_V = r_{V,T,\varphi} \in \mathcal{R}$ such that for all $S \in \mathcal{T}_0^+(\underline{\lambda})$, we have

$$\varphi_{\mathrm{ST}}\varphi \equiv \sum_{\mathrm{V}\in\mathfrak{T}_0^+(\underline{\lambda})} r_{\mathrm{V}}\varphi_{\mathrm{SV}} \ \mathrm{mod} \ \mathrm{YS}_{r,n}^{d,\rhd}\underline{\lambda}.$$

In particular, this basis $\{\varphi_{ST}\}$ is a cellular basis of $YS_n^{r,d}$.

Now we can apply the general theory of cellular algebras in view of Theorem 6.11. For example, we can easily give a complete set of non-isomorphic irreducible $YS_n^{r,d}$ -modules over an arbitrary field, and further prove that $YS_n^{r,d}$ is a quasi-hereditary algebra. For the cyclotomic Yokonuma-Schur algebra $YS_n^{r,d}$, we can also define the Schur functor from the category of $YS_n^{r,d}$ -modules to the category of $Y_{r,n}^d$ -modules and the tilting modules for it in exactly the same way as in Sections 4 and 5, we skip all the details and leave them to the reader.

References

- [AK] S. Ariki and K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations, Adv. Math. **106** (1994) 216-243.
- [ChL] M. Chlouveraki and S. Lambropoulou, The Yokonuma-Hecke algebras and the HOMFLYPT polynomial, J. Knot Theory Ramifications 22 (2013) 35 pp.
- [ChPA1] M. Chlouveraki and L. Poulain d'Andecy, Representation theory of the Yokonuma-Hecke algebra, Adv. Math. 259 (2014) 134-172.
- [ChPA2] M. Chlouveraki and L. Poulain d'Andecy, Markov traces on affine and cyclotomic Yokonuma-Hecke algebras, arXiv: 1406.3207.
- [ChS] M. Chlouveraki and V. Sécherre, The affine Yokonuma-Hecke algebra and the pro-p-Iwahori-Hecke algebra, arXiv: 1504.04557.
- [CW] W. Cui and J. Wan, Modular representations and branching rules for affine and cyclotomic Yokonuma-Hecke algebras, arXiv: 1506.06570.
- [C1] W. Cui, Cellularity of cyclotomic Yokonuma-Hecke algebras, arXiv: 1506.07321.
- [C2] W. Cui, Affine cellularity of affine Yokonuma-Hecke algebras, arXiv: 1510.02647.

- [CR] C.W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras, Pure and Applied Mathematics, Vol. XI Interscience Publishers, a division of John Wiley & Sons, New York-London 1962 xiv+685 pp.
- [DJM] R. Dipper, G. James and A. Mathas, Cyclotomic q-Schur algebras, Math. Z. 229 (1998) 385-416.
- [ER] J. Espinoza and S. Ryom-Hansen, Cell structures for the Yokonuma-Hecke algebra and the algebra of braids and ties, arXiv: 1506.00715.
- [GL] J.J. Graham and G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1-34.
- [HM1] J. Hu and A. Mathas, Graded induction for Specht modules, Int. Math. Res. Not. 6 (2012) 1230-1263.
- [HM2] J. Hu and A. Mathas, Quiver Schur algebras for linear quivers, Proc. Lond. Math. Soc. (3) 110 (2015) 1315-1386.
- [JM] G.D. James and A. Mathas, The Jantzen sum formula for cyclotomic q-Schur algebras, Trans. Amer. Math. Soc. 352 (2000) 5381-5404.
- [JP] N. Jacon and L. Poulain d'Andecy, An isomorphism theorem for Yokonuma-Hecke algebras and applications to link invariants, arXiv: 1501.06389.
- [Ju1] J. Juyumaya, Sur les nouveaux générateurs de l'algèbre de Hecke $\mathcal{H}(G, U, 1)$. (French) On new generators of the Hecke algebra $\mathcal{H}(G, U, 1)$, J. Algebra **204** (1998) 49-68.
- [Ju2] J. Juyumaya, Markov trace on the Yokonuma-Hecke algebra, J. Knot Theory Ramifications 13 (2004) 25-39.
- [JuK] J. Juyumaya and S. Kannan, Braid relations in the Yokonuma-Hecke algebra, J. Algebra 239 (2001) 272-297.
- [JuL] J. Juyumaya and S. Lambropoulou, p-adic framed braids II. With an appendix by Paul Gérardin, Adv. Math. 234 (2013) 149-191.
- [Lu] G. Lusztig, Character sheaves on disconnected groups. VII, Represent. Theory (electronic) 9 (2005) 209-266.
- [Ma1] A. Mathas, Iwahori-Hecke algebras and Schur algebras of the symmetric group, University Lecture Series, 15. American Mathematical Society, Providence, RI, 1999. xiv+188 pp.
- [Ma2] A. Mathas, Tilting modules for cyclotomic Schur algebras, J. Reine Angew. Math. **562** (2003) 137-169.
- [Ma3] A. Mathas, Seminormal forms and Gram determinants for cellular algebras, J. Reine Angew. Math. 619 (2008) 141-173.
- [Ri] C. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208 (1991) 209-223.
- [Ro] R. Rouquier, q-Schur algebras and complex reflection groups, Mosc. Math. J. 8 (2008) 119-158.
- [SS] N. Sawada, T. Shoji, Modified Ariki-Koike algebras and cyclotomic q-Schur algebras, Math. Z. 249 (2005) 829-867.
- [Vi] M.-F. Vignéras, The pro-*p*-Iwahori-Hecke algebra of a reductive *p*-adic group I, preprint (2013).
- [Yo] T. Yokonuma, Sur la structure des anneaux de Hecke d'un groupe de Chevalley fini, C. R. Acad. Sci. Paris Ser. A-B 264 (1967) 344-347.

School of Mathematics, Shandong University, Jinan, Shandong 250100, P.R. China. *E-mail address:* cwdeng@amss.ac.cn