On the Markov-Dyck shifts of vertex type

Kengo Matsumoto Department of Mathematics, Joetsu University of Education, Joetsu 943-8512 Japan

Abstract

For a given finite directed graph G , there are two types of Markov-Dyck shifts, the Markov-Dyck shift D_G^V of vertex type and the Markov-Dyck shift D_G^E of edge type. It is shown that, if G does not have multi-edges, the former is a finite-to-one factor of the latter, and they have the same topological entropy. An expression for the zeta function of a Markov-Dyck shift of vertex type is given. It is different from that of the Markov-Dyck shift of edge type.

Keywords: Markov-Dyck shift, subshift, zeta function, entropy, Catalan numbers, AMS Subject Classification: Primary 37B10; Secondary 46L05, 05A15.

1 Introduction

Let Σ be a finite alphabet, and let σ be the left shift on $\Sigma^{\mathbb{Z}}$ defined by $\sigma((x_n)_{n\in\mathbb{Z}})$ = $(x_{n+1})_{n\in\mathbb{Z}}, (x_n)_{n\in\mathbb{Z}}\in \Sigma^{\mathbb{Z}}$. For a closed subset $\Lambda\subset \Sigma^{\mathbb{Z}}$ satisfying $\sigma(\Lambda)=\Lambda$, the topological dynamical system $(Λ, σ)$ is called a subshift. Denote by $B_n(Λ)$ the set of all admissible words appearing in Λ with length n, and by $P_n(\Lambda)$ the set of all n-periodic points of (Λ, σ) , respectively. Then the topological entropy $h_{top}(\Lambda)$ and the zeta function $\zeta_{\Lambda}(z)$ for (Λ,σ) is defined by

$$
h_{top}(\Lambda) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(\Lambda)|,
$$
\n(1.1)

$$
\zeta_{\Lambda}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{|P_n(\Lambda)| z^n}{n}\right). \tag{1.2}
$$

They are crucial topological conjugacy invariants of (Λ, σ) . For an introduction to their theory, which belongs to symbolic dynamics, we refer to [\[10\]](#page-20-0) and [\[15\]](#page-20-1).

W. Krieger in [\[11\]](#page-20-2) has introduced the Dyck shifts from automata theory and language theory in computer science. They are non-sofic subshifts defined by Dyck languages. In [\[7,](#page-20-3) [11,](#page-20-2) [12,](#page-20-4) [14,](#page-20-5) [17\]](#page-20-6), a class of non-sofic subshifts called Markov-Dyck shifts have been studied (cf. [\[8\]](#page-20-7)). The subshifts are generalization of Dyck shifts by using finite directed graphs. They have recently come to be studied by computer scientists (cf. $[1, 2]$ $[1, 2]$). For a given finite directed graph $G = (V, E)$, there are two types of Markov-Dyck shifts, the Markov-Dyck shift D_G^V of vertex type and the Markov-Dyck shift D_G^E of edge type. Both of them are

not sofic subshifts if G is irreducible and not permutive. In the papers $[7, 11, 12, 14]$ $[7, 11, 12, 14]$ $[7, 11, 12, 14]$ $[7, 11, 12, 14]$, the Markov-Dyck shifts mean the Markov-Dyck shifts of edge type. In [\[14\]](#page-20-5), formulae of topological entropy and zeta functions for Markov-Dyck shifts of edge type have been presented.

In the first part of the paper, we will study relationship between the two types of Markov-Dyck shifts for finite directed graphs, the Markov-Dyck shift D_G^V of vertex type and the Markov-Dyck shift D_G^E of edge type. We will show that, if G does not have multiedges, there exists a finite-to-one factor code from D_G^E to D_G^V (Proposition [2.9\)](#page-7-0). The factor code can never yield a topological conjugacy unless the transition matrix of the graph is permutation. They have the same topological entropy (Theorem [2.10\)](#page-8-0).

In the second part of the paper, we will present a formula of the zeta function of a Markov-Dyck shift of vertex type (Theorem [3.9\)](#page-15-0). The formula is regarded as a generalization of the formula for Markov-Dyck shifts of edge type [\[14,](#page-20-5) Theorem 2.3]. In the final section, the zeta function of the Fibonacci-Dyck shift of vertex type will be presented. It is different from that of the Fibonacci-Dyck shift of edge type. Hence the Fibonacci-Dyck shift of vertex type is not topologically conjugate to the Fibonacci-Dyck shift of edge type.

2 Markov-Dyck shifts

Throughout this paper N is a fixed positive integer larger than 1. For a finite set S , we denote by |S| its cardinality. We consider the Dyck shift D_N with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = {\alpha_1, ..., \alpha_N}$, $\Sigma^+ = {\beta_1, ..., \beta_N}$. The symbols α_i, β_i correspond to the brackets (i, i) respectively, and have the product relations of monoid as follows:

$$
\alpha_i \beta_j = \begin{cases} \n\mathbf{1} & \text{if } i = j, \\ \n0 & \text{otherwise} \n\end{cases} \tag{2.1}
$$

for $i, j = 1, ..., N$ (cf. [\[12,](#page-20-4) [13\]](#page-20-10)). For a word $\omega = \omega_1 \cdots \omega_n$ of Σ , we denote by $\tilde{\omega}$ its reduced form. Namely $\tilde{\omega}$ is a word of $\Sigma \cup \{0,1\}$ obtained after applying the relations [\(2.1\)](#page-1-0) in ω . Then a word ω of Σ is said to be forbidden in D_N if and only if $\tilde{\omega} = 0$. Denote by \mathfrak{F}_N the set of forbidden words. The Dyck shift D_N is defined in [\[11\]](#page-20-2) by a subshift over Σ whose forbidden words are \mathfrak{F}_N , namely

$$
D_N = \{(x_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mid \forall k \in \mathbb{Z}, m \in \mathbb{N}, (x_k, x_{k+1}, \dots, x_{k+m}) \notin \mathfrak{F}_N\}.
$$
 (2.2)

Let $A = [A(i, j)]_{i,j=1,\dots,N}$ be an $N \times N$ matrix with entries in $\{0, 1\}$. Throughout this paper, A is assumed to be essential which means that it has no zero rows or columns. Consider the Cuntz-Krieger algebra \mathcal{O}_A for the matrix A that is the universal C^* -algebra generated by N partial isometries t_1, \ldots, t_N subject to the following relations:

$$
\sum_{j=1}^{N} t_j t_j^* = 1, \qquad t_i^* t_i = \sum_{j=1}^{N} A(i,j) t_j t_j^* \quad \text{for } i = 1, ..., N
$$
 (2.3)

([\[4\]](#page-20-11)). Define a correspondence $\varphi_A : \Sigma \longrightarrow \{t_i^*, t_i \mid i = 1, ..., N\}$ by setting

$$
\varphi_A(\alpha_i) = t_i^*, \qquad \varphi_A(\beta_i) = t_i \quad \text{ for } i = 1, ..., N.
$$

We denote by Σ^* the set of all words $\gamma_1 \cdots \gamma_n$ of elements of Σ . Define the set

$$
\mathfrak{F}_A = \{ \gamma_1 \cdots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0 \}.
$$

Definition 2.1. The topological Markov Dyck shift for A is defined as a subshift over Σ whose forbidden words are \mathfrak{F}_A . It is written D_A and called the Markov-Dyck shift for A for brevity.

If A is irreducible and not any permutation matrix, the subshift D_A can never be sofic ([\[17,](#page-20-6) Proposition 2.1]). If all entries of A are 1's, the C^* -algebra \mathcal{O}_A becomes the Cuntz algebra \mathcal{O}_N of order N and the subshift D_A becomes the Dyck shift D_N with 2N brackets ([\[3\]](#page-20-12)). We note that $\alpha_i\beta_j \in \mathfrak{F}_A$ if $i \neq j$, and $\alpha_{i_n} \cdots \alpha_{i_1} \in \mathfrak{F}_A$ if and only if $\beta_{i_1} \cdots \beta_{i_n} \in \mathfrak{F}_A$.

Let $G = (V, E)$ be a finite directed graph with vertex set V and edge set E. We denote by $s(e)$ the initial vertex of $e \in E$ and by $t(e)$ the final vertex, respectively. We assume that the cardinalities of V and of E are both finite and write $V = \{v_1, \ldots, v_{N_0}\}\$ and $E = \{e_1, \ldots, e_{N_1}\}.$ We also assume that each vertex of G has at least one in-coming edge and at least one out-going edge. The edge matrix $A^G = [A^G(i,j)]_{i,j=1}^{N_1}$ for G is an $N_1 \times N_1$ transition matrix with entries in $\{0, 1\}$ which is defined by

$$
A^{G}(i,j) = \begin{cases} 1 & \text{if } t(e_i) = s(e_j), \\ 0 & \text{otherwise.} \end{cases}
$$
 (2.4)

In [\[14\]](#page-20-5), we have defined the Markov-Dyck shift D_G for the graph G as the Markov-Dyck shift D_{A^G} for the matrix A^G , and presented formulae of the zeta function $\zeta_{D_G}(z)$ and the topological entropy $h(D_G)$. A finite matrix M with entries in $\{0, 1\}$ does not necessarily arise from a finite graph as $M = A^G$. The lemma below is easy to prove. For the sake of completeness, we provide its proof.

Lemma 2.2. Let $M = [M(i, j)]_{i,j=1}^N$ be an essential $N \times N$ matrix with entries in $\{0, 1\}$. Let us denote by $M_r[i] = [M(i, j)]_{j=1}^N$ and $M_c[j] = [M(i, j)]_{i=1}^N$ the *i*th row vector and the *j*th column vector for $i, j = 1, ..., N$ respectively. Then the following three conditions are *equivalent:*

- (i) There exists a finite directed graph G such that $M = A^G$.
- (ii) *For any* $i_1, i_2 \in \{1, 2, ..., N\}$,

$$
M_r[i_1] = M_r[i_2]
$$
 or $\langle M_r[i_1] | M_r[i_2] \rangle = 0.$ (2.5)

(iii) *For any* $j_1, j_2 \in \{1, 2, ..., N\}$,

$$
M_c[j_1] = M_c[j_2] \quad or \quad \langle M_c[j_1] \mid M_c[j_2] \rangle = 0,
$$
\n(2.6)

where $\langle \cdot | \cdot \rangle$ *means the inner product of vectors.*

Proof. (i) \implies (ii): Suppose that there exists a finite directed graph G such that $M = A^G$. For two edges $e_{i_1}, e_{i_2} \in E$, if $t(e_{i_1}) = t(e_{i_2})$, then $M_r[i_1] = M_r[i_2]$, otherwise $\langle M_r[i_1] |$ $M_r[i_2]\rangle = 0.$

(iii) \implies (i): Assume that the $N \times N$ matrix M satisfies the condition [\(2.6\)](#page-2-0). We will construct a finite directed graph $G = (V, E)$ such that $M = A^G$ as follows. Define an equivalence relation $j_1 \sim j_2$ in $\{1, 2, ..., N\}$ by $M_c[j_1] = M_c[j_2]$. Denote by $[j]_c$ the equivalence class of $j \in \{1, 2, ..., N\}$. Then the vertex set V is defined by the set of equivalence classes $\{[j]_c | j \in \{1, 2, ..., N\}\}\.$ Define an edge labeled e_i from $[i]_c$ to $[j]_c$ if $M(i, j) = 1$. If there exist edges from $[i]_c$ to $[j_1]_c$ labeled e_i and $[i]_c$ to $[j_2]_c$ labeled e_i , then $M(i, j_1) = M(i, j_2) = 1$. By the condition [\(2.6\)](#page-2-0), one has $[j_1]_c = [j_2]_c$. Hence the labeled graph is well-defined. Then as $s(e_j) = [j]_c$, the condition $t(e_i) = s(e_j)$ is equivalent to the condition $M(i, j) = 1$. Hence we have $A^G = M$.

(ii) \implies (iii): Suppose that there exist distinct $j_1 \neq j_2 \in \{1, 2, ..., N\}$ such that $M_c[j_1] \neq M_c[j_2]$ and $\langle M_c[j_1] | M_c[j_2] \rangle \neq 0$. The condition $M_c[j_1] \neq M_c[j_2]$ implies that there exists i_1 such that $M(i_1, j_1) \neq M(i_1, j_2)$. The condition $\langle M_c[j_1] | M_c[j_2] \rangle \neq 0$ implies that there exists i_2 such that $M(i_2, j_1) = M(i_2, j_2) = 1$ so that $\langle M_r[i_1] | M_r[j_2] \rangle \neq 0$, a contradiction to the condition (ii). contradiction to the condition (ii).

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is called the Fibonacci matrix. It can not arise from a finite directed graph as an edge matrix.

For a finite directed $G = (V, E)$, we have another transition matrix A_G , which is an $N_0 \times N_0$ matrix $A_G = [A_G(i, j)]_{i, j=1}^{N_0}$ defined by

$$
A_G(i,j) = \begin{cases} 1 & \text{if there exists an edge from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}
$$
 (2.7)

The matrix A_G is called the vertex matrix for the graph G. It has its entries in $\{0,1\}$.

Definition 2.3. Let $G = (V, E)$ be an essential finite directed graph.

- (i) The Markov-Dyck shift D_{A^G} for the edge matrix A^G is called the *Markov-Dyck shift of edge type* for G , and written D_G^E .
- (ii) The Markov-Dyck shift D_{A_G} for the vertex matrix A_G is called the *Markov-Dyck shift of vertex type* for G , and written D_G^V .

It is obvious that any finite matrix M with entries in $\{0, 1\}$ can arise from a finite graph G such that $M = A_G$. By Lemma [2.2,](#page-2-1) one sees that the class of Markov-Dyck shifts of edge type is a subclass of Markov-Dyck shifts of vertex type. As is well-known that for a finite directed graph G the topological Markov shift X_{A} G defined by the edge matrix ${\cal A}^G$ is topologically conjugate to the topological Markov shift X_{A_G} defined by the vertex matrix A_G . The Markov-Dyck shifts however do not have this property. Let G_1 be the following graph (Figure 1). The vertex matrix A_{G_1} and the edge matrix A^{G_1} are written as

$$
A_{G_1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A^{G_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.8}
$$

respectively. Then the Markov-Dyck shift $D_{G_1}^V$ of vertex type is nothing but the Dyck shift D_2 , whereas the Markov-Dyck shift $D_{G_1}^E$ of edge type is not D_2 . Both $D_{G_1}^V$ and $D_{G_1}^E$

Figure 1:

have 4 fixed points as subshifts. The former $D_{G_1}^V$ has 4 periodic points with least period 2. The latter $D_{G_1}^E$ has 6 periodic points with least period 2. Hence $D_{G_1}^V$ is not topologically conjugate to $D_{G_1}^E$.

A Dyck n-path is a continuous broken directed line on the upper half plane consisting of vectors $(1, 1)$ called rise and $(1, -1)$ called fall. It starts at the origin with rise and ends at $(2n, 0)$ with fall (see [\[5,](#page-20-13) [6\]](#page-20-14), etc.). Let $\gamma = (\gamma_1, \ldots, \gamma_{2n})$ be a Dyck n-path. Hence each γ_i is a rise or a fall. If γ_i is a rise, there exists the smallest $k = 1, 2, ..., 2n - i$ satisfying the following two conditions:

- (i) γ_{i+k} is a fall.
- (ii) $(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{i+k-1})$ is a Dyck $\frac{k-1}{2}$ -path (hence $k-1$ is even), which starts at the terminal vertex of γ_i and ends at the source vertex of γ_{i+k} .

We call the edge γ_{i+k} the partner of γ_i .

Let $G = (V, E)$ be a finite directed graph. Denote by $G^* = (V^*, E^*)$ the transposed graph of G. The vertex set V^* is V and the edge set E^* consists of the edges reversing its direction of the edges of G. For an edge $e \in E$, we denote by e^* the edge of G^* obtained by reversing the direction of e, so that $t(e^*) = s(e), s(e^*) = t(e)$ for $e \in E$. Recall that the edge set E of G is denoted by $\{e_1, \ldots, e_{N_1}\}\$ and the edge set E^* of G^* is written as ${e_1^*, \ldots, e_{N_1}^*}$. Put $\Sigma^-_E = E^*, \Sigma^+_E = E$ and $\Sigma^E_G = \Sigma^-_E \cup \Sigma^+_E$. A *G-Dyck* n-path of edge *type* for $n = 1, 2, \ldots$ is a Dyck n-path (x_1, \ldots, x_{2n}) labeled elements of Σ_G^E satisfying the following rules:

- (1E) a rise is labeled e_i^* for some $i = 1, ..., N_1$,
- (2E) a fall is labeled e_i for some $i = 1, \ldots, N_1$,
- (3E) the partner of a rise labeled e_i^* is labeled e_i ,
- (4E) a rise labeled e_i^* follows a rise labeled e_j^* if and only if $t(e_j^*) = s(e_i^*),$
- (5E) a rise labeled e_i^* follows a fall labeled e_j if and only if $t(e_j) = s(e_i^*),$
- (6E) a fall labeled e_i follows a fall labeled e_j if and only if $t(e_j) = s(e_i)$,
- (7E) a fall labeled e_i follows a rise labeled e_j^* if and only if $e_j = e_i$.

Similarly, for a vertex $v \in V$, we denote by v^* the corresponding vertex of G^* obtained by the transposed graph $G^* = (V^*, E^*)$. The vertex matrix A_{G*} for G^* satisfy the relations

$$
A_{G^*}(i,j) = A_G(j,i) \quad \text{for } i, j \in \{1, 2, \dots, N_0\}.
$$

Recall that the vertex set V of G is denoted by $\{v_1, \ldots, v_{N_0}\}$ and the vertex set V^* of G^* is written as $\{v_1^*, \ldots, v_{N_0}^*\}$. Put $\Sigma_V^- = V^*, \Sigma_V^+ = V$ and $\Sigma_G^V = \Sigma_V^- \cup \Sigma_V^+$. A $G\text{-}Dyck$ *n-path of vertex type* for $n = 1, 2, \ldots$ is a Dyck *n*-path (x_1, \ldots, x_{2n}) labeled elements of Σ^V_G satisfying the following rules:

(1V) a rise is labeled v_i^* for some $i = 1, ..., N_0$,

- (2V) a fall is labeled v_i for some $i = 1, \ldots, N_0$,
- (3V) the partner of a rise labeled v_i^* is labeled v_i ,
- (4V) a rise labeled v_i^* follows a rise labeled v_j^* if and only if $A_{G^*}(j,i) = 1$,

(5V) a rise labeled v_i^* follows a fall labeled v_j if and only if $A_G(j,k) = A_{G^*}(k,i) = 1$ for some v_k ,

- (6V) a fall labeled v_i follows a fall labeled v_j if and only if $A_G(j, i) = 1$,
- (7V) a fall labeled v_i follows a rise labeled v_j^* if and only if $v_j = v_i$.

The Dyck shift D_G^E of edge type is regarded to have its symbols in $E^* \cup E$ under the identification $\Sigma^- = E^*$, $\Sigma^+ = E$, and the Dyck shift D_G^V of vertex type is regarded to have its symbols in $V^* \cup V$ under the identification $\Sigma^- = V^*, \Sigma^+ = V$.

We note the following lemma

Lemma 2.4. *Keep the above notations.*

- (i) Any admissible word of the Dyck shift D_G^E of edge type is regarded as a part of a *labeled broken directed line of* G*-Dyck path of edge type. Conversely a labeled broken* directed line of G-Dyck path of edge type is an admissible word of the Dyck shift D_G^E *of edge type.*
- (ii) Any admissible word of the Dyck shift D_G^V of vertex type is regarded as a part of *a labeled broken directed line of* G*-Dyck path of vertex type. Conversely a labeled broken directed line of* G*-Dyck path of vertex type is an admissible word of the Dyck* shift D_G^V of vertex type.

Proof. (i) is clear from the definition of admissible words of the Dyck shift D_G^E of edge type.

(ii) Let t_1, \ldots, t_{N_0} be the partial isometries satisfying the relations [\(2.3\)](#page-1-1) for the vertex matrix A_G of G. For $i, j = 1, 2, ..., N_0$, we have $\beta_j \alpha_i$ is admissible in D_G^V if and inly if $t_j t_i^* \neq 0$ by definition. Since $t_j t_i^* = t_j t_j^* t_j t_i^* t_i^*$, the condition $t_j t_i^* \neq 0$ is equivalent to the condition that $t_j^* t_j t_i^* t_i \neq 0$. As

$$
t_i^* t_i = \sum_{k=1}^{N_0} A_G(i,k) t_k t_k^* \qquad t_j^* t_j = \sum_{k=1}^{N_0} A_G(j,k) t_k t_k^*,
$$

the condition that $t_j^* t_j t_i^* t_i \neq 0$ is equivalent to the condition $A_G(i,k) = A_G(j,k) = 1$ for some $k = 1, ..., N_0$. This shows that the condition $\beta_j \alpha_i$ is admissibe in D_G^V is equivalent to the condition $(5V)$ of G-Dyck n-path of vertex type. It is direct to see that the other conditions $(1V)$, $(2V)$, $(3V)$, $(4V)$, $(6V)$, $(7V)$ are compatible to the definitions of giving admissible words of the Dyck shift D_G^V of vertex type. \Box

We remark that a finite path of vertices of a labeled broken directed line of the G-Dyck path of edge type is not necessarily an admissible word of the Dyck shift D_G^V of vertex type. Consider the following correspondences in G-Dyck paths:

$$
\begin{cases}\n\text{a fall } e \in E \quad \longrightarrow \text{ the source } s(e) \in V \text{ of } e, \\
\text{a rise } e^* \in E^* \quad \longrightarrow \text{ the terminal } t(e^*) \in V^* \text{ of } e^*. \n\end{cases} \tag{2.9}
$$

The rules $(1E), \ldots, (7E)$ and $(1V), \ldots, (7V)$ ensure us the following lemma.

Lemma 2.5. *Keep the above notations.*

- (i) *Any sequence of vertices of a* G*-Dyck* n*-path of edge type yields a labeled sequence by* Σ_G^V *of a G-Dyck n-path of vertex type by the correspondence* [\(2.9\)](#page-5-0)*.*
- (ii) *Any labeled sequence by* Σ_G^V *of a G-Dyck n-path of vertex type is realized as a sequence of vertices of a* G*-Dyck* n*-path of edge type by the correspondence* [\(2.9\)](#page-5-0)*.*

By the above lemma, it is reasonable to define a 1-block map $\Phi : E \cup E^* \longrightarrow V \cup V^*$ by

$$
\begin{cases}\n\Phi(e) & = s(e) \in V \quad \text{for } e \in E, \\
\Phi(e^*) & = t(e^*)(=s(e)) \in V^* \quad \text{for } e^* \in E^*\n\end{cases}
$$

Hence we have

Proposition 2.6. *The* 1*-block map* $\Phi: E \cup E^* \longrightarrow V \cup V^*$ *induces a factor code* $\varphi =$ $\Phi_{\infty}: D_G^E \longrightarrow D_G^V.$

For $e_{i,k} \in E$ with $s(e_{i,k}) = v_i \in V$ and $t(e_{i,k}) = v_k \in V$, and $e_{k,j}^* \in E^*$ with $s(e_{k,j}^*) = v_k$ $v_k^* \in V^*$ and $t(e_{k,j}^*) = v_j^* \in V^*$, then the word $(e_{i,k}, e_{k,j}^*)$ is admissible in D_G^E and the word (v_i, v_j^*) is admissible in D_G^V such that

$$
\Phi\left(\begin{array}{c}v_i\\&\searrow\\v_k\end{array}\right)=(i\searrow\nearrow j^*),\qquad \Phi(e_{i,k},e_{k,j}^*)=(v_i,v_j^*).
$$

In the above situation, we call the vertex $v_k (= v_k^*)$ a valley. Hence the factor map φ : $D_G^E \longrightarrow D_G^V$ erases the valleys. We will show that the factor map φ is finite-to-one, so that the equality of the topological entropy $h_{top}(D_G^E) = h_{top}(D_G^V)$ holds.

We provide the height functions on D_G^E . These functions on the Dyck shift D_N have been first introduced by W. Krieger in [\[11\]](#page-20-2). For $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$, we set the height function

$$
H_0(x) = 0,
$$

\n
$$
H_m(x) = \sum_{k=0}^{m-1} (\chi_-(x_k) - \chi_+(x_k)), \qquad m \in \mathbb{N},
$$

\n
$$
H_{-m}(x) = \sum_{k=-1}^{-m} (-\chi_-(x_k) + \chi_+(x_k)), \qquad m \in \mathbb{N}
$$

where

$$
\chi_{-}(x_k) = \begin{cases} 1 & \text{if } x_k \in \Sigma^-, \\ 0 & \text{if } x_k \in \Sigma^+, \end{cases} \qquad \chi_{+}(x_k) = \begin{cases} 0 & \text{if } x_k \in \Sigma^-, \\ 1 & \text{if } x_k \in \Sigma^+.\end{cases}
$$

Definition 2.7. For $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$,

- (i) a vertex $t(x_{m-1})(=s(x_m))$ is called a relative minimum in x if $x_{m-1} \in E$ and $x_m \in E^*$.
- (ii) a vertex $t(x_{m-1})(=s(x_m))$ is called a minimum in x if $H_m(x) \leq H_n(x)$ for all $n \in \mathbb{Z}$.

Lemma 2.8. *For* $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$,

- (i) *if a vertex* $t(x_{m-1})(=s(x_m))$ *is not a relative minimum in* x, the word $(\Phi(x_{m-1}), \Phi(x_m))$ $\sum_{i=1}^{V}$ *uniquely determines the vertex* $t(x_{m-1})$ *,*
- (ii) *if a vertex* $t(x_{m-1})(=s(x_m))$ *is not minimum in* x, the sequence $\varphi(x) \in D_G^V$ *uniquely determines the vertex* $t(x_{m-1})$ *,*
- (iii) *if two vertices* $t(x_{n-1})$ *and* $t(x_{m-1})$ *are both minimum in* x*, then* $t(x_{n-1}) = t(x_{m-1})$ *.*

Proof. (i) Since the vertex $t(x_{m-1})(=s(x_m))$ is not a relative minimum in x, we have two cases.

Case 1: $x_{m-1} \in E^*$.

Since $\Phi(x_{m-1})$ is in V^* , we take a vertex $v_i \in V$ such that $\Phi(x_{m-1}) = v_i^*$. We then have $t(x_{m-1}) = v_i^*$.

Case 2: $x_{m-1} \in E$.

The condition that the vertex $t(x_{m-1})(=s(x_m))$ is not a relative minimum in x implies that x_m belongs to E, so that $\Phi(x_m) = v_j \in V$ for some j. We then have $t(x_{m-1}) =$ $s(x_m) = v_j.$

(ii) Suppose that the vertex $t(x_{m-1})(=s(x_m))$ is not minimum in x. If $t(x_{m-1})$ is not a relative minimum in x, the above discussion implies that the word $(\Phi(x_{m-1}), \Phi(x_m))$ in D_G^V uniquely determines the vertex $t(x_{m-1})$. Hence we may assume that $t(x_{m-1})$ is a relative minimum in x. Since $t(x_{m-1})(=s(x_m))$ is not minimum in x, there exists $i \in \mathbb{Z}$ such that $H_i(x) < H_m(x)$. We have two cases.

Case 1: $i > m$.

There exists $k \in \mathbb{Z}$ with $m < k < i$ such that $x_{k-1}, x_k \in E$, and $H_m(x) = H_k(x)$. We take a vertex $v_i \in V$ such that $\Phi(x_k) = v_i$. We then have $t(x_{m-1}) = t(x_{k-1}) = v_i$. Case 2: $i < m$.

There exists $l \in \mathbb{Z}$ with $i < l < m$ such that $x_{l-1}, x_l \in E^*$, and $H_m(x) = H_l(x)$. We take a vertex $v_j \in V$ such that $\Phi(x_{l-1}) = v_j$. We then have $t(x_{m-1}) = t(x_{l-1}) = v_j$.

(iii) Suppose that two vertices $t(x_{n-1})$ and $t(x_{m-1})$ are both minimum in x, so that $H_n(x) = H_m(x)$. Assume that $n < m$. The word $(x_n, x_{n+1}, \ldots, x_{m-1})$ is a G-Dyck path of edge type so that the vertices $s(x_n)$ and $t(x_{m-1})$ are the same. This implies that $t(x_{m-1}) = t(x_{m-1})$. □ $t(x_{n-1}) = t(x_{m-1}).$

Proposition 2.9. Suppose that G does not have multi-edges. Let $\varphi : D_G^E \longrightarrow D_G^V$ be the *factor code defined in Proposition [2.6.](#page-6-0)* For $x = (x_n)_{n \in \mathbb{Z}} \in D_G^E$, we have

(i) *if* x does not have a minimum vertex, then φ *is injective at* x, that *is*,

$$
\varphi^{-1}(\varphi(x)) = x,
$$

(ii) *if* x *has a minimum vertex, then*

$$
|\varphi^{-1}(\varphi(x))| \le N_0 = |V|.
$$

Therefore $\varphi: D_G^E \longrightarrow D_G^V$ *is a finite-to-one factor code.*

Proof. (i) Suppose that $x = (x_n)_{n \in \mathbb{Z}}$ does not have a miniumum vertex. By (ii) of the above lemma, the sequence $\varphi(x)$ determines the sequence $t(x_n), n \in \mathbb{Z}$ of vertices. Each symbol x_n is an edge of E or of E^{*}, and an edge is determined by the vertices $t(x_n), t(x_{n-1}) (=$ $s(x_n)$, so that the code φ is injective at x.

(ii) Suppose that x has a minimum vertex at $t(x_{m-1})$ for some $m \in \mathbb{Z}$. Then the vertex $t(x_{m-1})$ is a valley and $x_{m-1} \in E$, $x_m \in E^*$. By (iii) of the above lemma, other minimum vertices are the same as the vertex $t(x_{m-1})$. Hence we have

$$
|\varphi^{-1}(\varphi(x))| = |\{k \in \{1, 2, \dots, N_0\} \mid A_G(s(x_{m-1}), k) = A_{G^*}(k, t(x_m)) = 1\}|
$$

\$\le N_0 = |V|.

Theorem 2.10. Suppose that G does not have multi-edges. We then have $h_{top}(D_G^V)$ = $h_{top}(D_G^E)$.

Proof. Since there exists a factor code $\varphi : D_G^E \longrightarrow D_G^V$, the inequality $h_{top}(D_G^V) \leq$ $h_{top}(D_G^E)$ is clear. The 1-block map Φ naturally induces a map $\Phi_*: B_*(D_G^E) \longrightarrow B_*(D_G^V)$ between admissible words. It is not necessarily one-to-one at minimal points of words. We then have

$$
|B_n(D_G^E)| \le N_0 \cdot |B_n(D_G^V)|, \qquad n \in \mathbb{N}
$$

Therefore we have $h_{top}(D_G^E) \leq h_{top}(D_G^V)$.

Concerning embedding of the Markov-Dyck shifts, we have the following proposition.

Proposition 2.11. *Suppose that* G *does not have multi-edges. There exists an embedding* of D_G^E *into the 3rd power shift of* D_G^V .

Proof. Let t_i , $i = 1, ..., N_0$ be partial isometries satisfying the relations [\(2.3\)](#page-1-1) for the vertex matrix A_G . For an edge $e_n \in E$ with $s(e_n) = v_i, t(e_n) = v_j$, define a partial isometry $S_n = t_i t_j t_j^*$. It is easy to see that the family S_1, \ldots, S_{N_1} satisfies the relations [\(2.3\)](#page-1-1) for the edge matrix A^G , This implies that the correspondence $\Psi: E \cup E^* \longrightarrow (V \cup V^*)^{[3]}$ defined by

$$
\Psi(e_n) = (v_i, v_j, v_j^*), \qquad \Psi(e_n^*) = (v_j, v_j^*, v_i^*)
$$

induces an embedding of D_G^E into the 3rd power shift $(D_G^V)^{[3]}$ of D_G^V .

3 The zeta functions of Mrkov-Dyck shifts of vertex type

In what follows, we fix an arbitrary $N \times N$ matrix $A = [A(i, j)]_{i,j=1}^N$ with entries in $\{0, 1\}$. We will study the Markov-Dyck shift D_A and present a formula of the zeta function $\zeta_{D_A}(z)$. In [\[14\]](#page-20-5), a formula of the zeta function of the Markov-Dyck shifts of edge type has been presented. The Markov-Dyck shifts of edge type form a subclass of the class of Markov-Dyck shifts. In this section, we will study general Markov-Dyck shift D_A and present a formula of its zeta function $\zeta_{D_A}(z)$. For the $N \times N$ matrix A, let v_1, \ldots, v_N be N-vertices. Define a directed edge from v_i to v_j if $A(i, j) = 1$. We then have a finite directed graph written $G = (V, E)$ such that its vertex matrix A_G coincides with the original matrix A.

 \Box

 \Box

 \Box

Throughout this section, we identify α_i with v_i^* and β_i with v_i for $i = 1, ..., N$, respectively. Let $w = (w_1, \ldots, w_{2n})$ be a G-Dyck n-path of vertex type. As in [\[16\]](#page-20-15), w is called a G-Catalan word and satisfies the following conditions:

$$
\sum_{k=1}^{m} (\chi_{-}(w_k) - \chi_{+}(w_k)) \ge 0 \qquad \text{for all } m = 1, 2, ..., 2n
$$

and

$$
\sum_{k=1}^{2n} (\chi_{-}(w_k) - \chi_{+}(w_k)) = 0.
$$

Denote by C_n^A the set of G-Dyck *n*-pathes of vertex type. For $i = 1, ..., N$, put

$$
C_n^A(i) = \{ (w_1, \ldots, w_{2n}) \in C_n^A \mid (\alpha_i, w_1, \ldots, w_{2n}, \beta_i) \in C_{n+1}^A \}.
$$

Denote by $c_n^A(i)$ the cardinarity $|C_n^A(i)|$ of the set $C_n^A(i)$. We set $c_0^A(i) = 1$. Combinatorial properties of the sequence $c_n^A(i)$, $n = 0, 1, \ldots$ have been studied in [\[16,](#page-20-15) Section 4]. For $i = 1, \ldots, N$, let $f_i^A(z)$ be the generating function of the sequence $c_n^A(i), n = 0, 1, 2, \cdots$:

$$
f_i^A(z) = \sum_{n=0}^{\infty} c_n^A(i) z^n.
$$

Since one knows ([\[16,](#page-20-15) Section 4])

$$
C_{n+1}^{A}(i) = \bigcup_{k=0}^{n} \bigcup_{\substack{j \\ A(j,i)=1}} C_k^{A}(j) \times C_{n-k}^{A}(i),
$$

we have

$$
c_{n+1}^A(i) = \sum_{k=0}^n \sum_{j=1}^N A(j,i)c_k^A(j)c_{n-k}^A(i),
$$

so that the identity

$$
f_i^A(z) = 1 + z f_i^A(z) \sum_{j=1}^N A(j, i) f_j^A(z)
$$
 (3.1)

holds ([\[16,](#page-20-15) Proposition 4.2]). Let X_A be the shift space over $\Sigma^+ = V$ of the topological Markov shift defined by the matrix A:

$$
X_A = \{(x_n)_{n \in \mathbb{Z}} \in (\Sigma^+)^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}.
$$

For $n, k \in \mathbb{N}$, we set

$$
C_{n,k}^{A,+} = \{ (w_1, \ldots, w_{2n}, \beta_{i_1}, \ldots, \beta_{i_k}) \in B_{2n+k}(D_A) \mid
$$

$$
(w_1, \ldots, w_{2n}) \in C_n^A, (\beta_{i_1}, \ldots, \beta_{i_k}) \in B_k(X_A) \}.
$$

For $(w_1, ..., w_{2n}, \beta_{i_1}, ..., \beta_{i_k}) \in C_{n,k}^{A,+}$, we set

$$
s((w_1, ..., w_{2n}, \beta_{i_1}, ..., \beta_{i_k})) = \beta_{i_1}, t((w_1, ..., w_{2n}, \beta_{i_1}, ..., \beta_{i_k})) = \beta_{i_k}.
$$

We put

$$
C_A^+ = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n,k}^{A,+}.
$$

We then see the following lemma.

Lemma 3.1. *For* $\mu, \nu \in C_A^+$ *, the word* $\mu\nu$ *is admissible in* D_A *if and only if* $A(t(\mu), s(\nu)) =$ 1*.*

Put $I = \{1, ..., N\} \times \{1, ..., N\}$. Define an $I \times I$ matrix $\tilde{A} = [\tilde{A}((i, j), (k, l))]_{(i, j), (k, l) \in I}$ by

$$
\tilde{A}((i,j),(k,l)) = A(j,k)
$$

and a map $r: C_A^+ \longrightarrow I$ by

$$
r((w_1,\ldots,w_{2n},\beta_{i_1},\ldots,\beta_{i_k}))=(\beta_{i_1},\beta_{i_k})\in I.
$$

Then the quadruplet $\mathcal{C}_A^+ = (C_A^+, I, \tilde{A}, r)$ is a circular Markov code in the sense of Keller [\[9\]](#page-20-16). We then associate the following shift-invariant subset $\Omega_{\mathcal{C}_A^+}$ by

$$
\Omega_{\mathcal{C}_A^+} = \{ x = (x_n)_{n \in \mathbb{Z}} \mid \text{ there are } \dots k_{-1} < k_0 \le 0 < k_1 < \dots \text{ in } \mathbb{Z} \}
$$
\n
$$
\text{such that } x_{[k_i, k_{i+1})} \in C_A^+ \text{ and } \tilde{A}(r(x_{[k_{i-1}, k_i)}), r(x_{[k_i, k_{i+1})})) = 1 \} \tag{9].
$$

The zeta function $\zeta(\Omega_{\mathcal{C}_A^+}, z)$ for a shift-invariant set $\Omega_{\mathcal{C}_A^+}$ is similarly defined to [\(1.2\)](#page-0-0) by using a sequence of cardinalities of periodic points of $\Omega_{\mathcal{C}_{A}^{+}}$. Following Keller [\[9\]](#page-20-16), define a sequence $D(C_A^+,m) = \text{diag}[d_{(i,j),(i,j)}(C_A^+,m)], 3 \leq m \in \mathbb{N}$ of $I \times I$ -diagonal matrices with diagonal entries $d_{(i,j),(i,j)}(\mathcal{C}^+_A,m),(i,j) \in I$ by

$$
d_{(i,j),(i,j)}(\mathcal{C}_A^+,m) = |\{(w_1,\ldots,w_{2n},\beta_{i_1},\ldots,\beta_{i_k}) \in C_A^+ | i_i = i, i_k = j\}|
$$

$$
(= c_n^A(i)A^{k-1}(i,j))
$$

for $m = 2n + k$, and a matrix-valued generating function $F(\mathcal{C}_A^+, z)$ by

$$
F(\mathcal{C}_A^+, z) = \sum_{m=1}^{\infty} D(\mathcal{C}_A^+, m) \tilde{A} z^m.
$$

Denote by I_{N^2} the identity matrix of size N^2 . By using [\[9,](#page-20-16) Theorem 1], we have Proposition 3.2. $\zeta(\Omega_{\mathcal{C}_A^+}, z) = \det(I_{N^2} - F(\mathcal{C}_A^+, z))$

We then have for $(i, j), (p, q) \in I$

$$
F(C_A^+, z)((i, j), (p, q)) = \sum_{m=1}^{\infty} D(C_A^+, m) \tilde{A} z^m((i, j), (p, q))
$$

\n
$$
= \sum_{m=1}^{\infty} \sum_{\substack{n,k \ n \neq m}} D(C_A^+, 2n + k) \tilde{A} z^{2n+k}((i, j), (p, q))
$$

\n
$$
= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_n^A(i) A^{k-1}(i, j) \tilde{A}((i, j), (p, q)) z^{2n+k}
$$

\n
$$
= \sum_{n=1}^{\infty} c_n^A(i) z^{2n} \sum_{k=1}^{\infty} A^{k-1}(i, j) z^k \tilde{A}((i, j), (p, q))
$$

\n
$$
= (f_i^A(z^2) - 1) z \sum_{l=0}^{\infty} (z A)^l(i, j) A(j, p)
$$

\n
$$
= (f_i^A(z^2) - 1) z (1_N - z A)^{-1}(i, j) \cdot A(j, p).
$$

We define $N \times N$ matrices $F^A = [F^A(i,j)]_{i,j=1}^N$ and $H(C_A^+, z)$ by

$$
F^{A}(i,j) = (f_i^{A}(z^2) - 1)z(1_N - zA)^{-1}(i,j) \text{ and } H(C_A^+, z) = F^A \cdot A
$$

so that

$$
F(C_A^+, z)((i, j), (p, q)) = F^A(i, j)A(j, p) \text{ and } H(C_A^+, z)(i, p) = \sum_{j=1}^N F(C_A^+, z)((i, j), (p, q)).
$$

Lemma 3.3. det $(I_{N^2} - F(C_A^+, z)) = \det(I_N - H(C_A^+, z)).$

Proof. Let $U = [U((i, j), (p, q))]_{(i, j), (p, q) \in I}$ and $V = [V((i, j), (p, q))]_{(i, j), (p, q) \in I}$ be $I \times I$ matrices defined by

$$
U((i, j), (p, q)) = \begin{cases} 1 & \text{if } (i, j) = (p, q), \\ 1 & \text{if } i = p, j = N, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
V((i, j), (p, q)) = \begin{cases} 1 & \text{if } (i, j) = (p, q), \\ -1 & \text{if } i = p, j = N, q < N, \\ 0 & \text{otherwise}. \end{cases}
$$

The matrix $(I_{N^2} - F(\mathcal{C}_A^+, z))V$ is obtained from $(I_{N^2} - F(\mathcal{C}_A^+, z))$ by adding the minus of the (i, N) th column to the (i, j) th column for all $j = 1, 2, \ldots, N - 1$ and $i = 1, 2, \ldots, N$, and the matrix $U(I_{N^2} - F(\mathcal{C}_A^+, z))V$ is obtained from $(I_{N^2} - F(\mathcal{C}_A^+, z))V$ by adding the (i, j) th rows to the (i, N) th row for all $j = 1, 2, ..., N - 1$ and $i = 1, 2, ..., N$. Hence we see

$$
U(I_{N^2} - F(\mathcal{C}_A^+, z))V((i, j), (p, q)) = \begin{cases} 1 & \text{if } (i, j) = (p, q), q < N, \\ 0 & \text{if } (i, j) \neq (p, q), q < N, \\ 1 - \sum_{k=1}^N F^A(i, k)A(k, p) & \text{if } (i, j) = (p, q), q = N, \\ -F^A(i, j)A(j, p) & \text{if } j < N, q = N, \\ 0 & \text{otherwise.} \end{cases}
$$

Each (p, q) th column for $q < N$ of the matrix $U(I_{N^2} - F(\mathcal{C}^+_A, z))V$ has 1 on diagonal and zero elsewhere. Since

$$
1 - \sum_{k=1}^{N} F^{A}(i,k)A(k,p) = 1 - H(C_A^+,z)(i,p),
$$

by expanding the matrix $U(I_{N^2}-F(\mathcal{C}_A^+, z))V$ along the (p, q) th columns for $p = 1, 2, ..., N$ with $q < N$, we have

$$
\det(U(I_{N^2} - F(\mathcal{C}^+_A, z))V) = \det(I_N - H(C^+_A, z)).
$$

As $\det(U) = \det(V) = 1$, we get the desired equality.

Therefore we have

Proposition 3.4.

$$
\zeta(\Omega_{\mathcal{C}_A^+}, z) = \frac{\det(I_N - zA)}{\det(I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)}.
$$
\n(3.2)

 \Box

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Proof. Since

$$
H(C_A^+, z) = \text{diag}[f_1^A(z^2) - 1, \dots, f_N^A(z^2) - 1]zA(I_N - zA)^{-1},
$$

we have

$$
I_N - H(C_A^+, z) = I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA(I_N - zA)^{-1} + zA(I_N - zA)^{-1}
$$

= $(I_N - zA)^{-1} - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA(I_N - zA)^{-1}$
= $(I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)(I_N - zA)^{-1}$

so that the desired equality holds.

For
$$
j \in \{1, 2, ..., N\}
$$
 with $A(i, j) = 1$, we put
\n
$$
C_n^A[i; \{j\}] = \{(\alpha_i, w_1, ..., w_{2n-2}, \beta_i) \in C_n^A(j) \mid (w_1, ..., w_{2n-2}) \in C_{n-1}^A(i)\}
$$

and

$$
C_n^A[j] = \bigcup_{\substack{i=1 \ A(i,j)=1}}^N C_n^A[i; \{j\}], \qquad C^A[j] = \bigcup_{n=1}^\infty C_n^A[j].
$$

We set $c_n^A[j] = |C_n^A[j]|$. As $|C_n^A[i; \{j\}]| = c_{n-1}^A(i)$ if $A(i, j) = 1$, we have

$$
c_n^A[j] = \sum_{i=1}^N A(i,j)c_{n-1}^A(i).
$$
\n(3.3)

Similarly for a subset $\{j_1, ..., j_k\} \subset \{1, 2, ..., N\}$ with $A(i, j_1) = \cdots = A(i, j_k) = 1$, we put

$$
C_n^A[i; \{j_1, \ldots, j_k\}] = \bigcap_{m=1}^k C_n^A[i; \{j_m\}]
$$

and

$$
C_n^A[\{j_1, \ldots, j_k\}] = \bigcup_{\substack{i=1 \ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = A(i,j_k) = 1}}^N C_n^A[i; \{j_1, \ldots, j_k\}],
$$

$$
C^A[\{j_1, \ldots, j_k\}] = \bigcup_{n=1}^{\infty} C_n^A[\{j_1, \ldots, j_k\}].
$$

We set $c_n^A[\{j_1, ..., j_k\}] = |C_n^A[\{j_1, ..., j_k\}]|$ so that

$$
c_n^A[\{j_1, \dots, j_k\}] = \sum_{i=1}^N A(i, j_1) \cdots A(i, j_k) c_{n-1}^A(i). \tag{3.4}
$$

For a subset $\{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$ if there exists $i \in \{1, 2, \ldots, N\}$ such that $A(i, j_1) = \cdots = A(i, j_k) = 1$, we call the set $C^A[\{j_1, \ldots, j_k\}]$ the *Markov-Dyck code* with support $\{j_1, \ldots, j_k\}$. It is easy to see that the set $C^A[\{j_1, \ldots, j_k\}]$ is a circular code. Denote by $C^A[\{j_1,\ldots,j_k\}]^{\infty}$ the set of all two-sided sequences of alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ consisting of free concatenations of words of $C^A[\{j_1,\ldots,j_k\}]$. Let $g_{C^A[\{j_1,\ldots,j_k\}]}(z)$ be the generating function for the sequence $c_n^A[\{j_1, \ldots, j_k\}], n = 1, 2, \ldots$ defined by

$$
g_{C^A[\{j_1,\ldots,j_k\}]}(z) = \sum_{n=1}^{\infty} c_n^A[\{j_1,\ldots,j_k\}] z^{2n}.
$$

Lemma 3.5. (i) The generating function $g_{C^A[\{j_1,...,j_k\}]}(z)$ satisfies

$$
g_{C^A[\{j_1,\ldots,j_k\}]}(z) = z^2 \sum_{i=1}^N A(i,j_1)A(i,j_2)\cdots A(i,j_k)f_i^A(z^2). \tag{3.5}
$$

(ii) *The zeta function* $\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty}, z)$ *of the shift-invariant set* $C^A[\{j_1,\ldots,j_k\}]^{\infty} \subset \Sigma^{\mathbb{Z}}$ is

$$
\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty},z) = \frac{1}{1 - g_{C^A[\{j_1,\ldots,j_k\}]}(z)}.\tag{3.6}
$$

In particular for $j \in \{1, 2, \ldots, N\}$ *, we have*

$$
\zeta(C^A[\{j\}]^{\infty}, z) = \frac{1}{1 - g_{C^A[\{j\}]}(z)} = f_j^A(z^2). \tag{3.7}
$$

Proof. (i) By [\(3.4\)](#page-13-0), we have

$$
g_{C^{A}[\{j_{1},...,j_{k}\}]}(z) = \sum_{n=1}^{\infty} \sum_{i=1}^{N} A(i, j_{1}) \cdots A(i, j_{k}) c_{n-1}^{A}(i) z^{2n}
$$

= $z^{2} \sum_{i=1}^{N} A(i, j_{1}) \cdots A(i, j_{k}) \sum_{n=1}^{\infty} c_{n-1}^{A}(i) z^{2(n-1)}$
= $z^{2} \sum_{i=1}^{N} A(i, j_{1}) \cdots A(i, j_{k}) f_{i}^{A}(z^{2}).$

(ii) The set $C^A[\{j_1,\ldots,j_k\}]$ is a circular code, and the set $C^A[\{j_1,\ldots,j_k\}]^{\infty}$ consisting of the two-sided sequences of free concatenations of words of $C^A[\{j_1,\ldots,j_k\}]$. Hence a well-known theorem of combinatorics (cf. [\[18,](#page-20-17) Proposition 4.7.11]) ensures us the equality

$$
\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty},z)=\frac{1}{1-g_{C^A[\{j_1,\ldots,j_k\}]}(z)}.
$$

In particular we have

$$
g_{C^A[\{j\}]}(z) = z^2 \sum_{i=1}^N A(i,j) f_i^A(z^2) = \frac{f_j^A(z^2) - 1}{f_j^A(z^2)} = 1 - \frac{1}{f_j^A(z^2)}
$$

so that

$$
\zeta(C^A[\{j\}]^{\infty}, z) = \frac{1}{1 - g_{C^A[\{j\}]}(z)} = f_j^A(z^2). \tag{3.8}
$$

We call a subset $\{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$ *a support subset* if for any $i \in \{1, 2, \ldots, N\}$ there exists $l = 1, ..., k$ such that $A(i, j_l) = 1$. The set $\{1, 2, ..., N\}$ itself is a support subset. For a shift-invariant subset C of D_A , denote by $P_n(C)$ the set of n-periodic points of C. We set

$$
C^{A^{\infty}} = \bigcup_{\{j_1,\ldots,j_k\} \subset \{1,\ldots,N\}} C^A[\{j_1,\ldots,j_k\}]^{\infty} \subset \Sigma^{\mathbb{Z}}.
$$
 (3.9)

By the principle of inclusion of exclusion in combinatorics (cf. [\[18,](#page-20-17) 2.1]), we have

Lemma 3.6. Let $J = \{j_1, \ldots, j_k\}$ be a support subset of $\{1, 2, \ldots, N\}$. Then we have

$$
P_n(C^{A^{\infty}})
$$

= $\bigcup_{l=1}^k P_n(C^A[\{j_l\}]^{\infty}) - \bigcup_{\{j_1,j_2\} \subset J} P_n(C^A[\{j_1,j_2\}]^{\infty})$
 $\cdots (-1)^{m+1} \bigcup_{\{j_1,\ldots,j_m\} \subset J} P_n(C^A[\{j_1,\ldots,j_m\}]^{\infty}) \cdots (-1)^{k+1} \bigcup P_n(C^A[\{j_1,\ldots,j_k\}]^{\infty}),$

 $where$ $(-1)^{m+1} \bigcup_{\{j_1,\dots,j_m\} \subset J}$ *means* $\bigcup_{\{j_1,\dots,j_m\} \subset J}$ *if m is odd.*

Hence we have

Proposition 3.7. Let $J = \{j_1, \ldots, j_k\}$ be a support subset of $\{1, 2, \ldots, N\}$. Then we have

$$
\zeta(C^{A^{\infty}}, z)
$$
\n
$$
= \prod_{l=1}^{k} \zeta(C^{A}[\{j_l\}]^{\infty}, z) \cdot \prod_{\{j_1, j_2\} \subset J} \zeta(C^{A}[\{j_1, j_2\}]^{\infty}, z)^{-1}
$$
\n
$$
\cdots \prod_{\{j_1, \dots, j_m\} \subset J} \zeta(C^{A}[\{j_1, \dots, j_m\}]^{\infty}, z)^{(-1)^{m+1}} \cdots \zeta(C^{A}[\{j_1, \dots, j_k\}]^{\infty}, z)^{(-1)^{k+1}}.
$$

Corollary 3.8. Suppose that there exists $j_0 \in \{1, 2, ..., N\}$ such that $A(i, j_0) = 1$ for all $i = 1, 2, ..., N$. Then $\zeta(C^{A^{\infty}}, z) = f_{j_0}^A(z^2)$.

We reach the following formula of the zeta function of a Markov-Dyck shift of vertex type.

Theorem 3.9. Let A be an $N \times N$ essential matrix with entries in $\{0, 1\}$. Then the zeta $function \zeta_{D_A}(z)$ *of the Markov-Dyck shift* D_A *is given by the following formula:*

$$
\zeta_{D_A}(z) = \frac{\zeta(C^{A^{\infty}}, z)}{\det(I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}
$$
(3.10)

where

$$
\zeta(C^{A^{\infty}}, z) = \prod_{\{j_1, \dots, j_k\} \subset \{1, 2, \dots, N\}} \zeta(C^{A}[\{j_1, \dots, j_k\}]^{\infty}, z)^{(-1)^{k+1}},
$$

 $the\ products\ \prod_{\{j_1,\ldots,j_k\}\subset\{1,2,\ldots,N\}}\ run\ over\ all\ subsets\ of\ \{1,2,\ldots,N\},\ and\ the\ zeta function$ $\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty},z)$ *is given by*

$$
\zeta(C^A[\{j_1,\ldots,j_k\}]^{\infty},z)=\frac{1}{1-g_{C^A[\{j_1,\ldots,j_k\}]}(z)},
$$

where

$$
g_{C^A[\{j_1,\ldots,j_k\}]}(z) = z^2 \sum_{i=1}^N A(i,j_1) \cdots A(i,j_k) f_i^A(z^2),
$$

and the functions $f_i^A(z^2)$, $i = 1, 2, ..., N$ satisfiy the relations [\(3.1\)](#page-9-0).

Proof. For $n, k \in \mathbb{N}$, we define the following set $C_{n,k}^{A,-}$ similarly to $C_{n,k}^{A,+}$ by

$$
C_{n,k}^{A,-} = \{(\alpha_{i_1}, \ldots, \alpha_{i_k}, w_1, \ldots, w_{2n}) \in B_{2n+k}(D_A) \mid
$$

$$
(w_1, \ldots, w_{2n}) \in C_n^A, (\alpha_{i_1}, \ldots, \alpha_{i_k}) \in B_k(X_{A^t})\}.
$$

Similarly to the previous discussion, we have a circular Markov code $\mathcal{C}_A^- = (C_A^-, I, \tilde{A}^t, r)$ and the formula [\(3.2\)](#page-12-0) for $\zeta(\Omega_{\mathcal{C}_A^-}, z)$. We then have a disjoint union of periodic points

$$
P_n(D_A) = P_n(\Omega_{\mathcal{C}_A^+}) \cup P_n(\Omega_{\mathcal{C}_A^-}) \cup P_n(C^{A^{\infty}}) \cup P_n(X_A) \cup P_n(X_{A^t}).
$$

Since $\zeta(\Omega_{\mathcal{C}^+_A}, z) = \zeta(\Omega_{\mathcal{C}^-_A}, z)$, Proposition [3.4](#page-12-1) ensures us

$$
\zeta_{D_A}(z) = \zeta(\Omega_{\mathcal{C}_A^+}, z) \cdot \zeta(\Omega_{\mathcal{C}_A^-}, z) \cdot \zeta(C^{A^{\infty}}, z) \cdot \frac{1}{\det(I_N - zA)} \cdot \frac{1}{\det(I_N - zA^t)}
$$

$$
= \frac{\zeta(C^{A^{\infty}}, z)}{\det(I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}.
$$

 \Box

For a finite directed graph $G = (V, E)$ the above formula gives us the formula for the zeta function of the Markov-Dyck shift of vertex type.

Corollary 3.10. *Suppose that there exists* $j_0 \in \{1, 2, ..., N\}$ *such that* $A(i, j_0) =$ *for all* $i = 1, 2, ..., N$ *. Then*

$$
\zeta_{D_A}(z) = \frac{f_{j_0}^A(z^2)}{\det(I_N - \text{diag}[f_1^A(z^2), \dots, f_N^A(z^2)]zA)^2}.
$$

4 The zeta functions of Markov-Dyck shifts of edge type

The Markov-Dyck shifts in the paper [\[14\]](#page-20-5) are the Markov-Dyck shifts of edge type. In [\[14\]](#page-20-5), a formula of the zeta functions of Markov-Dyck shifts of edge type has been presented. In this section, we present the formula [\[14,](#page-20-5) Theorem 2.3] from Theorem [3.9.](#page-15-0) We need the following lemma.

Lemma 4.1. For a finite directed graph $G = (V, E)$ with $|V| = N_0$ and $|E| = N_1$. Let $f_1^V(x), \cdots, f_{N_0}^V(x)$ and $f_1^E(x), \cdots, f_{N_1}^E(x)$ be the functions satisfying the relations respec*tively*

$$
f_i^V(z) = 1 + z f_i^V(z) \sum_{j=1}^{N_0} A_G(j, i) f_j^V(z).
$$
 (4.1)

$$
f_i^E(z) = 1 + z f_i^E(z) \sum_{j=1}^{N_1} A^G(j, i) f_j^E(z).
$$
 (4.2)

Then we have

$$
\det(I_{N_0} - \text{diag}[f_1^V(z^2), \dots, f_{N_0}^V(z^2)]zA_G)
$$

=
$$
\det(I_{N_1} - \text{diag}[f_1^E(z^2), \dots, f_{N_1}^E(z^2)]zA^G).
$$

Proof. Put the sets $I_0 = \{1, 2, \ldots, N_0\}$, $I_1 = \{1, 2, \ldots, N_1\}$ and the diagonal matrices $D^{V}(z^2) = \text{diag}[f_1^{V}(z^2), \dots, f_{N_0}^{V}(z^2)]$ and $D^{E}(z^2) = \text{diag}[f_1^{E}(z^2), \dots, f_{N_1}^{E}(z^2)]$. Define the $N_0 \times N_1$ matrix $S = [S(i, j)]_{i \in I_0, j \in I_1}$ and the $N_1 \times N_0$ matrix $R = [R(j, i)]_{j \in I_1, i \in I_0}$ by

$$
S(i,j) = \begin{cases} 1 & \text{if } v_i = s(e_j), \\ 0 & \text{otherwise,} \end{cases} \qquad R(j,i) = \begin{cases} 1 & \text{if } t(e_j) = v_i, \\ 0 & \text{otherwise,} \end{cases}
$$

so that $A_G = SR$ and $A^G = RS$. For a vertex $v_i \in V$ and en edge $e_j \in E$, we set

$$
C_n^{A_G}(v_i) = \{ (w_1, \dots, w_{2n}) \in C_n^{A_G} \mid (v_i^*, w_1, \dots, w_{2n}, v_i) \in C_{n+1}^{A_G} \},
$$

$$
C_n^{A^G}(e_j) = \{ (g_1, \dots, g_{2n}) \in C_n^{A^G} \mid (e_j^*, g_1, \dots, g_{2n}, e_j) \in C_{n+1}^{A^G} \}.
$$

Let us denote by $c_n^G(v_i)$ and $c_n^G(e_j)$ their cardinalities $|C_n^{A}(\varepsilon_i)|$ and $|C_n^{A}(\varepsilon_j)|$ respectively $([16, \text{ pages } 8, 9])$ $([16, \text{ pages } 8, 9])$ $([16, \text{ pages } 8, 9])$. Then we have

$$
f_i^V(z) = \sum_{n=0}^{\infty} c_n^G(v_i) z^n
$$
, $f_j^E(z) = \sum_{n=0}^{\infty} c_n^G(e_j) z^n$

so that $f_j^E(z) = f_i^V(z)$ when $s(e_j) = v_i$. Hence we have

$$
f_i^V(z^2)S(i,j) = S(i,j)f_j^E(z^2)
$$

which implies that $D^V(z^2)S = SD^E(z^2)$. It then follows that

$$
zD^{V}(z^{2})A_{G} = zD^{V}(z^{2})SR = zS \cdot D^{E}(z^{2})R,
$$

\n
$$
zD^{E}(z^{2})A^{G} = zD^{E}(z^{2})RS = D^{E}(z^{2})R \cdot zS.
$$

Hence the matrices $zD^{V}(z^2)A_G$ and $zD^{E}(z^2)A^G$ are elementary equivalent (see [\[15,](#page-20-1) Definition 7.2.1]), so that $\det(I_{N_0} - zD^V(z^2)A_G) = \det(I_{N_1} - zD^E(z^2)A^G)$. \Box

Therefore we have

Proposition 4.2 ([\[14,](#page-20-5) Theorem 2.3]). *If a matrix* A *is an edge matrix* $A^G = [A^G(e, f)]_{e, f \in E}$ *defined by a finite directed graph* $G = (V, E)$ *with* $|V| = N_0$ *, then the zeta function of the Markov-Dyck shift* $D_G(= D_{AG})$ *of edge type is given by the following formula:*

$$
\zeta_{D_G}(z) = \frac{\Pi_{i=1}^{N_0} f_i^G(z^2)}{\det(I_N - \text{diag}[f_1^G(z^2), \dots, f_{N_0}^G(z^2)] z A_G)^2}
$$
(4.3)

where $f_1^G(z^2), \ldots, f_{N_0}^G(z^2)$ are the functions satisfying

$$
f_i^G(z) = 1 + z f_i^G(z) \sum_{j=1}^{N_0} A_G(j, i) f_j^G(z).
$$
 (4.4)

Proof. Since $f_i^G(x) = f_i^V(x), i = 1, ..., N_0$ and

$$
\zeta(C^{A^{\infty}}, z) = \prod_{i=1}^{N_0} \frac{1}{1 - g_{C^A[\{j\}]}(z)} = \prod_{i=1}^{N_0} f_i^G(z^2)
$$

(cf. (cf. [\[18,](#page-20-17) Proposition 4.7.11]), the preceding lemma implies the equality [\(4.3\)](#page-17-0) from Theorem [3.9.](#page-15-0) \Box

Figure 2:

5 The Fibonacci-Dyck shift of vertex type

Let G_2 be the finite directed graph defined in the Figure 2. The edge matrix A^{G_2} and the vertex matrix A_{G_2} are written as

$$
A^{G_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad A_{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
$$
 (5.1)

respectively. We then have

Proposition 5.1. $D_{G_2}^V$ is not topologically conjugate to $D_{G_2}^E$.

Proof. It is easy to see that the number of the 2-periodic points of $D_{G_2}^V$ is 6, whereas that of $D_{G_2}^E$ is 7. □

The Fibonacci-Dyck shift $D_{G_2}^E$ of edge type is a subshift $D_{A^{G_2}}$ over six symbols which correspond to the edges of the directed graphs G_2 and G_2^* of Figure 2. The Fibonacci-Dyck shift $D_{A_{G_2}}$ of vertex type is a subshift $D_{A_{G_2}}$ over four symbols which correspond to the vertices of the directed graphs of G_2 and G_2^* of Figure 2. Let us denote by α_1, α_2 and β_1, β_2 the symbols of $D_{A_{G_2}}$. They have the following algebraic relations from the relations

[\(2.3\)](#page-1-1) of operators for $A = A_{G_2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$:

$$
\alpha_1 \beta_1 = \beta_1 \alpha_1 + \beta_2 \alpha_2 = 1, \qquad \alpha_2 \beta_2 = \beta_1 \alpha_1, \qquad \beta_2 \alpha_2 \beta_2 = \beta_2,
$$

A word $\gamma = (\gamma_1, \ldots, \gamma_m)$ of $\Sigma = {\alpha_1, \alpha_2, \beta_1, \beta_2}$ is forbidden if $\gamma_1 \cdots \gamma_m = 0$. The Fibonacci-Dyck shift $D_{A_{G_2}}$ of vertex type is defined as a subshift over Σ whose forbidden words are defined in this sense.

We will compute the zeta function $\zeta_{D_{G_2}^V}(z)$ by using Corollary [3.10.](#page-16-0) Let $f_1(z)$, $f_2(z)$ be the functions $f_1^V(z)$, $f_2^V(z)$ which satisfy the following relations:

$$
f_1(z) - 1 = z(f_1(z) + f_2(z))f_1(z),
$$

\n
$$
f_2(z) - 1 = zf_1(z)f_2(z)
$$

so that the equalities

$$
f_2(z)^2 = f_1(z), \qquad z f_2(z)^3 - f_2(z) + 1 = 0
$$

hold (see [\[16,](#page-20-15) Section 7]). We then have

$$
\det(I_2 - \text{diag}[f_1(z^2), f_2(z^2)]zA_{G_2}) = \det\left(\begin{bmatrix} 1 - zf_1(z^2) & -zf_1(z^2) \\ -zf_1(z^2) & 1 \end{bmatrix}\right)
$$

= $1 - zf_1(z^2) - z^2f_1(z^2)f_2(z^2)$
= $2 - zf_1(z^2) - f_2(z^2)$.

Proposition 5.2. *The zeta function* $\zeta_{D_{G_2}^V}(z)$ *of the Fibonacci-Dyck shift of vertex type is*

$$
\zeta_{D_{G_2}^V}(z) = \frac{1}{(2\xi(z)^2 + \xi(z) - 1)^2} \tag{5.2}
$$

where $\xi(z) = \frac{2}{\sqrt{2}}$ $\frac{1}{3}\sin(\frac{1}{3}\arcsin\frac{3\sqrt{3}}{2})$ $\frac{\sqrt{3}}{2}z$) *for* $0 \le z \le \frac{2}{3\sqrt{3}}$.

Proof. By Corollary [3.10](#page-16-0) with the above discussions, we have

$$
\zeta_{D_{G_2}^V}(z) = \frac{f_1(z^2)}{(2 - z f_1(z^2) - f_2(z^2))^2}
$$

=
$$
\left(\frac{f_2(z^2)}{(2f_2(z^2) - 2z^2(f_2(z^2)^3) - z f_2(z^2)^2 - f_2(z^2))}\right)^2
$$

=
$$
\frac{1}{(1 - 2(z f_2(z^2)^2 - z f_2(z^2))^2}.
$$

By putting $\xi(z) = z f_2(z^2)$, we have

$$
\zeta_{D_{G_2}^V}(z) = \frac{1}{(2\xi(z)^2 + \xi(z) - 1)^2} \tag{5.3}
$$

 \Box

and $\xi(z)^3 - \xi(z) + z = 0$. As in [\[14,](#page-20-5) (4.10), (4.13)], we have

$$
\xi(z) = \frac{2}{\sqrt{3}} \sin(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z)
$$
 for $0 \le z \le \frac{2}{3\sqrt{3}}$.

We remark that the zeta function $\zeta_{D_{G_2}^E}(z)$ of the Fibonacci-Dyck shift of edge type is

$$
\zeta_{D_{G_2}^E}(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2} \qquad ([14, Section 7])
$$

which is different from (5.3) .

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