# MULTIPLE ZETA-STAR VALUES AND MULTIPLE INTEGRALS

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ABSTRACT. We prove a kind of integral expressions for finite multiple harmonic sums and multiple zeta-star values. Moreover, we introduce a class of multiple integrals, associated with some combinatorial data (called 2-labeled posets). This class includes both multiple zeta and zeta-star values of Euler-Zagier type, and also several other types of multiple zeta values. We show that these integrals can be used to obtain some relations among such zeta values quite transparently.

## 1. INTEGRAL EXPRESSION OF FINITE MULTIPLE HARMONIC SUMS

We begin with the finite multiple harmonic sums

$$s_{\mathbf{k}}(N) = \sum_{N = m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ is an $n$-tuple of positive integers and $N$ is a positive integer. When $k_1 \geq 2$, by definition, their sum gives the multiple zeta-star values (MZSVs for short):

$$(1.1) \qquad \zeta^\star(\mathbf{k}) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} = \sum_{N=1}^{\infty} s_{\mathbf{k}}(N).$$

One of the basic properties of these finite multiple sums is the following relation, called the duality:

**Theorem 1.1** ([H, K]). *For any index* $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^n$ *and* $N \geq 0$, *we have*

$$(1.2) \qquad \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} s_{\mathbf{k}}(i+1) = s_{\mathbf{k}^*}(N).$$

*Here* $\mathbf{k}^*$ *denotes the 'transpose' of* $\mathbf{k}$ *(see below).*

For an index $\mathbf{k} = (k_1, \ldots, k_n)$, we set

$$|\mathbf{k}| := k_1 + \cdots + k_n, \quad A(\mathbf{k}) := \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{n-1}\}.$$
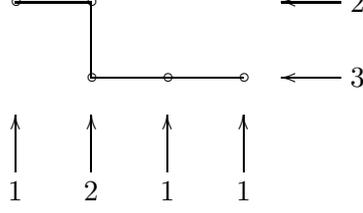
Then the transpose $\mathbf{k}^*$ is the index determined by the property

$$|\mathbf{k}| = |\mathbf{k}^*|, \quad \{1, \ldots, |\mathbf{k}| - 1\} = A(\mathbf{k}) \amalg A(\mathbf{k}^*).$$

For example, the transpose of $(2,3)$ is $(1,2,1,1)$. It can be illustrated by the following picture:



The identity (1.2) is somewhat analogous to the well-known duality

$$(1.3) \quad \zeta(a_1 + 1, \underbrace{1, \ldots, 1}_{b_1-1}, \ldots, a_s + 1, \underbrace{1, \ldots, 1}_{b_s-1}) = \zeta(b_s + 1, \underbrace{1, \ldots, 1}_{a_s-1}, \ldots, b_1 + 1, \underbrace{1, \ldots, 1}_{a_1-1})$$

of the multiple zeta values (MZVs)

$$\zeta(\mathbf{k}) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Since the latter duality follows immediately from the iterated integral expression

$$\zeta(a_1 + 1, \underbrace{1, \ldots, 1}_{b_1-1}, \ldots, a_s + 1, \underbrace{1, \ldots, 1}_{b_s-1})$$

$$(1.4) \quad = \int_0^1 \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{a_1} \circ \underbrace{\frac{dt}{1-t} \circ \cdots \circ \frac{dt}{1-t}}_{b_1} \circ \cdots \circ \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{a_s} \circ \underbrace{\frac{dt}{1-t} \circ \cdots \circ \frac{dt}{1-t}}_{b_s},$$

it is natural to ask for a similar integral expression of finite multiple sum $s_{\mathbf{k}}(N)$ from which (1.2) follows. Here is an answer:

**Theorem 1.2.** *Let* $\mathbf{k} = (k_1, \ldots, k_n)$ *be an index, and put* $k = |\mathbf{k}| = k_1 + \cdots + k_n$. *Moreover, define*

$$J(\mathbf{k}) = \{0, k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{n-1}\} = A(\mathbf{k}) \cup \{0\},$$

$$\Delta(\mathbf{k}) = \left\{ (t_1, \ldots, t_k) \in [0,1]^k \, \middle| \, \begin{array}{l} t_j < t_{j+1} \text{ if } j \notin J(\mathbf{k}), \\ t_j > t_{j+1} \text{ if } j \in J(\mathbf{k}) \end{array} \right\}.$$

*Then, for* $N \geq 1$, *we have*

$$(1.5) \qquad\qquad s_{\mathbf{k}}(N) = \int_{\Delta(\mathbf{k})} t_1^{N-1} dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k),$$

*where* $\omega_0(t) = \frac{dt}{t}$, $\omega_1(t) = \frac{dt}{1-t}$ *and*

$$(1.6) \qquad\qquad \delta(j) = \begin{cases} 0 & (j - 1 \notin J(\mathbf{k})), \\ 1 & (j - 1 \in J(\mathbf{k})). \end{cases}$$

*Remark.* We include 0 in the set $J(\mathbf{k})$ to set $\delta(1) = 1$ in (1.6), though this value is not used in the above theorem. We need it in Corollary 1.3.

*Proof.* We consider the case $\mathbf{k} = (2,1,2)$ as an example, and then the general pattern will be understood. In this case, we should prove

$$s_{(2,1,2)}(N) = \int_{t_1 < t_2 > t_3 > t_4 < t_5} t_1^{N-1} dt_1 \frac{dt_2}{t_2} \frac{dt_3}{1-t_3} \frac{dt_4}{1-t_4} \frac{dt_5}{t_5}.$$

(Here, implicitly, the inequalities $0 \le t_i \le 1$ are assumed.) The right-hand side is computed by repeating single integrals, i.e.,

$$\int_0^{t_2} t_1^{N-1} dt_1 = \frac{t_2^N}{N},$$

$$\frac{1}{N} \int_{t_3}^1 t_2^N \frac{dt_2}{t_2} = \frac{1-t_3^N}{N^2},$$

$$\frac{1}{N^2} \int_{t_4}^1 (1-t_3^N) \frac{dt_3}{1-t_3} = \sum_{N \ge m \ge 1} \frac{1-t_4^m}{N^2 m},$$

$$\sum_{N \ge m \ge 1} \frac{1}{N^2 m} \int_0^{t_5} (1-t_4^m) \frac{dt_4}{1-t_4} = \sum_{N \ge m \ge l \ge 1} \frac{t_5^l}{N^2 ml},$$

$$\sum_{N \ge m \ge l \ge 1} \frac{1}{N^2 ml} \int_0^1 t_5^l \frac{dt_5}{t_5} = \sum_{N \ge m \ge l \ge 1} \frac{1}{N^2 ml^2}.$$

The last sum is exactly $s_{(2,1,2)}(N)$. $\qquad\square$

To deduce Theorem 1.1 from Theorem 1.2, note that there is a bijection

(1.7) $$\Delta(\mathbf{k}) \ni (t_1, \dots, t_k) \longmapsto (1-t_1, \dots, 1-t_k) \in \Delta(\mathbf{k}^*)$$

and that the map $\delta^*$ associated with $\mathbf{k}^*$ as in (1.6) satisfies $\delta^*(j) = 1 - \delta(j)$ for $j = 2, \dots, k$. Hence, by changing of the integral variables $s_j = 1 - t_j$, we obtain

$$\begin{aligned}
s_{\mathbf{k}^*}(N) &= \int_{\Delta(\mathbf{k}^*)} s_1^{N-1} ds_1 \, \omega_{\delta^*(2)}(s_2) \cdots \omega_{\delta^*(k)}(s_k) \\
&= \int_{\Delta(\mathbf{k})} (1-t_1)^{N-1} dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k) \\
&= \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} \int_{\Delta(\mathbf{k})} t_1^i dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k) \\
&= \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} s_{\mathbf{k}}(i+1).
\end{aligned}$$

By (1.1) and (1.5), we also obtain an integral expression of MZSVs.

**Corollary 1.3.** *In the same notation as in Theorem 1.2, assume $k_1 \ge 2$. Then*

(1.8) $$\zeta^\star(\mathbf{k}) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k)}(t_k).$$

**Example 1.4.** For $\mathbf{k} = (2,1)$, we have

(1.9) $$\zeta^\star(2,1) = \int_{t_1 < t_2 > t_3} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{1-t_3}.$$

This integral can be divided into two parts:

$$\int_{t_1<t_2>t_3} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{1-t_3} = \left( \int_{t_1<t_3<t_2} + \int_{t_3<t_1<t_2} \right) \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{1-t_3}.$$

By the iterated integral expression (1.4) of MZVs, the right-hand side is equal to $\zeta(2,1)+\zeta(2,1)$. Therefore, from the integral expression (1.9), one obtains

$$\zeta^\star(2,1) = 2\zeta(2,1).$$

Note that this is different from the relation

$$\zeta^\star(2,1) = \zeta(2,1) + \zeta(3)$$

obtained from the series expressions. By comparing these two relations, one proves Euler's famous relation $\zeta(2,1) = \zeta(3)$.

More generally, from the integral and series expressions of $\zeta^\star(k-1,1)$, one can show the sum formula for double zeta values

$$\zeta(k-1,1) + \zeta(k-2,2) + \cdots + \zeta(2,k-2) = \zeta(k) \quad (k \geq 3)$$

in a similar manner.

## 2. MULTIPLE INTEGRALS ASSOCIATED WITH 2-LABELED FINITE POSETS

Now we define a class of integrals which includes both MZVs (1.4) and MZSVs (1.8). Recall that a finite poset is a finite set endowed with a partial order. In the following, we omit the word 'finite' since we only consider finite posets.

**Definition 2.1.**   (1) A *2-labeled poset* is a pair $X = (X, \delta_X)$ consisting of a poset $X$ and a map $\delta_X \colon X \to \{0,1\}$, called the *labeling map*. The *weight*, denoted by $|X|$, is the number of elements of the underlying set $X$, and the *depth*, denoted by $\mathrm{dep}(X)$, is the number of $x \in X$ such that $\delta(x) = 1$.
   (2) A 2-labeled poset $X$ is said *admissible* if $\delta_X(x) = 1$ for all minimal $x \in X$ and $\delta_X(x) = 0$ for all maximal $x \in X$.
   (3) For any poset $X$, we put

$$\Delta(X) := \{ (t_x)_{x \in X} \in [0,1]^X \mid t_x < t_y \text{ if } x < y \}.$$

   (4) For an admissible 2-labeled poset $X$, we define the associated integral by

$$(2.1) \qquad\qquad I(X) := \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta_X(x)}(t_x).$$

Here $\omega_0(t) = \frac{dt}{t}$ and $\omega_1(t) = \frac{dt}{1-t}$ are the same notation as in Theorem 1.2.

*Remark.* For the empty 2-labeled poset, denoted $\varnothing$, we put $I(\varnothing) = 1$. This is compatible with the usual definition $\zeta(\varnothing) = \zeta^\star(\varnothing) = 1$, where $\varnothing$ denotes the index of length 0.

We use Hasse diagrams to indicate 2-labeled posets, with vertices $\circ$ and $\bullet$ corresponding to $\delta(x) = 0$ and 1, respectively. For example,

$$X = \{1 < 2 < 3 < 4 < 5\} \text{ and } \big(\delta(1), \ldots, \delta(5)\big) = (1, 0, 1, 0, 0)$$

is represented as the diagram

This 2-labeled poset is admissible, and we have

$$I(X) = \int_{t_1 < t_2 < t_3 < t_4 < t_5} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{t_4} \frac{dt_5}{t_5} = \zeta(3, 2).$$

In general, the iterated integral expression of a MZV is equal to the integral $I(X)$ associated with an admissible 2-labeled *totally ordered* set $X$, and the weight and the depth of the MZV coincide with those of $X$.

Another example: Corollary 1.3 for $\mathbf{k} = (2, 3)$ gives

$$\zeta^\star(2, 3) = I \left( \diagup\!\!\!\diagdown \right).$$

Here we collect some basic constructions on 2-labeled posets, and relations between the associated integrals.

**Definition 2.2.**    (1) For 2-labeled posets $X$ and $Y$, one can naturally define their direct sum $X \amalg Y$: Its underlying poset is the direct sum of finite sets $X$ and $Y$ endowed with the partial order

$$x \leq y \text{ in } X \amalg Y \iff x, y \in X \text{ and } x \leq y \text{ in } X \text{ or}$$
$$x, y \in Y \text{ and } x \leq y \text{ in } Y.$$

The map $\delta_{X \amalg Y} \colon X \amalg Y \to \{0, 1\}$ is the direct sum of the maps $\delta_X \colon X \to \{0, 1\}$ and $\delta_Y \colon Y \to \{0, 1\}$.

(2) Let $X = (X, \leq)$ be a poset, and $a, b \in X$ not comparable, i.e., neither $a \leq b$ nor $b \leq a$ hold. Then we denote by $X_a^b$ the poset with the same underlying set $X$ and endowed with the order $\leq_a^b$ defined by

$$x \leq_a^b y \iff x \leq y, \text{ or } x \leq a \text{ and } b \leq y.$$

We call $X_a^b$ the refinement of $X$ obtained by imposing $a < b$. If $X$ is a 2-labeled poset, then $X_a^b$ also becomes a 2-labeled poset with the same labeling map.

(3) For a 2-labeled poset $X$, we define its *transpose* $X^*$ as the 2-labeled poset consisting of the same underlying set as $X$ endowed with the reversed order (i.e. $x \leq y$ in $X^*$ if and only if $y \leq x$ in $X$), and the labeling map $\delta_{X^*}(x) = 1 - \delta_X(x)$.

**Proposition 2.3.**    (1) *If $X$ and $Y$ are admissible 2-labeled posets, then $X \amalg Y$ is admissible and*

$$(2.2) \qquad\qquad I(X \amalg Y) = I(X) \cdot I(Y).$$

(2) *If $X$ is an admissible 2-labeled poset, and $a$ and $b \in X$ are not comparable, then both $X_a^b$ and $X_b^a$ are admissible and*

$$(2.3) \qquad\qquad I(X) = I(X_a^b) + I(X_b^a).$$

(3) *If $X$ is an admissible 2-labeled poset, then the transpose $X^*$ is admissible and*

$$(2.4) \qquad\qquad I(X^*) = I(X).$$

*Proof.* All assertions are easily verified. Note that (2.4) is shown by making the change of variables

$$\Delta(X) \ni (t_x) \longmapsto (1 - t_x) \in \Delta(X^*),$$

which is a generalization of (1.7).                                                                    $\square$

*Remark.* The shuffle relation for the MZVs can be derived from the identities (2.2) and (2.3) (see also the remark after Corollary 2.4). On the other hand, the identity (2.4) is a natural generalization of the duality (1.3) for the MZVs.

**Corollary 2.4.** *For any 2-labeled poset $X$, the integral $I(X)$ can be expressed as the sum of a finite number of MZVs of weight $|X|$ and depth $\mathrm{dep}(X)$.*

*Proof.* By using (2.3) several times, we express $I(X)$ as a sum of the integrals associated with 2-labeled totally ordered sets, i.e., the integral expressions of MZVs. Each of these 2-labeled totally ordered sets consists of the same underlying set and labeling map as $X$ and a total order extending the partial order of $X$. In particular, these have the same weight and depth as $X$.                                                        $\square$

*Remark.* There is an algebraic formalism for the integrals $I(X)$, similar to the well-known pair of the shuffle algebra $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$ and the homomorphism $Z \colon \mathfrak{H}^0 = \mathbb{Q} \oplus x\mathfrak{H}y \to \mathbb{R}$.

We write $\mathfrak{P}$ for the $\mathbb{Q}$-vector space generated by all isomorphism classes of 2-labeled posets, and define a product on it by $[X] \cdot [Y] = [X \amalg Y]$. Then, the subspace $\mathfrak{P}^0$ generated by admissible 2-labeled posets is a subalgebra, and the map $I \colon \mathfrak{P}^0 \to \mathbb{R}$, defined by linearity, is indeed a $\mathbb{Q}$-algebra homomorphism.

In fact, there exists a surjective $\mathbb{Q}$-algebra homomorphism $\rho \colon \mathfrak{P} \to \mathfrak{H}$ satisfying $\rho(\mathfrak{P}^0) = \mathfrak{H}^0$ and $Z \circ \rho = I$. This is defined as the unique homomorphism whose kernel is generated by $[X] - [X_a^b] - [X_b^a]$ and which sends totally ordered $[X]$ to a monomial in $\mathfrak{H}$ encoding $\delta_X$ appropriately.

## 3. OTHER EXAMPLES

In this section, we consider some values representable by the integrals $I(X)$, other than MZVs and MZSVs.

3.1. **Arakawa-Kaneko zeta values.** The Arakawa-Kaneko multiple zeta function [AK] is defined as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{-t} t^{s-1} dt$$

for an integer $k > 0$, where $Li_k(x) = \sum_{n=1}^\infty \frac{x^n}{n^k}$ is the $k$-th polylogarithm function. By making the variable change $x = 1 - e^{-t}$, we can write it as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^1 Li_k(x) \big(-\log(1 - x)\big)^{s-1} \frac{dx}{x}.$$

Now we use integral expressions

$$Li_k(x) = \int_{x > t_1 > \cdots > t_k > 0} \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \frac{dt_k}{1 - t_k}$$

and

$$-\log(1-x) = \int_0^x \frac{du}{1-u},$$

to deduce for any integer $n > 0$

$$(3.1) \qquad \xi_k(n) = \frac{1}{(n-1)!} I\left( \vphantom{\Big(} \right).$$

Moreover, there are exactly $(n-1)!$ ways to impose a total order on the $n-1$ black vertices. Thus we have the identity

$$\xi_k(n) = I\left( \vphantom{\Big(} \right).$$

Therefore, by (1.8), we obtain Ohno's relation [O, Theorem 2]

$$\xi_k(n) = \zeta^\star(k+1, \underbrace{1, \ldots, 1}_{n-1}).$$

3.2. **Mordell-Tornheim zeta values.** Next, we consider the values of the Mordell-Tornheim multiple zeta functions [M]

$$\zeta_{MT,r}(s_1, \ldots, s_r; s) = \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s}.$$

For positive integers $k_1, \ldots, k_r, k$, it is easy to show that

$$(3.2) \qquad \zeta_{MT,r}(k_1, \ldots, k_r; k) = I\left( \vphantom{\Big(} \right).$$

For example,

$$
\begin{aligned}
I\left( \vcenter{\hbox{}} \right) &= \int_{1>t_1>t_2>0} \frac{dt_1}{t_1}\frac{dt_2}{t_2}\left(\int_0^{t_2}\frac{du}{1-u}\right)\left(\int_0^{t_2}\frac{dv}{1-v}\right) \\
&= \int_0^1 \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\left(\sum_{m>0}\frac{t_2^m}{m}\right)\left(\sum_{n>0}\frac{t_2^n}{n}\right) \\
&= \sum_{m,n>0}\frac{1}{mn}\int_0^1\frac{dt_1}{t_1}\int_0^{t_1}t_2^{m+n}\frac{dt_2}{t_2} \\
&= \sum_{m,n>0}\frac{1}{mn(m+n)}\int_0^1 t_1^{m+n}\frac{dt_1}{t_1} \\
&= \sum_{m,n>0}\frac{1}{mn(m+n)^2} = \zeta_{MT,2}(1,1;2).
\end{aligned}
$$

The identity (3.2) implies, in particular, the result by Bradley-Zhou [BZ] that the Mordell-Tornheim zeta value $\zeta_{MT,r}(k_1,\ldots,k_r;k)$ is expressed as a finite sum of MZVs of weight $k_1 + \cdots + k_r + k$ and depth $r$.

3.3. **Certain zeta values of root systems of type $A$.** The third class of examples is a certain type of special values of zeta functions of root systems of type $A_N$, considered by Komori, Matsumoto and Tsumura [KMT] in a study of shuffle relations of MZVs. Explicitly, these values are written as

$$
\zeta\left(p_1,\ldots,p_a;\ \begin{matrix} q_1,\ldots,q_b \\ r_1,\ldots,r_c \end{matrix}\right) = \sum_{\substack{l_1>\cdots>l_a>m_1+n_1 \\ m_1>\cdots>m_b>0 \\ n_1>\cdots>n_c>0}} \frac{1}{l_1^{p_1}\cdots l_a^{p_a}m_1^{q_1}\cdots m_b^{q_b}n_1^{r_1}\cdots n_c^{r_c}},
$$

for three sequences $(p_1,\ldots,p_a)$, $(q_1,\ldots,q_b)$ and $(r_1,\ldots,r_c)$ of positive integers. To describe the corresponding diagram, we introduce an abbreviation: For a sequence $\mathbf{k} = (k_1,\ldots,k_n)$ of positive integers, we write



for the vertical diagram



so that

$$
\zeta(\mathbf{k}) = I\left( \vcenter{\hbox{}}\ \mathbf{k}\ \right).
$$

Using this notation, one can verify that

$$(3.3) \qquad \zeta(\mathbf{p};\mathbf{q};\mathbf{r}) = I\left( \vcenter{\hbox{}} \right)$$

for $\mathbf{p} = (p_1,\ldots,p_a)$, $\mathbf{q} = (q_1,\ldots,q_b)$ and $\mathbf{r} = (r_1,\ldots,r_c)$.

In [KMT], the following relation plays an important role:

$$
\begin{aligned}
(3.4) \quad & \zeta\!\left( p_1,\ldots,p_a;\ \begin{matrix} q_1,\ldots,q_b \\ r_1,\ldots,r_c \end{matrix} \right) \\
&= \sum_{j=0}^{q_1-1} \binom{r_1-1+j}{j} \zeta\!\left( p_1,\ldots,p_a,r_1+j;\ \begin{matrix} q_1-j,q_2,\ldots,q_b \\ r_2,\ldots,r_c \end{matrix} \right) \\
&\quad + \sum_{j=0}^{r_1-1} \binom{q_1-1+j}{j} \zeta\!\left( p_1,\ldots,p_a,q_1+j;\ \begin{matrix} q_2,\ldots,q_b \\ r_1-j,r_2,\ldots,r_c \end{matrix} \right).
\end{aligned}
$$

We point out that our expression (3.3) implies this relation quite naturally. To do this, we denote by $X$ the 2-labeled poset indicated in (3.3), and name some vertices as follows:
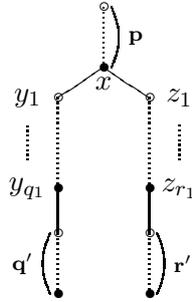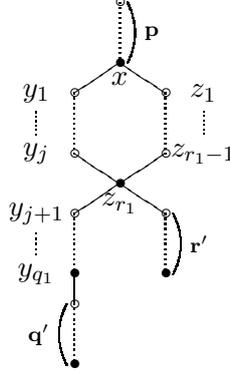


where $\mathbf{q}' = (q_2,\ldots,q_b)$ and $\mathbf{r}' = (r_2,\ldots,r_c)$. By (2.3), one has

$$(3.5) \qquad I(X) = I(X_{y_{q_1}}^{z_{r_1}}) + I(X_{z_{r_1}}^{y_{q_1}}).$$

In $X_{y_{q_1}}^{z_{r_1}}$, the inequalities $x > z_{r_1} > y_{q_1}$ and $x > y_1 > \cdots > y_{q_1}$ hold, hence one can consider $q_1$ further refinements by imposing $y_j > z_{r_1} > y_{j+1}$ for $j = 0,\ldots,q_1-1$ (here we write $y_0 = x$). Thus the identity

$$(3.6) \qquad I(X_{y_{q_1}}^{z_{r_1}}) = \sum_{j=0}^{q_1-1} I(X_j)$$

holds, where $X_j$ is represented by the diagram



Moreover, since there are $\binom{r_1-1+j}{j}$ ways to impose a total order on the white vertices $y_1, \ldots, y_j,\ z_1, \ldots, z_{r_1-1}$, one has

$$(3.7) \qquad I(X_j) = \binom{r_1 - 1 + j}{j} \zeta\left(p_1, \ldots, p_a, r_1 + j;\ \begin{matrix} q_1 - j, q_2, \ldots, q_b \\ r_2, \ldots, r_c \end{matrix}\ \right).$$

The identities (3.6) and (3.7) expresses the first term of (3.5) as desired in (3.4). The second is obtained in the same way.

*Remark.* In [KMT], using partial fraction decompositions, the identity (3.4) is proved with some variables (irrelevant to the decomposition) *complex valued*, not necessarily positive integral. It seems difficult to apply our method in this paper to such functional relations.

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