## ON THE OPTIMAL REGULARITY OF WEAK GEODESICS IN THE SPACE OF METRICS ON A POLARIZED MANIFOLD

ROBERT J. BERMAN

ABSTRACT. Let (X, L) be a polarized compact manifold, i.e. L is an ample line bundle over X and denote by  $\mathcal{H}$  the infinite dimensional space of all positively curved Hermitian metrics on L equipped with the Mabuchi metric. In this short note we show, using Bedford-Taylor type envelope techniques developed in the authors previous work [3], that Chen's weak geodesic connecting any two elements in  $\mathcal{H}$  are  $C^{1,1}$ -smooth, i.e. the real Hessian is bounded, for any fixed time t, thus improving the original bound on the Laplacins due to Chen. This also gives a partial generalization of Blocki's refinement of Chen's regularity result. More generally, a regularity result for complex Monge-Ampère equations over  $X \times D$ , for D a pseudconvex domain in  $\mathbb{C}^n$  is given.

### 1. INTRODUCTION

Let X be an n-dimensional compact complex manifold equipped with a Kähler form  $\omega$  and denote by  $[\omega]$  the corresponding cohomology class in  $H^2(X, \mathbb{R})$ . The space of all Kähler metrics in  $[\omega]$  may be identified with the space  $\mathcal{H}(X, \omega)$  of all Kähler potentials, modulo constants, i.e. the space of all functions u on X such that

$$\omega_u := \omega + dd^c u, \quad (dd^c := \frac{i}{2\pi} \partial \bar{\partial})$$

is positive, i.e. defines a Kähler form on X. Mabuchi introduced a natural Riemannian metric on  $\mathcal{H}(X, \omega)$  [21], where the squared norm of a tangent vector  $v \in C^{\infty}(X)$ at u is defined by

(1.1) 
$$g_{|u}(v,v) := \int_X v^2 \omega_v^r$$

The main case of geometric interest is when the cohomology class  $[\omega]$  is integral, which equivalently means that it can be realized as the first Chern class  $c_1(L)$  of an ample line bundle L over the projective algebraic manifold X. Then the space  $\mathcal{H}(X,\omega)$  may be identified with the space  $\mathcal{H}(L)$  of all positively curved metrics  $\phi$ on the line bundle L and as pointed by Donaldson [15] the space  $\mathcal{H}(L)$  may then be interpreted as the symmetric space dual of the group  $\operatorname{Ham}(X,\omega)$  of Hamiltonian diffeomorphisms of  $(X,\omega)$ . Under this (formal) correspondence the geodesics in  $\mathcal{H}(X,\omega)$  correspond to one-parameter subgroups in the (formal) complexification of  $\operatorname{Ham}(X,\omega)$  and this motivated Donaldson's conjecture concerning the existence of geodesics in  $\mathcal{H}(X,\omega)$ , connecting any two given elements.

However, Donaldson's existence problem has turned out to be quite subtle. In fact, according to the recent counter-examples in [20, 11] the existence of bona fide geodesic segments fails in general. On the other hand, there always exists a (unique) weak geodesic  $u_t$  connecting given points  $u_0$  and  $u_1$  in  $\mathcal{H}(X, \omega)$  defined as follows. First recall that, by an important observation of Semmes [23] and Donaldson [15],

after a complexification of the variable t, the geodesic equation for  $u_t$  on  $X \times [0, 1]$ may be written as the following complex Monge-Ampère equation on a domain  $M := X \times D$  in  $X \times \mathbb{C}$  for the function  $U(x, t) := u_t(x)$ :

(1.2) 
$$(\pi^* \omega + dd^c U)^{n+1} = 0.$$

As shown by Chen [9], with complements by Blocki [8], for any smoothly bounded domain D in  $\mathbb{C}$  the corresponding boundary value problem on M admits a unique solution U such that  $\pi^* \omega + dd^c U$  is a positive current with coefficients in  $L^{\infty}$ , satisfying the equation 1.2 almost everywhere. In particular, when D is an annulus in  $\mathbb C$  this construction gives rise to the notion of a weak geodesic curve  $u_t$  in the space of all functions u such that  $\omega_u$  is a positive current with coefficients in  $L^{\infty}$ (the latter regularity equivalently means that the Laplacian of u is in  $L^{\infty}$ ). In particular, by standard linear elliptic estimates, U is "almost  $C^{1,1}$ " in the sense that U is in the Hölder class  $C^{1,\alpha}$  for any  $\alpha < 1$ . As shown by Blocki [8], in the case when X admits a Kähler metric with non-negative holomorphic bisectional curvature Chen's regularity result can be improved to give that U is  $C^{1,1}$ -smooth. However, the assumption on X appearing in Blocki's result is very strong and essentially implies that X is a homogenuous manifold. In this short note we point out that, in the case when the given Kähler class  $[\omega]$  is an integral the function  $u_t$ on X is in general, for any fixed t, in  $C^{1,1}(X)$ , i.e. its first derivatives are Lipschitz continuous. More precisely, the real Hessian of  $u_t$  has bounded coefficients with a a bound which is independent of t:

**Theorem 1.1.** For any integral Kähler class  $[\omega]$  the weak geodesic  $u_t$  connecting any two points  $u_0$  and  $u_1$  in the space  $\mathcal{H}(X,\omega)$  of  $\omega$ -Kähler potentials has the property that, for any fixed t, the function  $u_t$  is in  $C^{1,1}(X)$ . More precisely, the upper bound on the sup norm on X of the real Hessian of  $u_t$  only depending on an upper bound of sup norms of the real Hessians of  $u_0$  and  $u_1$ .

This regularity result should be compared with recent results of Darvas-Lempert [11] showing that the solution  $U(x,t) := u_t(x)$  is not, in general,  $C^2$ -smooth up to the boundary of M in (more precisely  $dd^cU$  is not represented by a continuous form). However, the argument in [11], which is inspired by a similar argument in the case of M = D for a pseudoconvex domain D in  $\mathbb{C}^2$  due to Bedford-Fornaess [1], does not seem to exclude the possibility that U be  $C^2$ -smooth in the *interior* of M. Anyway, the latter scenario appears to be highly unlikely in view of the explicit counter-example of Gamelin-Sibony [17] to interior  $C^2$ -regularity for the case when D is the unit-ball in  $\mathbb{C}^2$ . Note also that, since the bounds on the real Hessian of  $u_t$  are controlled by the Hessians of  $u_0$  and  $u_1$  the previous theorem shows that  $PSH(X,\omega) \cap C^{1,1}(X)$  is closed with respect to weak geodesics. By the very recent work of Darvas [10] and Guedj [18] this the latter property equivalently means that  $PSH(X,\omega) \cap C^{1,1}(X)$  defines a geodesic subspace of the metric completion of the space  $\mathcal{H}$  equipped with the Mabuchi metric.

The starting point of the proof of Theorem 1.1 is the well-known Perron type envelope representation of the solution to the Dirichlet problem for the complex Monge-Ampère operator. The proof, which is inspired by Bedford-Taylor's approach in their seminal paper [2], proceeds by a straight-forward generalization of the technique used in [3] to establish the corresponding regularity result for certain envelopes of positively curved metrics in a line bundle  $L \to X$  (which can be viewed as solutions to a free boundary value problem for the complex Monge-Ampère equation on X). In fact, the situation here is considerably simpler than the one in [3] which covers the case when the line bundle L is merely big (the  $C^{1,1}$ -regularity then holds on the ample locus of L in X) and one of the motivations for the present note is to highlight the simplicity of the approach in [3] in the present situation (see also [22] for other generalizations of [3]). But it should be stressed that, just as in [3], the results can be generalized to more general line bundles. For example, by passing to a smooth resolution, Theorem 1.1 be generalized to show that the weak geodesic connecting any two smooth metrics with non-negative curvature current on an ample line bundle L over a singular compact normal complex variety X is  $C^{1,1}$ -smooth on the regular locus of X (for a fixed "time").

As it turns out one can formulate a general result (Theorem 2.1 below) which contains both Theorem 1.1 and the corresponding regularity result in [3]. In particular, the latter result covers the case when the domain D is the unit-disc (or more generally, the unit-ball in  $\mathbb{C}^n$ , where the following more precise regularity result holds:

**Theorem 1.2.** For any integral Kähler class  $[\omega]$  on a compact complex manifold X the solution U to the Dirichlet problem for the complex Monge-Ampère equation 1.2 with  $C^2$ -boundary data,  $\omega$ -psh along the slices  $\{t\} \times X$ , is  $C^{1,1}$ -smooth in the interior of  $X \times D$ , if D is the unit-disc in  $\mathbb{C}$ .

As pointed out by Donaldson [15] the boundary value problem appearing in the previous theorem can be viewed as an infinite dimensional analog of a standard boundary value problem for holomorphic discs in the complexification of a compact Lie group G or more precisely the classical factorization theorem for loops in G (recall that the role of G in the present infinite dimensional setting is played by the group  $\operatorname{Ham}(X,\omega)$  of Hamiltonian diffeomorphisms). As shown by Donaldson [16] the solution U is in general not smooth and Donaldson raised the problem of studying the singularities of Chen's weak solution; the paper can thus be seen as one step in this direction.

One potentially useful consequence of the regularity results in Theorems 1.1, 1.2 is that, for a fixed "time" t the differential of  $u_t$  (which geometrically represents the connection one form of the corresponding metric on the line bundle L) is Lipschitz continuous and in particular differentiable on X - E, where the exceptional set E is a null set for the Lebesgue measure. For example, it then follows from the results in [3] that the corresponding scaled Bergman kernel  $B_k(x,x)/k^n$ , attached to high tensor powers  $L^{\otimes k}$ , converges when  $k \to \infty$  point-wise on X - E to the density of  $\omega_{n_{i}}^{n}$ . By a circle of ideas going back to Yau such Bergman kernels can be used to approximate differential geometric objects in Kähler geometry. Accordingly, the precise  $C^{1,1}$ -regularity established in the present paper will hopefully find applications in Kähler geometry in the future. In fact, one of the initial motivations for writing the present note came from a very recent joint work with Bo Berndtsson [5] where Bergman kernel asymptotics are used to establish the convexity of Mabuchi's K-energy along weak geodesics and where the precise  $C^{1,1}$ -regularity was needed at an early stage of the work. Eventually it turned that Chen's regularity, or more precisely the fact that  $u_t$  has a bounded Laplacian, is sufficient to get the point-wise convergence of  $B_k/k^n$  for some subsequence away from some (non-explicit) null set E (see Theorem 2.1 in [5]) which is enough to run the approximation argument.

But with a bit of imagination one could envisage future situations where the more precise  $C^{1,1}$ -regularity would be needed.

Let us finally point out that in a very recent article Darvas and Rubinstein [12] consider psh-envelopes of functions of the form  $f = \min\{f_1, f_2, ..., f_m\}$ . Such envelopes appear in the Legendre transform type formula for weak geodesics introduced in [12] which has remarkable applications to the study of the completion of the Mabuchi metric space [10]. The same technique from [3] we describe here implies  $C^{1,1}$  regularity of such envelopes in the case the Kähler class is integral (see the first point in Section 2.3). In [12] the authors give a different proof of this result (still using [3]) and also prove a Laplacian bound in the case of a general Kähler class.

Acknowledgments. I am grateful to Jean-Pierre Demailly for many stimulating and illuminating discussions on the topic of the present note. Also thanks to Slawomir Dinew for comments and Yanir Rubinstein for discussions related to [12]. This work has been supported by grants from the Swedish and European Research Councils and the Wallenberg Foundation.

# 2. $C^{1,1}$ -regularity of solutions to complex Monge-Ampère equations over products

2.1. Notation: quasi-psh functions vs metrics on line bundles. Here we will briefly recall the notion for (quasi-) psh functions and metrics on line bundles that we will use. Let  $(X, \omega_0)$  be a compact complex manifold of dimension n equipped with a fixed Kähler form  $\omega_0$ , i.e. a smooth real positive closed (1, 1)-form on X. Denote by  $PSH(X, \omega_0)$  be the space of all  $\omega_0$ -psh functions u on X, i.e.  $u \in L^1(X)$ and u is (strongly) upper-semicontinuos (usc) and

$$\omega_u := \omega_0 + \frac{i}{2\pi} \partial \bar{\partial} u := \omega_0 + dd^c u \ge 0,$$

holds in the sense of currents. We will write  $\mathcal{H}(X, \omega_0)$  for the interior of  $PSH(X, \omega_0) \cap \mathcal{C}^{\infty}(X)$ , i.e. the space of all Kähler potentials (w.r.t  $\omega_0$ ). In the *integral case*, i.e. when  $[\omega] = c_1(L)$  for a holomorphic line bundle  $L \to X$ , the space  $PSH(X, \omega_0)$  may be identified with the space  $\mathcal{H}_L$  of (singular) Hermitian metrics on L with positive curvature current. We will use additive notion for metrics on L, i.e. we identify an Hermitian metric  $\|\cdot\|$  on L with its "weight"  $\phi$ . Given a covering  $(U_i, s_i)$  of X with local trivializing sections  $s_i$  of  $L_{|U_i}$  the object  $\phi$  is defined by the collection of open functions  $\phi_{|U_i}$  defined by

$$|s_i||^2 = e^{-\phi_{|U_i|}}.$$

The (normalized) curvature  $\omega$  of the metric  $\|\cdot\|$  is the globally well-defined (1, 1)-current defined by the following local expression:

 $\omega = dd^c \phi_{|U_i}.$ 

The identification between  $\mathcal{H}_L$  and  $PSH(X, \omega_0)$  referred to above is obtained by fixing  $\phi_0$  and identifying  $\phi$  with the function  $u := \phi - \phi_0$ , so that  $dd^c \phi = \omega_u$ .

2.2. The  $C^{1,1}$ -regularity of weak geodesics. Let  $(X, \omega)$  be a compact Kähler manifold and D a domain in  $\mathbb{C}^n$ . Set  $M := X \times D$  and denote by  $\pi$  the natural projection from M to X. Given a continuous function f on  $\partial M (= X \times \partial D)$  we define the following point-wise Perron type upper envelope on the interior of M:

(2.1) 
$$U := P(f) := \sup\{V : V \in \mathcal{F}\},\$$

where  $\mathcal{F}$  denotes the set of all  $V \in PSH(M, \pi^*\omega)$  such that  $V_{|\partial M} \leq f$  on the boundary  $\partial M$  (in a point-wise limiting sense). In the case when D is a smoothly bounded pseudoconvex domain and f is  $\omega$ -psh in the "X-directions", i.e.  $f(\cdot, t) \in$  $PSH(X, \omega)$  it was shown in [4] that P(f) is continuous up to the boundary of M and U then coincides with the unique solution of the Dirichlet problem for the corresponding complex Monge-Ampère operator with boundary data f, in the weak sense of pluripotential theory [2]. Here we will establish the following higher order regularity result for the envelope P(f):

**Theorem 2.1.** Let  $(X, \omega_0)$  be an n-dimensional integral compact Kähler manifold manifold and D a bounded domain in  $\mathbb{C}^m$  and set  $M := X \times D$ . Then, given fa function on  $\partial M$  such that  $f(\cdot, \tau)$  is in  $C^{1,1}(X)$ , with a uniform bound on the corresponding real Hessians, the function  $u_{\tau} := P(f)_{|X \times \{\tau\}}$  is in  $C^{1,1}(X)$  and satisfies

$$\sup_{X} |\nabla^2 u_{\tau}|_{\omega_0} \le C,$$

where  $|\nabla^2 u_{\tau}|_{\omega_0}$  denotes the point-wise norm of the real Hessian matrix of the function  $u_{\tau}$  on X defined with respect to the Kähler metric  $\omega_0$ . Moreover, the constant C only depends on an upper bound on the sup norm of the real Hessians of  $f_{\tau}$  for  $\tau \in \partial D$ . In the case when D is the unit-ball the function  $U(x, \tau)$  is in  $C_{loc}^{1,1}$  in the interior of M.

2.2.1. Proof of Theorem 2.1. In the course of the proof of the theorem we will identify an  $\pi^*\omega$ -psh function U on M with a positively curved metric  $\Phi$  on the line bundle  $\pi^*L \to M$ . The case when D is a point is the content of Theorem 1.1 in [3] and as will be next explained the general case can be proved in completely analogous manner. First recall that the argument in [3] is modelled on Bedford-Taylor's proof of the case when X is a point and D is the unit-ball [2] (see also Demailly's simplifications [13]). The latter proof uses that B is a homogenuous domain. In order to explain the idea of the proof of Theorem 2.1 first consider the case when (X, L) is homogenuous, i.e. the group Aut (X, L) of all biholomorphic automorphisms of X lifting to L acts transitively on X. In particular, there exists a family  $F_{\lambda}$  in Aut (X, L) parametrized by  $\lambda \in \mathbb{C}^n$  such that, for any fixed point  $x \in X$ , the map  $\lambda \mapsto F_{\lambda}(x)$  is a biholomorphism (onto its image) from a sufficiently small ball centered at the origin in  $\mathbb{C}^n$ . Given a metric  $\phi$  on L we set  $\phi^{\lambda} := F_{\lambda}^*\phi$ . Similarly, given a metric  $\Phi(=\Phi(x, \tau))$  on  $\pi^*L$  we set

$$\Phi^{\lambda} := (F_{\lambda} \times I)^* \Phi.$$

Since  $F_{\lambda}$  is holomorphic the metric  $\Phi^{\lambda}$  has positive curvature iff  $\Phi$  has positive curvature. Now to first prove a Lipschitz bound on  $P\Phi_f$ , where  $\Phi_f$  is the metric on  $L \to \partial M$  corresponding to the given boundary data f, we take any candidate  $\Psi$  for the sup defining  $P\Phi_f$  and note that, on  $\partial M$ , i.e. for  $\tau \in \partial D$ :

(2.2) 
$$\Psi^{\lambda} \le \Phi_f^{\lambda} \le \Phi_f + C_1 |\lambda|,$$

where  $C_1$  only depends on the Lipschitz bounds in the "X-direction" of the given function f on  $X \times \partial D$ . But this means that  $\Psi^{\lambda} - C_1|\lambda|$  is also a candidate for sup

defining  $P\Phi_f$  and hence  $\Psi^{\lambda} - C_1|\lambda| \leq P\Phi_f$  on all of  $X \times D$ . Finally, taking the sup over all candidates  $\Psi$  gives, on  $X \times D$ , that

$$(P\Phi_f)^{\lambda} \le (P\Phi_f) + C_1|\lambda|$$

Since this holds for any  $\lambda$  and in particular for  $-\lambda$  this concludes the proof of the desired Lipschitz bound on  $P\Phi_f$ . Next, to prove the bound on the real Hessian one first replaces  $\Psi^{\lambda}$  in the previous argument with  $\frac{1}{2}(\Psi^{\lambda} + \Psi^{-\lambda})$  and deduces, precisely as before, that

$$\frac{1}{2}\left( (P\Phi_f)^{\lambda} + (P\Phi_f)^{-\lambda} \right) \le (P\Phi_f) + C_2 |\lambda|^2,$$

where now  $C_2$  depends on the upper bound in the "X-direction" of the real Hessian of the function f on  $X \times \partial D$ . The previous inequality implies an upper bound on the real Hessians of the local regularizations  $\Psi_{\epsilon}$  of  $P\Phi_f$  defined by local convolutions. Moreover, since  $dd^c\Psi_{\epsilon} \geq 0$  it follows from basic linear algebra that a lower bound on the real Hessians also holds. Hence, letting  $\epsilon \to 0$  shows that  $P\Phi_f$  is in  $C_{loc}^{1,1}$ in the "X-direction" with a uniform upper bound on the real Hessians (compare [2, 13]).

Of course, a general polarized manifold (X, L) may not admit even a single (non-trivial) holomorphic vector field. But as shown in [3] this problem can be circumvented by passing to the total space Y of the dual line bundle  $L^* \to X$ , which does admit an abundance of holomorphic vector fields. The starting point is the standard correspondence between positively curved metrics  $\phi$  on L and psh "log-homogenuous" functions  $\chi$  on Y induced by the following formula:

$$\chi(z, w) = \phi(z) + \log |w|^2,$$

where z denotes a vector of local coordinates on X and (z, w) denote the corresponding local coordinates on Y induced by a local trivialization of L. Accordingly, the envelope  $P\Phi_f$  on X corresponds to an envelope construction on Y, defined w.r.t the class of psh log-homogenous functions on Y. Fixing a metric  $\phi_0$  on L we denote by K the compact set in Y defined by the corresponding unit-circle bundle. By homogenity any function  $\chi$  as above is uniquely determined by its restriction to K. Now, for any fixed point  $y_0$  in K there exists an (n+1)-tuple of global holomorphic vector fields  $V_i$  on Y defining a frame in a neighborhood of  $y_0$ :

**Lemma 2.2.** Given any point  $y_0$  in the space  $Y^*$  defined as the complement of the zero-section in the total space of  $L^*$  there exist holomorphic vector fields  $V_1, ..., V_{n+1}$  on  $Y^*$  which are linearly independent close to  $y_0$ .

Proof. This follows from Lemma 3.7 in [3]. For completeness and since we do not need the explicit estimates furnished by Lemma 3.7 in [3] we give a short direct proof here. Set  $Z := \mathbb{P}(L^* \oplus \mathbb{C})$ , viewed as the fiber-wise  $\mathbb{P}^1$ -compactification of Y. Denote by  $\pi$  the natural projection from Z to X and by  $\mathcal{O}(1)$  the relative (fiberwise) hyper plane line bundle on Z. As is well-known, for any sufficiently positive integer the line bundle  $L_m := (\pi^*L) \otimes \mathcal{O}(1)^{\otimes m}$  on Z is ample and holomorphically trivial on  $Y^*$ . As a consequence, the rank n + 1- vector bundle  $E := TZ \otimes L_m^{\otimes k}$  is globally generated for k sufficiently large, i.e. any point  $z_0$  in Z there exists global holomorphic sections  $S_1, ..., S_{n+1}$  spanning  $E_{|z_0}$ . Since,  $L_m$  is holomorphically trivial on  $Y^* \subset Z$  this concludes the proof. Now, integrating the (short-time) flow of the holomorphic vector field  $V(\lambda) := \sum \lambda_i V_i$  gives a family of holomorphic maps  $F_{\lambda}(y)$  defined for  $y \in K$  and  $\lambda$  in a sufficiently small ball B centered at the origin in  $\mathbb{C}^{n+1}$  such that  $\lambda \mapsto F_{\lambda}(y_0)$  is a biholomorphism. However, the problem is that the corresponding function  $\chi^{\lambda} := F_{\lambda}^* \chi$  is only defined in a neighborhood of K in Y (and not log-homogenuous). But this issue can be bypassed by replacing  $\chi^{\lambda}$  with a new function that we will denote by  $T(\chi^{\lambda})$ , where T(f), for f a function on K, is obtained by first taking the sup of f over the orbits of the standard  $S^{1-}$  action on Y to get an  $S^{1-}$ -invariant function  $g := \hat{f}$  and then replacing g with its log-homogenuous extension  $\tilde{g}$ , i.e.

$$T(f) := \widetilde{\left(\widehat{\chi^{\lambda}}\right)}.$$

The following lemma follows from basic properties of plurisubharmonic functions (see [3] for a proof):

**Lemma 2.3.** If f is the restriction to the unit-circle bundle  $K \subset Y$  of a psh function, then T(f) is a psh log homogenuous function on Y

Now performing the previous constructions for any fixed  $\tau \in D$  and identifying a candidate  $\Psi$  with a function  $\chi$  on  $Y \times D$ , as above, gives

(2.3) 
$$\chi^{\lambda}(y_0) \le \widehat{\chi^{\lambda}}(y_0) = \left(\widehat{\chi^{\lambda}}\right)(y_0) := T(\chi^{\lambda})(y_0).$$

But, by construction, for  $\tau \in \partial D$  we have  $T(\chi^{\lambda}) \leq T(\chi^{\lambda}_{\Phi_f})$  and since  $f_{\tau}$  is assumed Lipschitz for  $\tau \in \partial D$  we also have that

$$T(\chi_{\Phi_f}^{\lambda}) \le T(\chi_{\Phi_f}) + C_1|\lambda| = \chi_{\Phi_f} + C_1|\lambda|.$$

But this means that  $T(\chi^{\lambda}) - C_1|\lambda|$  is a candidate for the sup in question and hence bounded from above by  $\chi_{P\Phi_f}$ , which combined with the inequality 2.3 gives

$$\chi^{\lambda}(y_0) - C_1|\lambda| \le \chi_{\Phi_f}(y_0).$$

Taking the sup over all candidates  $\chi$  and replacing  $\lambda$  with  $-\lambda$  hence gives the desired Lipschitz bound on  $P\Phi_f$  at the given point  $y_0$  and hence, by compactness, for any point in K. The estimate on the Hessian then proceeds precisely as above.

Finally, in the case when B is the unit-ball one can exploit that B is homogenuous (under the action of the Möbius group), replacing the holomorphic maps  $(x, \tau) \mapsto (F_{\lambda}(x), \tau)$  used above with  $(x, \tau) \mapsto (F_{\lambda}(x), G_a(\tau))$ , where  $G_a$  is a suitable family of Möbius transformations (the case when X is point is precisely the original situation in [2]). Then the proof proceeds precisely as before.

### 2.3. Further remarks.

- The proof of the previous theorem also applies in the more general situation where f may be written as  $f = \inf_{\alpha \in A} f_{\alpha}$  for a given family of functions  $f_{\alpha}$ , as long as the Hessians of  $f_{\alpha}(\tau, \cdot)$  are uniformly bounded on X (by a constant C independent of  $\tau$  and  $\alpha$ )) and similarly for the Lipschitz bound. Indeed, then equation 2.2 holds with f replaced by  $f_{\alpha}$  for any  $\alpha \in A$  with the same constant C. For D equal to a point this result has been obtained in [12] using a different proof.
- As shown in [4] (using a different pluripotential method), in the case of a general, possibly non-integral, Kähler class  $[\omega]$  a bounded Laplacian in the X-directions of the boundary data f results in a bounded Laplacian

of the corresponding envelope. In the case of geodesics this result has also recently been obtained in [19] by refining Chen's proof.

• By the proof of the previous theorem, the Lipschitz norm  $||u_t||_{C^{0,1}(X)}$  of a weak geodesic  $u_t$  only depends on an upper bound on the Lipschitz norms of  $u_0$  and  $u_1$ . Since the Lipschitz norm in the *t*-variable is controlled by the  $C^0$ -norm of  $u_0 - u_1$  [6] it follows that the Lipschitz norm  $||U||_{C^{0,1}(X \times A)}$  of the corresponding solution U on  $X \times A$  is controlled by the Lipschitz norms of  $u_0$  and  $u_1$  and the  $C^0$ -norm of  $u_0 - u_1$ . For a general Kähler class this result also follows from Blocki's gradient estimate [7, 8].

#### References

- Bedford, E; Fornæss, J-E: Counterexamples to regularity for the complex Monge-Ampère equation. Invent. Math. 50 (1978/79), no. 2, 129–134
- [2] Bedford, Eric; Taylor, B. A. The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. 37 (1976), no. 1, 1–44.
- [3] Berman, R.J: Bergman kernels and equilibrium measures for line bundles over projective manifolds. The Amer. J. of Math., Vol. 131, Nr. 5, October 2009
- [4] Berman, R.J; Demailly, J-P: Regularity of plurisubharmonic upper envelopes in big cohomology classes. In "Perspectives in Analysis, Geometry, and Topology", Springer-Verlag,
- [5] Berman, R.J; Berndtsson, B: Convexity of the K-energy on the space of Kähler metrics. arXiv:1405.0401
- [6] Berndtsson, B: A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. arXiv:1303.4975
- [7] Blocki, Z: A gradient estimate in the Calabi-Yau theorem , Math.Ann. 344 (2009), 317-327
- [8] Blocki, Z: On geodesics in the space of Kähler metrics, Proceedings of the "Conference in Geometry" dedicated to Shing-Tung Yau (Warsaw, April 2009), in "Advances in Geometric Analysis", ed. S. Janeczko, J. Li, D. Phong, Advanced Lectures in Mathematics 21, pp. 3-20, International Press, 2012
- [9] Chen, X.X: The space of K"ahler metrics, J. Diff. Geom. 56 (2000), 189-234
- [10] Darvas, T: Envelopes, Singularity Types and Geodesics in E(X,\omega). Preprint (2014).
- [11] Darvas, T; Lempert, L: Weak geodesics in the space of Kähler metrics. Math. Res. Lett. 19 (2012), no. 5, 1127–1135.
- [12] Darvas, T; Rubinstein, Y.A: Kiselman's principle, the Dirichlet problem for the Monge-Ampere equation, and rooftop obstacle problems. Preprint in 2014.
- [13] Demailly, J-P: Potential Theory in Several Complex Variables. Notes from the Trenot conference in 1992. http://www-fourier.ujf-grenoble.fr/~demailly/documents.html
- [14] Donaldson, S. K: Remarks on gauge theory, complex geometry and 4-manifold topology. Fields Medallists' lectures, 384–403, World Sci. Ser. 20th Century Math., 5, World Sci. Publ., River Edge, NJ, 1997.
- [15] Donaldson, S. K: Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In Northern California Symplectic Geometry Seminar, volume 196 of Amer. Math. Soc. Transl. Ser. 2, pages 13–33. Amer. Math. Soc., Providence, RI, 1999
- [16] Donaldson, S. K: Holomorphic discs and the complex Monge-Ampère equation. J. Symplectic Geom. 1 (2002), no. 2, 171–196.
- [17] Gamelin, W; Sibony, N: Subharmonicity for uniform algebras, J. of Funct. Analysis 35(1980), 64–108
- [18] Guedj, V: The metric completion of the Riemannian space of Kähler metrics. arXiv:1401.7857
- [19] He, W: On the space of Kahler potentials. arXiv:1208.1021
- [20] Lempert,L; Vivas, L: Geodesics in the space of Kähler metrics. Duke Math. J. Volume 162, Number 7 (2013), 1369-138
- [21] Mabuchi, T: Some symplectic geometry on compact K"a hler manifolds. I. Osaka J. Math., 24(2):227–252, 1987.
- [22] Ross, J; Witt Nystrom, D: Envelopes of positive metrics with prescribed singularities
- [23] Semmes, S: Complex Monge-Amp'ere and symplectic ma nifolds. Amer. J. Math. 114(3):495–550, 1992