# On the Markov-Dyck shifts of vertex type

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#### Abstract

For a given finite directed graph G, there are two types of Markov-Dyck shifts, the Markov-Dyck shift  $D_G^V$  of vertex type and the Markov-Dyck shift  $D_G^E$  of edge type. It is shown that, if G does not have multi-edges, the former is a finite-to-one factor of the latter, and they have the same topological entropy. An expression for the zeta function of a Markov-Dyck shift of vertex type is given. It is different from that of the Markov-Dyck shift of edge type.

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### 1 Introduction

Let  $\Sigma$  be a finite alphabet, and let  $\sigma$  be the left shift on  $\Sigma^{\mathbb{Z}}$  defined by  $\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}, (x_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ . For a closed subset  $\Lambda \subset \Sigma^{\mathbb{Z}}$  satisfying  $\sigma(\Lambda) = \Lambda$ , the topological dynamical system  $(\Lambda, \sigma)$  is called a subshift. Denote by  $B_n(\Lambda)$  the set of all admissible words appearing in  $\Lambda$  with length n, and by  $P_n(\Lambda)$  the set of all n-periodic points of  $(\Lambda, \sigma)$ , respectively. Then the topological entropy  $h_{top}(\Lambda)$  and the zeta function  $\zeta_{\Lambda}(z)$  for  $(\Lambda, \sigma)$  is defined by

$$h_{top}(\Lambda) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(\Lambda)|, \qquad (1.1)$$

$$\zeta_{\Lambda}(z) = \exp(\sum_{n=1}^{\infty} \frac{|P_n(\Lambda)|z^n}{n}).$$
(1.2)

They are crucial topological conjugacy invariants of  $(\Lambda, \sigma)$ . For an introduction to their theory, which belongs to symbolic dynamics, we refer to [10] and [15].

W. Krieger in [11] has introduced the Dyck shifts from automata theory and language theory in computer science. They are non-sofic subshifts defined by Dyck languages. In [7, 11, 12, 14, 17], a class of non-sofic subshifts called Markov-Dyck shifts have been studied (cf. [8]). The subshifts are generalization of Dyck shifts by using finite directed graphs. They have recently come to be studied by computer scientists (cf. [1, 2]). For a given finite directed graph G = (V, E), there are two types of Markov-Dyck shifts, the Markov-Dyck shift  $D_G^V$  of vertex type and the Markov-Dyck shift  $D_G^E$  of edge type. Both of them are not sofic subshifts if G is irreducible and not permutive. In the papers [7, 11, 12, 14], the Markov-Dyck shifts mean the Markov-Dyck shifts of edge type. In [14], formulae of topological entropy and zeta functions for Markov-Dyck shifts of edge type have been presented.

In the first part of the paper, we will study relationship between the two types of Markov-Dyck shifts for finite directed graphs, the Markov-Dyck shift  $D_G^V$  of vertex type and the Markov-Dyck shift  $D_G^E$  of edge type. We will show that, if G does not have multiedges, there exists a finite-to-one factor code from  $D_G^E$  to  $D_G^V$  (Proposition 2.9). The factor code can never yield a topological conjugacy unless the transition matrix of the graph is permutation. They have the same topological entropy (Theorem 2.10).

In the second part of the paper, we will present a formula of the zeta function of a Markov-Dyck shift of vertex type (Theorem 3.9). The formula is regarded as a generalization of the formula for Markov-Dyck shifts of edge type [14, Theorem 2.3]. In the final section, the zeta function of the Fibonacci-Dyck shift of vertex type will be presented. It is different from that of the Fibonacci-Dyck shift of edge type. Hence the Fibonacci-Dyck shift of vertex type is not topologically conjugate to the Fibonacci-Dyck shift of edge type.

#### 2 Markov-Dyck shifts

Throughout this paper N is a fixed positive integer larger than 1. For a finite set S, we denote by |S| its cardinality. We consider the Dyck shift  $D_N$  with alphabet  $\Sigma = \Sigma^- \cup \Sigma^+$  where  $\Sigma^- = \{\alpha_1, \ldots, \alpha_N\}, \Sigma^+ = \{\beta_1, \ldots, \beta_N\}$ . The symbols  $\alpha_i, \beta_i$  correspond to the brackets  $(i, )_i$  respectively, and have the product relations of monoid as follows:

$$\alpha_i \beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$
(2.1)

for i, j = 1, ..., N (cf. [12, 13]). For a word  $\omega = \omega_1 \cdots \omega_n$  of  $\Sigma$ , we denote by  $\tilde{\omega}$  its reduced form. Namely  $\tilde{\omega}$  is a word of  $\Sigma \cup \{0, \mathbf{1}\}$  obtained after applying the relations (2.1) in  $\omega$ . Then a word  $\omega$  of  $\Sigma$  is said to be forbidden in  $D_N$  if and only if  $\tilde{\omega} = 0$ . Denote by  $\mathfrak{F}_N$  the set of forbidden words. The Dyck shift  $D_N$  is defined in [11] by a subshift over  $\Sigma$  whose forbidden words are  $\mathfrak{F}_N$ , namely

$$D_N = \{ (x_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mid \forall k \in \mathbb{Z}, m \in \mathbb{N}, (x_k, x_{k+1}, \dots, x_{k+m}) \notin \mathfrak{F}_N \}.$$
(2.2)

Let  $A = [A(i, j)]_{i,j=1,...,N}$  be an  $N \times N$  matrix with entries in  $\{0, 1\}$ . Throughout this paper, A is assumed to be essential which means that it has no zero rows or columns. Consider the Cuntz-Krieger algebra  $\mathcal{O}_A$  for the matrix A that is the universal  $C^*$ -algebra generated by N partial isometries  $t_1, \ldots, t_N$  subject to the following relations:

$$\sum_{j=1}^{N} t_j t_j^* = 1, \qquad t_i^* t_i = \sum_{j=1}^{N} A(i,j) t_j t_j^* \quad \text{for } i = 1, \dots, N$$
(2.3)

([4]). Define a correspondence  $\varphi_A : \Sigma \longrightarrow \{t_i^*, t_i \mid i = 1, \dots, N\}$  by setting

$$\varphi_A(\alpha_i) = t_i^*, \qquad \varphi_A(\beta_i) = t_i \quad \text{for } i = 1, \dots, N.$$

We denote by  $\Sigma^*$  the set of all words  $\gamma_1 \cdots \gamma_n$  of elements of  $\Sigma$ . Define the set

$$\mathfrak{F}_A = \{\gamma_1 \cdots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0\}.$$

**Definition 2.1.** The topological Markov Dyck shift for A is defined as a subshift over  $\Sigma$  whose forbidden words are  $\mathfrak{F}_A$ . It is written  $D_A$  and called the Markov-Dyck shift for A for brevity.

If A is irreducible and not any permutation matrix, the subshift  $D_A$  can never be sofic ([17, Proposition 2.1]). If all entries of A are 1's, the C\*-algebra  $\mathcal{O}_A$  becomes the Cuntz algebra  $\mathcal{O}_N$  of order N and the subshift  $D_A$  becomes the Dyck shift  $D_N$  with 2N brackets ([3]). We note that  $\alpha_i\beta_j \in \mathfrak{F}_A$  if  $i \neq j$ , and  $\alpha_{i_n} \cdots \alpha_{i_1} \in \mathfrak{F}_A$  if and only if  $\beta_{i_1} \cdots \beta_{i_n} \in \mathfrak{F}_A$ .

Let G = (V, E) be a finite directed graph with vertex set V and edge set E. We denote by s(e) the initial vertex of  $e \in E$  and by t(e) the final vertex, respectively. We assume that the cardinalities of V and of E are both finite and write  $V = \{v_1, \ldots, v_{N_0}\}$  and  $E = \{e_1, \ldots, e_{N_1}\}$ . We also assume that each vertex of G has at least one in-coming edge and at least one out-going edge. The edge matrix  $A^G = [A^G(i, j)]_{i,j=1}^{N_1}$  for G is an  $N_1 \times N_1$  transition matrix with entries in  $\{0, 1\}$  which is defined by

$$A^{G}(i,j) = \begin{cases} 1 & \text{if } t(e_{i}) = s(e_{j}), \\ 0 & \text{otherwise.} \end{cases}$$
(2.4)

In [14], we have defined the Markov-Dyck shift  $D_G$  for the graph G as the Markov-Dyck shift  $D_{A^G}$  for the matrix  $A^G$ , and presented formulae of the zeta function  $\zeta_{D_G}(z)$  and the topological entropy  $h(D_G)$ . A finite matrix M with entries in  $\{0, 1\}$  does not necessarily arise from a finite graph as  $M = A^G$ . The lemma below is easy to prove. For the sake of completeness, we provide its proof.

**Lemma 2.2.** Let  $M = [M(i,j)]_{i,j=1}^N$  be an essential  $N \times N$  matrix with entries in  $\{0,1\}$ . Let us denote by  $M_r[i] = [M(i,j)]_{j=1}^N$  and  $M_c[j] = [M(i,j)]_{i=1}^N$  the *i*th row vector and the *j*th column vector for i, j = 1, ..., N respectively. Then the following three conditions are equivalent:

- (i) There exists a finite directed graph G such that  $M = A^G$ .
- (ii) For any  $i_1, i_2 \in \{1, 2, \dots, N\}$ ,

$$M_r[i_1] = M_r[i_2] \quad or \quad \langle M_r[i_1] \mid M_r[i_2] \rangle = 0.$$
(2.5)

(iii) For any  $j_1, j_2 \in \{1, 2, \dots, N\}$ ,

$$M_{c}[j_{1}] = M_{c}[j_{2}] \quad or \quad \langle M_{c}[j_{1}] \mid M_{c}[j_{2}] \rangle = 0, \tag{2.6}$$

where  $\langle \cdot | \cdot \rangle$  means the inner product of vectors.

*Proof.* (i)  $\Longrightarrow$  (ii): Suppose that there exists a finite directed graph G such that  $M = A^G$ . For two edges  $e_{i_1}, e_{i_2} \in E$ , if  $t(e_{i_1}) = t(e_{i_2})$ , then  $M_r[i_1] = M_r[i_2]$ , otherwise  $\langle M_r[i_1] | M_r[i_2] \rangle = 0$ . (iii)  $\implies$  (i): Assume that the  $N \times N$  matrix M satisfies the condition (2.6). We will construct a finite directed graph G = (V, E) such that  $M = A^G$  as follows. Define an equivalence relation  $j_1 \sim j_2$  in  $\{1, 2, \ldots, N\}$  by  $M_c[j_1] = M_c[j_2]$ . Denote by  $[j]_c$  the equivalence class of  $j \in \{1, 2, \ldots, N\}$ . Then the vertex set V is defined by the set of equivalence classes  $\{[j]_c \mid j \in \{1, 2, \ldots, N\}\}$ . Define an edge labeled  $e_i$  from  $[i]_c$  to  $[j]_c$  if M(i, j) = 1. If there exist edges from  $[i]_c$  to  $[j_1]_c$  labeled  $e_i$  and  $[i]_c$  to  $[j_2]_c$  labeled  $e_i$ , then  $M(i, j_1) = M(i, j_2) = 1$ . By the condition (2.6), one has  $[j_1]_c = [j_2]_c$ . Hence the labeled graph is well-defined. Then as  $s(e_j) = [j]_c$ , the condition  $t(e_i) = s(e_j)$  is equivalent to the condition M(i, j) = 1. Hence we have  $A^G = M$ .

(ii)  $\Longrightarrow$  (iii): Suppose that there exist distinct  $j_1 \neq j_2 \in \{1, 2, ..., N\}$  such that  $M_c[j_1] \neq M_c[j_2]$  and  $\langle M_c[j_1] \mid M_c[j_2] \rangle \neq 0$ . The condition  $M_c[j_1] \neq M_c[j_2]$  implies that there exists  $i_1$  such that  $M(i_1, j_1) \neq M(i_1, j_2)$ . The condition  $\langle M_c[j_1] \mid M_c[j_2] \rangle \neq 0$  implies that there exists  $i_2$  such that  $M(i_2, j_1) = M(i_2, j_2) = 1$  so that  $\langle M_r[i_1] \mid M_r[j_2] \rangle \neq 0$ , a contradiction to the condition (ii).

The matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  is called the Fibonacci matrix. It can not arise from a finite directed graph as an edge matrix.

For a finite directed G = (V, E), we have another transition matrix  $A_G$ , which is an  $N_0 \times N_0$  matrix  $A_G = [A_G(i, j)]_{i,j=1}^{N_0}$  defined by

$$A_G(i,j) = \begin{cases} 1 & \text{if there exists an edge from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$
(2.7)

The matrix  $A_G$  is called the vertex matrix for the graph G. It has its entries in  $\{0, 1\}$ .

**Definition 2.3.** Let G = (V, E) be an essential finite directed graph.

- (i) The Markov-Dyck shift  $D_{A^G}$  for the edge matrix  $A^G$  is called the *Markov-Dyck shift* of edge type for G, and written  $D_G^E$ .
- (ii) The Markov-Dyck shift  $D_{A_G}$  for the vertex matrix  $A_G$  is called the *Markov-Dyck* shift of vertex type for G, and written  $D_G^V$ .

It is obvious that any finite matrix M with entries in  $\{0, 1\}$  can arise from a finite graph G such that  $M = A_G$ . By Lemma 2.2, one sees that the class of Markov-Dyck shifts of edge type is a subclass of Markov-Dyck shifts of vertex type. As is well-known that for a finite directed graph G the topological Markov shift  $X_{AG}$  defined by the edge matrix  $A^G$  is topologically conjugate to the topological Markov shift  $X_{AG}$  defined by the vertex matrix  $A_G$ . The Markov-Dyck shifts however do not have this property. Let  $G_1$  be the following graph (Figure 1). The vertex matrix  $A_{G_1}$  and the edge matrix  $A^{G_1}$  are written as

$$A_{G_1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A^{G_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
(2.8)

respectively. Then the Markov-Dyck shift  $D_{G_1}^V$  of vertex type is nothing but the Dyck shift  $D_2$ , whereas the Markov-Dyck shift  $D_{G_1}^E$  of edge type is not  $D_2$ . Both  $D_{G_1}^V$  and  $D_{G_1}^E$ 



Figure 1:

have 4 fixed points as subshifts. The former  $D_{G_1}^V$  has 4 periodic points with least period 2. The latter  $D_{G_1}^E$  has 6 periodic points with least period 2. Hence  $D_{G_1}^V$  is not topologically conjugate to  $D_{G_1}^E$ .

A Dyck *n*-path is a continuous broken directed line on the upper half plane consisting of vectors (1, 1) called rise and (1, -1) called fall. It starts at the origin with rise and ends at (2n, 0) with fall (see [5, 6], etc.). Let  $\gamma = (\gamma_1, \ldots, \gamma_{2n})$  be a Dyck *n*-path. Hence each  $\gamma_i$  is a rise or a fall. If  $\gamma_i$  is a rise, there exists the smallest  $k = 1, 2, \ldots, 2n - i$  satisfying the following two conditions:

- (i)  $\gamma_{i+k}$  is a fall.
- (ii)  $(\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{i+k-1})$  is a Dyck  $\frac{k-1}{2}$ -path (hence k-1 is even), which starts at the terminal vertex of  $\gamma_i$  and ends at the source vertex of  $\gamma_{i+k}$ .

We call the edge  $\gamma_{i+k}$  the partner of  $\gamma_i$ .

Let G = (V, E) be a finite directed graph. Denote by  $G^* = (V^*, E^*)$  the transposed graph of G. The vertex set  $V^*$  is V and the edge set  $E^*$  consists of the edges reversing its direction of the edges of G. For an edge  $e \in E$ , we denote by  $e^*$  the edge of  $G^*$  obtained by reversing the direction of e, so that  $t(e^*) = s(e), s(e^*) = t(e)$  for  $e \in E$ . Recall that the edge set E of G is denoted by  $\{e_1, \ldots, e_{N_1}\}$  and the edge set  $E^*$  of  $G^*$  is written as  $\{e_1^*, \ldots, e_{N_1}^*\}$ . Put  $\Sigma_E^- = E^*, \Sigma_E^+ = E$  and  $\Sigma_G^E = \Sigma_E^- \cup \Sigma_E^+$ . A G-Dyck n-path of edge type for  $n = 1, 2, \ldots$  is a Dyck n-path  $(x_1, \ldots, x_{2n})$  labeled elements of  $\Sigma_G^E$  satisfying the following rules:

- (1E) a rise is labeled  $e_i^*$  for some  $i = 1, \ldots, N_1$ ,
- (2E) a fall is labeled  $e_i$  for some  $i = 1, \ldots, N_1$ ,
- (3E) the partner of a rise labeled  $e_i^*$  is labeled  $e_i$ ,
- (4E) a rise labeled  $e_i^*$  follows a rise labeled  $e_i^*$  if and only if  $t(e_i^*) = s(e_i^*)$ ,
- (5E) a rise labeled  $e_i^*$  follows a fall labeled  $e_j$  if and only if  $t(e_j) = s(e_i^*)$ ,
- (6E) a fall labeled  $e_i$  follows a fall labeled  $e_j$  if and only if  $t(e_j) = s(e_i)$ ,
- (7E) a fall labeled  $e_i$  follows a rise labeled  $e_i^*$  if and only if  $e_j = e_i$ .

Similarly, for a vertex  $v \in V$ , we denote by  $v^*$  the corresponding vertex of  $G^*$  obtained by the transposed graph  $G^* = (V^*, E^*)$ . The vertex matrix  $A_{G^*}$  for  $G^*$  satisfy the relations

$$A_{G^*}(i,j) = A_G(j,i)$$
 for  $i, j \in \{1, 2, \dots, N_0\}$ .

Recall that the vertex set V of G is denoted by  $\{v_1, \ldots, v_{N_0}\}$  and the vertex set  $V^*$  of  $G^*$  is written as  $\{v_1^*, \ldots, v_{N_0}^*\}$ . Put  $\Sigma_V^- = V^*, \Sigma_V^+ = V$  and  $\Sigma_G^V = \Sigma_V^- \cup \Sigma_V^+$ . A G-Dyck *n*-path of vertex type for  $n = 1, 2, \ldots$  is a Dyck *n*-path  $(x_1, \ldots, x_{2n})$  labeled elements of  $\Sigma_G^V$  satisfying the following rules:

(1V) a rise is labeled  $v_i^*$  for some  $i = 1, \ldots, N_0$ ,

- (2V) a fall is labeled  $v_i$  for some  $i = 1, \ldots, N_0$ ,
- (3V) the partner of a rise labeled  $v_i^*$  is labeled  $v_i$ ,
- (4V) a rise labeled  $v_i^*$  follows a rise labeled  $v_j^*$  if and only if  $A_{G^*}(j,i) = 1$ ,

(5V) a rise labeled  $v_i^*$  follows a fall labeled  $v_j$  if and only if  $A_G(j,k) = A_{G^*}(k,i) = 1$  for some  $v_k$ ,

- (6V) a fall labeled  $v_i$  follows a fall labeled  $v_j$  if and only if  $A_G(j,i) = 1$ ,
- (7V) a fall labeled  $v_i$  follows a rise labeled  $v_i^*$  if and only if  $v_j = v_i$ .

The Dyck shift  $D_G^E$  of edge type is regarded to have its symbols in  $E^* \cup E$  under the identification  $\Sigma^- = E^*, \Sigma^+ = E$ , and the Dyck shift  $D_G^V$  of vertex type is regarded to have its symbols in  $V^* \cup V$  under the identification  $\Sigma^- = V^*, \Sigma^+ = V$ .

We note the following lemma

#### Lemma 2.4. Keep the above notations.

- (i) Any admissible word of the Dyck shift D<sup>E</sup><sub>G</sub> of edge type is regarded as a part of a labeled broken directed line of G-Dyck path of edge type. Conversely a labeled broken directed line of G-Dyck path of edge type is an admissible word of the Dyck shift D<sup>E</sup><sub>G</sub> of edge type.
- (ii) Any admissible word of the Dyck shift D<sup>V</sup><sub>G</sub> of vertex type is regarded as a part of a labeled broken directed line of G-Dyck path of vertex type. Conversely a labeled broken directed line of G-Dyck path of vertex type is an admissible word of the Dyck shift D<sup>V</sup><sub>G</sub> of vertex type.

*Proof.* (i) is clear from the definition of admissible words of the Dyck shift  $D_G^E$  of edge type.

(ii) Let  $t_1, \ldots, t_{N_0}$  be the partial isometries satisfying the relations (2.3) for the vertex matrix  $A_G$  of G. For  $i, j = 1, 2, \ldots, N_0$ , we have  $\beta_j \alpha_i$  is admissible in  $D_G^V$  if and inly if  $t_j t_i^* \neq 0$  by definition. Since  $t_j t_i^* = t_j t_j^* t_j t_i^* t_i t_i^*$ , the condition  $t_j t_i^* \neq 0$  is equivalent to the condition that  $t_j^* t_j t_i^* t_i \neq 0$ . As

$$t_i^* t_i = \sum_{k=1}^{N_0} A_G(i,k) t_k t_k^* \qquad t_j^* t_j = \sum_{k=1}^{N_0} A_G(j,k) t_k t_k^*,$$

the condition that  $t_j^* t_j t_i^* t_i \neq 0$  is equivalent to the condition  $A_G(i, k) = A_G(j, k) = 1$  for some  $k = 1, \ldots, N_0$ . This shows that the condition  $\beta_j \alpha_i$  is admissible in  $D_G^V$  is equivalent to the condition (5V) of *G*-Dyck *n*-path of vertex type. It is direct to see that the other conditions (1V), (2V), (3V), (4V), (6V), (7V) are compatible to the definitions of giving admissible words of the Dyck shift  $D_G^V$  of vertex type.

We remark that a finite path of vertices of a labeled broken directed line of the G-Dyck path of edge type is not necessarily an admissible word of the Dyck shift  $D_G^V$  of vertex type. Consider the following correspondences in G-Dyck paths:

$$\begin{cases} a \text{ fall } e \in E & \longrightarrow \text{ the source } s(e) \in V \text{ of } e, \\ a \text{ rise } e^* \in E^* & \longrightarrow \text{ the terminal } t(e^*) \in V^* \text{ of } e^*. \end{cases}$$
(2.9)

The rules  $(1E), \ldots, (7E)$  and  $(1V), \ldots, (7V)$  ensure us the following lemma.

Lemma 2.5. Keep the above notations.

- (i) Any sequence of vertices of a G-Dyck n-path of edge type yields a labeled sequence by  $\Sigma_G^V$  of a G-Dyck n-path of vertex type by the correspondence (2.9).
- (ii) Any labeled sequence by  $\Sigma_G^V$  of a G-Dyck n-path of vertex type is realized as a sequence of vertices of a G-Dyck n-path of edge type by the correspondence (2.9).

By the above lemma, it is reasonable to define a 1-block map  $\Phi:E\cup E^*\longrightarrow V\cup V^*$  by

$$\begin{cases} \Phi(e) &= s(e) \in V & \text{for } e \in E, \\ \Phi(e^*) &= t(e^*)(=s(e)) \in V^* & \text{for } e^* \in E^* \end{cases}$$

Hence we have

**Proposition 2.6.** The 1-block map  $\Phi : E \cup E^* \longrightarrow V \cup V^*$  induces a factor code  $\varphi = \Phi_{\infty} : D_G^E \longrightarrow D_G^V$ .

For  $e_{i,k} \in E$  with  $s(e_{i,k}) = v_i \in V$  and  $t(e_{i,k}) = v_k \in V$ , and  $e_{k,j}^* \in E^*$  with  $s(e_{k,j}^*) = v_k^* \in V^*$  and  $t(e_{k,j}^*) = v_j^* \in V^*$ , then the word  $(e_{i,k}, e_{k,j}^*)$  is admissible in  $D_G^E$  and the word  $(v_i, v_i^*)$  is admissible in  $D_G^V$  such that

$$\Phi\left(\begin{array}{cc} v_i & v_j^* \\ & \searrow & \swarrow \\ v_k & \swarrow \end{array}\right) = (i \searrow \nearrow j^*), \qquad \Phi(e_{i,k}, e_{k,j}^*) = (v_i, v_j^*).$$

In the above situation, we call the vertex  $v_k (= v_k^*)$  a valley. Hence the factor map  $\varphi$ :  $D_G^E \longrightarrow D_G^V$  erases the valleys. We will show that the factor map  $\varphi$  is finite-to-one, so that the equality of the topological entropy  $h_{top}(D_G^E) = h_{top}(D_G^V)$  holds.

We provide the height functions on  $D_G^E$ . These functions on the Dyck shift  $D_N$  have been first introduced by W. Krieger in [11]. For  $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$ , we set the height function

$$H_0(x) = 0,$$
  

$$H_m(x) = \sum_{k=0}^{m-1} (\chi_-(x_k) - \chi_+(x_k)), \qquad m \in \mathbb{N},$$
  

$$H_{-m}(x) = \sum_{k=-1}^{-m} (-\chi_-(x_k) + \chi_+(x_k)), \qquad m \in \mathbb{N}$$

where

$$\chi_{-}(x_{k}) = \begin{cases} 1 & \text{if } x_{k} \in \Sigma^{-}, \\ 0 & \text{if } x_{k} \in \Sigma^{+}, \end{cases} \qquad \chi_{+}(x_{k}) = \begin{cases} 0 & \text{if } x_{k} \in \Sigma^{-}, \\ 1 & \text{if } x_{k} \in \Sigma^{+}. \end{cases}$$

**Definition 2.7.** For  $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$ ,

- (i) a vertex  $t(x_{m-1})(=s(x_m))$  is called a relative minimum in x if  $x_{m-1} \in E$  and  $x_m \in E^*$ .
- (ii) a vertex  $t(x_{m-1})(=s(x_m))$  is called a minimum in x if  $H_m(x) \leq H_n(x)$  for all  $n \in \mathbb{Z}$ .

Lemma 2.8. For  $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$ ,

- (i) if a vertex  $t(x_{m-1})(=s(x_m))$  is not a relative minimum in x, the word  $(\Phi(x_{m-1}), \Phi(x_m))$ in  $D_G^V$  uniquely determines the vertex  $t(x_{m-1})$ ,
- (ii) if a vertex  $t(x_{m-1})(=s(x_m))$  is not minimum in x, the sequence  $\varphi(x) \in D_G^V$  uniquely determines the vertex  $t(x_{m-1})$ ,
- (iii) if two vertices  $t(x_{n-1})$  and  $t(x_{m-1})$  are both minimum in x, then  $t(x_{n-1}) = t(x_{m-1})$ .

*Proof.* (i) Since the vertex  $t(x_{m-1})(=s(x_m))$  is not a relative minimum in x, we have two cases.

Case 1:  $x_{m-1} \in E^*$ .

Since  $\Phi(x_{m-1})$  is in  $V^*$ , we take a vertex  $v_i \in V$  such that  $\Phi(x_{m-1}) = v_i^*$ . We then have  $t(x_{m-1}) = v_i^*$ .

Case 2:  $x_{m-1} \in E$ .

The condition that the vertex  $t(x_{m-1})(=s(x_m))$  is not a relative minimum in x implies that  $x_m$  belongs to E, so that  $\Phi(x_m) = v_j \in V$  for some j. We then have  $t(x_{m-1}) = s(x_m) = v_j$ .

(ii) Suppose that the vertex  $t(x_{m-1})(=s(x_m))$  is not minimum in x. If  $t(x_{m-1})$  is not a relative minimum in x, the above discussion implies that the word  $(\Phi(x_{m-1}), \Phi(x_m))$ in  $D_G^V$  uniquely determines the vertex  $t(x_{m-1})$ . Hence we may assume that  $t(x_{m-1})$  is a relative minimum in x. Since  $t(x_{m-1})(=s(x_m))$  is not minimum in x, there exists  $i \in \mathbb{Z}$ such that  $H_i(x) < H_m(x)$ . We have two cases.

Case 1: i > m.

There exists  $k \in \mathbb{Z}$  with m < k < i such that  $x_{k-1}, x_k \in E$ , and  $H_m(x) = H_k(x)$ . We take a vertex  $v_j \in V$  such that  $\Phi(x_k) = v_j$ . We then have  $t(x_{m-1}) = t(x_{k-1}) = v_j$ . Case 2: i < m.

There exists  $l \in \mathbb{Z}$  with i < l < m such that  $x_{l-1}, x_l \in E^*$ , and  $H_m(x) = H_l(x)$ . We take a vertex  $v_j \in V$  such that  $\Phi(x_{l-1}) = v_j$ . We then have  $t(x_{m-1}) = t(x_{l-1}) = v_j$ .

(iii) Suppose that two vertices  $t(x_{n-1})$  and  $t(x_{m-1})$  are both minimum in x, so that  $H_n(x) = H_m(x)$ . Assume that n < m. The word  $(x_n, x_{n+1}, \ldots, x_{m-1})$  is a G-Dyck path of edge type so that the vertices  $s(x_n)$  and  $t(x_{m-1})$  are the same. This implies that  $t(x_{n-1}) = t(x_{m-1})$ .

**Proposition 2.9.** Suppose that G does not have multi-edges. Let  $\varphi : D_G^E \longrightarrow D_G^V$  be the factor code defined in Proposition 2.6. For  $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$ , we have

(i) if x does not have a minimum vertex, then  $\varphi$  is injective at x, that is,

$$\varphi^{-1}(\varphi(x)) = x,$$

(ii) if x has a minimum vertex, then

$$|\varphi^{-1}(\varphi(x))| \le N_0 = |V|.$$

Therefore  $\varphi: D_G^E \longrightarrow D_G^V$  is a finite-to-one factor code.

*Proof.* (i) Suppose that  $x = (x_n)_{n \in \mathbb{Z}}$  does not have a minimum vertex. By (ii) of the above lemma, the sequence  $\varphi(x)$  determines the sequence  $t(x_n), n \in \mathbb{Z}$  of vertices. Each symbol  $x_n$  is an edge of E or of  $E^*$ , and an edge is determined by the vertices  $t(x_n), t(x_{n-1})(=s(x_n))$ , so that the code  $\varphi$  is injective at x.

(ii) Suppose that x has a minimum vertex at  $t(x_{m-1})$  for some  $m \in \mathbb{Z}$ . Then the vertex  $t(x_{m-1})$  is a valley and  $x_{m-1} \in E, x_m \in E^*$ . By (iii) of the above lemma, other minimum vertices are the same as the vertex  $t(x_{m-1})$ . Hence we have

$$\begin{aligned} |\varphi^{-1}(\varphi(x))| &= |\{k \in \{1, 2, \dots, N_0\} \mid A_G(s(x_{m-1}), k) = A_{G^*}(k, t(x_m)) = 1\}| \\ &\leq N_0 = |V|. \end{aligned}$$

**Theorem 2.10.** Suppose that G does not have multi-edges. We then have  $h_{top}(D_G^V) = h_{top}(D_G^E)$ .

*Proof.* Since there exists a factor code  $\varphi : D_G^E \longrightarrow D_G^V$ , the inequality  $h_{top}(D_G^V) \leq h_{top}(D_G^E)$  is clear. The 1-block map  $\Phi$  naturally induces a map  $\Phi_* : B_*(D_G^E) \longrightarrow B_*(D_G^V)$  between admissible words. It is not necessarily one-to-one at minimal points of words. We then have

$$|B_n(D_G^E)| \le N_0 \cdot |B_n(D_G^V)|, \qquad n \in \mathbb{N}$$

Therefore we have  $h_{top}(D_G^E) \leq h_{top}(D_G^V)$ .

Concerning embedding of the Markov-Dyck shifts, we have the following proposition.

**Proposition 2.11.** Suppose that G does not have multi-edges. There exists an embedding of  $D_G^E$  into the 3rd power shift of  $D_G^V$ .

Proof. Let  $t_i, i = 1, ..., N_0$  be partial isometries satisfying the relations (2.3) for the vertex matrix  $A_G$ . For an edge  $e_n \in E$  with  $s(e_n) = v_i, t(e_n) = v_j$ , define a partial isometry  $S_n = t_i t_j t_j^*$ . It is easy to see that the family  $S_1, \ldots, S_{N_1}$  satisfies the relations (2.3) for the edge matrix  $A^G$ , This implies that the correspondence  $\Psi : E \cup E^* \longrightarrow (V \cup V^*)^{[3]}$  defined by

$$\Psi(e_n) = (v_i, v_j, v_j^*), \qquad \Psi(e_n^*) = (v_j, v_j^*, v_i^*)$$

induces an embedding of  $D_G^E$  into the 3rd power shift  $(D_G^V)^{[3]}$  of  $D_G^V$ .

### 3 The zeta functions of Mrkov-Dyck shifts of vertex type

In what follows, we fix an arbitrary  $N \times N$  matrix  $A = [A(i, j)]_{i,j=1}^N$  with entries in  $\{0, 1\}$ . We will study the Markov-Dyck shift  $D_A$  and present a formula of the zeta function  $\zeta_{D_A}(z)$ . In [14], a formula of the zeta function of the Markov-Dyck shifts of edge type has been presented. The Markov-Dyck shifts of edge type form a subclass of the class of Markov-Dyck shifts. In this section, we will study general Markov-Dyck shift  $D_A$  and present a formula of its zeta function  $\zeta_{D_A}(z)$ . For the  $N \times N$  matrix A, let  $v_1, \ldots, v_N$  be N-vertices. Define a directed edge from  $v_i$  to  $v_j$  if A(i, j) = 1. We then have a finite directed graph written G = (V, E) such that its vertex matrix  $A_G$  coincides with the original matrix A.

Throughout this section, we identify  $\alpha_i$  with  $v_i^*$  and  $\beta_i$  with  $v_i$  for  $i = 1, \ldots, N$ , respectively. Let  $w = (w_1, \ldots, w_{2n})$  be a *G*-Dyck *n*-path of vertex type. As in [16], *w* is called a *G*-Catalan word and satisfies the following conditions:

$$\sum_{k=1}^{m} (\chi_{-}(w_k) - \chi_{+}(w_k)) \ge 0 \quad \text{for all } m = 1, 2, \dots, 2n$$

and

$$\sum_{k=1}^{2n} (\chi_{-}(w_k) - \chi_{+}(w_k)) = 0$$

Denote by  $C_n^A$  the set of G-Dyck *n*-pathes of vertex type. For i = 1, ..., N, put

$$C_n^A(i) = \{ (w_1, \dots, w_{2n}) \in C_n^A \mid (\alpha_i, w_1, \dots, w_{2n}, \beta_i) \in C_{n+1}^A \}.$$

Denote by  $c_n^A(i)$  the cardinarity  $|C_n^A(i)|$  of the set  $C_n^A(i)$ . We set  $c_0^A(i) = 1$ . Combinatorial properties of the sequence  $c_n^A(i)$ ,  $n = 0, 1, \ldots$  have been studied in [16, Section 4]. For  $i = 1, \ldots, N$ , let  $f_i^A(z)$  be the generating function of the sequence  $c_n^A(i)$ ,  $n = 0, 1, 2, \cdots$ :

$$f_i^A(z) = \sum_{n=0}^{\infty} c_n^A(i) z^n.$$

Since one knows ([16, Section 4])

$$C_{n+1}^{A}(i) = \bigcup_{k=0}^{n} \bigcup_{\substack{j \\ A(j,i)=1}} C_{k}^{A}(j) \times C_{n-k}^{A}(i),$$

we have

$$c_{n+1}^{A}(i) = \sum_{k=0}^{n} \sum_{j=1}^{N} A(j,i) c_{k}^{A}(j) c_{n-k}^{A}(i),$$

so that the identity

$$f_i^A(z) = 1 + z f_i^A(z) \sum_{j=1}^N A(j,i) f_j^A(z)$$
(3.1)

holds ([16, Proposition 4.2]). Let  $X_A$  be the shift space over  $\Sigma^+ = V$  of the topological Markov shift defined by the matrix A:

$$X_A = \{ (x_n)_{n \in \mathbb{Z}} \in (\Sigma^+)^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z} \}.$$

For  $n, k \in \mathbb{N}$ , we set

$$C_{n,k}^{A,+} = \{ (w_1, \dots, w_{2n}, \beta_{i_1}, \dots, \beta_{i_k}) \in B_{2n+k}(D_A) \mid \\ (w_1, \dots, w_{2n}) \in C_n^A, \ (\beta_{i_1}, \dots, \beta_{i_k}) \in B_k(X_A) \}.$$

For  $(w_1, \ldots, w_{2n}, \beta_{i_1}, \ldots, \beta_{i_k}) \in C_{n,k}^{A,+}$ , we set

$$s((w_1,\ldots,w_{2n},\beta_{i_1},\ldots,\beta_{i_k})) = \beta_{i_1},$$
  
$$t((w_1,\ldots,w_{2n},\beta_{i_1},\ldots,\beta_{i_k})) = \beta_{i_k}.$$

We put

$$C_A^+ = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n,k}^{A,+}$$

We then see the following lemma.

**Lemma 3.1.** For  $\mu, \nu \in C_A^+$ , the word  $\mu\nu$  is admissible in  $D_A$  if and only if  $A(t(\mu), s(\nu)) = 1$ .

Put  $I = \{1, ..., N\} \times \{1, ..., N\}$ . Define an  $I \times I$  matrix  $\tilde{A} = [\tilde{A}((i, j), (k, l))]_{(i, j), (k, l) \in I}$  by

$$\dot{A}((i,j),(k,l)) = A(j,k)$$

and a map  $r:C^+_A\longrightarrow I$  by

$$r((w_1,\ldots,w_{2n},\beta_{i_1},\ldots,\beta_{i_k})) = (\beta_{i_1},\beta_{i_k}) \in I$$

Then the quadruplet  $C_A^+ = (C_A^+, I, \tilde{A}, r)$  is a circular Markov code in the sense of Keller [9]. We then associate the following shift-invariant subset  $\Omega_{C_A^+}$  by

$$\Omega_{\mathcal{C}_{A}^{+}} = \{ x = (x_{n})_{n \in \mathbb{Z}} \mid \text{ there are } \dots k_{-1} < k_{0} \le 0 < k_{1} < \dots \text{ in } \mathbb{Z}$$
  
such that  $x_{[k_{i},k_{i+1})} \in C_{A}^{+}$  and  $\tilde{A}(r(x_{[k_{i-1},k_{i})}), r(x_{[k_{i},k_{i+1})})) = 1 \}$  ([9]).

The zeta function  $\zeta(\Omega_{\mathcal{C}^+_A}, z)$  for a shift-invariant set  $\Omega_{\mathcal{C}^+_A}$  is similarly defined to (1.2) by using a sequence of cardinalities of periodic points of  $\Omega_{\mathcal{C}^+_A}$ . Following Keller [9], define a sequence  $D(\mathcal{C}^+_A, m) = \text{diag}[d_{(i,j),(i,j)}(\mathcal{C}^+_A, m)], 3 \leq m \in \mathbb{N}$  of  $I \times I$ -diagonal matrices with diagonal entries  $d_{(i,j),(i,j)}(\mathcal{C}^+_A, m), (i,j) \in I$  by

$$d_{(i,j),(i,j)}(\mathcal{C}^+_A, m) = |\{(w_1, \dots, w_{2n}, \beta_{i_1}, \dots, \beta_{i_k}) \in C^+_A \mid i_i = i, i_k = j\}|$$
$$(= c_n^A(i)A^{k-1}(i,j))$$

for m = 2n + k, and a matrix-valued generating function  $F(\mathcal{C}^+_A, z)$  by

$$F(\mathcal{C}_A^+, z) = \sum_{m=1}^{\infty} D(\mathcal{C}_A^+, m) \tilde{A} z^m.$$

Denote by  $I_{N^2}$  the identity matrix of size  $N^2$ . By using [9, Theorem 1], we have **Proposition 3.2.**  $\zeta(\Omega_{\mathcal{C}_A^+}, z) = \det(I_{N^2} - F(\mathcal{C}_A^+, z))$  We then have for  $(i, j), (p, q) \in I$ 

$$\begin{split} F(\mathcal{C}^+_A, z)((i, j), (p, q)) &= \sum_{m=1}^{\infty} D(\mathcal{C}^+_A, m) \tilde{A} z^m((i, j), (p, q)) \\ &= \sum_{m=1}^{\infty} \sum_{\substack{n,k\\2n+k=m}} D(\mathcal{C}^+_A, 2n+k) \tilde{A} z^{2n+k}((i, j), (p, q)) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_n^A(i) A^{k-1}(i, j) \tilde{A}((i, j), (p, q)) z^{2n+k} \\ &= \sum_{n=1}^{\infty} c_n^A(i) z^{2n} \sum_{k=1}^{\infty} A^{k-1}(i, j) z^k \tilde{A}((i, j), (p, q)) \\ &= (f_i^A(z^2) - 1) z \sum_{l=0}^{\infty} (zA)^l (i, j) A(j, p) \\ &= (f_i^A(z^2) - 1) z (1_N - zA)^{-1}(i, j) \cdot A(j, p). \end{split}$$

We define  $N\times N$  matrices  $F^A=[F^A(i,j)]_{i,j=1}^N$  and  $H(C^+_A,z)$  by

$$F^{A}(i,j) = (f_{i}^{A}(z^{2}) - 1)z(1_{N} - zA)^{-1}(i,j)$$
 and  $H(C_{A}^{+}, z) = F^{A} \cdot A$ 

so that

$$F(\mathcal{C}_{A}^{+}, z)((i, j), (p, q)) = F^{A}(i, j)A(j, p) \quad \text{and} \quad H(\mathcal{C}_{A}^{+}, z)(i, p) = \sum_{j=1}^{N} F(\mathcal{C}_{A}^{+}, z)((i, j), (p, q)).$$

**Lemma 3.3.** det $(I_{N^2} - F(\mathcal{C}^+_A, z)) = det(I_N - H(C^+_A, z)).$ 

*Proof.* Let  $U = [U((i, j), (p, q))]_{(i,j), (p,q) \in I}$  and  $V = [V((i, j), (p, q))]_{(i,j), (p,q) \in I}$  be  $I \times I$  matrices defined by

$$\begin{split} U((i,j),(p,q)) &= \begin{cases} 1 & \text{if } (i,j) = (p,q), \\ 1 & \text{if } i = p, \ j = N, \\ 0 & \text{otherwise}, \end{cases} \\ V((i,j),(p,q)) &= \begin{cases} 1 & \text{if } (i,j) = (p,q), \\ -1 & \text{if } i = p, \ j = N, \ q < N, \\ 0 & \text{otherwise}. \end{cases} \end{split}$$

The matrix  $(I_{N^2} - F(\mathcal{C}_A^+, z))V$  is obtained from  $(I_{N^2} - F(\mathcal{C}_A^+, z))$  by adding the minus of the (i, N)th column to the (i, j)th column for all j = 1, 2, ..., N - 1 and i = 1, 2, ..., N, and the matrix  $U(I_{N^2} - F(\mathcal{C}_A^+, z))V$  is obtained from  $(I_{N^2} - F(\mathcal{C}_A^+, z))V$  by adding the (i, j)th rows to the (i, N)th row for all j = 1, 2, ..., N - 1 and i = 1, 2, ..., N. Hence we

see

$$U(I_{N^2} - F(\mathcal{C}_A^+, z))V((i, j), (p, q)) = \begin{cases} 1 & \text{if } (i, j) = (p, q), \ q < N, \\ 0 & \text{if } (i, j) \neq (p, q), \ q < N, \\ 1 - \sum_{k=1}^N F^A(i, k)A(k, p) & \text{if } (i, j) = (p, q), \ q = N, \\ -F^A(i, j)A(j, p) & \text{if } j < N, \ q = N, \\ 0 & \text{otherwise.} \end{cases}$$

Each (p,q)th column for q < N of the matrix  $U(I_{N^2} - F(\mathcal{C}^+_A, z))V$  has 1 on diagonal and zero elsewhere. Since

$$1 - \sum_{k=1}^{N} F^{A}(i,k)A(k,p) = 1 - H(C_{A}^{+},z)(i,p),$$

by expanding the matrix  $U(I_{N^2} - F(\mathcal{C}^+_A, z))V$  along the (p, q)th columns for p = 1, 2, ..., N with q < N, we have

$$\det(U(I_{N^2} - F(\mathcal{C}^+_A, z))V) = \det(I_N - H(\mathcal{C}^+_A, z)).$$

As det(U) = det(V) = 1, we get the desired equality.

Therefore we have

#### Proposition 3.4.

$$\zeta(\Omega_{\mathcal{C}_{A}^{+}}, z) = \frac{\det(I_{N} - zA)}{\det(I_{N} - \operatorname{diag}[f_{1}^{A}(z^{2}), \dots, f_{N}^{A}(z^{2})]zA)}.$$
(3.2)

*Proof.* Since

$$H(C_A^+, z) = \text{diag}[f_1^A(z^2) - 1, \dots, f_N^A(z^2) - 1]zA(I_N - zA)^{-1},$$

we have

$$I_N - H(C_A^+, z) = I_N - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)] z A(I_N - zA)^{-1} + z A(I_N - zA)^{-1}$$
  
=  $(I_N - zA)^{-1} - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)] z A(I_N - zA)^{-1}$   
=  $(I_N - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)] z A)(I_N - zA)^{-1}$ 

so that the desired equality holds.

For 
$$j \in \{1, 2, \dots, N\}$$
 with  $A(i, j) = 1$ , we put  
 $C_n^A[i; \{j\}] = \{(\alpha_i, w_1, \dots, w_{2n-2}, \beta_i) \in C_n^A(j) \mid (w_1, \dots, w_{2n-2}) \in C_{n-1}^A(i)\}$ 

and

$$C_n^A[j] = \bigcup_{\substack{i=1\\A(i,j)=1}}^N C_n^A[i;\{j\}], \qquad C^A[j] = \bigcup_{n=1}^\infty C_n^A[j].$$

We set  $c_n^A[j] = |C_n^A[j]|$ . As  $|C_n^A[i; \{j\}]| = c_{n-1}^A(i)$  if A(i, j) = 1, we have

$$c_n^A[j] = \sum_{i=1}^N A(i,j) c_{n-1}^A(i).$$
(3.3)

Similarly for a subset  $\{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$  with  $A(i, j_1) = \cdots = A(i, j_k) = 1$ , we put

$$C_n^A[i; \{j_1, \dots, j_k\}] = \bigcap_{m=1}^k C_n^A[i; \{j_m\}]$$

and

$$C_n^A[\{j_1, \dots, j_k\}] = \bigcup_{\substack{i=1\\A(i,j_1)=\dots=A(i,j_k)=1}}^N C_n^A[i; \{j_1, \dots, j_k\}],$$
$$C^A[\{j_1, \dots, j_k\}] = \bigcup_{n=1}^\infty C_n^A[\{j_1, \dots, j_k\}].$$

We set  $c_n^A[\{j_1, ..., j_k\}] = |C_n^A[\{j_1, ..., j_k\}]|$  so that

$$c_n^A[\{j_1,\ldots,j_k\}] = \sum_{i=1}^N A(i,j_1)\cdots A(i,j_k)c_{n-1}^A(i).$$
(3.4)

For a subset  $\{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$  if there exists  $i \in \{1, 2, \ldots, N\}$  such that  $A(i, j_1) = \cdots = A(i, j_k) = 1$ , we call the set  $C^A[\{j_1, \ldots, j_k\}]$  the *Markov-Dyck code with support*  $\{j_1, \ldots, j_k\}$ . It is easy to see that the set  $C^A[\{j_1, \ldots, j_k\}]$  is a circular code. Denote by  $C^A[\{j_1, \ldots, j_k\}]^{\infty}$  the set of all two-sided sequences of alphabet  $\Sigma = \Sigma^- \cup \Sigma^+$  consisting of free concatenations of words of  $C^A[\{j_1, \ldots, j_k\}]$ . Let  $g_{C^A[\{j_1, \ldots, j_k\}]}(z)$  be the generating function for the sequence  $c_n^A[\{j_1, \ldots, j_k\}], n = 1, 2, \ldots$  defined by

$$g_{C^{A}[\{j_{1},\dots,j_{k}\}]}(z) = \sum_{n=1}^{\infty} c_{n}^{A}[\{j_{1},\dots,j_{k}\}]z^{2n}$$

**Lemma 3.5.** (i) The generating function  $g_{C^{A}[\{j_{1},...,j_{k}\}]}(z)$  satisfies

$$g_{C^{A}[\{j_{1},\dots,j_{k}\}]}(z) = z^{2} \sum_{i=1}^{N} A(i,j_{1})A(i,j_{2})\cdots A(i,j_{k})f_{i}^{A}(z^{2}).$$
(3.5)

(ii) The zeta function  $\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty},z)$  of the shift-invariant set  $C^A[\{j_1,\ldots,j_k\}]^{\infty} \subset \Sigma^{\mathbb{Z}}$  is

$$\zeta(C^{A}[\{j_{1},\ldots,j_{k}\}]^{\infty},z) = \frac{1}{1 - g_{C^{A}[\{j_{1},\ldots,j_{k}\}]}(z)}.$$
(3.6)

In particular for  $j \in \{1, 2, ..., N\}$ , we have

$$\zeta(C^{A}[\{j\}]^{\infty}, z) = \frac{1}{1 - g_{C^{A}[\{j\}]}(z)} = f_{j}^{A}(z^{2}).$$
(3.7)

*Proof.* (i) By (3.4), we have

$$g_{C^{A}[\{j_{1},\dots,j_{k}\}]}(z) = \sum_{n=1}^{\infty} \sum_{i=1}^{N} A(i,j_{1}) \cdots A(i,j_{k}) c_{n-1}^{A}(i) z^{2n}$$
$$= z^{2} \sum_{i=1}^{N} A(i,j_{1}) \cdots A(i,j_{k}) \sum_{n=1}^{\infty} c_{n-1}^{A}(i) z^{2(n-1)}$$
$$= z^{2} \sum_{i=1}^{N} A(i,j_{1}) \cdots A(i,j_{k}) f_{i}^{A}(z^{2}).$$

(ii) The set  $C^{A}[\{j_{1}, \ldots, j_{k}\}]$  is a circular code, and the set  $C^{A}[\{j_{1}, \ldots, j_{k}\}]^{\infty}$  consisting of the two-sided sequences of free concatenations of words of  $C^{A}[\{j_{1}, \ldots, j_{k}\}]$ . Hence a well-known theorem of combinatorics (cf. [18, Proposition 4.7.11]) ensures us the equality

$$\zeta(C^{A}[\{j_{1},\ldots,j_{k}\}]^{\infty},z) = \frac{1}{1 - g_{C^{A}[\{j_{1},\ldots,j_{k}\}]}(z)}$$

In particular we have

$$g_{C^{A}[\{j\}]}(z) = z^{2} \sum_{i=1}^{N} A(i,j) f_{i}^{A}(z^{2}) = \frac{f_{j}^{A}(z^{2}) - 1}{f_{j}^{A}(z^{2})} = 1 - \frac{1}{f_{j}^{A}(z^{2})}$$

so that

$$\zeta(C^{A}[\{j\}]^{\infty}, z) = \frac{1}{1 - g_{C^{A}[\{j\}]}(z)} = f_{j}^{A}(z^{2}).$$
(3.8)

We call a subset  $\{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$  a support subset if for any  $i \in \{1, 2, \ldots, N\}$ there exists  $l = 1, \ldots, k$  such that  $A(i, j_l) = 1$ . The set  $\{1, 2, \ldots, N\}$  itself is a support subset. For a shift-invariant subset C of  $D_A$ , denote by  $P_n(C)$  the set of *n*-periodic points of C. We set

$$C^{A^{\infty}} = \bigcup_{\{j_1,\dots,j_k\} \subset \{1,\dots,N\}} C^A[\{j_1,\dots,j_k\}]^{\infty} \subset \Sigma^{\mathbb{Z}}.$$
(3.9)

By the principle of inclusion of exclusion in combinatorics (cf. [18, 2.1]), we have

**Lemma 3.6.** Let  $J = \{j_1, ..., j_k\}$  be a support subset of  $\{1, 2, ..., N\}$ . Then we have

$$P_n(C^{A^{\infty}}) = \bigcup_{l=1}^k P_n(C^A[\{j_l\}]^{\infty}) - \bigcup_{\{j_1, j_2\} \subset J} P_n(C^A[\{j_1, j_2\}]^{\infty}) \\ \cdots (-1)^{m+1} \bigcup_{\{j_1, \dots, j_m\} \subset J} P_n(C^A[\{j_1, \dots, j_m\}]^{\infty}) \cdots (-1)^{k+1} \bigcup P_n(C^A[\{j_1, \dots, j_k\}]^{\infty}),$$

where  $(-1)^{m+1} \bigcup_{\{j_1,...,j_m\} \subset J} means \bigcup_{\{j_1,...,j_m\} \subset J} if m is odd.$ 

Hence we have

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**Proposition 3.7.** Let  $J = \{j_1, \ldots, j_k\}$  be a support subset of  $\{1, 2, \ldots, N\}$ . Then we have

$$\zeta(C^{A^{\infty}}, z) = \prod_{l=1}^{k} \zeta(C^{A}[\{j_{l}\}]^{\infty}, z) \cdot \prod_{\{j_{1}, j_{2}\} \subset J} \zeta(C^{A}[\{j_{1}, j_{2}\}]^{\infty}, z)^{-1} \cdots \prod_{\{j_{1}, \dots, j_{m}\} \subset J} \zeta(C^{A}[\{j_{1}, \dots, j_{m}\}]^{\infty}, z)^{(-1)^{m+1}} \cdots \zeta(C^{A}[\{j_{1}, \dots, j_{k}\}]^{\infty}, z)^{(-1)^{k+1}}.$$

**Corollary 3.8.** Suppose that there exists  $j_0 \in \{1, 2, ..., N\}$  such that  $A(i, j_0) = 1$  for all i = 1, 2, ..., N. Then  $\zeta(C^{A^{\infty}}, z) = f_{j_0}^A(z^2)$ .

We reach the following formula of the zeta function of a Markov-Dyck shift of vertex type.

**Theorem 3.9.** Let A be an  $N \times N$  essential matrix with entries in  $\{0, 1\}$ . Then the zeta function  $\zeta_{D_A}(z)$  of the Markov-Dyck shift  $D_A$  is given by the following formula:

$$\zeta_{D_A}(z) = \frac{\zeta(C^{A^{\infty}}, z)}{\det(I_N - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}$$
(3.10)

where

$$\zeta(C^{A^{\infty}}, z) = \prod_{\{j_1, \dots, j_k\} \subset \{1, 2, \dots, N\}} \zeta(C^A[\{j_1, \dots, j_k\}]^{\infty}, z)^{(-1)^{k+1}},$$

the products  $\prod_{\{j_1,\ldots,j_k\}\subset\{1,2,\ldots,N\}}$  run over all subsets of  $\{1,2,\ldots,N\}$ , and the zeta function  $\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty}, z)$  is given by

$$\zeta(C^{A}[\{j_{1},\ldots,j_{k}\}]^{\infty},z) = \frac{1}{1 - g_{C^{A}[\{j_{1},\ldots,j_{k}\}]}(z)},$$

where

$$g_{C^{A}[\{j_{1},\dots,j_{k}\}]}(z) = z^{2} \sum_{i=1}^{N} A(i,j_{1}) \cdots A(i,j_{k}) f_{i}^{A}(z^{2}),$$

and the functions  $f_i^A(z^2), i = 1, 2, ..., N$  satisfy the relations (3.1).

*Proof.* For  $n, k \in \mathbb{N}$ , we define the following set  $C_{n,k}^{A,-}$  similarly to  $C_{n,k}^{A,+}$  by

$$C_{n,k}^{A,-} = \{ (\alpha_{i_1}, \dots, \alpha_{i_k}, w_1, \dots, w_{2n}) \in B_{2n+k}(D_A) \mid (w_1, \dots, w_{2n}) \in C_n^A, (\alpha_{i_1}, \dots, \alpha_{i_k}) \in B_k(X_{A^t}) \}.$$

Similarly to the previous discussion, we have a circular Markov code  $C_A^- = (C_A^-, I, \tilde{A}^t, r)$ and the formula (3.2) for  $\zeta(\Omega_{C_A^-}, z)$ . We then have a disjoint union of periodic points

$$P_n(D_A) = P_n(\Omega_{\mathcal{C}_A^+}) \cup P_n(\Omega_{\mathcal{C}_A^-}) \cup P_n(C^{A^\infty}) \cup P_n(X_A) \cup P_n(X_{A^t}).$$

Since  $\zeta(\Omega_{\mathcal{C}^+_A}, z) = \zeta(\Omega_{\mathcal{C}^-_A}, z)$ , Proposition 3.4 ensures us

$$\begin{aligned} \zeta_{D_A}(z) &= \zeta(\Omega_{\mathcal{C}_A^+}, z) \cdot \zeta(\Omega_{\mathcal{C}_A^-}, z) \cdot \zeta(C^{A^\infty}, z) \cdot \frac{1}{\det(I_N - zA)} \cdot \frac{1}{\det(I_N - zA^t)} \\ &= \frac{\zeta(C^{A^\infty}, z)}{\det(I_N - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}. \end{aligned}$$

For a finite directed graph G = (V, E) the above formula gives us the formula for the zeta function of the Markov-Dyck shift of vertex type.

**Corollary 3.10.** Suppose that there exists  $j_0 \in \{1, 2, ..., N\}$  such that  $A(i, j_0) = for$  all i = 1, 2, ..., N. Then

$$\zeta_{D_A}(z) = \frac{f_{j_0}^A(z^2)}{\det(I_N - \operatorname{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}.$$

## 4 The zeta functions of Markov-Dyck shifts of edge type

The Markov-Dyck shifts in the paper [14] are the Markov-Dyck shifts of edge type. In [14], a formula of the zeta functions of Markov-Dyck shifts of edge type has been presented. In this section, we present the formula [14, Theorem 2.3] from Theorem 3.9. We need the following lemma.

**Lemma 4.1.** For a finite directed graph G = (V, E) with  $|V| = N_0$  and  $|E| = N_1$ . Let  $f_1^V(x), \dots, f_{N_0}^V(x)$  and  $f_1^E(x), \dots, f_{N_1}^E(x)$  be the functions satisfying the relations respectively

$$f_i^V(z) = 1 + z f_i^V(z) \sum_{j=1}^{N_0} A_G(j,i) f_j^V(z).$$
(4.1)

$$f_i^E(z) = 1 + z f_i^E(z) \sum_{j=1}^{N_1} A^G(j, i) f_j^E(z).$$
(4.2)

Then we have

$$\det(I_{N_0} - \operatorname{diag}[f_1^V(z^2), \dots, f_{N_0}^V(z^2)]zA_G) \\ = \det(I_{N_1} - \operatorname{diag}[f_1^E(z^2), \dots, f_{N_1}^E(z^2)]zA^G).$$

*Proof.* Put the sets  $I_0 = \{1, 2, ..., N_0\}$ ,  $I_1 = \{1, 2, ..., N_1\}$  and the diagonal matrices  $D^V(z^2) = \text{diag}[f_1^V(z^2), ..., f_{N_0}^V(z^2)]$  and  $D^E(z^2) = \text{diag}[f_1^E(z^2), ..., f_{N_1}^E(z^2)]$ . Define the  $N_0 \times N_1$  matrix  $S = [S(i, j)]_{i \in I_0, j \in I_1}$  and the  $N_1 \times N_0$  matrix  $R = [R(j, i)]_{j \in I_1, i \in I_0}$  by

$$S(i,j) = \begin{cases} 1 & \text{if } v_i = s(e_j), \\ 0 & \text{otherwise,} \end{cases} \qquad R(j,i) = \begin{cases} 1 & \text{if } t(e_j) = v_i, \\ 0 & \text{otherwise,} \end{cases}$$

so that  $A_G = SR$  and  $A^G = RS$ . For a vertex  $v_i \in V$  and en edge  $e_j \in E$ , we set

$$C_n^{A_G}(v_i) = \{ (w_1, \dots, w_{2n}) \in C_n^{A_G} \mid (v_i^*, w_1, \dots, w_{2n}, v_i) \in C_{n+1}^{A_G} \},\$$
  
$$C_n^{A^G}(e_j) = \{ (g_1, \dots, g_{2n}) \in C_n^{A^G} \mid (e_j^*, g_1, \dots, g_{2n}, e_j) \in C_{n+1}^{A^G} \}.$$

Let us denote by  $c_n^G(v_i)$  and  $c_n^G(e_j)$  their cardinalities  $|C_n^{A_G}(v_i)|$  and  $|C_n^{A^G}(e_j)|$  respectively ([16, pages 8,9]). Then we have

$$f_{i}^{V}(z) = \sum_{n=0}^{\infty} c_{n}^{G}(v_{i})z^{n}, \qquad f_{j}^{E}(z) = \sum_{n=0}^{\infty} c_{n}^{G}(e_{j})z^{n}$$

so that  $f_j^E(z) = f_i^V(z)$  when  $s(e_j) = v_i$ . Hence we have

$$f_i^V(z^2)S(i,j) = S(i,j)f_j^E(z^2)$$

which implies that  $D^V(z^2)S = SD^E(z^2)$ . It then follows that

$$zD^{V}(z^{2})A_{G} = zD^{V}(z^{2})SR = zS \cdot D^{E}(z^{2})R,$$
  
$$zD^{E}(z^{2})A^{G} = zD^{E}(z^{2})RS = D^{E}(z^{2})R \cdot zS.$$

Hence the matrices  $zD^V(z^2)A_G$  and  $zD^E(z^2)A^G$  are elementary equivalent (see [15, Definition 7.2.1]), so that  $\det(I_{N_0} - zD^V(z^2)A_G) = \det(I_{N_1} - zD^E(z^2)A^G)$ .

Therefore we have

**Proposition 4.2** ([14, Theorem 2.3]). If a matrix A is an edge matrix  $A^G = [A^G(e, f)]_{e,f \in E}$ defined by a finite directed graph G = (V, E) with  $|V| = N_0$ , then the zeta function of the Markov-Dyck shift  $D_G(=D_{A^G})$  of edge type is given by the following formula:

$$\zeta_{D_G}(z) = \frac{\prod_{i=1}^{N_0} f_i^G(z^2)}{\det(I_N - \operatorname{diag}[f_1^G(z^2), \dots, f_{N_0}^G(z^2)]zA_G)^2}$$
(4.3)

where  $f_1^G(z^2), \ldots, f_{N_0}^G(z^2)$  are the functions satisfying

$$f_i^G(z) = 1 + z f_i^G(z) \sum_{j=1}^{N_0} A_G(j, i) f_j^G(z).$$
(4.4)

*Proof.* Since  $f_i^G(x) = f_i^V(x), i = 1, \dots, N_0$  and

$$\zeta(C^{A^{\infty}}, z) = \prod_{i=1}^{N_0} \frac{1}{1 - g_{C^A[\{j\}]}(z)} = \prod_{i=1}^{N_0} f_i^G(z^2)$$

(cf. (cf. [18, Proposition 4.7.11]), the preceding lemma implies the equality (4.3) from Theorem 3.9.  $\hfill \Box$ 



Figure 2:

### 5 The Fibonacci-Dyck shift of vertex type

Let  $G_2$  be the finite directed graph defined in the Figure 2. The edge matrix  $A^{G_2}$  and the vertex matrix  $A_{G_2}$  are written as

$$A^{G_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad A_{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
(5.1)

respectively. We then have

**Proposition 5.1.**  $D_{G_2}^V$  is not topologically conjugate to  $D_{G_2}^E$ .

*Proof.* It is easy to see that the number of the 2-periodic points of  $D_{G_2}^V$  is 6, whereas that of  $D_{G_2}^E$  is 7.

The Fibonacci-Dyck shift  $D_{G_2}^E$  of edge type is a subshift  $D_{A^{G_2}}$  over six symbols which correspond to the edges of the directed graphs  $G_2$  and  $G_2^*$  of Figure 2. The Fibonacci-Dyck shift  $D_{A_{G_2}}$  of vertex type is a subshift  $D_{A_{G_2}}$  over four symbols which correspond to the vertices of the directed graphs of  $G_2$  and  $G_2^*$  of Figure 2. Let us denote by  $\alpha_1, \alpha_2$  and  $\beta_1, \beta_2$  the symbols of  $D_{A_{G_2}}$ . They have the following algebraic relations from the relations

(2.3) of operators for  $A = A_{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ :

$$\alpha_1\beta_1 = \beta_1\alpha_1 + \beta_2\alpha_2 = 1, \qquad \alpha_2\beta_2 = \beta_1\alpha_1, \qquad \beta_2\alpha_2\beta_2 = \beta_2,$$

A word  $\gamma = (\gamma_1, \ldots, \gamma_m)$  of  $\Sigma = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$  is forbidden if  $\gamma_1 \cdots \gamma_m = 0$ . The Fibonacci-Dyck shift  $D_{A_{G_2}}$  of vertex type is defined as a subshift over  $\Sigma$  whose forbidden words are defined in this sense.

We will compute the zeta function  $\zeta_{D_{G_2}^V}(z)$  by using Corollary 3.10. Let  $f_1(z), f_2(z)$  be the functions  $f_1^V(z), f_2^V(z)$  which satisfy the following relations:

$$f_1(z) - 1 = z(f_1(z) + f_2(z))f_1(z),$$
  

$$f_2(z) - 1 = zf_1(z)f_2(z)$$

so that the equalities

$$f_2(z)^2 = f_1(z), \qquad zf_2(z)^3 - f_2(z) + 1 = 0$$

hold (see [16, Section 7]). We then have

$$\det(I_2 - \operatorname{diag}[f_1(z^2), f_2(z^2)]zA_{G_2}) = \det\left(\begin{bmatrix}1 - zf_1(z^2) & -zf_1(z^2)\\ -zf_1(z^2) & 1\end{bmatrix}\right)$$
$$= 1 - zf_1(z^2) - z^2f_1(z^2)f_2(z^2)$$
$$= 2 - zf_1(z^2) - f_2(z^2).$$

**Proposition 5.2.** The zeta function  $\zeta_{D_{G_2}^V}(z)$  of the Fibonacci-Dyck shift of vertex type is

$$\zeta_{D_{G_2}^V}(z) = \frac{1}{(2\xi(z)^2 + \xi(z) - 1)^2}$$
(5.2)

where  $\xi(z) = \frac{2}{\sqrt{3}} \sin(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z)$  for  $0 \le z \le \frac{2}{3\sqrt{3}}$ .

Proof. By Corollary 3.10 with the above discussions, we have

$$\begin{split} \zeta_{D_{G_2}^V}(z) &= \frac{f_1(z^2)}{(2 - z f_1(z^2) - f_2(z^2))^2} \\ &= \left(\frac{f_2(z^2)}{(2 f_2(z^2) - 2 z^2 (f_2(z^2)^3) - z f_2(z^2)^2 - f_2(z^2))}\right)^2 \\ &= \frac{1}{(1 - 2(z f_2(z^2)^2 - z f_2(z^2))^2}. \end{split}$$

By putting  $\xi(z) = z f_2(z^2)$ , we have

$$\zeta_{D_{G_2}^V}(z) = \frac{1}{(2\xi(z)^2 + \xi(z) - 1)^2}$$
(5.3)

and  $\xi(z)^3 - \xi(z) + z = 0$ . As in [14, (4.10), (4.13)], we have

$$\xi(z) = \frac{2}{\sqrt{3}}\sin(\frac{1}{3}\arcsin\frac{3\sqrt{3}}{2}z)$$
 for  $0 \le z \le \frac{2}{3\sqrt{3}}$ .

We remark that the zeta function  $\zeta_{D^E_{G_2}}(z)$  of the Fibonacci-Dyck shift of edge type is

$$\zeta_{D_{G_2}^E}(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2} \qquad ([14, Section7])$$

which is different from (5.3).

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