

# Subset-lex: did we miss an order?

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## Abstract

We generalize a well-known algorithm for the generation of all subsets of a set in lexicographic order with respect to the sets as lists of elements (subset-lex order). We obtain algorithms for various combinatorial objects such as the subsets of a multiset, compositions and partitions represented as lists of parts, and for certain restricted growth strings. The algorithms are often loopless and require at most one extra variable for the computation of the next object. The performance of the algorithms is very competitive even when not loopless. A Gray code corresponding to the subset-lex order and a Gray code for compositions that was found during this work are described.

Appendix II about ranking and unranking methods for mixed radix words added January 2, 2024 at the very end. The text is otherwise unchanged.

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0:	$[ \cdot \cdot \cdot \cdot \cdot ]$	$\{ \}$	$[ \cdot \cdot \cdot \cdot \cdot ]$	$\{ \}$
1:	$[ 1 \cdot \cdot \cdot \cdot \cdot ]$	$\{ 0 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 4 \}$
2:	$[ 1 1 \cdot \cdot \cdot \cdot ]$	$\{ 0, 1 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 3 \}$
3:	$[ 1 1 1 \cdot \cdot \cdot ]$	$\{ 0, 1, 2 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 3, 4 \}$
4:	$[ 1 1 1 1 \cdot \cdot ]$	$\{ 0, 1, 2, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 2 \}$
5:	$[ 1 1 1 1 1 \cdot ]$	$\{ 0, 1, 2, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 2, 4 \}$
6:	$[ 1 1 1 \cdot 1 \cdot ]$	$\{ 0, 1, 2, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 2, 3 \}$
7:	$[ 1 1 \cdot 1 \cdot 1 \cdot ]$	$\{ 0, 1, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 2, 3, 4 \}$
8:	$[ 1 1 \cdot 1 1 \cdot \cdot ]$	$\{ 0, 1, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1 \}$
9:	$[ 1 1 \cdot \cdot 1 \cdot \cdot ]$	$\{ 0, 1, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 4 \}$
10:	$[ 1 \cdot 1 \cdot \cdot \cdot \cdot ]$	$\{ 0, 2 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 3 \}$
11:	$[ 1 \cdot 1 \cdot 1 \cdot \cdot \cdot ]$	$\{ 0, 2, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 3, 4 \}$
12:	$[ 1 \cdot \cdot 1 1 1 \cdot \cdot ]$	$\{ 0, 2, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 2 \}$
13:	$[ 1 \cdot \cdot 1 \cdot 1 \cdot \cdot ]$	$\{ 0, 2, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 2, 4 \}$
14:	$[ 1 \cdot \cdot \cdot 1 \cdot \cdot \cdot ]$	$\{ 0, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 2, 3 \}$
15:	$[ 1 \cdot \cdot \cdot 1 1 \cdot \cdot ]$	$\{ 0, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 1, 2, 3, 4 \}$
16:	$[ 1 \cdot \cdot \cdot \cdot 1 \cdot \cdot ]$	$\{ 0, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0 \}$
17:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot ]$	$\{ 1 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 4 \}$
18:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot ]$	$\{ 1, 2 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 3 \}$
19:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 1 \cdot ]$	$\{ 1, 2, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 3, 4 \}$
20:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 1 1 \cdot ]$	$\{ 1, 2, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 2 \}$
21:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 1 \cdot 1 ]$	$\{ 1, 2, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 2, 4 \}$
22:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot 1 \cdot ]$	$\{ 1, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 2, 3 \}$
23:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot 1 1 \cdot ]$	$\{ 1, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 2, 3, 4 \}$
24:	$[ \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot \cdot 1 ]$	$\{ 1, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1 \}$
25:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot ]$	$\{ 2 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 4 \}$
26:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 1 \cdot ]$	$\{ 2, 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 3 \}$
27:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 1 1 \cdot ]$	$\{ 2, 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 3, 4 \}$
28:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot 1 \cdot ]$	$\{ 2, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 2 \}$
29:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 \cdot ]$	$\{ 3 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 2, 4 \}$
30:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 1 \cdot ]$	$\{ 3, 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 2, 3 \}$
31:	$[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 1 1 \cdot ]$	$\{ 4 \}$	$[ \cdot \cdot \cdot \cdot \cdot 1 ]$	$\{ 0, 1, 2, 3, 4 \}$

Figure 1: Two lexicographic orders for the subsets of a 5-element set: ordering using the sets as lists of elements (left) and ordering using the characteristic words (right). Dots are used to denote zeros in the characteristic words.

## 1 Two lexicographic orders for binary words

Two simple ways to represent subsets of a set  $\{0, 1, 2, \dots, n - 1\}$  are as lists of elements and as the characteristic word, the binary word with ones at the positions corresponding to the elements in the subset. Sorting all subsets by list of characteristic words gives the right columns of figure 1, sorting by the list of elements gives the left columns. We will concern ourselves with the generalization of the ordering by the lists which we call *subset-lex* order.

The algorithms for computing the successor or predecessor of a (nonempty) subset  $S$  in subset-lex order are well-known<sup>1</sup>. In the following let  $z$  be the last, and  $y$  the next to last element in  $S$ . We call an element minimal or maximal if it is respectively the smallest or greatest element in the superset

**Algorithm 1** (Next-SL2). *Compute the successor of the subset  $S$ .*

1. If there is just one element, and it is maximal, stop.
2. If  $z$  is not maximal, append  $z + 1$ .
3. Otherwise remove  $z$  and  $y$ , then append  $y + 1$ .

**Algorithm 2** (Prev-SL2). *Compute the predecessor of the subset  $S$ .*

1. If there is just one element, and it is minimal, stop.

---

<sup>1</sup>For example, an algorithm for the  $k$ -subsets of an  $n$ -set is given in [25, Algorithm LEXSUB, p.18], see section 11 for an implementation.

2. If  $z - 1 \in S$ , remove  $z$ .
3. Otherwise remove  $z$ , append  $z - 1$  and the maximal element.

C++ implementations of all algorithms discussed here are given in the FXT library [1]. Here and elsewhere we give slightly simplified versions of the actual code, like omitting modifiers such as `public`, `protected`, and `private`. The type `ulong` is short for `unsigned long`.

```

1 // FILE: src/comb/subset-lex.h
2 class subset_lex
3 // Nonempty subsets of the set {0,1,2,...,n-1} in subset-lex order.
4 // Representation as list of parts.
5 // Loopless generation.
6 {
7     ulong n_;    // number of elements in set, should have n>=1
8     ulong k_;    // index of last element in subset
9     ulong n1_;   // == n - 1 for n >= 1, and == 0 for n==0
10    ulong *x_;  // x[0...k-1]: subset of {0,1,2,...,n-1}
11
12    subset_lex(ulong n)
13    // Should have n>=1,
14    // for n==0 one set with the element zero is generated.
15    {
16        n_ = n;
17        n1_ = (n_ ? n_ - 1 : 0 );
18        x_ = new ulong[n_ + (n_==0)];
19        first();
20    }

```

We will always use the method names `first()` and `last()` for the first and last element in the respective list of combinatorial objects.

```

1     ulong first()
2     {
3         k_ = 0;
4         x_[0] = 0;
5         return k_ + 1;
6     }
7
8     ulong last()
9     {
10        k_ = 0;
11        x_[0] = n1_;
12        return k_ + 1;
13    }

```

The methods for computing the successor and predecessor will always be called `next()` and `prev()` respectively.

```

1     ulong next()
2     // Return number of elements in subset.
3     // Return zero if current is last.
4     {
5         if ( x_[k_] == n1_ ) // last element is max?
6         {
7             if ( k_==0 ) return 0;
8
9                 --k_;      // remove last element
10                x_[k_] += 1; // increment last element
11            }
12            else // add next element from set:
13            {
14                ++k_;
15                x_[k_] = x_[k_-1] + 1;
16            }
17
18        return k_ + 1;
19    }
20
21
22     ulong prev()
23     // Return number of elements in subset.
24     // Return zero if current is first.
25

```

```

26     if ( k_ == 0 ) // only one element ?
27     {
28         if ( x_[0]==0 ) return 0;
29         x_[0] --; // decrement last (and only) element
30         ++k_;
31         x_[k_] = n1_; // append maximal element
32     }
33     else
34     {
35         if ( x_[k_] == x_[k_-1] + 1 ) --k_; // remove last element
36         else
37         {
38             x_[k_] --; // decrement last element
39             ++k_;
40             x_[k_] = n1_; // append maximal element
41         }
42     }
43 }
44
45     return k_ + 1;
46 }
```

A program demonstrating usage of the class is

```

1 // FILE: demo/comb/subset-lex-demo.cc
2     ulong n = 6;
3     subset_lex S(n);
4     do
5     {
6         // visit subset
7     }
8     while ( S.next() );
```

## 2 Subset-lex order for multisets

For the internal representation of the subsets of a multiset, we could use lists of pairs  $(e, m)$  where  $e$  is the element and  $m$  its multiplicity. This choice would lead to loopless algorithms as will become clear in a moment. The disadvantage of this representation, however, is that the generalizations we will find would have a slightly more complicated form. We will instead use generalized characteristic words, where nonzero entries can be at most the multiplicity of the element in question. For a multiset  $\{0^{m(0)}, 1^{m(1)}, \dots, k^{m(k)}\}$  these are the mixed radix numbers with radix vector  $[m(0)+1, m(1)+1, \dots, m(k)+1]$ . The subsets of the set  $\{0^1, 1^2, 2^2, 3^3\}$  in subset-lex order are shown in figure 2.

The algorithms for computing the successor or predecessor of a (nonempty) subset  $S$  of a multiset in subset-lex are now given. In the following let  $z$  be the last element in  $S$ .

**Algorithm 3** (Next-SL). *Compute the successor of the subset  $S$ .*

1. If the multiplicity of  $z$  is not maximal, increase it; return.
2. If  $z$  is not maximal, append  $z + 1$  with multiplicity 1; return.
3. If  $z$  is the only element, stop.
4. Remove  $z$  and decrease the multiplicity of next to last nonzero element  $y$ , then append  $y + 1$  with multiplicity 1.

The description omits the case of the empty set, the implementation takes care of this case by intially pointing to the leftmost digit of the characteristic word, which is zero. With the first call to the routine it is changed to 1.

**Algorithm 4** (Prev-SL). *Compute the predecessor of the subset  $S$ .*

0:	[ . . . . ]	{ }
1:	[ . 1 . . . ]	{ { 0 } }
2:	[ . 1 1 . . ]	{ { 0, 1 } }
3:	[ . 1 2 . . ]	{ { 0, 1, 1 } }
4:	[ . 1 2 1 . ]	{ { 0, 1, 1, 2 } }
5:	[ . 1 2 2 . ]	{ { 0, 1, 1, 2, 2 } }
6:	[ . 1 2 2 1 ]	{ { 0, 1, 1, 2, 2, 3 } }
7:	[ . 1 2 2 2 ]	{ { 0, 1, 1, 2, 2, 3, 3 } }
8:	[ . 1 2 2 3 ]	{ { 0, 1, 1, 2, 3, 3, 3 } }
9:	[ . 1 2 1 1 ]	{ { 0, 1, 1, 2, 3 } }
10:	[ . 1 2 1 2 ]	{ { 0, 1, 1, 2, 3, 3 } }
11:	[ . 1 2 1 3 ]	{ { 0, 1, 1, 2, 3, 3, 3 } }
12:	[ . 1 2 . 1 ]	{ { 0, 1, 1, 3 } }
13:	[ . 1 2 . 2 ]	{ { 0, 1, 1, 3, 3 } }
14:	[ . 1 2 . 3 ]	{ { 0, 1, 1, 3, 3, 3 } }
15:	[ . 1 1 1 . ]	{ { 0, 1, 2 } }
16:	[ . 1 1 2 . ]	{ { 0, 1, 2, 2 } }
17:	[ . 1 1 2 1 ]	{ { 0, 1, 2, 2, 3 } }
18:	[ . 1 1 2 2 ]	{ { 0, 1, 2, 2, 3, 3 } }
19:	[ . 1 1 2 3 ]	{ { 0, 1, 2, 2, 3, 3, 3 } }
20:	[ . 1 1 1 1 ]	{ { 0, 1, 2, 3 } }
21:	[ . 1 1 1 2 ]	{ { 0, 1, 2, 3, 3 } }
22:	[ . 1 1 1 3 ]	{ { 0, 1, 2, 3, 3, 3 } }
23:	[ . 1 1 . 1 ]	{ { 0, 1, 3 } }
24:	[ . 1 1 . 2 ]	{ { 0, 1, 3, 3 } }
25:	[ . 1 1 . 3 ]	{ { 0, 1, 3, 3, 3 } }
26:	[ . 1 . 1 . ]	{ { 0, 2 } }
27:	[ . 1 . 2 . ]	{ { 0, 2, 2 } }
28:	[ . 1 . 2 1 ]	{ { 0, 2, 2, 3 } }
29:	[ . 1 . 2 2 ]	{ { 0, 2, 2, 3, 3 } }
30:	[ . 1 . 2 3 ]	{ { 0, 2, 2, 3, 3, 3 } }
31:	[ . 1 . 1 1 ]	{ { 0, 2, 3 } }
32:	[ . 1 . 1 2 ]	{ { 0, 2, 3, 3 } }
33:	[ . 1 . 1 3 ]	{ { 0, 2, 3, 3, 3 } }
34:	[ . 1 . . 1 ]	{ { 0, 3 } }
35:	[ . 1 . . 2 ]	{ { 0, 3, 3 } }
36:	[ . 1 . . 3 ]	{ { 0, 3, 3, 3 } }
37:	[ . . 1 . . ]	{ { 1 } }
38:	[ . . 2 . . ]	{ { 1, 1 } }
39:	[ . . 2 1 . ]	{ { 1, 1, 2 } }
40:	[ . . 2 2 . ]	{ { 1, 1, 2, 2 } }
41:	[ . . 2 2 1 ]	{ { 1, 1, 2, 2, 3 } }
42:	[ . . 2 2 2 ]	{ { 1, 1, 2, 2, 3, 3 } }
43:	[ . . 2 2 3 ]	{ { 1, 1, 2, 2, 3, 3, 3 } }
44:	[ . . 2 1 1 ]	{ { 1, 1, 2, 3 } }
45:	[ . . 2 1 2 ]	{ { 1, 1, 2, 3, 3 } }
46:	[ . . 2 1 3 ]	{ { 1, 1, 2, 3, 3, 3 } }
47:	[ . . 2 . 1 ]	{ { 1, 1, 3 } }
48:	[ . . 2 . 2 ]	{ { 1, 1, 3, 3 } }
49:	[ . . 2 . 3 ]	{ { 1, 1, 3, 3, 3 } }
50:	[ . . 1 1 . ]	{ { 1, 2 } }
51:	[ . . 1 2 . ]	{ { 1, 2, 2 } }
52:	[ . . 1 2 1 ]	{ { 1, 2, 2, 3 } }
53:	[ . . 1 2 2 ]	{ { 1, 2, 2, 3, 3 } }
54:	[ . . 1 2 3 ]	{ { 1, 2, 2, 3, 3, 3 } }
55:	[ . . 1 1 1 ]	{ { 1, 2, 3 } }
56:	[ . . 1 1 2 ]	{ { 1, 2, 3, 3 } }
57:	[ . . 1 1 3 ]	{ { 1, 2, 3, 3, 3 } }
58:	[ . . 1 . 1 ]	{ { 1, 3 } }
59:	[ . . 1 . 2 ]	{ { 1, 3, 3 } }
60:	[ . . 1 . 3 ]	{ { 1, 3, 3, 3 } }
61:	[ . . . 1 . ]	{ { 2 } }
62:	[ . . . 2 . ]	{ { 2, 2 } }
63:	[ . . . 2 1 ]	{ { 2, 2, 3 } }
64:	[ . . . 2 2 ]	{ { 2, 2, 3, 3 } }
65:	[ . . . 2 3 ]	{ { 2, 2, 3, 3, 3 } }
66:	[ . . . 1 1 ]	{ { 2, 3 } }
67:	[ . . . 1 2 ]	{ { 2, 3, 3 } }
68:	[ . . . 1 3 ]	{ { 2, 3, 3, 3 } }
69:	[ . . . . 1 ]	{ { 3 } }
70:	[ . . . . 2 ]	{ { 3, 3 } }
71:	[ . . . . 3 ]	{ { 3, 3, 3 } }

Figure 2: Subsets of the multiset  $\{0^1, 1^2, 2^2, 3^3\}$  in subset-lex order. Dots are used to denote zeros in the (generalized) characteristic words.

1. If the set is empty, stop.
2. Decrease the multiplicity of  $z$ .
3. If the new multiplicity is nonzero, return.
4. Increase the multiplicity of  $z - 1$  and append the maximal element.

Clearly both algorithms are loopless for the representation using a list of pairs. With the representation as characteristic words algorithm 3 involves a loop in the last step, a scan for the next to last nonzero element. This could be prevented by maintaining an additional list of positions where the characteristic word is nonzero. Algorithm 4 does stay loopless, the position of the maximal element does not need to be sought. Our implementations will always use a variable holding the position of the last nonzero position in the characteristic word. This is called the “current track” in the implementations.

We will not make the equivalents of algorithm 3 loopless, as the maintenance of the additional list would render algorithm 4 slower (at least as long as mixed calls to both shall be allowed), but see section 3 for one such example.

A step in either direction has either one or three positions in the characteristic word changed. The number of transitions with one change are more frequent with higher multiplicities. The worst case occurs if all multiplicities are one (corresponding to binary words), when (about) half of the steps involve only one transition.

The C++ implementation uses the sentinel technique to reduce the number of conditional branches.

```

1 // FILE: src/comb/mixedradix-subset-lex.h
2 class mixedradix_subset_lex
3 {
4     ulong n_;      // number of digits (n kinds of elements in multiset)
5     ulong tr_;    // aux: current track
6     ulong *a_;    // digits of mixed radix number
7             // (multiplicities in subset).
8     ulong *m1_;   // nines (radix minus one) for each digit
9             // (multiplicity of kind k in superset).
10
11    mixedradix_subset_lex(ulong n, ulong mm, const ulong *m=0)
12    {
13        n_ = n;
14        a_ = new ulong[n_+2]; // two sentinels, one left, one right
15        a_[0] = 1; a_[n_+1] = 1;
16        ++a_; // nota bene
17        m1_ = new ulong[n_+2];
18        m1_[0] = 0; m1_[n_+1] = 0; // sentinel with n==0
19        ++m1_; // nota bene
20
21        mixedradix_init(n_, mm, m, m1_); // set up m1[], omitted
22
23        first();
24    }

```

The omitted routine `mixedradix_init()` sets all elements of the array `m1[]` to `mm` (fixed radix case), unless the pointer `m` is nonzero, then the elements behind `m` are copied into `m1[]` (mixed radix case).

```

1     void first()
2     {
3         for (ulong k=0; k<n_; ++k) a_[k] = 0;
4         tr_ = 0; // start by looking at leftmost (zero) digit
5     }
6
7     void last()
8     {
9         for (ulong k=0; k<n_; ++k) a_[k] = 0;

```

```

10     ulong n1 = ( n_ ? n_ - 1 : 0 );
11     a_[n1] = m1_[n1];
12     tr_ = n1;
13 }

```

Note the while-loop in the `next()` method.

```

1     bool next()
2 {
3     ulong j = tr_;
4     if ( a_[j] < m1_[j] ) // easy case 1: increment
5     {
6         a_[j] += 1;
7         return true;
8     }
9     // here a_[j] == m1_[j]
10    if ( j+1 < n_ ) // easy case 2: append (move track to the right)
11    {
12        ++j;
13        a_[j] = 1;
14        tr_ = j;
15        return true;
16    }
17    a_[j] = 0;
18    // find first nonzero digit to the left:
19    --j;
20    while ( a_[j] == 0 ) { --j; } // may read sentinel a_[-1]
21    if ( (long)j < 0 ) return false; // current is last
22    a_[j] -= 1; // decrement digit to the left
23    ++j;
24    a_[j] = 1;
25    tr_ = j;
26    return true;
27 }

```

The method `prev()` is indeed loopless:

```

1     bool prev()
2 {
3     ulong j = tr_;
4     if ( a_[j] > 1 ) // easy case 1: decrement
5     {
6         a_[j] -= 1;
7         return true;
8     }
9     else
10    {
11        if ( tr_ == 0 )
12        {
13            if ( a_[0] == 0 ) return false; // current is first
14            a_[0] = 0; // now word is first (all zero)
15            return true;
16        }
17        a_[j] = 0;
18        --j; // now looking at next track to the left
19        if ( a_[j] == m1_[j] ) // easy case 2: move track to left
20        {
21            tr_ = j; // move track one left
22        }
23        else
24        {
25            a_[j] += 1; // increment digit to the left
26            j = n_ - 1;
27            a_[j] = m1_[j]; // set rightmost digit = nine
28            tr_ = j; // move to rightmost track
29        }
30    }
31    return true;
32 }
33

```

	colex	Gray code	subset-lex	iset
0:	[ . . . . ]	[ 4 . 2 . ]	[ . . . . ]	{ }
1:	[ 1 . . . ]	[ 3 . 2 . ]	[ 1 . . . ]	{ 0 }
2:	[ 2 . . . ]	[ 2 . 2 . ]	[ 2 . . . ]	{ 0 }
3:	[ 3 . . . ]	[ 1 . 2 . ]	[ 3 . . . ]	{ 0 }
4:	[ 4 . . . ]	[ . . 2 . ]	[ 4 . . . ]	{ 0 }
5:	[ . 1 . . ]	[ . . 1 . ]	[ 4 . 1 . ]	{ 0, 2 }
6:	[ . 2 . . ]	[ 1 . 1 . ]	[ 4 . 2 . ]	{ 0, 2 }
7:	[ . 3 . . ]	[ 2 . 1 . ]	[ 4 . . 1 ]	{ 0, 3 }
8:	[ . . 1 . ]	[ 3 . 1 . ]	[ 3 . 1 . ]	{ 0, 2 }
9:	[ 1 . 1 . . ]	[ 4 . 1 . ]	[ 3 . 2 . ]	{ 0, 2 }
10:	[ 2 . 1 . . ]	[ 4 . . . ]	[ 3 . . 1 ]	{ 0, 3 }
11:	[ 3 . 1 . . ]	[ 3 . . . ]	[ 2 . 1 . ]	{ 0, 2 }
12:	[ 4 . 1 . . ]	[ 2 . . . ]	[ 2 . 2 . ]	{ 0, 2 }
13:	[ . . 2 . . ]	[ 1 . . . ]	[ 2 . . 1 ]	{ 0, 3 }
14:	[ 1 . 2 . . ]	[ . . . . ]	[ 1 . 1 . ]	{ 0, 2 }
15:	[ 2 . 2 . . ]	[ . . 1 . . ]	[ 1 . . 2 ]	{ 0, 2 }
16:	[ 3 . 2 . . ]	[ . 2 . . . ]	[ 1 . . . 1 ]	{ 0, 3 }
17:	[ 4 . 2 . . ]	[ . . 3 . . ]	[ . 1 . . . ]	{ 1 }
18:	[ . . . 1 ]	[ . . 3 . 1 ]	[ . 2 . . . ]	{ 1 }
19:	[ 1 . . . 1 ]	[ . . 2 . 1 ]	[ . . 3 . . ]	{ 1 }
20:	[ 2 . . . 1 ]	[ . . 1 . 1 ]	[ . . 3 . 1 ]	{ 1, 3 }
21:	[ 3 . . . 1 ]	[ . . . . 1 ]	[ . . 2 . 1 ]	{ 1, 3 }
22:	[ 4 . . . 1 ]	[ . 1 . . 1 ]	[ . . 1 . 1 ]	{ 1, 3 }
23:	[ . 1 . . 1 ]	[ 2 . . . 1 ]	[ . . . 1 . ]	{ 2 }
24:	[ . 2 . . 1 ]	[ 3 . . . 1 ]	[ . . . 2 . ]	{ 2 }
25:	[ . 3 . . 1 ]	[ 4 . . . 1 ]	[ . . . . 1 ]	{ 3 }

Figure 3: Mixed radix numbers of length 4 in falling factorial base that are non-adjacent forms (NAF), in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the sets of positions of nonzero digits).

34      }

The performance of the generator is quite satisfactory. A system based on an AMD Phenom™ II X4 945 processor clocked at 3.0 GHz and the GCC C++ compiler version 4.9.0 [12] were used for measuring. The updates using `next()` cost about 12 cycles for base 2 (the worst case) and 9 cycles for base 16. Using `prev()` takes about 8.5 cycles with base 2 and 4.2 cycles with base 16, the latter figure corresponding to a rate of 780 million subsets per second.

These and all following such figures are average values, obtained by measuring the total time for the generation of all words of a certain size and dividing by the number of words.

### 3 Loopless computation of the successor for non-adjacent forms

The method of using a list of nonzero positions in the words to obtain loopless methods for both `next()` and `prev()` has been implemented for words where no two adjacent digits are nonzero (non-adjacent forms, NAF).

In the following algorithm for the computation of the successor let  $a_i$  be the digits of the length- $n$  NAF where  $0 \leq i < n$ .

**Algorithm 5** (Next-NAF-SL). *Compute the successor of a non-adjacent form in subset-lex order.*

1. Let  $j$  be the position of the last nonzero digit.
2. If  $a_j$  is not maximal, increment it and return.

0:	[ 4 . 2 . ]	(0, 4) (1, 3) (2)	[ 4 3 2 1 0 ]
1:	[ 3 . 2 . ]	(0, 3) (1, 4) (2)	[ 3 4 2 0 1 ]
2:	[ 2 . 2 . ]	(0, 2) (1, 4) (3)	[ 2 4 0 3 1 ]
3:	[ 1 . 2 . ]	(0, 1) (2, 4) (3)	[ 1 0 4 3 2 ]
4:	[ . . 2 . ]	(0) (1) (2, 4) (3)	[ 0 1 4 3 2 ]
5:	[ . . 1 . ]	(0) (1) (2, 3) (4)	[ 0 1 3 2 4 ]
6:	[ 1 . 1 . ]	(0, 1) (2, 3) (4)	[ 1 0 3 2 4 ]
7:	[ 2 . 1 . ]	(0, 2) (1, 3) (4)	[ 2 3 0 1 4 ]
8:	[ 3 . 1 . ]	(0, 3) (1, 2) (4)	[ 3 2 1 0 4 ]
9:	[ 4 . 1 . ]	(0, 4) (1, 2) (3)	[ 4 2 1 3 0 ]
10:	[ 4 . . . ]	(0, 4) (1) (2) (3)	[ 4 1 2 3 0 ]
11:	[ 3 . . . ]	(0, 3) (1) (2) (4)	[ 3 1 2 0 4 ]
12:	[ 2 . . . ]	(0, 2) (1) (3) (4)	[ 2 1 0 3 4 ]
13:	[ 1 . . . ]	(0, 1) (2) (3) (4)	[ 1 0 2 3 4 ]
14:	[ . . . . ]	(0) (1) (2) (3) (4)	[ 0 1 2 3 4 ]
15:	[ . 1 . . ]	(0) (1, 2) (3) (4)	[ 0 2 1 3 4 ]
16:	[ . 2 . . ]	(0) (1, 3) (2) (4)	[ 0 3 2 1 4 ]
17:	[ . 3 . . ]	(0) (1, 4) (2) (3)	[ 0 4 2 3 1 ]
18:	[ . 3 . 1 ]	(0) (1, 4) (2, 3)	[ 0 4 3 2 1 ]
19:	[ . 2 . 1 ]	(0) (1, 3) (2, 4)	[ 0 3 4 1 2 ]
20:	[ . 1 . 1 ]	(0) (1, 2) (3, 4)	[ 0 2 1 4 3 ]
21:	[ . . . 1 ]	(0) (1) (2) (3, 4)	[ 0 1 2 4 3 ]
22:	[ 1 . . 1 ]	(0, 1) (2) (3, 4)	[ 1 0 2 4 3 ]
23:	[ 2 . . 1 ]	(0, 2) (1) (3, 4)	[ 2 1 0 4 3 ]
24:	[ 3 . . 1 ]	(0, 3) (1) (2, 4)	[ 3 1 4 0 2 ]
25:	[ 4 . . 1 ]	(0, 4) (1) (2, 3)	[ 4 1 3 2 0 ]

Figure 4: Gray code for the length-4 non-adjacent forms in falling factorial base (left), together with the corresponding involutions in cycle form (middle) and in array form (right).

3. If  $j + 2 < n$ , set  $a_{j+2} = 1$  (append new digit) and return.
4. Set  $a_j = 0$  (set last nonzero digit to zero).
5. If  $j + 1 < n$ , set  $a_{j+1} = 1$  (move digit right) and return.
6. If there is just one nonzero digit, stop.
7. Let  $k$  be the position of the nearest nonzero digit left of  $j$ .
8. Set  $a_k = a_k - 1$  (decrement digit to the left).
9. If  $a_k = 0$ , set  $a_{k+1} = 1$  (move digit right) and return.
10. Otherwise,  $a_{k+2} = 1$  (append digit two positions to the right).

In the following implementation the array `iset[]` holds the positions of the nonzero digits. We only read the last or second last element, the adjustments of the length (variable `ni`) have been left out in the description above. The implementation handles all  $n \geq 0$  correctly, for the all-zero word `iset[]` is of length one and its only element points to the leftmost digit.

```

1 // FILE: src/comb/mixedradix-naf-subset-lex.h
2 class mixedradix_naf_subset_lex
3 {
4     ulong *iset_; // Set of positions of nonzero digits
5     ulong *a_; // digits
6     ulong *m1_; // nines (radix minus one) for each digit
7     ulong ni_; // number of elements in iset[]
8     ulong n_; // number of digits
9
10    void first()
11    {
12        for (ulong k=0; k<n_; ++k) a_[k] = 0;
13        // iset[] initially with one element zero:
14        iset_[0] = 0;
15        ni_ = 1;
16    }

```

```

17     if ( n==0 ) // make things work for n == 0
18     {
19         m1_[0] = 0;
20         a_[0] = 0;
21     }
22 }
23 }
```

The computation of the successor is

```

1     bool next()
2     {
3         ulong j = iset_[ni_-1];
4         const ulong aj = a_[j] + 1;
5         if ( aj <= m1_[j] ) // can increment last digit
6         {
7             a_[j] = aj;
8             return true;
9         }
10        if ( j + 2 < n_ ) // can append new digit
11        {
12            iset_[ni_] = j + 2;
13            a_[j+2] = 1; // assume all m1[] are nonzero
14            ++ni_;
15            return true;
16        }
17        a_[j] = 0; // set last nonzero digit to zero
18        if ( j + 1 < n_ ) // can move last digit to the right
19        {
20            a_[j+1] = 1;
21            iset_[ni_-1] = j + 1;
22            return true;
23        }
24        if ( ni_ == 1 ) return false; // current is last
25
26        // Now we look to the left:
27        const ulong k = iset_[ni_-2]; // nearest nonzero digit to the left
28        const ulong ak = a_[k] - 1; // decrement digit to the left
29        a_[k] = ak;
30        if ( ak == 0 ) // move digit one to the right
31        {
32            a_[k+1] = 1;
33            iset_[ni_-2] = k + 1;
34            --ni_;
35            return true;
36        }
37        else // append digit two positions to the right
38        {
39            a_[k+2] = 1;
40            iset_[ni_-1] = k + 2;
41            return true;
42        }
43    }
44 }
```

The update via `next()` takes about 7 cycles. The updates for co-lexicographic order and the Gray code respectively take about 15 and 21 cycles.

We note that the falling factorial NAFs of length  $n$  are one-to-one with involutions (self-inverse permutations) of  $n+1$  elements. Process the NAF from left to right; for  $a_i = 0$  let the next unused element of the permutation be a fixed point (and mark it as used); for  $a_i \neq 0$  put the next unused element in a cycle with the  $a_i$ th unused element (and mark both as used). A (simplistic) implementation of this method is given in the program `demo/comb/perm-involution-naf-demo.cc`.

The binary NAFs of length 8 are shown in figure 5. Algorithms for the generation of these NAFs as binary words for both lexicographic order and the Gray code are

	colex	Gray code	subset-lex
0:	.....	.1..1..1	1..... = { 0 }
1:	....1	.1..1...	1.1.... = { 0, 2 }
2:	....1.	.1..1.1.	1.1.1... = { 0, 2, 4 }
3:	....1..	.1....1.	1.1.1.1. = { 0, 2, 4, 6 }
4:	....1.1	.1.....	1.1.1..1 = { 0, 2, 4, 7 }
5:	....1...	.1....1	1.1..1... = { 0, 2, 5 }
6:	....1..1	.1..1.1	1.1..1.1 = { 0, 2, 5, 7 }
7:	....1.1.	.1....1	1.1....1 = { 0, 2, 6 }
8:	...1....	.1.1.1..	1.1....1 = { 0, 2, 7 }
9:	...1..1..1	.1.1.1.1	1....1... = { 0, 3 }
10:	...1..1..1	.1.1....1	1....1..1 = { 0, 3, 5 }
11:	...1..1..1	.1....1	1....1.1 = { 0, 3, 5, 7 }
12:	...1..1.1	.1.1....1	1....1..1 = { 0, 3, 6 }
13:	...1.....	....1..1.	1....1..1 = { 0, 3, 7 }
14:	...1....1	....1....	1....1... = { 0, 4 }
15:	...1....1.	....1....1	1....1..1 = { 0, 4, 6 }
16:	...1....1..	....1.1..1	1....1..1 = { 0, 4, 7 }
17:	...1..1.1	....1.1..	1....1..1 = { 0, 5 }
18:	...1..1...	....1..1..	1....1..1 = { 0, 5, 7 }
19:	...1..1..1	....1..1.1	1....1.1.. = { 0, 6 }
20:	...1..1..1.	....1....1	1....1....1 = { 0, 7 }
21:	...1.....	.....1..	1..... = { 1 }
22:	...1.....1	.....1..1	1.1.... = { 1, 3 }
23:	...1....1.	....1..1..	1.1....1 = { 1, 3, 5 }
24:	...1....1..	....1....1	1.1....1.1 = { 1, 3, 5, 7 }
25:	...1....1.1	....1....1	1.1....1.. = { 1, 3, 6 }
26:	...1....1..1	....1.1..1	1.1....1..1 = { 1, 3, 7 }
27:	...1..1..1..1	....1.1..1..	1....1... = { 1, 4 }
28:	...1..1..1..1	....1.1..1..1	1....1..1..1 = { 1, 4, 6 }
29:	...1..1....1	....1....1..1	1....1..1..1 = { 1, 4, 7 }
30:	...1..1..1..1	....1....1..1	1....1....1 = { 1, 5 }
31:	...1..1..1..1	....1....1..1	1....1..1..1 = { 1, 5, 7 }
32:	...1..1..1..1	....1....1.1	1....1....1 = { 1, 6 }
33:	...1..1..1..1	....1....1..1	1....1....1 = { 1, 7 }
34:	...1.....1	....1....1..1	1....1....1 = { 2 }
35:	...1.....1	....1....1.1	1....1....1 = { 2, 4 }
36:	...1.....1	....1....1.1..1	1....1....1..1 = { 2, 4, 6 }
37:	...1....1..1	....1....1..1..1	1....1....1..1..1 = { 2, 4, 7 }
38:	...1....1..1..1	....1....1..1..1..1	1....1....1..1..1..1 = { 2, 5 }
39:	...1....1..1..1..1	....1....1..1..1..1..1	1....1....1..1..1..1..1 = { 2, 5, 7 }
40:	...1....1..1..1..1	....1....1..1..1..1..1	1....1....1..1..1..1..1 = { 2, 6 }
41:	...1....1..1..1..1	....1....1..1..1..1..1	1....1....1..1..1..1..1 = { 2, 7 }
42:	...1....1..1..1..1..1	....1....1..1..1..1..1..1	1....1....1..1..1..1..1..1 = { 3 }
43:	...1....1..1..1..1..1	....1....1..1..1..1..1..1	1....1....1..1..1..1..1..1 = { 3, 5 }
44:	...1....1..1..1..1..1	....1....1..1..1..1..1..1	1....1....1..1..1..1..1..1 = { 3, 5, 7 }
45:	...1....1..1..1..1..1	....1....1..1..1..1..1..1	1....1....1..1..1..1..1..1 = { 3, 6 }
46:	...1....1..1..1..1..1..1	....1....1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1 = { 3, 7 }
47:	...1....1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1 = { 4 }
48:	...1....1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1 = { 4, 6 }
49:	...1....1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1..1 = { 4, 7 }
50:	...1....1..1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1..1..1 = { 5 }
51:	...1....1..1..1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1..1..1..1 = { 5, 7 }
52:	...1....1..1..1..1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1..1..1..1..1 = { 6 }
53:	...1....1..1..1..1..1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1..1..1..1..1	1....1....1..1..1..1..1..1..1..1..1..1..1..1..1..1 = { 7 }
54:	...1....1..1..1..1..1..1..1..1..1..1..1..1..1..1	....1....1..1..1..1..1..1..1..1..1..1..1..1..1..1..1	

Figure 5: Binary non-adjacent forms of length 8 in co-lexicographic order, minimal-change order (Gray code), and subset-lex order (with the corresponding sets without consecutive elements).

given in [2, p.75-77]. Here we give the implementations for computing successor and predecessor for the nonempty NAFs in subset-lex order.

The routine for the successor needs to start with the word that has a single set bit at the highest position of the desired word length.

```

1 // FILE: src/bits/fibre-subset-lexrev.h
2 ulong next_subset_lexrev_fib(ulong x)
3 {
4     ulong x0 = x & -x; // lowest bit
5     ulong xs = x0 >> 2;
6     if ( xs != 0 ) // easy case: set bit right of lowest bit
7     {
8         x |= xs;
9         return x;
10    }
11    else // lowest bit at index 0 or 1
12    {
13        if ( x0 == 2 ) // at index 1
14        {
15            x -= 1;
16            return x;
17        }
18        x ^= x0; // clear lowest bit
19        x0 = x & -x; // new lowest bit ...
20        x0 >>= 1; x -= x0; // ... is moved one to the right
21        return x;
22    }
23 }
24 }
```

The all-zero word is returned as successor of the word whose value is 1. The routine for the predecessor can be started with the all-zero word.

```

1 ulong prev_subset_lexrev_fib(ulong x)
2 {
3     ulong x0 = x & -x; // lowest bit
4     if ( x & (x0<<2) ) // easy case: next higher bit is set
5     {
6         x ^= x0; // clear lowest bit
7         return x;
8     }
9     else
10    {
11        x += x0; // move lowest bit to the left and
12        x |= (x0!=1); // set rightmost bit unless blocked by next bit
13        return x;
14    }
15 }
```

0:	.....	[ 1 1 1 1 1 1 1 ]
1:	....1	[ 1 1 1 1 1 2 ]
2:	...1.	[ 1 1 1 1 2 1 ]
3:	...11	[ 1 1 1 1 3 ]
4:	...1..	[ 1 1 1 2 1 1 ]
5:	...1.1	[ 1 1 1 2 2 ]
6:	...11.	[ 1 1 1 3 1 ]
7:	...111	[ 1 1 1 4 ]
8:	..1...	[ 1 1 2 1 1 1 ]
9:	..1..1	[ 1 1 2 1 2 ]
10:	..1.1.	[ 1 1 2 2 1 ]
11:	..1.11	[ 1 1 2 3 ]
12:	..11..	[ 1 1 3 1 1 ]
13:	..11.1	[ 1 1 3 2 ]
14:	..111.	[ 1 1 4 1 ]
15:	..1111	[ 1 1 5 ]
16:	.1....	[ 1 2 1 1 1 1 ]
17:	.1...1	[ 1 2 1 1 2 ]
18:	.1..1.	[ 1 2 1 2 1 ]
19:	.1..11	[ 1 2 1 3 ]
20:	.1.1..	[ 1 2 2 1 1 ]
21:	.1.1.1	[ 1 2 2 2 ]
22:	.1.11.	[ 1 2 3 1 ]
23:	.1.111	[ 1 2 4 ]
24:	.11...	[ 1 3 1 1 1 ]
25:	.11..1	[ 1 3 1 2 ]
26:	.11.1.	[ 1 3 2 1 ]
27:	.11.11	[ 1 3 3 ]
28:	.111..	[ 1 4 1 1 ]
29:	.111.1	[ 1 4 2 ]
30:	.1111.	[ 1 5 1 ]
31:	.11111	[ 1 6 ]
32:	1.....	[ 2 1 1 1 1 1 ]
33:	1....1	[ 2 1 1 1 2 ]
34:	1...1.	[ 2 1 1 2 1 ]
35:	1...11	[ 2 1 1 3 ]
36:	1..1..	[ 2 1 2 1 1 ]
37:	1..1.1	[ 2 1 2 2 ]
38:	1..11.	[ 2 1 3 1 ]
39:	1..111	[ 2 1 4 ]
40:	1.1...	[ 2 2 1 1 1 ]
41:	1.1..1	[ 2 2 1 2 ]
42:	1.1.1.	[ 2 2 2 1 ]
43:	1.1.11	[ 2 2 3 ]
44:	1.11..	[ 2 3 1 1 ]
45:	1.11.1	[ 2 3 2 ]
46:	1.111.	[ 2 4 1 ]
47:	1.1111	[ 2 5 ]
48:	11....	[ 3 1 1 1 1 ]
49:	11...1	[ 3 1 1 2 ]
50:	11..1.	[ 3 1 2 1 ]
51:	11..11	[ 3 1 3 ]
52:	11.1..	[ 3 2 1 1 ]
53:	11.1.1	[ 3 2 2 ]
54:	11.11.	[ 3 3 1 ]
55:	11.111	[ 3 4 ]
56:	111...	[ 4 1 1 1 ]
57:	111..1	[ 4 1 2 ]
58:	111.1.	[ 4 2 1 ]
59:	111.11	[ 4 3 ]
60:	1111..	[ 5 1 1 ]
61:	1111.1	[ 5 2 ]
62:	11111.	[ 6 1 ]
63:	111111	[ 7 ]

Figure 6: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), lexicographic order. A succession of  $k$  ones, followed by a zero, in the run-length encoding stands for a part  $k+1$ , one trailing zero is implied.

## 4 Compositions

One of the most simple algorithms in combinatorial generation may be the computation of the successor of a composition represented as a list of parts for the lexicographic ordering. In the following let  $z$  be the last element of the composition.

**Algorithm 6** (Next-Comp). *Compute the successor of a composition in lexicographic order.*

1. If there is just one part, stop (this is the last composition).
2. Add 1 to the second last part (and remove  $z$ ) and append  $z - 1$  ones at the end.

**Algorithm 7** (Prev-Comp). *Compute the predecessor of a composition in lexicographic order.*

1. If the number of parts is maximal (composition into all ones), stop.
2. If  $z > 1$ , replace  $z$  by  $z - 1, 1$  (move one unit right); return
3. Otherwise, replace the tail  $y, 1^q$  ( $y$  followed by  $q$  ones) by  $y - 1, q + 1$ .

Figure 6 shows the compositions of 7 in lexicographic order. The corresponding run-length encodings appear in lexicographic order as well.

The algorithm for the successor can be made loopless when care is taken that only ones are left beyond the end of the current composition.

```

1 // FILE: src/comb/composition-nz.h
2 class composition_nz
3 // Compositions of n into positive parts, lexicographic order.
4 {
5     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
6     ulong n_; // composition of n
7     ulong m_; // current composition is into m parts
8
9     ulong next()
10    // Return number of parts of generated composition.
11    // Return zero if the current is the last composition.
12    {
13        if (m_ <= 1) return 0; // current is last
14
15        // [* , Y, Z] --> [* , Y+1, 1, 1, 1, ..., 1] (Z-1 trailing ones)
16        a_[m_-1] += 1;
17        const ulong z = a_[m_];
18        a_[m_] = 1;
19        // all parts a[m+1], a[m+2], ..., a[n] are already ==1
20        m_ += z - 2;
21
22        return m_;
23    }

```

Figure 7 shows two orderings for the compositions. The ordering corresponding to the (complemented) binary Gray code is shown in the left columns. We will call this order *RL-order*, as the compositions correspond to the run-lengths of the binary words in lexicographic order.

0:	111111	[ 7 ]	.	.....	[ 7 ]
1:	11111.	[ 6 1 ]	1.	1.....	[ 1 6 ]
2:	1111..	[ 5 1 1 ]	11.	11.....	[ 1 1 5 ]
3:	1111.1	[ 5 2 ]	111.	111...	[ 1 1 1 4 ]
4:	111..1	[ 4 1 2 ]	1111.	1111..	[ 1 1 1 1 3 ]
5:	111...1	[ 4 1 1 1 ]	11111.	11111..	[ 1 1 1 1 1 2 ]
6:	111.1.	[ 4 2 1 ]	111111.	111111.	[ 1 1 1 1 1 1 1 ]
7:	111.11	[ 4 3 ]	1111.1	1111.1	[ 1 1 1 1 2 1 ]
8:	11..11	[ 3 1 3 ]	1111.11	1111.11	[ 1 1 1 2 2 ]
9:	11..1.	[ 3 1 2 1 ]	1111..1	1111..1	[ 1 1 1 2 1 1 ]
10:	11....	[ 3 1 1 1 1 ]	111..1	111..1	[ 1 1 1 3 1 ]
11:	11...1	[ 3 1 1 2 ]	11..1.	11..1.	[ 1 1 2 3 ]
12:	11..1.1	[ 3 2 2 ]	11..11.	11..11.	[ 1 1 2 1 2 ]
13:	11..1..	[ 3 2 1 1 ]	11..111.	11..111.	[ 1 1 2 1 1 1 ]
14:	11..11.	[ 3 3 1 ]	11..111.	11..111.	[ 1 1 2 2 1 ]
15:	11..111.	[ 3 4 ]	11..1.	11..1.	[ 1 1 3 2 ]
16:	1..1111	[ 2 1 4 ]	11..11.	11..11.	[ 1 1 3 1 1 ]
17:	1..111.	[ 2 1 3 1 ]	11..111.	11..111.	[ 1 1 4 1 ]
18:	1..11..	[ 2 1 2 1 1 ]	1.1...	1.1...	[ 1 2 4 ]
19:	1..1.1.	[ 2 1 2 2 ]	1.11..	1.11..	[ 1 2 1 3 ]
20:	1....1	[ 2 1 1 1 2 ]	1..111.	1..111.	[ 1 2 1 1 2 ]
21:	1....1	[ 2 1 1 1 1 1 ]	1..1111.	1..1111.	[ 1 2 1 1 1 1 ]
22:	1...1.	[ 2 1 1 2 1 ]	1..111.	1..111.	[ 1 2 1 2 1 ]
23:	1...11.	[ 2 1 1 3 ]	1..1.1.	1..1.1.	[ 1 2 2 2 ]
24:	1..1.11	[ 2 2 3 ]	1..1..11	1..1..11	[ 1 2 2 1 1 ]
25:	1..1..1.	[ 2 2 2 1 ]	1..1..1.	1..1..1.	[ 1 2 3 1 ]
26:	1..1...	[ 2 2 1 1 1 ]	1..1..1.	1..1..1.	[ 1 3 3 ]
27:	1..1..1.	[ 2 2 1 2 ]	1..11..	1..11..	[ 1 3 1 2 ]
28:	1..11..1	[ 2 3 2 ]	1..111..	1..111..	[ 1 3 1 1 1 ]
29:	1..11..	[ 2 3 1 1 ]	1..1..1.	1..1..1.	[ 1 3 2 1 ]
30:	1..111.	[ 2 4 1 ]	1....1.	1....1.	[ 1 4 2 ]
31:	1..1111.	[ 2 5 ]	1....11.	1....11.	[ 1 4 1 1 ]
32:	..11111.	[ 1 1 5 ]	1....1.	1....1.	[ 1 5 1 ]
33:	..1111.	[ 1 1 4 1 ]	.1....	.1....	[ 2 5 ]
34:	..11..	[ 1 1 3 1 1 ]	.11...	.11...	[ 2 1 4 ]
35:	..11..1	[ 1 1 3 2 ]	.111..	.111..	[ 2 1 1 3 ]
36:	..1...1	[ 1 1 2 1 2 ]	.1111..	.1111..	[ 2 1 1 1 2 ]
37:	..1...1	[ 1 1 2 1 1 1 ]	.11111..	.11111..	[ 2 1 1 1 1 1 ]
38:	..1..1.	[ 1 1 2 2 1 ]	.111..1	.111..1	[ 2 1 1 2 1 ]
39:	..1..11	[ 1 1 2 3 ]	.11..1.	.11..1.	[ 2 1 2 2 ]
40:	....11	[ 1 1 1 1 3 ]	.11..11	.11..11	[ 2 1 2 1 1 ]
41:	....1.	[ 1 1 1 1 2 1 ]	.11..1.	.11..1.	[ 2 1 3 1 ]
42:	....1.	[ 1 1 1 1 1 1 1 ]	.1..1..	.1..1..	[ 2 2 3 ]
43:	....1.	[ 1 1 1 1 1 2 ]	.1..11.	.1..11.	[ 2 2 1 2 ]
44:	....1.1	[ 1 1 1 2 2 ]	.1..111.	.1..111.	[ 2 2 1 1 1 ]
45:	....1..	[ 1 1 1 2 1 1 ]	.1..1..1	.1..1..1	[ 2 2 2 1 ]
46:	....11.	[ 1 1 1 3 1 ]	.1..1..1.	.1..1..1.	[ 2 3 2 ]
47:	....111.	[ 1 1 1 4 ]	.1..11..	.1..11..	[ 2 3 1 1 ]
48:	.1..111	[ 1 2 4 ]	.1..1..1.	.1..1..1.	[ 2 4 1 ]
49:	.1..11..	[ 1 2 3 1 ]	.1..1..1..	.1..1..1..	[ 3 4 ]
50:	.1..1..1.	[ 1 2 2 1 1 ]	.1..11..	.1..11..	[ 3 1 3 ]
51:	.1..1..1.	[ 1 2 2 2 ]	.1..111..	.1..111..	[ 3 1 1 2 ]
52:	.1...1..1	[ 1 2 1 1 2 ]	.1..1111..	.1..1111..	[ 3 1 1 1 1 ]
53:	.1....1.	[ 1 2 1 1 1 1 ]	.1..11..1	.1..11..1	[ 3 1 2 1 ]
54:	.1...1..1.	[ 1 2 1 2 1 ]	.1..1..1.	.1..1..1.	[ 3 2 2 ]
55:	.1..11..1.	[ 1 2 1 3 ]	.1..1..11..	.1..1..11..	[ 3 2 1 1 ]
56:	.11..111.	[ 1 3 3 ]	.1..1..11.	.1..1..11.	[ 3 3 1 ]
57:	.11..1..1	[ 1 3 2 1 ]	.1...1..1.	.1...1..1.	[ 4 3 ]
58:	.11...1..	[ 1 3 1 1 1 ]	.1...11..1.	.1...11..1.	[ 4 1 2 ]
59:	.11..1..1	[ 1 3 1 2 ]	.1...111..1.	.1...111..1.	[ 4 1 1 1 ]
60:	.111..1..1	[ 1 4 2 ]	.1...1..11..1.	.1...1..11..1.	[ 4 2 1 ]
61:	.111..1..	[ 1 4 1 1 ]	.1...1..1.	.1...1..1.	[ 5 2 ]
62:	.1111..1..	[ 1 5 1 ]	.1...11..1..1.	.1...11..1..1.	[ 5 1 1 ]
63:	.11111..1	[ 1 6 ]	.1...11..1..1..1.	.1...11..1..1..1..1.	[ 6 1 ]

Figure 7: The compositions of 7 together with their run-length encodings as binary words (where dots denote zeros), in an order corresponding to the complemented binary Gray code (left) and in subset-lex order (right). A succession of  $k$  ones, followed by a zero, in the run-length encoding stands for a part  $k+1$ , one trailing zero is implied (left). The roles of ones and zeros are reversed for the subset-lex order (right).

For comparison with the new algorithm we give the (loopless) algorithms. In the following let  $m$  be the number of parts in the composition and  $x, y, z$  the last three parts.

**Algorithm 8** (Next-Comp-RL). *Compute the successor of a composition in RL-order.*

1. If  $m$  is odd: if  $z \geq 2$ , replace  $z$  by  $z - 1, 1$  and return; otherwise ( $z = 1$ ) replace  $y, 1$  by  $y + 1$  and return.
2. If  $m$  is even: if  $y \geq 2$ , replace  $y, z$  by  $y - 1, 1, z$  and return; otherwise ( $y = 1$ ) replace  $x, 1, z$  by  $x + 1, z$  and return.

The next algorithm is obtained from the previous simply by swapping “even” and “odd” in the description.

**Algorithm 9** (Prev-Comp-RL). *Compute the predecessor of a composition in RL-order.*

1. If  $m$  is even: if  $z \geq 2$ , replace  $z$  by  $z - 1, 1$  and return; otherwise ( $z = 1$ ) replace  $y, 1$  by  $y + 1$  and return.
2. If  $m$  is odd: if  $y \geq 2$ , replace  $y, z$  by  $y - 1, 1, z$  and return; otherwise ( $y = 1$ ) replace  $x, 1, z$  by  $x + 1, z$  and return.

With each transition, at most three parts (at the end of the composition) are changed. The number of parts changes by 1 with each step<sup>2</sup>.

An alternative algorithm for this ordering is given in [19, ex.12, sect.7.2.1.1, p.308], see also [24].

Now we give the algorithms for subset-lex order (the first is essentially given in [23]).

**Algorithm 10** (Next-Comp-SL). *Compute the successor of a composition in subset-lex order.*

1. If  $z = 1$  and there are at most two parts, stop.
2. If  $z \geq 2$ , replace  $z$  by  $1, z - 1$  (move all but one unit to the right); return.
3. Otherwise ( $z = 1$ ) add 1 to the third last part and remove  $z$  (move one unit two places left:  $x, y, 1$  is replaced by  $x + 1, y$ ).

For the next algorithm let  $y, z$  be the two last elements.

**Algorithm 11** (Prev-Comp-SL). *Compute the predecessor of a composition in subset-lex order.*

1. If there is just one part, stop.
2. If  $y = 1$ , replace  $y, z$  by  $z + 1$  (add  $z$  to the left); return.
3. Otherwise ( $y \geq 2$ ) replace  $y, z$  by  $y - 1, z, 1$  (move one unit two places right).

At most two parts are changed by either method and these span at most those three parts at the end of the composition. Again, the number of parts changes by 1 with each step.

Note that also the initializations are loopless for the preceding two orderings.

We give the crucial parts of the implementation.

---

<sup>2</sup>See `src/comb/composition-nz-rl.h` for an implementation.

```

1 // FILE: src/comb/composition-nz-subset-lex.h
2 class composition_nz_subset_lex
3 {
4     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // composition of n
6     ulong m_; // current composition is into m parts
7
8     void first()
9     {
10         a_[0] = 0;
11         a_[1] = n_;
12         m_ = ( n_ ? 1 : 0 );
13     }
14
15     void last()
16     {
17         if ( n_ >= 2 )
18         {
19             a_[1] = n_ - 1;
20             a_[2] = 1;
21             for (ulong j=2; j<=n_; ++j) a_[j] = 1;
22             m_ = 2;
23         }
24         else
25         {
26             a_[1] = n_;
27             m_ = n_;
28         }
29     }

```

For the methods `next()` and `prev()` we give one of the two implementations found in the file. Both methods return the number of parts in the generated composition and return zero if there are no more compositions.

```

1     ulong next()
2     {
3         const ulong z = a_[m_];
4         if ( z<=1 ) // move one unit two places left
5         { // [*, X, Y, 1] --> [*, X+1, Y]
6             if ( m_ <= 2 ) return 0; // current is last
7             m_ -= 1;
8             a_[m_-1] += 1;
9             return m_;
10        }
11        else // move all but one unit right
12        { // [*, Y, Z] --> [*, Y, 1, Z-1]
13            a_[m_] = 1;
14            m_ += 1;
15            a_[m_] = z - 1;
16            return m_;
17        }
18    }
19
20    ulong prev()
21    {
22        if ( m_ <= 1 ) return 0; // current is first
23        const ulong y = a_[m_-1];
24        if ( y==1 ) // add Z to left place
25        { // [*, 1, Z] --> [*, Z+1]
26            const ulong z = a_[m_];
27            a_[m_-1] = z + 1;
28            a_[m_] = 1;
29            m_ -= 1;
30            return m_;
31        }
32        else // move one unit two places right
33        { // [*, Y, Z] --> [*, Y-1, Z, 1]
34            a_[m_-1] = y - 1;
35            m_ += 1;
36            // a[m] == 1 already
37            return m_;
38        }
39    }

```

The method `next()` takes about 5 cycles for lexicographic order, 9 cycles for RL-order, and 6 cycles for subset-lex order. The method `prev()` takes about 10.5 cycles for lexicographic order and is identical in performance to `next()` for the other orders.

Note that the last parts in the successive compositions in lexicographic order give the (one-based) ruler function (sequence [A001511](#) in [28]), see `src/comb/ruler-func1.h` for the trivial implementation, compare to `src/comb/ruler-func.h` (giving sequence [A007814](#) in [28]) which uses the techniques from [10] and [3] as described in [19, Algorithm L, p.290].

An loopless implementation for the generation of the compositions into odd parts in subset-lex order is given in `src/comb/composition-nz-odd-subset-lex.h`.

### Ranking and unranking

An unranking algorithm for the subset-lex order is obtained by observing (see figure 7) that the composition into one part has rank 0 and otherwise the first part and the remaining parts are easily determined.

**Algorithm 12** (Unrank-Comp-SL). *Recursive routine  $U(r, n, C[], m)$  for the computation of the composition  $C[]$  of  $n$  with rank  $r$  in subset-lex order. The auxiliary variable  $m \geq 0$  is the index of the part about to be written in  $C[]$ . The initial call is  $F(r, n, C[], 0)$ .*

1. If  $r = 0$ , set  $C[m] = n$  and return  $m + 1$  (the number of parts).
2. Set  $f = 0$  (the tentative first part).
3. Set  $f = f + 1$  and  $t = 2^{n-f-1}$ .
4. If  $r < t$  (can use part  $f$ ), set  $C[m] = f$  and return  $F(r, n - f, C[], m + 1)$ .
5. Set  $r = r - t$  and go to step 3.

We give an iterative implementation.

```

1 // FILE:  src/comb/composition-nz-rank.cc
2 ulong composition_nz_subset_lex_unrank(ulong r, ulong *x, ulong n)
3 {
4     if ( n==0 )  return 0;
5     ulong m = 0;
6     while ( true )
7     {
8         if ( r==0 ) // composition into one part
9         {
10             x[m++] = n;
11             return m;
12         }
13         r -= 1;
14         ulong t = 1UL << (n-1);
15         for (ulong f=1; f<n; ++f) // find first part f >= 1
16         {
17             t >>= 1; // == 2**((n-f)-1)
18             // == number of compositions of n with first part f
19
20             if ( r < t ) // first part is f
21             {
22                 x[m++] = f;
23                 n -= f;
24                 break;
25             }
26             r -= t;
27         }
28     }
29 }
30 }
```

The following algorithm for computing the rank is modeled as inverse of the method above.

**Algorithm 13** (Rank-Comp-SL). *Computation of the rank  $r$  of a composition  $C[]$  of  $n$  in subset-lex order.*

1. Set  $r = 0$  and  $e = 0$  (position of part under consideration).
2. Set  $f = C[e]$  (part under consideration).
3. If  $f = n$ , return  $r$ .
4. Set  $r = r + 1$ ,  $t = 2^{n-1}$ , and  $n = n - f$ .
5. While  $f > 1$ , set  $t = t/2$ ,  $r = r + t$ , and  $f = f - 1$ .
6. Set  $e = e + 1$  and go to step 2

An implementation is

```

1  ulong composition_nz_subset_lex_rank(const ulong *x, ulong m, ulong n)
2  {
3      ulong r = 0; // rank
4      ulong e = 0; // position of first part
5      while (e < m)
6      {
7          ulong f = x[e];
8          if (f==n) return r;
9          r += 1;
10         ulong t = 1UL << (n-1);
11         n -= f;
12         while (f > 1)
13         {
14             t >= 1;
15             r += t;
16             f -= 1;
17         }
18         e += 1;
19     }
20     return r; // return r==0 for the empty composition
21 }
```

Conversion functions between binary words in lexicographic order and subset-lex order are shown in [2, pp.71-72].

```

1 // FILE: src/bits/bitlex.h
2 ulong negidx2lexrev(ulong k)
3 {
4     ulong z = 0;
5     ulong h = highest_one(k);
6     while (k)
7     {
8         while (0==(h&k)) h >>= 1;
9         z ^= h;
10        ++k;
11        k &= h - 1;
12    }
13    return z;
14 }
15
16 ulong lexrev2negidx(ulong x)
17 {
18     if (0==x) return 0;
19     ulong h = x & -x; // lowest one
20     ulong r = (h-1);
21     while (x^=h)
22     {
23         r += (h-1);
24         h = x & -x; // next higher one
25     }
26     r += h; // highest bit
27     return r;
```

```
29 }
```

Based on these, alternative ranking and unranking functions can be given<sup>3</sup>.

Ranking and unranking methods for the compositions into odd parts can be obtained by replacing  $2^{n-f-1}$  (number of compositions of  $n$  with first part  $f$ ) in the algorithms 12 and 13 by the Fibonacci numbers  $F_{n-f-1}$  (number of compositions of  $n$  into odd parts with first part  $f$ ), where  $n \geq 1$ ,  $f < n$ , and  $f$  odd (see sequence A242086 in [28]).

## 5 Partitions as weakly increasing lists of parts

We now give algorithms for the computation of the successor for partitions represented as weakly increasing lists of parts. The generation of all partitions in this representation is the subject of [16].

### 5.1 All partitions

Figure 8 shows the partitions of 11 as weakly increasing lists of parts, in lexicographic order (left) and in subset-lex order (right). We first give a description of the computation of the successor in lexicographic order. The three last parts of the partition are denoted by  $x$ ,  $y$ , and  $z$ .

**Algorithm 14** (Next-Part-Asc). *Compute the successor of a partition in lexicographic order.*

1. If there is just one part, stop.
2. If  $z - 1 < y + 1$ , replace  $y, z$  by  $y + z$ ; return.
3. Otherwise, change  $y$  to  $y + 1$  and append parts  $y + 1$  as long as there are at least  $y + 1$  units left.
4. Add the remaining units to the last part.

The second step of the algorithm can be merged into the other steps, as it is a special case of them.

```
1 // FILE: src/comb/partition-asc.h
2 class partition_asc
3 {
4     ulong *a_; // partition: a[1] + a[2] + ... + a[m_] = n
5     ulong n_; // integer partitions of n
6     ulong m_; // current partition has m parts
```

The implementation is correct for all  $n \geq 0$ , for  $n = 0$  the empty list of parts is generated.

```
1     ulong next()
2     // Return number of parts of generated partition.
3     // Return zero if the current partition is the last.
4     {
5         if (m_ <= 1) return 0; // current is last
6
7         ulong z1 = a_[m_] - 1; // take one unit from last part
8         m_ -= 1;
9         const ulong y1 = a_[m_] + 1; // add one unit to previous part
10
11        while (y1 <= z1) // can put part Y+1
12        {
13            a_[m_] = y1;
```

---

<sup>3</sup>See `src/comb/composition-nz-rank.cc` where the corresponding routines for all three orders shown here are implemented.

1:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$	$[11]$
2:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2]$	$[1 \ 10]$
3:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 3]$	$[1 \ 1 \ 9]$
4:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2]$	$[1 \ 1 \ 1 \ 1 \ 8]$
5:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 4]$	$[1 \ 1 \ 1 \ 1 \ 7]$
6:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 3]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 6]$
7:	$[1 \ 1 \ 1 \ 1 \ 1 \ 5]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 5]$
8:	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 4]$
9:	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 4]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 3]$
10:	$[1 \ 1 \ 1 \ 1 \ 1 \ 3 \ 3]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2]$
11:	$[1 \ 1 \ 1 \ 1 \ 6]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$
12:	$[1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 3]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2]$
13:	$[1 \ 1 \ 1 \ 1 \ 2 \ 5]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 3]$
14:	$[1 \ 1 \ 1 \ 1 \ 3 \ 4]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 4]$
15:	$[1 \ 1 \ 1 \ 1 \ 7]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2]$
16:	$[1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 3 \ 3]$
17:	$[1 \ 1 \ 1 \ 2 \ 2 \ 4]$	$[1 \ 1 \ 1 \ 1 \ 2 \ 5]$
18:	$[1 \ 1 \ 1 \ 2 \ 3 \ 3]$	$[1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 3]$
19:	$[1 \ 1 \ 1 \ 2 \ 6]$	$[1 \ 1 \ 1 \ 1 \ 3 \ 4]$
20:	$[1 \ 1 \ 1 \ 3 \ 5]$	$[1 \ 1 \ 1 \ 2 \ 6]$
21:	$[1 \ 1 \ 1 \ 4 \ 4]$	$[1 \ 1 \ 1 \ 2 \ 2 \ 4]$
22:	$[1 \ 1 \ 1 \ 8]$	$[1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2]$
23:	$[1 \ 1 \ 2 \ 2 \ 2 \ 3]$	$[1 \ 1 \ 1 \ 2 \ 3 \ 3]$
24:	$[1 \ 1 \ 2 \ 2 \ 5]$	$[1 \ 1 \ 1 \ 3 \ 5]$
25:	$[1 \ 1 \ 2 \ 3 \ 4]$	$[1 \ 1 \ 1 \ 4 \ 4]$
26:	$[1 \ 1 \ 2 \ 7]$	$[1 \ 1 \ 2 \ 7]$
27:	$[1 \ 1 \ 3 \ 3 \ 3]$	$[1 \ 1 \ 2 \ 2 \ 5]$
28:	$[1 \ 1 \ 3 \ 6]$	$[1 \ 1 \ 2 \ 2 \ 2 \ 3]$
29:	$[1 \ 1 \ 4 \ 5]$	$[1 \ 1 \ 2 \ 3 \ 4]$
30:	$[1 \ 1 \ 9]$	$[1 \ 1 \ 3 \ 6]$
31:	$[1 \ 2 \ 2 \ 2 \ 2 \ 2]$	$[1 \ 1 \ 3 \ 3 \ 3]$
32:	$[1 \ 2 \ 2 \ 2 \ 4]$	$[1 \ 1 \ 4 \ 5]$
33:	$[1 \ 2 \ 2 \ 3 \ 3]$	$[1 \ 2 \ 8]$
34:	$[1 \ 2 \ 2 \ 6]$	$[1 \ 2 \ 2 \ 6]$
35:	$[1 \ 2 \ 3 \ 5]$	$[1 \ 2 \ 2 \ 2 \ 4]$
36:	$[1 \ 2 \ 4 \ 4]$	$[1 \ 2 \ 2 \ 2 \ 2 \ 2]$
37:	$[1 \ 2 \ 8]$	$[1 \ 2 \ 2 \ 3 \ 3]$
38:	$[1 \ 3 \ 3 \ 4]$	$[1 \ 2 \ 3 \ 5]$
39:	$[1 \ 3 \ 7]$	$[1 \ 2 \ 4 \ 4]$
40:	$[1 \ 4 \ 6]$	$[1 \ 3 \ 7]$
41:	$[1 \ 5 \ 5]$	$[1 \ 3 \ 3 \ 4]$
42:	$[1 \ 10]$	$[1 \ 4 \ 6]$
43:	$[2 \ 2 \ 2 \ 2 \ 3]$	$[1 \ 5 \ 5]$
44:	$[2 \ 2 \ 2 \ 5]$	$[2 \ 9]$
45:	$[2 \ 2 \ 3 \ 4]$	$[2 \ 2 \ 7]$
46:	$[2 \ 2 \ 7]$	$[2 \ 2 \ 2 \ 5]$
47:	$[2 \ 3 \ 3 \ 3]$	$[2 \ 2 \ 2 \ 2 \ 3]$
48:	$[2 \ 3 \ 6]$	$[2 \ 2 \ 3 \ 4]$
49:	$[2 \ 4 \ 5]$	$[2 \ 3 \ 6]$
50:	$[2 \ 9]$	$[2 \ 3 \ 3 \ 3]$
51:	$[3 \ 3 \ 5]$	$[2 \ 4 \ 5]$
52:	$[3 \ 4 \ 4]$	$[3 \ 8]$
53:	$[3 \ 8]$	$[3 \ 3 \ 5]$
54:	$[4 \ 7]$	$[3 \ 4 \ 4]$
55:	$[5 \ 6]$	$[4 \ 7]$
56:	$[11]$	$[5 \ 6]$

Figure 8: The partitions of 11 as weakly increasing lists of parts, lexicographic order (left) and subset-lex order (right).

```

14         z1 -= y1;
15         m_ += 1;
16     }
17     a_[m_] = y1 + z1; // add remaining units to last part
18
19 } return m_;
20

```

The computation of the successor in subset-lex order is loopless.

**Algorithm 15** (Next-Part-Asc-SL). *Compute the successor of a partition in subset-lex order. A sentinel zero shall precede list of elements.*

1. If  $z \geq 2y$ , replace  $y, z$  by  $y, y, z - y$  (extend to the right); return.
2. If  $z - 1 \geq y + 1$ , replace  $y, z$  by  $y + 1, z - 1$  (add one unit to the left); return.
3. If  $z = 1$  (the all-ones partition) do the following. Stop if the number of parts is  $\leq 3$ , otherwise replace the tail  $[1, 1, 1, 1]$  by  $[2, 2]$  and return.
4. If the number of parts is 2, stop.
5. Replace  $x, y, z$  by  $x + 1, y + z - 1$  (add one unit to second left, add rest to end) and return.

Note that with each update all but the last two parts are the same as in the predecessor. This can be an advantage over the lexicographic order for computations where partial results for prefixes can be reused.

```

1 // FILE: src/comb/partition-asc-subset-lex.h
2 class partition_asc_subset_lex
3 {
4     ulong *a_; // partition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // partition of n
6     ulong m_; // current partition has m parts
7
8     explicit partition_asc_subset_lex(ulong n)
9     {
10         n_ = n;
11         a_ = new ulong[n_+1+(n_==0)];
12         // sentinel a[0] set in first()
13         first();
14     }
15
16     void first()
17     {
18         a_[0] = 1; // sentinel: read (once) by test for right-extension
19         a_[1] = n_ + (n_==0); // use partitions of n=1 for n=0 internally
20         m_ = 1;
21     }

```

The implementation is again correct for all  $n \geq 0$ .

```

1     ulong next()
2     // Loopless algorithm.
3     {
4         ulong y = a_[m_-1]; // may read sentinel a[0]
5         ulong z = a_[m_];
6
7         if (z >= 2*y) // extend to the right:
8             { // [* , Y, Z] --> [* , Y, Y, Z-Y]
9                 a_[m_] = y;
10                a_[m_+1] = z - y; // >= y
11                ++m_;
12                return m_;
13            }
14
15         z -= 1; y += 1;
16         if (z >= y) // add one unit to the left:
17             { // [* , Y, Z] --> [* , Y+1, Z-1]
18                 a_[m_-1] = y;

```

```

19         a_[m_] = z;
20         return m_;
21     }
22
23     if ( z == 0 ) // all-ones partition
24     {
25         if ( n_ <= 3 ) return 0; // current is last
26         // [1, ..., 1, 1, 1] --> [1, ..., 2, 2]
27         m_ -= 2;
28         a_[m_] = 2; a_[m_-1] = 2;
29         return m_;
30     }
31
32     // add one unit to second left, add rest to end:
33     // [* , X, Y, Z] --> [* , X+1, Y+Z-1]
34     a_[m_-2] += 1;
35     a_[m_-1] += z;
36     m_ -= 1;
37     m_ -= (m_ == 1); // last if partition is into one part
38     return m_;
39 }
40

```

The updates via `next()` take about 15 cycles for lexicographic order and about 13 cycles for subset-lex order.

## 5.2 Partitions into odd and into distinct parts

It is not very difficult to adapt the algorithms to specialized partitions. We give the algorithms for partitions into distinct and odd parts by their implementations.

Computation of the successor for partitions into distinct parts in lexicographic order:

```

1 // FILE: src/comb/partition-dist-asc.h
2 ulong next()
3 {
4     if ( m_ <= 1 ) return 0; // current is last
5
6     ulong s = a_[m_] + a_[m_-1];
7     ulong k = a_[m_-1] + 1;
8     m_ -= 1;
9     // split s into k, k+1, k+2, ..., y, z where z >= y + 1:
10    while ( s >= ( k + (k+1) ) )
11    {
12        a_[m_] = k;
13        s -= k;
14        k += 1;
15        m_ += 1;
16    }
17
18    a_[m_] = s;
19
20    return m_;
21 }

```

Computation of the successor for partitions into odd parts in lexicographic order:

```

1 // FILE: src/comb/partition-odd-asc.h
2 ulong next()
3 {
4     const ulong z = a_[m_]; // can read sentinel a[0] if n==0
5     const ulong y = a_[m_-1]; // can read sentinel a[0] (a[-1] for n==0)
6     ulong s = y + z; // sum of parts we scan over
7
8     ulong k; // min value of next term
9     if ( z >= y+4 ) // add last 2 terms
10    {
11        if ( m_ == 1 ) return 0; // current is last
12        k = y + 2;
13        a_[m_-1] = k;
14        s -= k;
15    }
16    else // add last 3 terms

```

0:	[ 19 ]	[ 19 ]
1:	[ 1 18 ]	[ 1 1 17 ]
2:	[ 1 2 16 ]	[ 1 1 1 1 15 ]
3:	[ 1 2 3 13 ]	[ 1 1 1 1 1 13 ]
4:	[ 1 2 3 4 9 ]	[ 1 1 1 1 1 1 11 ]
5:	[ 1 2 3 5 8 ]	[ 1 1 1 1 1 1 1 9 ]
6:	[ 1 2 3 6 7 ]	[ 1 1 1 1 1 1 1 1 7 ]
7:	[ 1 2 4 12 ]	[ 1 1 1 1 1 1 1 1 1 1 5 ]
8:	[ 1 2 4 5 7 ]	[ 1 1 1 1 1 1 1 1 1 1 1 1 3 ]
9:	[ 1 2 5 11 ]	[ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 ]
10:	[ 1 2 6 10 ]	[ 1 1 1 1 1 1 1 1 1 1 1 1 3 3 ]
11:	[ 1 2 7 9 ]	[ 1 1 1 1 1 1 1 1 1 3 5 ]
12:	[ 1 3 15 ]	[ 1 1 1 1 1 1 1 1 3 3 3 ]
13:	[ 1 3 4 11 ]	[ 1 1 1 1 1 1 1 1 3 7 ]
14:	[ 1 3 4 5 6 ]	[ 1 1 1 1 1 1 1 1 5 5 ]
15:	[ 1 3 5 10 ]	[ 1 1 1 1 1 1 1 3 5 ]
16:	[ 1 3 6 9 ]	[ 1 1 1 1 1 1 1 3 9 ]
17:	[ 1 3 7 8 ]	[ 1 1 1 1 1 1 1 1 3 3 3 3 ]
18:	[ 1 4 14 ]	[ 1 1 1 1 1 1 1 1 5 7 ]
19:	[ 1 4 5 9 ]	[ 1 1 1 1 1 1 1 3 3 7 ]
20:	[ 1 4 6 8 ]	[ 1 1 1 1 1 1 3 5 5 ]
21:	[ 1 5 13 ]	[ 1 1 1 1 1 3 11 ]
22:	[ 1 5 6 7 ]	[ 1 1 1 1 1 1 3 3 3 5 ]
23:	[ 1 6 12 ]	[ 1 1 1 1 1 5 9 ]
24:	[ 1 7 11 ]	[ 1 1 1 1 1 7 7 ]
25:	[ 1 8 10 ]	[ 1 1 1 1 1 3 3 9 ]
26:	[ 2 17 ]	[ 1 1 1 1 1 3 3 3 3 3 ]
27:	[ 2 3 14 ]	[ 1 1 1 1 1 3 5 7 ]
28:	[ 2 3 4 10 ]	[ 1 1 1 1 1 5 5 5 ]
29:	[ 2 3 5 9 ]	[ 1 1 1 1 3 13 ]
30:	[ 2 3 6 8 ]	[ 1 1 1 1 3 3 3 7 ]
31:	[ 2 4 13 ]	[ 1 1 1 1 3 3 5 5 ]
32:	[ 2 4 5 8 ]	[ 1 1 1 5 11 ]
33:	[ 2 4 6 7 ]	[ 1 1 1 7 9 ]
34:	[ 2 5 12 ]	[ 1 1 3 3 11 ]
35:	[ 2 6 11 ]	[ 1 1 3 3 3 3 5 ]
36:	[ 2 7 10 ]	[ 1 1 3 5 9 ]
37:	[ 2 8 9 ]	[ 1 1 3 7 7 ]
38:	[ 3 16 ]	[ 1 1 5 5 7 ]
39:	[ 3 4 12 ]	[ 1 3 15 ]
40:	[ 3 4 5 7 ]	[ 1 3 3 3 9 ]
41:	[ 3 5 11 ]	[ 1 3 3 3 3 3 3 ]
42:	[ 3 6 10 ]	[ 1 3 3 5 7 ]
43:	[ 3 7 9 ]	[ 1 3 5 5 5 ]
44:	[ 4 15 ]	[ 1 5 13 ]
45:	[ 4 5 10 ]	[ 1 7 11 ]
46:	[ 4 6 9 ]	[ 1 9 9 ]
47:	[ 4 7 8 ]	[ 3 3 13 ]
48:	[ 5 14 ]	[ 3 3 3 3 7 ]
49:	[ 5 6 8 ]	[ 3 3 3 5 5 ]
50:	[ 6 13 ]	[ 3 5 11 ]
51:	[ 7 12 ]	[ 3 7 9 ]
52:	[ 8 11 ]	[ 5 5 9 ]
53:	[ 9 10 ]	[ 5 7 7 ]

Figure 9: The partitions of 19 into distinct parts (left) and odd parts (right) in subset-lex order.

```

17     {
18         if ( m_ <= 2 )  return 0; // current is last
19         const ulong x = a_[m_-2];
20         s += x;
21         k = x + 2;
22         m_ -= 2;
23     }
24
25     const ulong k2 = k + k;
26     while ( s >= k2 + k )
27     {
28         a_[m_] = k;  s -= k;  m_ += 1;
29         a_[m_] = k;  s -= k;  m_ += 1;
30     }
31
32     a_[m_] = s;
33     return m_;
34 }
```

All routines in this section return the number of parts in the generated partition and return zero if there are no more partitions.

Computation of the successor for partitions into distinct parts in subset-lex order:

```

1 // FILE: src/comb/partition-dist-asc-subset-lex.h
2     ulong next()
3     // Loopless algorithm.
4     {
5         ulong y = a_[m_-1]; // may read sentinel a[0]
6         ulong z = a_[m_];
7
8         if ( z >= 2*y + 3 ) // can extend to the right
9         { // [*, Y, Z] --> [*, Y, Y+1, Z-1]
10            y += 1;
11            a_[m_] = y;
12            a_[m_+1] = z - y; // >= y
13            ++m_;
14            return m_;
15        }
16        else // add to the left
17        {
18            z -= 1; y += 1;
19
20            if ( z > y ) // add one unit to the left
21            { // [*, Y, Z] --> [*, Y+1, Z-1]
22
23                if ( m_<=1 ) return 0; // current is last
24
25                a_[m_-1] = y;
26                a_[m_] = z;
27                return m_;
28            }
29            else // add to one unit second left
30            // and combine last with second last
31            { // [*, X, Y, Z] --> [*, X+1, Y+Z]
32
33                if ( m_<=2 ) return 0; // current is last
34
35                a_[m_-2] += 1;
36                a_[m_-1] += z;
37                --m_;
38                return m_;
39            }
40        }
41    }
```

Computation of the successor for partitions into odd parts in subset-lex order:

```

1 // FILE: src/comb/partitions-odd-asc-subset-lex.h
2     ulong next()
3     // Loopless algorithm.
4     {
5         ulong y = a_[m_-1]; // may read sentinel a[0]
6         ulong z = a_[m_];
7
```

```

8      if ( z >= 3*y ) // can extend to the right
9      { // [*, Y, Z] --> [*, Y, Y, Y, Z-2*Y]
10         a_[m_] = y;
11         a_[m_+1] = y;
12         a_[m_+2] = z - 2 * y;
13         m_ += 2;
14         return m_;
15     }
16
17     if ( m_ >= n_ ) // all-ones partition
18     {
19         if ( n_ <= 5 ) return 0; // current is last
20         // [1, ..., 1, 1, 1, 1, 1] --> [1, ..., 3, 3]
21         m_ -= 4;
22         a_[m_] = 3;
23         a_[m_-1] = 3;
24         return m_;
25     }
26
27     ulong z2 = z - 2;
28     ulong y2 = y + 2;
29
30     if ( z2 >= y2 ) // add 2 units to the left
31     { // [*, Y, Z] --> [*, Y+2, Z-2]
32         a_[m_-1] = y2;
33         a_[m_] = z2;
34         return m_;
35     }
36
37     if ( m_==2 ) // current is last (happens only for n even)
38         return 0;
39
40     // here m >= 3; add to second or third left:
41
42     ulong x2 = a_[m_-2] + 2;
43     ulong s = z + y - 2;
44
45     if ( x2 <= z2 )
46         // add 2 units to third left, repeat part, put rest at end
47         { // [*, X, Y, Z] --> [*, X+2, X+2, Y+Z-2-X-2]
48             a_[m_-2] = x2;
49             a_[m_-1] = x2;
50             a_[m_] = s - x2;
51             return m_;
52         }
53
54     if ( m_==3 ) // current is last (happens only for n odd)
55         return 0;
56
57     // add 2 units to third left, combine rest into second left
58     // [*, W, X, Y, Z] --> [*, W+2, X+Y+Z-2]
59     a_[m_-3] += 2;
60     a_[m_-2] += s;
61     m_ -= 2;
62     return m_;
63 }

```

The updates via `next()` of partitions into distinct parts take about 11 cycles for lexicographic order and about 9 cycles for subset-lex order. The respective figures for the partitions into odd parts are 13 and 13.5 cycles.

Algorithms for ranking and unranking are obtained by replacing  $2^{n-f-1}$  (the number of compositions of  $n$  with first part  $f$ ) in algorithms 12 and 13 by the expression for the number of partitions of  $n$  (of the desired kind) with first part  $f$ .

## 6 Subset-lex order for restricted growth strings (RGS)

By modifying the algorithms for the successor and predecessor (3 and 4) for subset-lex order to adhere to certain conditions, one obtains the equivalents for RGS. These can often be given without any difficulty. We give two examples, RGS for set partitions and RGS for  $k$ -ary Dyck words.

### 6.1 RGS for set partitions

We consider the restricted growth strings (RGS)  $a_0, a_1, \dots, a_{n-1}$  such that  $a_0 = 0$  and  $a_k \leq 1 + \max(a_0, a_1, \dots, a_{k-1})$ . These RGS are counted by the Bell numbers, see sequence [A000110](#) in [28]. Figure 10 shows the all such RGS of length 5 in lexicographic and subset-lex order. The set partitions are obtained by putting all  $i$  such that  $a_i = k$  into the same set.

In the following algorithm we assume that  $m_k = 1 + \max(a_0, a_1, \dots, a_{k-1})$  and that there is a sentinel  $a_{-1} = 1$ .

**Algorithm 16** (Next-Setpart-RGS). *Compute the successor in subset-lex order.*

1. Let  $j$  be the index of the last nonzero digit ( $j = 1$  for the all-zero RGS).
2. If  $a_j < m_j$ , set  $a_j = a_j + 1$  and return.
3. If  $j + 1 < n$ , set  $a_{j+1} = 1$  and return.
4. Set  $a_j = 0$ .
5. Set  $j$  to the position of the next nonzero digit to the left. If  $j = -1$ , stop.
6. Set  $a_j = a_j - 1$  and  $a_{j+1} = 1$ .

The implementation has to take care of updating the array  $m[]$  which has an additional (write-only) element at its end.

```

1 // FILE: src/comb/setpart-rgs-subset-lex.h
2 class setpart_rgssubset_lex
3 {
4     ulong *a_; // digits of the RGS
5     ulong *m_; // maximum + 1 in prefix
6     ulong tr_; // current track
7     ulong n_; // number of digits in RGS

```

The computation of the successor is correct for all  $n \geq 0$ , we omit the constructor, which sets  $m_0 = 1$  for  $n = 0$  to cover this case.

```

1     bool next()
2     {
3         ulong j = tr_;
4         if ( a_[j] < m_[j] ) // easy case 1: can increment track
5         {
6             if ( n_ <= 1 ) return false; // handle n <= 1 correctly
7             a_[j] += 1;
8             return true;
9         }
10        const ulong j1 = j + 1;
11        if ( j1 < n_ ) // easy case 2: can attach
12        {
13            m_[j1] = m_[j] + 1;
14            a_[j1] = +1;
15            tr_ = j1;
16            return true;
17        }
18        a_[j] = 0;
19        m_[j] = m_[j-1];
20    }

```

	lexicographic	subset-lex
1:	[ . . . . . ]	{1 2 3 4 5}
2:	[ . . . . 1 ]	{1 2 3 4} {5}
3:	[ . . . 1 . ]	{1 2 3 5} {4}
4:	[ . . . 1 1 ]	{1 2 3} {4 5}
5:	[ . . . 1 2 ]	{1 2 3} {4} {5}
6:	[ . . 1 . . ]	{1 2 4 5} {3}
7:	[ . . 1 . 1 ]	{1 2 4} {3 5}
8:	[ . . 1 . 2 ]	{1 2 4} {3} {5}
9:	[ . . 1 1 . ]	{1 2 5} {3 4}
10:	[ . . 1 1 1 ]	{1 2} {3 4 5}
11:	[ . . 1 1 2 ]	{1 2} {3 4} {5}
12:	[ . . 1 2 . ]	{1 2 5} {3} {4}
13:	[ . . 1 2 1 ]	{1 2} {3 5} {4}
14:	[ . . 1 2 2 ]	{1 2} {3} {4 5}
15:	[ . . 1 2 3 ]	{1 2} {3} {4} {5}
16:	[ . 1 . . . ]	{1 3 4 5} {2}
17:	[ . 1 . . 1 ]	{1 3 4} {2 5}
18:	[ . 1 . . 2 ]	{1 3 4} {2} {5}
19:	[ . 1 . 1 . ]	{1 3 5} {2 4}
20:	[ . 1 . 1 1 ]	{1 3} {2 4 5}
21:	[ . 1 . 1 2 ]	{1 3} {2 4} {5}
22:	[ . 1 . 2 . ]	{1 3 5} {2} {4}
23:	[ . 1 . 2 1 ]	{1 3} {2 5} {4}
24:	[ . 1 . 2 2 ]	{1 3} {2} {4 5}
25:	[ . 1 . 2 3 ]	{1 3} {2} {4} {5}
26:	[ . 1 1 . . ]	{1 4 5} {2 3}
27:	[ . 1 1 . 1 ]	{1 4} {2 3 5}
28:	[ . 1 1 . 2 ]	{1 4} {2 3} {5}
29:	[ . 1 1 1 . ]	{1 5} {2 3 4}
30:	[ . 1 1 1 1 ]	{1} {2 3 4 5}
31:	[ . 1 1 1 2 ]	{1} {2 3 4} {5}
32:	[ . 1 1 2 . ]	{1 5} {2 3} {4}
33:	[ . 1 1 2 1 ]	{1} {2 3 5} {4}
34:	[ . 1 1 2 2 ]	{1} {2 3} {4 5}
35:	[ . 1 1 2 3 ]	{1} {2 3} {4} {5}
36:	[ . 1 2 . . ]	{1 4 5} {2} {3}
37:	[ . 1 2 . 1 ]	{1 4} {2 5} {3}
38:	[ . 1 2 . 2 ]	{1 4} {2} {3 5}
39:	[ . 1 2 . 3 ]	{1 4} {2} {3} {5}
40:	[ . 1 2 1 . ]	{1 5} {2 4} {3}
41:	[ . 1 2 1 1 ]	{1} {2 4 5} {3}
42:	[ . 1 2 1 2 ]	{1} {2 4} {3 5}
43:	[ . 1 2 1 3 ]	{1} {2 4} {3} {5}
44:	[ . 1 2 2 . ]	{1 5} {2} {3 4}
45:	[ . 1 2 2 1 ]	{1} {2 5} {3 4}
46:	[ . 1 2 2 2 ]	{1} {2} {3 4 5}
47:	[ . 1 2 2 3 ]	{1} {2} {3 4} {5}
48:	[ . 1 2 3 . ]	{1 5} {2} {3} {4}
49:	[ . 1 2 3 1 ]	{1} {2 5} {3} {4}
50:	[ . 1 2 3 2 ]	{1} {2} {3 5} {4}
51:	[ . 1 2 3 3 ]	{1} {2} {3} {4 5}
52:	[ . 1 2 3 4 ]	{1} {2} {3} {4} {5}

Figure 10: Restricted growth strings and corresponding set partitions in lexicographic and subset-lex order.

```

22
23     // Find nonzero track to the left:
24     do { --j; } while ( a_[j] == 0 ); // can read sentinel
25
26     if ( (long)j < 0 ) return false; // current is last
27
28     if ( a_[j] == m_[j] ) m_[j+1] = m_[j];
29     a_[j] -= 1;
30
31     ++j;
32     a_[j] = 1;
33     tr_ = j;
34
35     return true;
}

```

An update takes about 9.5 cycles for subset-lex order and 12.5 cycles for lexicographic order.

## 6.2 RGS for $k$ -ary Dyck words

Figure 11 shows the 55 RGS for the 3-ary Dyck words of length 12, in both lexicographic and subset-lex order. The  $j$ th value in each RGS contains the distance of the position  $j$ th one in the Dyck word from its maximal value  $k \cdot j$ . The sequences of numbers of  $k$ -ary Dyck words are entries [A000108](#) ( $k = 2$ , Catalan numbers), [A001764](#) ( $k = 3$ ), [A002293](#) ( $k = 4$ ), and [A002294](#) ( $k = 5$ ) in [28].

It shall suffice to give the implementation for the (loopless) computation of the predecessor in subset-lex order. A sentinel  $a[-1]=+1$  is used.

```

1 // FILE: src/comb/dyck-rgs-subset-lex.h
2 class dyck_rgssubset_lex
3 {
4     ulong *a_; // digits of the RGS: a_[k] <= as[k-1] + 1
5     ulong tr_; // current track
6     ulong n_; // number of digits in RGS
7     ulong i_; // k-ary Dyck words: i = k - 1
8
9     void last()
10    {
11        for (ulong k=0; k<n_; ++k) a_[k] = 0;
12        tr_ = n_ - 1;
13        // make things work for n <= 1:
14        if ( n_==0 )
15        {
16            tr_ = 0;
17            a_[0] = 1;
18        }
19        if ( n_>=2 ) a_[tr_] = i_;
20    }
}

```

All cases  $n \geq 0$  are handled correctly.

```

1     bool prev()
2     // Loopless algorithm.
3     {
4         if ( n_<=1 ) return false; // just one RGS
5
6         ulong j = tr_;
7         if ( a_[j] > 1 ) // can decrement track
8         {
9             a_[j] -= 1;
10            return true;
11        }
12
13        const ulong aj = a_[j]; // zero or one
14
15        a_[j] = 0;
16        --j;
17
18        if ( a_[j] == a_[j-1] + i_ ) // move track to the left
19        {

```

1:	[ . . . . ]	1..1..1..1..	[ . . . . ]	1..1..1..1..
2:	[ . . . 1 ]	1..1..1.1...	[ . 1 . . ]	1.1..1..1..
3:	[ . . . 2 ]	1..1..11...	[ . 2 . . ]	11...1..1..
4:	[ . . 1 . ]	1..1..1..1..	[ . 2 1 . ]	11..1..1..1..
5:	[ . . 1 1 ]	1..1..1..1..	[ . 2 2 . ]	11..1..1..1..
6:	[ . . 1 2 ]	1..1..1..1....	[ . 2 3 . ]	11..1....1..
7:	[ . . 1 3 ]	1..1..11...	[ . 2 4 . ]	111.....1..
8:	[ . . 2 . ]	1..11..1..	[ . 2 4 1 ]	111....1..
9:	[ . . 2 1 ]	1..11..1..	[ . 2 4 2 ]	111....1..
10:	[ . . 2 2 ]	1..11..1....	[ . 2 4 3 ]	111...1....
11:	[ . . 2 3 ]	1..11..1....	[ . 2 4 4 ]	111..1....
12:	[ . . 2 4 ]	1..111....	[ . 2 4 5 ]	111.1....
13:	[ . 1 . . ]	1..1..1..1..	[ . 2 4 6 ]	1111.....
14:	[ . 1 .. 1 ]	1..1..1..1..	[ . 2 3 1 ]	11..1....1..
15:	[ . 1 .. 2 ]	1..1..11...	[ . 2 3 2 ]	11..1..1....
16:	[ . 1 1 .. ]	1..1..1..1..	[ . 2 3 3 ]	11..1..1....
17:	[ . 1 1 1 ]	1..1..1..1..	[ . 2 3 4 ]	11..1..1....
18:	[ . 1 1 2 ]	1..1..1..1....	[ . 2 3 5 ]	11..11.....
19:	[ . 1 1 3 ]	1..1..11...	[ . 2 2 1 ]	11..1..1..1..
20:	[ . 1 2 .. ]	1..1..1..1..	[ . 2 2 2 ]	11..1..1..1..
21:	[ . 1 2 1 ]	1..1..1..1..	[ . 2 2 3 ]	11..1..1..1..
22:	[ . 1 2 2 ]	1..1..1..1....	[ . 2 2 4 ]	11..11.....
23:	[ . 1 2 3 ]	1..1..1..1....	[ . 2 1 1 ]	11...1..1..
24:	[ . 1 2 4 ]	1..1..11....	[ . 2 1 2 ]	11...1..1..1..
25:	[ . 1 3 .. ]	1..11....1..	[ . 2 1 3 ]	11...11....
26:	[ . 1 3 1 ]	1..11....1..	[ . 2 .. 1 ]	11...1..1..1..
27:	[ . 1 3 2 ]	1..11....1..	[ . 2 .. 2 ]	11...11....
28:	[ . 1 3 3 ]	1..11....1..	[ . 1 1 .. ]	1..1..1..1..1..
29:	[ . 1 3 4 ]	1..11..1....	[ . 1 2 .. ]	1..1..1..1..1..
30:	[ . 1 3 5 ]	1..111....	[ . 1 3 .. ]	1..11..1..1..
31:	[ . 2 .. . ]	11...1..1..	[ . 1 3 1 ]	1..11..1..1..
32:	[ . 2 .. 1 ]	11...1..1..1..	[ . 1 3 2 ]	1..11..1..1..
33:	[ . 2 .. 2 ]	11...11....	[ . 1 3 3 ]	1..11..1..1..
34:	[ . 2 1 .. ]	11..1..1..1..	[ . 1 3 4 ]	1..11..1....
35:	[ . 2 1 1 ]	11..1..1..1..	[ . 1 3 5 ]	1..111..1....
36:	[ . 2 1 2 ]	11..1..1..1..	[ . 1 2 1 ]	1..1..1..1..1..
37:	[ . 2 1 3 ]	11..11....	[ . 1 2 2 ]	1..1..1..1..1..
38:	[ . 2 2 .. ]	11..1..1..1..	[ . 1 2 3 ]	1..1..1..1..1..
39:	[ . 2 2 1 ]	11..1..1..1..	[ . 1 2 4 ]	1..1..11.....
40:	[ . 2 2 2 ]	11..1..1..1..	[ . 1 1 1 ]	1..1..1..1..1..
41:	[ . 2 2 3 ]	11..1..1..1..	[ . 1 1 2 ]	1..1..1..1..1..
42:	[ . 2 2 4 ]	11..11....	[ . 1 1 3 ]	1..1..11....
43:	[ . 2 3 .. ]	11..1..1..1..	[ . 1 .. 1 ]	1..1..1..1..1..
44:	[ . 2 3 1 ]	11..1..1..1..	[ . 1 .. 2 ]	1..1..11....
45:	[ . 2 3 2 ]	11..1..1..1..	[ . .. 1 ]	1..1..1..1..1..
46:	[ . 2 3 3 ]	11..1..1..1..	[ . .. 2 ]	1..11..1..1..
47:	[ . 2 3 4 ]	11..1..1..1..	[ . .. 2 1 ]	1..11..1..1..
48:	[ . 2 3 5 ]	11..11....	[ . .. 2 2 ]	1..11..1....
49:	[ . 2 4 .. ]	111....1..	[ . .. 2 3 ]	1..11..1....
50:	[ . 2 4 1 ]	111....1..	[ . .. 2 4 ]	1..111..1....
51:	[ . 2 4 2 ]	111....1....	[ . .. 1 1 ]	1..1..1..1..1..
52:	[ . 2 4 3 ]	111...1....	[ . .. 1 2 ]	1..1..1..1..1..
53:	[ . 2 4 4 ]	111..1....	[ . .. 1 3 ]	1..1..11....
54:	[ . 2 4 5 ]	111..1....	[ . .. 1 .. ]	1..1..1..1..1..
55:	[ . 2 4 6 ]	1111.....	[ . .. 2 .. ]	1..1..11....

Figure 11: The 55 RGS for the 3-ary Dyck words of length 12, in lexicographic order (left) and subset-lex order (right).

```

20         --tr_;
21         return true;
22     }
23
24     if ( j==0 ) // current or next is last
25     {
26         if ( aj == 0 ) return false;
27         return true;
28     }
29
30     a_[j] += 1; // increment left digit
31     tr_ = n_ - 1; // move to right end
32     a_[tr_] = a_[tr_-1] + i_; // set to max value
33     return true;
34 }
```

One update takes about 10 cycles for both lexicographic and subset-lex order.

For the generation of other restricted growth strings in subset-lex order, see the following files.

```

// Ascent sequences, see OEIS sequence A022493:
src/comb/ascent-rgs-subset-lex.h
src/comb/ascent-rgs.h // lexicographic order

// RGS for Catalan objects, see OEIS sequence A000108:
src/comb/catalan-rgs-subset-lex.h (loopless prev())
src/comb/catalan-rgs.h // lexicographic order

// Standard Young tableaux, represented as ballot sequences,
// see OEIS sequence A000085:
src/comb/young-tab-rgs-subset-lex.h
src/comb/young-tab-rgs.h // lexicographic order
```

## 7 A variant of the subset-lex order

An order obtained by processing the positions in the characteristic words as for subset-lex order but with reversed priorities is shown in figure 12. We will call this ordering *subset-lexrev*. The algorithms for computing successor and predecessor are easily obtained from those for the subset-lex order, we just give the implementation for `prev()`, which is (again) loopless.

```

1  // FILE: src/comb/mixedradix-subset-lexrev.h
2  bool prev()
3  {
4      ulong j = tr_;
5      if ( a_[j] > 1 ) // easy case: just decrement
6      {
7          a_[j] -= 1;
8          return true;
9      }
10
11     a_[j] = 0;
12     ++j; // now looking at next track to the right
13
14     if ( j >= n_ ) // was on rightmost track (last two steps)
15     {
16         bool q = ( a_[j] != 0 );
17         a_[j] = 0;
18         return q;
19     }
20
21     if ( a_[j] == m1_[j] ) // semi-easy case: move track to left
22     {
23         tr_ = j; // move track one right
24         return true;
25     }
26     else
27     {
28         a_[j] += 1; // increment digit to the right
29         j = 0;
```

0:	[ . . . . ]	{ }
1:	[ . . . . 1 ]	{ 3 }
2:	[ . . . . 2 ]	{ 3, 3 }
3:	[ . . . . 3 ]	{ 3, 3, 3 }
4:	[ . . . 1 3 ]	{ 3, 3, 3, 2 }
5:	[ . . . 2 3 ]	{ 3, 3, 3, 2, 2 }
6:	[ . . 1 2 3 ]	{ 3, 3, 3, 2, 2, 1 }
7:	[ . 2 2 3 ]	{ 3, 3, 3, 2, 2, 1, 1 }
8:	[ 1 2 2 3 ]	{ 3, 3, 3, 2, 2, 1, 1, 0 }
9:	[ 1 1 2 3 ]	{ 3, 3, 3, 2, 2, 1, 0 }
10:	[ 1 . 2 3 ]	{ 3, 3, 3, 2, 2, 0 }
11:	[ . 1 1 3 ]	{ 3, 3, 3, 2, 1 }
12:	[ . 2 1 3 ]	{ 3, 3, 3, 2, 1, 1 }
13:	[ 1 2 1 3 ]	{ 3, 3, 3, 2, 1, 1, 0 }
14:	[ 1 1 1 3 ]	{ 3, 3, 3, 2, 1, 0 }
15:	[ 1 . 1 3 ]	{ 3, 3, 3, 2, 0 }
16:	[ . 1 . 3 ]	{ 3, 3, 3, 1 }
17:	[ . 2 . 3 ]	{ 3, 3, 3, 1, 1 }
18:	[ 1 2 . 3 ]	{ 3, 3, 3, 1, 1, 0 }
19:	[ 1 1 . 3 ]	{ 3, 3, 3, 1, 0 }
20:	[ 1 . . 3 ]	{ 3, 3, 3, 0 }
21:	[ . . 1 2 ]	{ 3, 3, 2 }
22:	[ . . 2 2 ]	{ 3, 3, 2, 2 }
23:	[ . 1 2 2 ]	{ 3, 3, 2, 2, 1 }
24:	[ . 2 2 2 ]	{ 3, 3, 2, 2, 1, 1 }
25:	[ 1 2 2 2 ]	{ 3, 3, 2, 2, 1, 1, 0 }
26:	[ 1 1 2 2 ]	{ 3, 3, 2, 2, 1, 0 }
27:	[ 1 . 2 2 ]	{ 3, 3, 2, 2, 0 }
28:	[ . 1 1 2 ]	{ 3, 3, 2, 1 }
29:	[ . 2 1 2 ]	{ 3, 3, 2, 1, 1 }
30:	[ 1 2 1 2 ]	{ 3, 3, 2, 1, 1, 0 }
31:	[ 1 1 1 2 ]	{ 3, 3, 2, 1, 0 }
32:	[ 1 . 1 2 ]	{ 3, 3, 2, 0 }
33:	[ . 1 . 2 ]	{ 3, 3, 1 }
34:	[ . 2 . 2 ]	{ 3, 3, 1, 1 }
35:	[ 1 2 . 2 ]	{ 3, 3, 1, 1, 0 }
36:	[ 1 1 . 2 ]	{ 3, 3, 1, 0 }
37:	[ 1 . . 2 ]	{ 3, 3, 0 }
38:	[ . . 1 1 ]	{ 3, 2 }
39:	[ . . 2 1 ]	{ 3, 2, 2 }
40:	[ . 1 2 1 ]	{ 3, 2, 2, 1 }
41:	[ . 2 2 1 ]	{ 3, 2, 2, 1, 1 }
42:	[ 1 2 2 1 ]	{ 3, 2, 2, 1, 1, 0 }
43:	[ 1 1 2 1 ]	{ 3, 2, 2, 1, 0 }
44:	[ 1 . 2 1 ]	{ 3, 2, 2, 0 }
45:	[ . 1 1 1 ]	{ 3, 2, 1 }
46:	[ . 2 1 1 ]	{ 3, 2, 1, 1 }
47:	[ 1 2 1 1 ]	{ 3, 2, 1, 1, 0 }
48:	[ 1 1 1 1 ]	{ 3, 2, 1, 0 }
49:	[ 1 . 1 1 ]	{ 3, 2, 0 }
50:	[ . 1 . 1 ]	{ 3, 1 }
51:	[ . 2 . 1 ]	{ 3, 1, 1 }
52:	[ 1 2 . 1 ]	{ 3, 1, 1, 0 }
53:	[ 1 1 . 1 ]	{ 3, 1, 0 }
54:	[ 1 . . 1 ]	{ 3, 0 }
55:	[ . . 1 . ]	{ 2 }
56:	[ . . 2 . ]	{ 2, 2 }
57:	[ . 1 2 . ]	{ 2, 2, 1 }
58:	[ . 2 2 . ]	{ 2, 2, 1, 1 }
59:	[ 1 2 2 . ]	{ 2, 2, 1, 1, 0 }
60:	[ 1 1 2 . ]	{ 2, 2, 1, 0 }
61:	[ 1 . 2 . ]	{ 2, 2, 0 }
62:	[ . 1 1 . ]	{ 2, 1 }
63:	[ . 2 1 . ]	{ 2, 1, 1 }
64:	[ 1 2 1 . ]	{ 2, 1, 1, 0 }
65:	[ 1 1 1 . ]	{ 2, 1, 0 }
66:	[ 1 . 1 . ]	{ 2, 0 }
67:	[ . 1 . . ]	{ 1 }
68:	[ . 2 . . ]	{ 1, 1 }
69:	[ 1 2 . . ]	{ 1, 1, 0 }
70:	[ 1 1 . . ]	{ 1, 0 }
71:	[ 1 . . . ]	{ 0 }

Figure 12: Subsets of the set  $\{0^1, 1^2, 2^2, 3^3\}$  in subset-lexrev order. Dots denote zeros in the (generalized) characteristic words. Note the sets are printed starting with the largest element.

	lex	colex	subset-lexrev	subset-lex
1:	[ . . . . . ]	[ . . . . . ]	[ . . . . . ]	[ . . . . . ]
2:	[ . . . . 1 ]	[ . . . . 1 ]	[ . . . . 1 ]	[ . 1 2 3 3 ]
3:	[ . . . . 2 ]	[ . . . . 1 1 ]	[ . . . . 2 ]	[ . 1 2 3 4 ]
4:	[ . . . . 3 ]	[ . . . 1 1 1 ]	[ . . . . 3 ]	[ . 1 2 2 2 ]
5:	[ . . . . 4 ]	[ . . 1 1 1 1 ]	[ . . . . 4 ]	[ . 1 2 2 3 ]
6:	[ . . . 1 1 ]	[ . . . . 2 ]	[ . . . . 1 4 ]	[ . 1 2 2 4 ]
7:	[ . . . 1 2 ]	[ . . . . 1 2 ]	[ . . . . 2 4 ]	[ . 1 1 3 3 ]
8:	[ . . . 1 3 ]	[ . . . 1 1 2 ]	[ . . . . 3 4 ]	[ . 1 1 3 4 ]
9:	[ . . . 1 4 ]	[ . . 1 1 1 2 ]	[ . . . . 1 3 4 ]	[ . 1 1 2 2 ]
10:	[ . . . 2 2 ]	[ . . . . 2 2 ]	[ . . . . 2 3 4 ]	[ . 1 1 2 3 ]
11:	[ . . . 2 3 ]	[ . . 1 2 2 2 ]	[ . . . . 1 2 3 4 ]	[ . 1 1 2 4 ]
12:	[ . . . 2 4 ]	[ . 1 1 2 2 2 ]	[ . . . . 1 1 3 4 ]	[ . 1 1 1 1 ]
13:	[ . . . 3 3 ]	[ . 2 2 2 2 2 ]	[ . . . . 1 2 4 ]	[ . 1 1 1 2 ]
14:	[ . . . 3 4 ]	[ . 1 2 2 2 2 ]	[ . . . . 2 2 4 ]	[ . 1 1 1 3 ]
15:	[ . . . 1 1 1 ]	[ . . . . 3 ]	[ . . . . 1 2 2 4 ]	[ . 1 1 1 4 ]
16:	[ . . . 1 1 2 ]	[ . . . . 1 3 ]	[ . . . . 1 1 2 4 ]	[ . 2 3 3 ]
17:	[ . . . 1 1 3 ]	[ . . . . 1 1 3 ]	[ . . . . 1 1 4 ]	[ . 2 3 4 ]
18:	[ . . . 1 1 4 ]	[ . . . 1 1 1 3 ]	[ . . . . 1 1 1 4 ]	[ . 2 2 2 ]
19:	[ . . . 1 2 2 ]	[ . . . . 2 3 ]	[ . . . . 1 3 ]	[ . 2 2 3 ]
20:	[ . . . 1 2 3 ]	[ . . . . 1 2 3 ]	[ . . . . 2 3 ]	[ . 2 2 4 ]
21:	[ . . . 1 2 4 ]	[ . . 1 1 2 3 ]	[ . . . . 3 3 ]	[ . 1 3 3 ]
22:	[ . . . 1 3 3 ]	[ . . . 2 2 3 ]	[ . . . . 1 3 3 ]	[ . 1 3 4 ]
23:	[ . . . 1 3 4 ]	[ . . 1 2 2 3 ]	[ . . . . 2 3 3 ]	[ . 1 2 2 ]
24:	[ . . . 2 2 2 ]	[ . . . . 3 3 ]	[ . . . . 1 2 3 3 ]	[ . 1 2 3 ]
25:	[ . . . 2 2 3 ]	[ . . . . 1 3 3 ]	[ . . . . 1 1 3 3 ]	[ . 1 2 4 ]
26:	[ . . . 2 2 4 ]	[ . . . . 1 1 3 3 ]	[ . . . . 1 2 3 ]	[ . 1 1 1 ]
27:	[ . . . 2 3 3 ]	[ . . . . 2 3 3 ]	[ . . . . 2 2 3 ]	[ . 1 1 2 ]
28:	[ . . . 2 3 4 ]	[ . . . . 1 2 3 3 ]	[ . . . . 1 2 2 3 ]	[ . 1 1 3 ]
29:	[ . . . 1 1 1 1 ]	[ . . . . 4 ]	[ . . . . 1 1 2 3 ]	[ . 1 1 4 ]
30:	[ . . . 1 1 1 2 ]	[ . . . . 1 4 ]	[ . . . . 1 1 3 ]	[ . 3 3 ]
31:	[ . . . 1 1 1 3 ]	[ . . . . 1 1 4 ]	[ . . . . 1 1 1 3 ]	[ . 3 4 ]
32:	[ . . . 1 1 1 4 ]	[ . . . . 1 1 1 4 ]	[ . . . . 1 2 ]	[ . 2 2 ]
33:	[ . . . 1 1 2 2 ]	[ . . . . 2 4 ]	[ . . . . 2 2 ]	[ . 2 3 ]
34:	[ . . . 1 1 2 3 ]	[ . . . . 1 2 4 ]	[ . . . . 1 2 2 ]	[ . 2 4 ]
35:	[ . . . 1 1 2 4 ]	[ . . . . 1 1 2 4 ]	[ . . . . 2 2 2 ]	[ . 1 1 ]
36:	[ . . . 1 1 3 3 ]	[ . . . . 2 2 4 ]	[ . . . . 1 2 2 2 ]	[ . 1 2 ]
37:	[ . . . 1 1 3 4 ]	[ . . . . 1 2 2 4 ]	[ . . . . 1 1 2 2 ]	[ . 1 3 ]
38:	[ . . . 1 2 2 2 ]	[ . . . . 3 4 ]	[ . . . . 1 1 2 ]	[ . 1 4 ]
39:	[ . . . 1 2 2 3 ]	[ . . . . 1 3 4 ]	[ . . . . 1 1 1 2 ]	[ . 1 ]
40:	[ . . . 1 2 2 4 ]	[ . . . . 1 1 3 4 ]	[ . . . . 1 1 ]	[ . 2 ]
41:	[ . . . 1 2 3 3 ]	[ . . . . 2 3 4 ]	[ . . . . 1 1 1 ]	[ . 3 ]
42:	[ . . . 1 2 3 4 ]	[ . . . . 1 2 3 4 ]	[ . . . . 1 1 1 1 ]	[ . 4 ]

Figure 13: RGS corresponding to certain lattice paths (see text) in lexicographic, co-lexicographic, subset-lexrev, and subset-lex order.

```

30      a_[j] = m1_[j]; // set leftmost digit = nine
31      tr_ = j; // move to leftmost track
32      return true;
33  }
34 }
```

One use of the subset-lexrev order is the development of algorithms for the generation of orderings for objects where the subset-lex order exhibits no simple structure.

As an example we consider the restricted growth strings corresponding to lattice paths from  $(0,0)$  to  $(n,n)$  with steps  $(+1,0)$  and  $(+1,+1)$  that do not go below the diagonal. These RGS are words  $a_0, a_1, \dots, a_{n-1}$  such that  $a_0 = 0$ ,  $a_j \leq j$ , and  $a_{j+1} \geq a_j$  (the final  $a_n = n$  is omitted). These RGS are counted by the Catalan numbers, see sequence [A000108](#) in [28].

Figure 13 shows the all such RGS of length 5 in lexicographic, co-lexicographic, subset-lexrev, and subset-lex order. The listing in subset-lex order does not seem to have any apparently useful features, while the listing in subset-lexrev order suggests the following method of generation.

We now assume a sentinel  $a_n = +\infty$  at the end of the word.

**Algorithm 17** (Next-Catalan-Step-RGS). *Compute the successor in subset-lexrev order.*

1. Let  $a_t$  be the first nonzero digit.
2. If  $a_t < a_{t+1}$  and  $a_t < t$ , increment  $a_t$  and return.
3. If  $t \geq 2$ , set  $a_{t-1} = 1$  and return.
4. Remove all leading ones (setting them to zero), let  $j$  be the position of the first digit  $a_j \neq 1$ .
5. If the word is all-zero (that is,  $j = n$ ), stop.
6. Set  $a_j = a_j - 1$  and set  $a_{j-1} = 1$ .

The implementation correctly handles all cases  $n \geq 0$ .

```

1 // FILE: src/comb/catalan-step-rgs-subset-lexrev.h
2 class catalan_step_rgssubset_lexrev
3 {
4     ulong *a_; // RGS
5     ulong n2_; // aux: min(n,2).
6     ulong tr_; // aux: track we are looking at
7     ulong n_; // length of RGS

```

The routine for the successor returns the position of the rightmost change.

```

1     ulong next()
2     {
3         const ulong a0 = a_[tr_];
4         if ( a0 < a_[tr_+1] ) // may read sentinel
5         {
6             if ( a0 < tr_ ) // can increment
7             {
8                 a_[tr_] = a0 + 1;
9                 return tr_;
10            }
11        }
12        if ( tr_ != 1 ) // can move left and increment (from 0 to 1)
13        {
14            --tr_;
15            a_[tr_] = 1;
16            return tr_;
17        }
18        // remove ones:
19        ulong j = tr_;
20        do { a_[j] = 0; } while ( a_[++j] == 1 );
21        if ( j==n2_ ) return 0; // current was last
22        // decrement first value != 1:
23        ulong aj = a_[j] - 1;
24        a_[j] = aj;
25        // move left and restore the 1:
26        tr_ = j - 1;
27        a_[tr_] = 1;
28        return j; // rightmost change at j==tr+1
29    }
30
31
32
33
34
35

```

An update takes about 10.5 cycles, whereas the updates for lexicographic and co-lexicographic order respectively take 9 and 8 cycles.

We mention that the subset-lexrev order coincides for certain objects with well-known orderings. For example, for partitions as lists of parts as shown in figure 14, both orderings are by falling length of partitions as major order, the minor order is lexi-

0:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$
1:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2]$	$[2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$
2:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 3 \ .]$	$[2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
3:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ .]$	$[3 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
4:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 4 \ .]$	$[2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
5:	$[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 3 \ .]$	$[3 \ 2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
6:	$[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ .]$	$[4 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
7:	$[1 \ 1 \ 1 \ 1 \ 1 \ 5 \ .]$	$[2 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ .]$
8:	$[1 \ 1 \ 1 \ 1 \ 2 \ 4 \ .]$	$[3 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ .]$
9:	$[1 \ 1 \ 1 \ 1 \ 3 \ 3 \ .]$	$[3 \ 3 \ 1 \ 1 \ 1 \ 1 \ .]$
10:	$[1 \ 1 \ 1 \ 2 \ 2 \ 3 \ .]$	$[4 \ 2 \ 1 \ 1 \ 1 \ 1 \ .]$
11:	$[1 \ 1 \ 2 \ 2 \ 2 \ 2 \ .]$	$[5 \ 1 \ 1 \ 1 \ 1 \ 1 \ .]$
12:	$[1 \ 1 \ 1 \ 1 \ 6 \ .]$	$[2 \ 2 \ 2 \ 2 \ 2 \ .]$
13:	$[1 \ 1 \ 1 \ 2 \ 5 \ .]$	$[3 \ 2 \ 2 \ 2 \ 1 \ .]$
14:	$[1 \ 1 \ 1 \ 3 \ 4 \ .]$	$[3 \ 3 \ 2 \ 1 \ 1 \ .]$
15:	$[1 \ 1 \ 2 \ 2 \ 4 \ .]$	$[4 \ 2 \ 2 \ 1 \ 1 \ .]$
16:	$[1 \ 1 \ 2 \ 3 \ 3 \ .]$	$[4 \ 3 \ 1 \ 1 \ 1 \ .]$
17:	$[1 \ 2 \ 2 \ 2 \ 3 \ .]$	$[5 \ 2 \ 1 \ 1 \ 1 \ .]$
18:	$[2 \ 2 \ 2 \ 2 \ 2 \ .]$	$[6 \ 1 \ 1 \ 1 \ 1 \ .]$
19:	$[1 \ 1 \ 1 \ 7 \ .]$	$[3 \ 3 \ 2 \ 2 \ .]$
20:	$[1 \ 1 \ 2 \ 6 \ .]$	$[4 \ 2 \ 2 \ 2 \ .]$
21:	$[1 \ 1 \ 3 \ 5 \ .]$	$[3 \ 3 \ 3 \ 1 \ .]$
22:	$[1 \ 2 \ 2 \ 5 \ .]$	$[4 \ 3 \ 2 \ 1 \ .]$
23:	$[1 \ 1 \ 4 \ 4 \ .]$	$[5 \ 2 \ 2 \ 1 \ .]$
24:	$[1 \ 2 \ 3 \ 4 \ .]$	$[4 \ 4 \ 1 \ 1 \ .]$
25:	$[2 \ 2 \ 2 \ 4 \ .]$	$[5 \ 3 \ 1 \ 1 \ .]$
26:	$[1 \ 3 \ 3 \ 3 \ .]$	$[6 \ 2 \ 1 \ 1 \ .]$
27:	$[2 \ 2 \ 3 \ 3 \ .]$	$[7 \ 1 \ 1 \ 1 \ .]$
28:	$[1 \ 1 \ 8 \ .]$	$[4 \ 3 \ 3 \ .]$
29:	$[1 \ 2 \ 7 \ .]$	$[4 \ 4 \ 2 \ .]$
30:	$[1 \ 3 \ 6 \ .]$	$[5 \ 3 \ 2 \ .]$
31:	$[2 \ 2 \ 6 \ .]$	$[6 \ 2 \ 2 \ .]$
32:	$[1 \ 4 \ 5 \ .]$	$[5 \ 4 \ 1 \ .]$
33:	$[2 \ 3 \ 5 \ .]$	$[6 \ 3 \ 1 \ .]$
34:	$[2 \ 4 \ 4 \ .]$	$[7 \ 2 \ 1 \ .]$
35:	$[3 \ 3 \ 4 \ .]$	$[8 \ 1 \ 1 \ .]$
36:	$[1 \ 9 \ .]$	$[5 \ 5 \ .]$
37:	$[2 \ 8 \ .]$	$[6 \ 4 \ .]$
38:	$[3 \ 7 \ .]$	$[7 \ 3 \ .]$
39:	$[4 \ 6 \ .]$	$[8 \ 2 \ .]$
40:	$[5 \ 5 \ .]$	$[9 \ 1 \ .]$
41:	$[10 \ . \ . \ . \ . \ .]$	$[10 \ . \ . \ . \ . \ .]$

Figure 14: The partitions of 10 in subset-lexrev order, as weakly increasing lists of parts (left) and as weakly decreasing lists of parts (right).

cographic in case of weakly increasing parts and reverse co-lexicographic for weakly decreasing parts.

Similar observations can be made (for example) for compositions into a fixed number of parts, for both subset-lex and subset-lexrev order.

## 8 SL-Gray order for binary words

A well-known construction for the binary reflected Gray code proceeds by successively reversing ranges having identical prefixes that end in a one, using increasingly longer prefixes, see top of figure 15.

The same construction, now using prefixes ending in a zero, gives a Gray code for subset-lex order, see bottom of figure 15. We will call this ordering the *SL-Gray order*.

For the computation of the successor we keep a variable  $t$  for the current track and a variable  $d$  indicating the direction in which the track will be moved if necessary. Initially  $t = 0$  and  $d = +1$ .

**Algorithm 18** (Next-SL-Gray). *Compute the successor in SL-Gray order.*

1. If  $d = +1$ , do the following (try to append trailing ones):
  - (a) If  $a_t = 0$ , set  $a_t = 1$ ,  $t = t + 1$ , and return.
  - (b) Otherwise, set  $d = -1$  (change direction),  $t = n - 1$  (move to rightmost track),  $j = n - 2$ ,  $a_j = 1 - a_j$ , and return.
2. Otherwise ( $d = -1$ ), do the following (try to remove trailing ones):
  - (a) If  $a_{t-1} = 1$ , set  $a_t = 0$ ,  $t = t - 1$ , and return.
  - (b) Otherwise, set  $d = +1$  (change direction),  $j = t - 2$ ,  $a_j = 1 - a_j$ ,  $t = t + 1$  (move right), and return.

The algorithm is loopless. In the implementation, the variables `tr` and `dt` respectively correspond to  $t$  and  $d$  in the algorithm. Two sentinels  $a_{-1} = a_n = +1$  are used.

```

1 // FILE: src/comb/binary-sl-gray.h
2 class binary_sl_gray
3 {
4     ulong n_;    // number of digits
5     ulong tr_;   // aux: current track (0 <= tr <= n)
6     ulong dt_;   // aux: direction in which track tries to move
7     ulong *a_;   // digits
8
9     void first()
10    {
11        for (ulong k=0; k<n_; ++k) a_[k] = 0;
12        tr_ = 0;
13        dt_ = +1;
14    }
15
16     explicit binary_sl_gray(ulong n)
17    {
18        n_ = n;
19        a_ = new ulong[n_+2]; // sentinels at both ends
20
21        a_[n_+1] = +1; // != 0
22        a_[0] = +1;
23
24        ++a_; // nota bene
25        first();
26    }
27
28    void first()
29

```

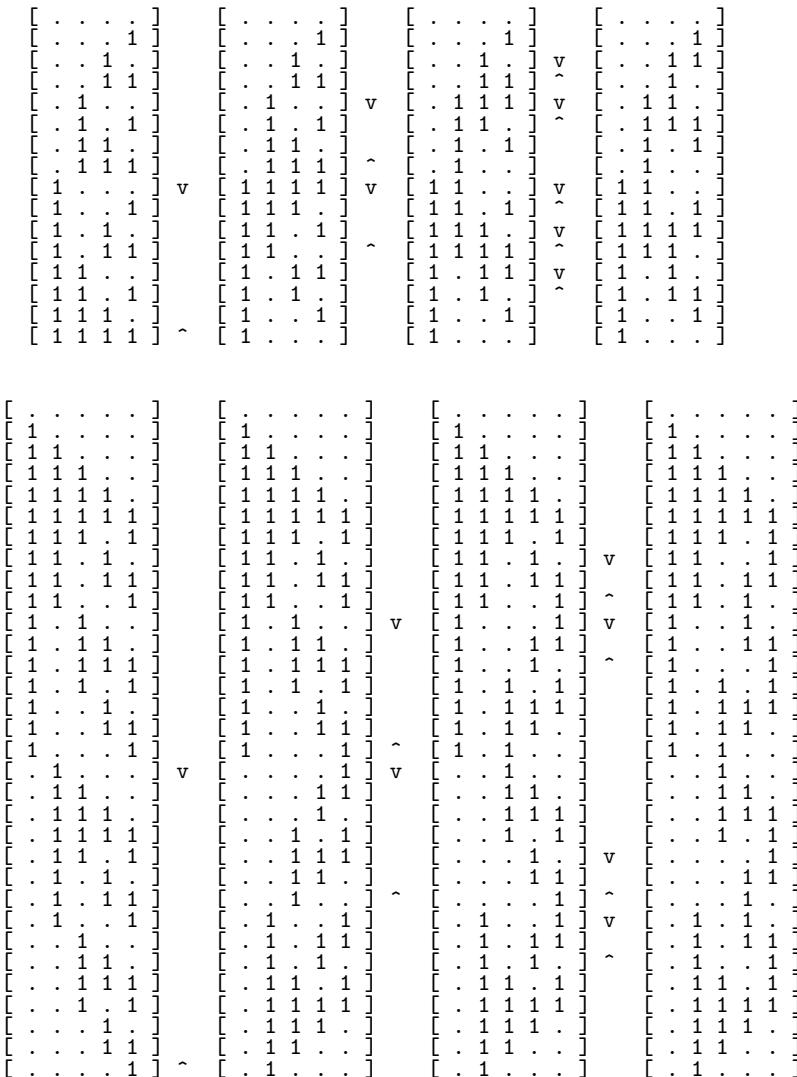


Figure 15: Construction of Gray codes by reversing sublists: for the binary reflected Gray code (top) and for the binary SL-Gray order (bottom). The symbols  $\vee$  and  $\wedge$  respectively mark begin and end of the reversed sublists.

```

30     {
31         for (ulong k=0; k<n_; ++k) a_[k] = 0;
32         tr_ = 0;
33         dt_ = +1;
34         j_ = ( n_>=2 ? 1 : 0); // wrt. last word
35         dm_ = -1; // wrt. last word
36     }
37 }
38

```

The routines for successor and predecessor handle all cases  $n \geq 0$  correctly.

```

1     bool next()
2     {
3         if ( dt_ == +1 ) // try to append trailing ones
4         {
5             if ( a_[tr_] == 0 ) // can append // may read sentinel a[n]
6             {
7                 a_[tr_] = 1;
8                 ++tr_;
9             }
10            else
11            {
12                dt_ = -1UL; // change direction
13                tr_ = n_ - 1;
14                ulong j_ = tr_ - 1;
15                // Is current last (only for n_ <= 1)?
16                if ( j_ > n_ ) return false;
17                a_[j_] = 1 - a_[j_];
18            }
19        }
20        else // dt_ == -1 // try to remove trailing ones
21        {
22            if ( a_[tr_-1] != 0 ) // can remove // tr - 1 >= 0
23            {
24                a_[tr_] = 0;
25                --tr_;
26            }
27            else
28            {
29                dt_ = +1; // change direction
30                ulong j_ = tr_ - 2;
31                if ( (long)j_ < 0 ) return false;
32                a_[j_] = 1 - a_[j_];
33                ++tr_;
34            }
35        }
36    }
37 }
38

```

The routine to compute the predecessor is obtained by (essentially) negating the direction the track tries to move (variable  $dt$ ):

```

1     bool prev()
2     {
3         if ( dt_ != +1 ) // dt== -1 // try to append trailing ones
4         {
5             if ( a_[tr_+1] == 0 ) // can append // may read sentinel a[n]
6             {
7                 a_[tr_+1] = 1;
8                 ++tr_;
9             }
10            else
11            {
12                dt_ = +1; // change direction
13                tr_ = n_ - 1;
14                ulong j_ = tr_ - 1;
15                a_[j_] = 1 - a_[j_];
16                ++tr_;
17            }
18        }
19        else // dt_ == +1 // try to remove trailing ones
20        {
21            if ( tr_ == 0 )

```

```

22         {
23             if ( a_[0]==0 )  return false; // (only for n_ <= 1)
24             a_[0] = 0;
25             return true;
26         }
27         // tr - 1 >= -1 (can read low sentinel)
28         if ( a_[tr_-2] != 0 ) // can remove
29         {
30             a_[tr_-1] = 0;
31             --tr_;
32         }
33     else
34     {
35         dt_ = -1UL; // change direction
36         ulong j_ = tr_ - 3;
37         a_[j_] = 1 - a_[j_];
38         --tr_;
39     }
40 }
41 }
42 }
43 return true;
44 }
```

One update in either direction takes about 7.5 cycles. An implementation for the generation of the SL-Gray order in a binary word is given in `src/bits/bit-sl-gray.h`.

The ranking algorithm is easily obtained by observing that if the highest bit is not set (and the word is nonzero), the order for the remaining word is reflected, as shown in figure 15.

**Algorithm 19** (Binary-SL-Gray-Rank). *Recursive routine  $F(k, n)$  for the computation of the rank of an  $n$ -bit word  $k$  in binary SL-Gray order.*

1. If  $k = 0$ , return 0.
2. Set  $w = k$  and unset the highest bit in  $w$ .
3. If the highest bit of  $k$  (at position  $n - 1$ ) is set, return  $1 + F(w, n - 1)$ .
4. Otherwise return  $2^n - F(w, n - 1)$ .

In the following implementation the variable `ldn` must give the number of bits in the SL-Gray code.

```

1 // FILE:  src/bits/bin-to-sl-gray.h
2 ulong sl_gray_to_bin(ulong k, ulong ldn)
3 {
4     if ( k==0 )  return 0;
5     ulong b = 1UL << (ldn-1); // mask for bit at end
6     ulong h = k & b; // bit at end
7     k ^= h;
8     ulong z = sl_gray_to_bin( k, ldn-1 ); // recursion
9     if ( h==0 )  return (b<<1) - z;
10    else          return 1 + z;
11 }
```

A routine for the conversion of a binary word into the corresponding word in SL-Gray order (unranking algorithm) is

```

1 ulong bin_to_sl_gray(ulong k, ulong ldn)
2 {
3     if ( ldn==0 )  return 0;
4     ulong b = 1UL << (ldn-1); // highest bit
5     ulong m = (b<<1) - 1; // mask for reversing direction
6     ulong z = b; // Gray code
7     k -= 1; // move all-zero word to begin
8     while ( b != 0 )
9     {
10         const ulong h = k & b; // bit under consideration
```

Figure 16: The delta sequence (top) of the binary SL-Gray order, starting after the initial slope and indexing positions from the end of the words.

```

12     z ^= h; // with one, switch bit in Gray code
13     if (!h) k ^= m; // reverse direction with zero
14     k += 1; // SL-Gray
15     b >>= 1; // next lower bit
16     m >>= 1; // next smaller mask
17   }
18
19   return z;
20 }

```

The delta sequence (sequence of positions of changes), starting after the initial slope and indexing positions from the end, is shown in figure 16, this is sequence A217262 in [28]. It can be obtained from the ruler function (sequence A007814) by replacing 0 by  $w = 01210$ , 1 by 3, 2 by 141, 3 by 12521, 4 by 1236321, ...,  $n$  by  $123\dots(n-1)(n+2)(n-1)\dots321$ :

```

0 1 0 2 0 1 0 3 0 1 0 2 0 1 0 4 0 ...
w 3 w 141 w 3 w 12521 w 3 w 141 w 3 w 1236321 w ... = (ruler function)
01210 3 01210 141 01210 3 01210 12521 01210 3 01210 141 01210 3 01210 ...

```

It is easy to see that (for  $n \geq 4$ ) the SL-Gray order for the  $2^n$  words of length  $n$  has  $2^{n-2} - 2$  successive transitions that are 3-close (have a distance of 3), the rest are 1-close (adjacent changes).

A Gray code where the maximal distance between successive transitions is  $k$  is called  $k$ -close. In [2, p.400] it has been observed that 1-close Gray codes exists for  $n \leq 6$  but not for  $n = 7$  (and it appears unlikely that for any  $n \geq 8$  such a Gray code exists). Does a 2-close Gray code exist? We remark that the SL-Gray order is 2-close if the radices for all digits are even and  $\geq 4$ .

## 9 SL-Gray order for mixed radix words

The generalization of the SL-Gray order to mixed radix words is obtained by successive reversion of the sublists with identical prefix for all prefixes ending in an even digit.

For the computation of the successor we keep a variable  $t$  for the current track and variables  $d_j = \pm 1$  (an array) for the direction the digit  $a_j$  is currently moving in (whether it is incremented or decremented). We further use  $m_j$  for the maximal value the digit  $a_j$  can have (the “nines”). Initially all digits  $a_k$  are zero, all directions  $d_k$  are  $+1$ , and  $t = 0$ .

**Algorithm 20** (Next-SL-Gray). *Compute the successor SL-Gray order.*

1. Set  $j = t$  and  $b = a_j + d_j$ .

0:	[ . . . . ]	[ + + + + ]	{ }
1:	[ 1 . . . ]	[ + + + + ]	{ 0 }
2:	[ 1 1 . . ]	[ - + + + ]	{ 0, 1 }
3:	[ 1 2 . . ]	[ - + + + ]	{ 0, 1, 1 }
4:	[ 1 2 1 . ]	[ - + + + ]	{ 0, 1, 1, 2 }
5:	[ 1 2 2 . ]	[ - - + + ]	{ 0, 1, 1, 2, 2 }
6:	[ 1 2 2 1 ]	[ - - + + ]	{ 0, 1, 1, 2, 2, 3 }
7:	[ 1 2 2 2 ]	[ - - - + ]	{ 0, 1, 1, 2, 2, 3, 3 }
8:	[ 1 2 2 3 ]	[ - - - + ]	{ 0, 1, 1, 2, 2, 3, 3, 3 }
9:	[ 1 2 1 3 ]	[ - - - - ]	{ 0, 1, 1, 2, 3, 3, 3 }
10:	[ 1 2 1 2 ]	[ - - - - ]	{ 0, 1, 1, 2, 3, 3 }
11:	[ 1 2 1 1 ]	[ - - - - ]	{ 0, 1, 1, 2, 3 }
12:	[ 1 2 . 1 ]	[ - - - + ]	{ 0, 1, 1, 3 }
13:	[ 1 2 . 2 ]	[ - - - + ]	{ 0, 1, 1, 3, 3 }
14:	[ 1 2 . 3 ]	[ - - - + ]	{ 0, 1, 1, 3, 3, 3 }
15:	[ 1 1 . 3 ]	[ - - + - ]	{ 0, 1, 3, 3, 3 }
16:	[ 1 1 . 2 ]	[ - - + - ]	{ 0, 1, 3, 3 }
17:	[ 1 1 . 1 ]	[ - - + - ]	{ 0, 1, 3 }
18:	[ 1 1 1 1 ]	[ - - + + ]	{ 0, 1, 2, 3 }
19:	[ 1 1 1 2 ]	[ - - + + ]	{ 0, 1, 2, 3, 3 }
20:	[ 1 1 1 3 ]	[ - - + + ]	{ 0, 1, 2, 3, 3, 3 }
21:	[ 1 1 2 3 ]	[ - - + - ]	{ 0, 1, 2, 2, 3, 3, 3 }
22:	[ 1 1 2 2 ]	[ - - + - ]	{ 0, 1, 2, 2, 3, 3 }
23:	[ 1 1 2 1 ]	[ - - + - ]	{ 0, 1, 2, 2, 3 }
24:	[ 1 1 2 . ]	[ - - - + ]	{ 0, 1, 2, 2 }
25:	[ 1 1 1 . ]	[ - - - + ]	{ 0, 1, 2 }
26:	[ 1 . 1 . ]	[ - - + + ]	{ 0, 2 }
27:	[ 1 . 2 . ]	[ - - + + ]	{ 0, 2, 2 }
28:	[ 1 . 2 1 ]	[ - - - + ]	{ 0, 2, 2, 3 }
29:	[ 1 . 2 2 ]	[ - - - + ]	{ 0, 2, 2, 3, 3 }
30:	[ 1 . 2 3 ]	[ - - - + ]	{ 0, 2, 2, 3, 3, 3 }
31:	[ 1 . 1 3 ]	[ - - - - ]	{ 0, 2, 3, 3, 3 }
32:	[ 1 . 1 2 ]	[ - - - - ]	{ 0, 2, 3, 3 }
33:	[ 1 . 1 1 ]	[ - - - - ]	{ 0, 2, 3 }
34:	[ 1 . . 1 ]	[ - - - + ]	{ 0, 3 }
35:	[ 1 . . 2 ]	[ - - - + ]	{ 0, 3, 3 }
36:	[ 1 . . 3 ]	[ - - - + ]	{ 0, 3, 3, 3 }
37:	[ . . . 3 ]	[ - + + - ]	{ 3, 3, 3 }
38:	[ . . . 2 ]	[ - + + - ]	{ 3, 3 }
39:	[ . . . 1 ]	[ - + + - ]	{ 3 }
40:	[ . . 1 1 ]	[ - + + + ]	{ 2, 3 }
41:	[ . . 1 2 ]	[ - + + + ]	{ 2, 3, 3 }
42:	[ . . 1 3 ]	[ - + + + ]	{ 2, 3, 3, 3 }
43:	[ . . 2 3 ]	[ - + + - ]	{ 2, 2, 3, 3, 3 }
44:	[ . . 2 2 ]	[ - + + - ]	{ 2, 2, 3, 3 }
45:	[ . . 2 1 ]	[ - + + - ]	{ 2, 2, 3 }
46:	[ . . 2 . ]	[ - + - + ]	{ 2, 2 }
47:	[ . . 1 . ]	[ - + - + ]	{ 2 }
48:	[ . 1 1 . ]	[ - + + + ]	{ 1, 2 }
49:	[ . 1 2 . ]	[ - + + + ]	{ 1, 2, 2 }
50:	[ . 1 2 1 ]	[ - + - + ]	{ 1, 2, 2, 3 }
51:	[ . 1 2 2 ]	[ - + - + ]	{ 1, 2, 2, 3, 3 }
52:	[ . 1 2 3 ]	[ - + - + ]	{ 1, 2, 2, 3, 3, 3 }
53:	[ . 1 1 3 ]	[ - + - - ]	{ 1, 2, 3, 3, 3 }
54:	[ . 1 1 2 ]	[ - + - - ]	{ 1, 2, 3, 3 }
55:	[ . 1 1 1 ]	[ - + - - ]	{ 1, 2, 3 }
56:	[ . 1 . 1 ]	[ - + - + ]	{ 1, 3 }
57:	[ . 1 . 2 ]	[ - + - + ]	{ 1, 3, 3 }
58:	[ . 1 . 3 ]	[ - + - + ]	{ 1, 3, 3, 3 }
59:	[ . 2 . 3 ]	[ - + + - ]	{ 1, 1, 3, 3, 3 }
60:	[ . 2 . 2 ]	[ - + + - ]	{ 1, 1, 3, 3 }
61:	[ . 2 . 1 ]	[ - + + - ]	{ 1, 1, 3 }
62:	[ . 2 1 1 ]	[ - + + + ]	{ 1, 1, 2, 3 }
63:	[ . 2 1 2 ]	[ - + + + ]	{ 1, 1, 2, 3, 3 }
64:	[ . 2 1 3 ]	[ - + + + ]	{ 1, 1, 2, 3, 3, 3 }
65:	[ . 2 2 3 ]	[ - + + - ]	{ 1, 1, 2, 2, 3, 3, 3 }
66:	[ . 2 2 2 ]	[ - + + - ]	{ 1, 1, 2, 2, 3, 3 }
67:	[ . 2 2 1 ]	[ - + + - ]	{ 1, 1, 2, 2, 3 }
68:	[ . 2 2 . ]	[ - + - + ]	{ 1, 1, 2, 2 }
69:	[ . 2 1 . ]	[ - + - + ]	{ 1, 1, 2 }
70:	[ . 2 . . ]	[ - - + + ]	{ 1, 1 }
71:	[ . 1 . . ]	[ - - + + ]	{ 1 }

Figure 17: Mixed radix words (left) and corresponding subsets (right) of the multiset  $\{0^1, 1^2, 2^2, 3^3\}$  in SL-Gray order, and the array of directions used in the computation (middle).

2. If  $b \neq 0$  and  $b \leq m_j$ , set  $a_j = b$  and return (easy case).
3. Set  $d_j = -d_j$  (change direction for digit  $j$ ).
4. If  $d_j = +1$  and  $a_{j+1} = 0$ , set  $a_{j+1} = +1$ ,  $t = j + 1$  (move track right), and return.
5. If  $d_j = -1$  and  $a_{j-1} = m_{j-1}$ , set  $a_j = 0$ ,  $d_{j-1} = -1$ ,  $t = j - 1$  (move track left), and return.
6. Find the position  $p$  of the nearest digit  $a_k$  to the left such that  $a_k + d_k$  is in the range  $0, 1, \dots, m_k$  (a valid digit). In the process, change  $d_k$  for all  $k$  where  $p < k < j$ .
7. Set  $a_p = a_p + d_p$  (change digit, keep track).

The details for termination have been omitted, this is handled with the help of sentinels in the implementation.

```

1 // FILE: src/comb/mixedradix-sl-gray.h
2 class mixedradix_sl_gray
3 {
4     ulong n_;      // number of digits
5                 // (n kinds of elements in multiset, n>=1)
6     ulong tr_;    // aux: current track
7     ulong *a_;    // digits of mixed radix number
8                 // (multiplicity of kind k in subset).
9     ulong *d_;    // directions (either +1 or -1)
10    ulong *m1_;   // nines (radix minus one) for each digit
11                 // (multiplicity of kind k in set).

```

Sentinels are used in all arrays at index  $-1$  and at index  $n$  to handle termination.

```

1     mixedradix_sl_gray(ulong n, ulong mm, const ulong *m=0)
2     // Must have n>=1
3     {
4         n_ = n;
5         a_ = new ulong[n_+2]; // all with sentinels at both ends
6         d_ = new ulong[n_+2];
7         m1_ = new ulong[n_+2];
8
9         // sentinels on the right:
10        a_[n_+1] = +1; // != 0
11        m1_[n_+1] = +1; // same as a_[n+1]
12        d_[n_+1] = +1; // positive
13
14         // sentinels on the left:
15        a_[0] = +2; // >= +2
16        m1_[0] = +2; // same as a_[0]
17        d_[0] = 0; // zero
18
19        ++a_; ++d_; ++m1_; // nota bene
20
21        mixedradix_init(n_, mm, m, m1_);
22        first();
23    }
24
25    void first()
26    {
27        for (ulong k=0; k<n_; ++k) a_[k] = 0;
28        for (ulong k=0; k<n_; ++k) d_[k] = +1;
29        tr_ = 0;
30    }

```

The method for the successor handles all cases  $n \geq 0$  correctly. The variable **a1** in the implementation corresponds to  $b$  in the algorithm.

```

1     bool next()
2     {
3         ulong j = tr_;
4         const ulong dj = d_[j];
5         const ulong a1 = a_[j] + dj; // a[j] +- 1

```

```

6      if ( (a1 != 0) && (a1 <= m1_[j]) ) // easy case
7      {
8          a_[j] = a1;
9          return true;
10     }
11
12     d_[j] = -dj; // change direction
13
14     if ( dj == +1 ) // so a_[j] == m1_[j] == nine
15     {
16         // Try to move track right with a[j] == nine:
17         const ulong j1 = j + 1;
18         if ( a_[j1] == 0 ) // can read high sentinel
19         {
20             a_[j1] = +1;
21             tr_ = j1;
22             return true;
23         }
24     }
25     else // here dj == -1, so a_[j] == 1
26     {
27         if ( (long)j <= 0 ) return false; // current is last
28
29         // Try to move track left with a[j] == 1:
30         const ulong j1 = j - 1;
31         if ( a_[j1] == m1_[j1] ) // can read low sentinel when n_ == 1
32         {
33             a_[j] = 0;
34             d_[j1] = -1UL;
35             tr_ = j1;
36             return true;
37         }
38     }
39
40     // find first changeable track to the left:
41     --j;
42     while ( a_[j] + d_[j] > m1_[j] ) // may read low sentinels
43     {
44         d_[j] = -d_[j]; // change direction
45         --j;
46     }
47
48     if ( (long)j < 0 ) return false; // current is last
49
50     // Change digit left but keep track:
51     a_[j] += d_[j];
52
53     return true;
54 }
55

```

## 10 A Gray code for compositions

We now describe a Gray code for compositions where at most one unit is moved in each step, all moves are 1-close or 2-close (these always cross a part 1), and all moves are at the end of the current composition.

The compositions of 7 in this ordering are shown in figure 18 (left), it can be obtained from the lexicographic order by successively reversing the sublists with identical prefixes that end in an odd part. To keep matters simple for the iterative algorithm, we arrange the compositions for  $n$  odd such that the first part is only decreasing (starting with the composition  $[n]$ ) and otherwise such that the first part is increasing (starting with the composition  $[1, n - 1]$ ). The compositions of 6 in this ordering are obtained by dropping the first part 1 in all compositions of 7 in the second half of the list, see figure 18.

A recursive routine can be obtained by switching between the recursive functions

0:	111111	[ 7 ]	0:	111.....	[ 4 1 1 1 1 1 1 ]
1:	11111.	[ 6 1 ]	1:	11..1...	[ 3 1 2 1 1 1 1 ]
2:	1111..	[ 5 1 1 ]	2:	11....1.	[ 3 1 1 1 2 1 1 ]
3:	1111.1	[ 5 2 ]	3:	11.....1	[ 3 1 1 1 1 1 2 ]
4:	111.11	[ 4 3 ]	4:	11...1..	[ 3 1 1 2 1 1 1 ]
5:	111.1.	[ 4 2 1 ]	5:	11.1....	[ 3 2 1 1 1 1 1 ]
6:	111...	[ 4 1 1 1 ]	6:	1.11....	[ 2 3 1 1 1 1 1 ]
7:	111..1	[ 4 1 2 ]	7:	1.1.1....	[ 2 2 2 1 1 1 1 ]
8:	11..11	[ 3 1 3 ]	8:	1.1...1.	[ 2 2 1 1 2 1 1 ]
9:	11..1.	[ 3 1 2 1 ]	9:	1.1....1	[ 2 2 1 1 1 1 2 ]
10:	11....	[ 3 1 1 1 1 ]	10:	1.1..1..	[ 2 2 1 2 1 1 1 ]
11:	11...1	[ 3 1 1 2 ]	11:	1...11..	[ 2 1 1 3 1 1 1 ]
12:	11.1..	[ 3 2 1 1 ]	12:	1...1.1.	[ 2 1 1 2 2 1 1 ]
13:	11.1.1	[ 3 2 2 ]	13:	1...1..1	[ 2 1 1 2 1 2 ]
14:	11..11.	[ 3 3 1 ]	14:	1....11	[ 2 1 1 1 1 1 3 ]
15:	11..111	[ 3 4 ]	15:	1....1.1	[ 2 1 1 1 2 2 2 ]
16:	1.1111	[ 2 5 ]	16:	1....11.	[ 2 1 1 1 3 1 ]
17:	1.111.	[ 2 4 1 ]	17:	1..1..1.	[ 2 1 2 1 2 1 ]
18:	1..11..	[ 2 3 1 1 ]	18:	1..1...1.	[ 2 1 2 1 1 2 ]
19:	1..11.1	[ 2 3 2 ]	19:	1..1.1..	[ 2 1 2 2 1 1 ]
20:	1..1.11	[ 2 2 3 ]	20:	1..11....	[ 2 1 3 1 1 1 ]
21:	1..1.1.	[ 2 2 2 1 ]	21:	..111...	[ 1 1 4 1 1 1 1 ]
22:	1..1...	[ 2 2 1 1 ]	22:	..11..1.	[ 1 1 3 1 2 1 ]
23:	1..1..1	[ 2 2 1 2 ]	23:	..11...1	[ 1 1 3 1 1 2 ]
24:	1...11	[ 2 1 1 3 ]	24:	..11.1..	[ 1 1 3 2 1 1 ]
25:	1...1.	[ 2 1 1 2 1 ]	25:	..1.11..	[ 1 1 2 3 1 1 ]
26:	1....1.	[ 2 1 1 1 1 ]	26:	..1.1.1.	[ 1 1 2 2 2 1 ]
27:	1....1	[ 2 1 1 1 2 ]	27:	..1.1..1.	[ 1 1 2 2 1 2 ]
28:	1...1..	[ 2 1 2 1 1 ]	28:	..1...11	[ 1 1 2 1 1 3 ]
29:	1...1.1	[ 2 1 2 2 ]	29:	..1..1.1	[ 1 1 2 1 2 2 ]
30:	1..11..	[ 2 1 3 1 ]	30:	..1..11.	[ 1 1 2 1 3 1 ]
31:	1..111	[ 2 1 4 ]	31:	...111.	[ 1 1 1 1 4 1 ]
32:	..11111	[ 1 5 ]	32:	....11.1	[ 1 1 1 1 3 2 ]
33:	..111.	[ 1 1 4 1 ]	33:	....1.11	[ 1 1 1 1 2 3 ]
34:	..11..	[ 1 1 3 1 ]	34:	....111	[ 1 1 1 1 1 4 ]
35:	..11.1	[ 1 1 3 2 ]	35:	...1..11	[ 1 1 1 2 1 3 ]
36:	..1.11	[ 1 1 2 3 ]	36:	...1.1.1	[ 1 1 1 2 2 2 ]
37:	.1.1.	[ 1 1 2 2 1 ]	37:	...1.11.	[ 1 1 1 2 3 1 ]
38:	.1...	[ 1 1 2 1 1 ]	38:	...11.1.	[ 1 1 1 3 2 1 ]
39:	.1..1.	[ 1 1 2 1 2 ]	39:	...11..1	[ 1 1 1 3 1 2 ]
40:	.1...11	[ 1 1 1 1 3 ]	40:	...111..	[ 1 1 1 4 1 1 ]
41:	.1...1.	[ 1 1 1 1 2 1 ]	41:	.1..11..	[ 1 2 1 3 1 1 ]
42:	.1....1.	[ 1 1 1 1 1 1 ]	42:	.1..1.1.	[ 1 2 1 2 2 1 ]
43:	.1...1.	[ 1 1 1 1 1 2 ]	43:	.1..1..1.	[ 1 2 1 2 1 2 ]
44:	.1...1..	[ 1 1 1 2 1 1 ]	44:	.1..1..11	[ 1 2 1 1 1 3 ]
45:	.1..1.1	[ 1 1 1 2 2 ]	45:	.1..1..1.	[ 1 2 1 1 2 2 ]
46:	.1..11.	[ 1 1 1 3 1 ]	46:	.1...11.	[ 1 2 1 1 3 1 ]
47:	.1..111	[ 1 1 1 4 ]	47:	.1..1..1.	[ 1 2 2 1 2 1 ]
48:	.1..111	[ 1 2 1 3 ]	48:	.1..1...1	[ 1 2 2 1 1 2 ]
49:	.1..1..1	[ 1 2 1 2 1 ]	49:	.1..1.1..	[ 1 2 2 2 1 1 ]
50:	.1....1	[ 1 2 1 1 1 ]	50:	.1..11....	[ 1 2 3 1 1 1 ]
51:	.1...1..1	[ 1 2 1 1 2 ]	51:	.11..1...	[ 1 3 2 1 1 1 ]
52:	.1..1..	[ 1 2 2 1 1 ]	52:	.11...1..	[ 1 3 1 1 2 1 ]
53:	.1..1.1	[ 1 2 2 2 ]	53:	.11....1	[ 1 3 1 1 1 2 ]
54:	.1..11.	[ 1 2 3 1 ]	54:	.11...1..	[ 1 3 1 2 1 1 ]
55:	.1..111	[ 1 2 4 ]	55:	.111....	[ 1 4 1 1 1 1 ]
56:	.11..11	[ 1 3 3 ]			
57:	.11..1.	[ 1 3 2 1 ]			
58:	.11...	[ 1 3 1 1 ]			
59:	.11..1	[ 1 3 1 2 ]			
60:	.111..	[ 1 4 1 1 ]			
61:	.111..1	[ 1 4 2 ]			
62:	.1111..	[ 1 5 1 ]			
63:	.111111	[ 1 6 ]			

Figure 18: The 64 compositions of 7 together with their binary encodings as in figure 6 (left) and the 56 compositions of 9 into exactly 6 parts (right).

for lexicographic and reversed lexicographic order whenever an odd part has been written. In the following a global array  $\mathbf{a}[]$  is used.

```

1 // FILE: src/comb/composition-nz-gray-rec-demo.cc
2 void F(ulong n, ulong m)
3 {
4     if ( n==0 ) { visit(m); return; }
5     for (ulong f=n; f!=0; --f) // first part decreasing
6     {
7         a[m] = f;
8         if ( 0 == (f & 1) ) F(n-f, m+1);
9         else                 B(n-f, m+1);
10    }
11 }
12 }
13 void B(ulong n, ulong m)
14 {
15     if ( n==0 ) { visit(m); return; }
16     for (ulong f=1; f<=n; ++f) // first part increasing
17     {
18         a[m] = f;
19         if ( 0 == (f & 1) ) B(n-f, m+1);
20         else                 F(n-f, m+1);
21     }
22 }
23 }
24 }
```

The initial call is  $F(n, 0)$  for  $n$  odd and  $G(n, 0)$  for  $n$  even.

For the iterative algorithm we use a function  $D(x)$  that shall return  $+1$  if  $x$  is odd and otherwise  $-1$ . We will refer to the last three parts as respectively  $x$ ,  $y$ , and  $z$ .

**Algorithm 21** (Next-Comp-Gray). *Compute the successor of a composition.*

1. If  $z = n - 1$  and  $n$  is odd, or  $z = n$  and  $n$  is even, stop.
2. If  $z = 1$ , do the following:
  - (a) If  $D(y) = +1$ , set  $y = y + 1$  and drop  $z$ ; return.
  - (b) Otherwise ( $D(y) = -1$ ), set  $y = y - 1$  and append a part 1; return.
3. Otherwise ( $z > 1$ ) do the following:
  - (a) If  $z$  is odd, set  $z = z - 1$  and append a part 1; return.
  - (b) If  $y > 1$ , set  $y = y - D(y)$  and  $z = z + D(y)$ ; return.
  - (c) If  $x > 1$ , set  $x = x + D(x)$  and  $z = z - D(x)$ ; return.
  - (d) Otherwise ( $x = 1$ ) set  $x = x + 1$  and  $z = z - 1$ .

The algorithm is loopless. Note that the initialization is loopless as well.

The implementation handles all  $n \geq$  correctly.

```

1 // FILE: src/comb/composition-nz-gray2.h
2 class composition_nz_gray2
3 {
4     ulong *a_; // composition: a[1] + a[2] + ... + a[m] = n
5     ulong n_; // compositions of n
6     ulong m_; // current composition has m parts
7     ulong e_; // aux: detection of last composition
8
9     explicit composition_nz_gray2(ulong n)
10    {
11        n_ = n;
12        a_ = new ulong[n_+1+(n_==0)];
13        a_[0] = 0; // returned by last_part() when n==0
14        a_[1] = 0; // returned by first_part() when n==0
15    }
```

```

16         if ( n_ <= 1 ) e_ = n_;
17     else             e_ = ( oddq(n_) ? n_ - 1 : n_ );
18
19         first();
20     }
21
22 void first()
23 {
24     if ( n_ <= 1 )
25     {
26         a_[1] = n_;
27         m_ = n_;
28     }
29     else
30     {
31         if ( oddq(n_) )
32         {
33             a_[1] = n_;
34             m_ = 1;
35         }
36         else
37         {
38             a_[1] = 1;
39             a_[2] = n_ - 1;
40             m_ = 2;
41         }
42     }
43 }
44

```

A few auxiliary routines are used.

```

1     bool oddq(ulong x) const { return 0 != (x & 1UL); }
2     bool evenq(ulong x) const { return 0 == (x & 1UL); }
3
4     ulong par_to_dir_odd(ulong x) const
5     {
6         if ( oddq(x) ) return +1;
7         else           return -1UL;
8     }
9
10    ulong par_to_dir_even(ulong x) const
11    {
12        if ( evenq(x) ) return +1;
13        else           return -1UL;
14    }

```

We split the update into routines for the cases  $z == 1$  and  $z > 1$  as in the algorithm.

```

1     ulong next_zeq1() // for Z == 1
2     {
3         const ulong y = a_[m_-1];
4         const ulong dy = par_to_dir_odd(y);
5
6         if ( dy == +1 )
7         { // [*, Y, 1 ] --> [*, Y+1 ]
8             a_[m_-1] = y + 1;
9             m_- = 1;
10            return m_;
11        }
12        else // dy == -1
13        { // [*, Y, 1 ] --> [*, Y-1, 1, 1 ]
14            a_[m_-1] = y - 1;
15            m_ += 1;
16            a_[m_] = 1;
17            return m_;
18        }
19    }
20
21    ulong next_zgt1() // for Z > 1
22    {
23        const ulong z = a_[m_];
24        const ulong y = a_[m_-1];
25
26        if ( oddq(z) )
27        { // [*, Z ] --> [*, Z-1, 1 ]

```

```

28         a_[m_] = z - 1;
29         m_ += 1;
30         a_[m_] = 1;
31         return m_;
32     }
33
34     if ( y != 1 ) // Y > 1
35     { // [*, Y, Z ] --> [*, Y+1, Z+1 ]
36         const ulong dy = par_to_dir_even(y);
37         a_[m_-1] = y + dy;
38         a_[m_] = z - dy;
39         return m_;
40     }
41     else // Y == 1
42     {
43         const ulong x = a_[m_-2];
44
45         if ( x != 1 )
46         { // [*, X, 1, Z ] --> [*, X+1, 1, Z+1 ]
47             const ulong dx = par_to_dir_odd(x);
48             a_[m_-2] = x + dx;
49             a_[m_] = z - dx;
50             return m_;
51         }
52         else // X == 1
53         { // [*, X, 1, Z ] --> [*, X+1, 1, Z-1 ]
54             a_[m_-2] = x + 1;
55             a_[m_] = z - 1;
56             return m_;
57         }
58     }
59 }
```

The routine `next()` returns number of parts in the new composition and zero if there are no more compositions.

```

1     ulong next()
2     {
3         ulong z = a_[m_];
4         if ( z == e_ ) return 0; // current is last
5         if ( z != 1 ) return next_zgt1();
6         else           return next_zeq1();
7     }
```

One update takes about 11 cycles.

We remark that the sublists of compositions into a fixed number of parts correspond to the combinations in *enup* order. The list of binary words on the right of figure 18 are those shown in [2, fig.6.6-B (left), p.189].

### Ranking and unranking

Recursive routines for ranking and unranking are obtained easily. The following implementations are for the ordering with the first part decreasing.

```

1 // FILE: src/comb/composition-nz-rank.cc
2 ulong
3 composition_nz_gray_rank(const ulong *x, ulong m, ulong n)
4 // Return rank r of composition x[], 0 <= r < 2**n-1
5 // where n is the sum of all parts.
6 {
7     if ( m <= 1 ) return 0;
8
9     ulong f = x[0]; // first part
10    ulong s = n - f; // remaining sum
11
12    ulong y = composition_nz_gray_rank(x+1, m-1, s);
13    if ( 0 == ( f & 1 ) ) // first part even
14        return ( 1UL << (s-1) ) + y;
15    else
16        return ( 1UL << s ) - 1 - y;
```

1:	1.....	{ 0 }	(continued)
2:	11....	{ 0, 1 }	.1.11. { 1, 3, 4 }
3:	111...	{ 0, 1, 2 }	.1.1.1 { 1, 3, 5 }
4:	11.1..	{ 0, 1, 3 }	.1..1. { 1, 4 }
5:	11..1.	{ 0, 1, 4 }	.1..11 { 1, 4, 5 }
6:	11...1	{ 0, 1, 5 }	.1...1 { 1, 5 }
7:	1.1...	{ 0, 2 }	.1... { 2 }
8:	1.11..	{ 0, 2, 3 }	.11... { 2, 3 }
9:	1.1.1.	{ 0, 2, 4 }	.111. { 2, 3, 4 }
10:	1.1..1	{ 0, 2, 5 }	.11.1 { 2, 3, 5 }
11:	1..1..	{ 0, 3 }	.1.1. { 2, 4 }
12:	1..11.	{ 0, 3, 4 }	.1.11 { 2, 4, 5 }
13:	1..1.1.	{ 0, 3, 5 }	.1..1 { 2, 5 }
14:	1....1.	{ 0, 4 }	....1.. { 3 }
15:	1...11.	{ 0, 4, 5 }	....11. { 3, 4 }
16:	1....1	{ 0, 5 }	....111 { 3, 4, 5 }
17:	.1....	{ 1 }	....1.1 { 3, 5 }
18:	.11...	{ 1, 2 }	....1. { 4 }
19:	.111..	{ 1, 2, 3 }	....11 { 4, 5 }
20:	.11.1.	{ 1, 2, 4 }	.....1 { 5 }
21:	.11..1	{ 1, 2, 5 }	
22:	.1.1..	{ 1, 3 }	

Figure 19: Nonempty subsets of the set  $\{0, 1, 2, \dots, 5\}$  with at most 3 elements.

```

17 }
18
19 ulong
20 composition_nz_gray_unrank(ulong r, ulong *x, ulong n)
21 // Generate composition x[] of n with rank r.
22 // Return number of parts m of generated composition, 0 <= m <= n.
23 {
24     if ( r == 0 )
25     {
26         if ( n==0 )  return 0;
27         x[0] = n;
28         return 1;
29     }
30
31     ulong h = highest_one_idx(r);
32     ulong f = n - 1 - h; // first part
33     x[0] = f;
34
35     ulong b = 1UL << h; // highest one
36     r ^= b; // delete highest one
37
38     bool p = f & 1; // first part f odd ?
39     if ( p ) r = b - 1 - r; // change direction with odd f
40
41     return 1 + composition_nz_gray_unrank( r , x+1, n-f );
42 }
```

## 11 Appendix I: Nonempty subsets with at most $k$ elements

The modifications needed in algorithm 1 for restricting the subsets to those with a prescribed maximal number of elements are quite small. Without ado, we give the implementation.

```

1 // FILE: src/comb/ksubset-lex.h
2 class ksubset_lex
3 {
4     ulong n_; // number of elements in set, should have n>=1
5     ulong j_; // number of elements in subset
6     ulong m_; // max number of elements in subsets
7     ulong *x_; // x[0...j-1]: subset of {0,1,2,...,n-1}
```

The computation of the successor is

```

1     ulong next()
2     {
3         ulong j1 = j_ - 1;
4         ulong z1 = x_[j1] + 1;
5         if ( z1 < n_ ) // last element is not max
6         {
7             if ( j_ < m_ ) // append element
8             {
9                 x_[j_] = x_[j1] + 1;
10                ++j_;
11                return j_;
12            }
13            x_[j1] = z1; // increment last element
14            return j_;
15        }
16        else // last element is max
17        {
18            if ( j1 == 0 ) return 0; // current is last
19            --j_;
20            x_[j_-1] += 1;
21            return j_;
22        }
23    }
24 }
25

```

We omit the routine for the predecessor. Updates take no more time than the updates for the unrestricted subsets.

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## 12 Appendix II: Ranking and unranking methods for mixed radix words

The following material was added January 2, 2024.

We give C++ code for ranking and unranking mixed radix words in Subset-lex order and in SL-Gray order.

### Subset-lex order

We put the routines into a C++ class because a table (variable `Jmp`) has to be used for ranking and unranking. It is computed only once, in the constructor.

```

1 // FILE: src/comb/mixedradix-subset-lex-rank.h
2 class mixedradix_subset_lex_rank
3 // Ranking and unranking function for subset-lex order.
4 {
5     const ulong * A;    // digits of mixed radix number
6     const ulong * m1;  // nines for mixed radix base
7     const ulong n;    // number of digits
8     ulong * Jmp {nullptr}; // jump sizes, see constructor
9
10    mixedradix_subset_lex_rank( const ulong *tA, // digits
11                                ulong tn,        // number of digits
12                                const ulong *tm1 // nines
13                                )
14        : A( tA ), m1( tm1 ), n( tn )
15    {
16        Jmp = new ulong[n];
17        for (ulong j=0; j < n; ++j)
18        {
19            ulong p = 1; // product of all radices from index j+1 to end
20            for (ulong k=j+1; k < n; ++k) p *= ( 1 + m1[k] );
21            Jmp[j] = p - 1;
22        }
23    }
24
25    ulong rank() const
26    // Return rank of mixed radix number A[].
27    {
28        // last index e such that A[e] != 0,
29        // e == 0 also for the all-zero word:
30        ulong e = 0;
31        for (ulong j=0; j<n; ++j)
32            if ( A[j] != 0 ) e = j;
33
34        ulong r = 0;
35        ulong j = 0; // track
36        while ( true )
37        {
38            const ulong & aj = A[j];
39            const ulong & nine = m1[j];
40            const ulong & jmp = Jmp[j];
41
42            if ( j == e ) // on last non-zero digit
43            {
44                // forward (digits increasing, fast)
45                r += aj;
46                break; // and return r
47            }
48            else // not on last track
49            {
50                // forward (digits decreasing, slow)
51                r += nine;
52                r += jmp * ( nine - aj );
53            }
54            ++j;
55        }
56    }
57
58    void unrank( ulong r )
59    // Compute mixed radix number A[] of rank r.
60    {
61        const ulong & one = m1[n];
62        const ulong & zero = m1[0];
63
64        for (ulong j=n-1; j>=0; --j)
65        {
66            const ulong & jmp = Jmp[j];
67
68            if ( r >= one )
69            {
70                r -= one;
71                A[j] = one;
72            }
73            else
74            {
75                r -= zero;
76                A[j] = zero;
77            }
78            r /= jmp;
79        }
80    }
81
82    void print() const
83    // Print mixed radix number A[].
84    {
85        for (ulong j=0; j<n; ++j)
86            cout << A[j];
87    }
88}
```

```

34         return r;
35     }
1      ulong unrank(ulong r, ulong *B) const
2 // Write mixed radix number with rank r into B[].
3 // Return index of last non-zero digit, 0 also for the all-zero word.
4 {
5     ulong j = 0; // track
6     while ( j < n )
7     {
8         const ulong & nine = m1[j];
9
10        if ( r <= nine ) // on last non-zero digit
11        {
12            // forward (digits increasing, fast)
13            B[j] = r;
14            break;
15        }
16        else // not on last track
17        {
18            // forward (digits decreasing, slow)
19            r -= nine;
20            const ulong & jmp = Jmp[j];
21            ulong d = nine; // new digit
22
23            while ( r > jmp )
24            {
25                r -= jmp;
26                d -= 1;
27            }
28
29            B[j] = d;
30        }
31
32        ++j;
33    }
34
35    const ulong tr = j;
36    while ( ++j < n ) { B[j] = 0; }
37    return tr;
38}

```

## SL-Gray order

```

1 // FILE: src/comb/mixedradix-sl-gray-rank.h
2 class mixedradix_sl_gray_rank
3 // Ranking and unranking function for SL-Gray order.
4 {
5     // Same data and constructor as in mixedradix_subset_lex_rank, omitted.
6
7     ulong rank() const
8     // Return rank of mixed radix number A[].
9     {
10        // last index e such that A[e] != 0,
11        // e == 0 also for the all-zero word:
12        ulong e = 0;
13        for (ulong j=0; j<n; ++j)
14            if ( A[j] != 0 ) e = j;
15
16        ulong r = 0;
17        bool fwq = true; // whether going forward
18        ulong j = 0; // track
19        while ( true )
20        {
21            const ulong & aj = A[j];
22            const ulong & nine = m1[j];
23            const ulong & jmp = Jmp[j];
24
25            if ( j == e ) // on last non-zero digit
26            {
27                if ( fwq ) // forward (digits increasing, fast)
28                {
29                    r += aj;
30
31                }
32            }
33        }
34    }
35
36    void unrank(ulong r, ulong *A)
37    {
38        const ulong & nine = m1[0];
39
40        if ( r == 0 ) // all-zero word
41        {
42            A[0] = 0;
43            return;
44        }
45
46        const ulong & jmp = Jmp[0];
47
48        ulong d = r % nine;
49        r /= nine;
50
51        A[0] = d;
52
53        for (ulong j=1; j<n; ++j)
54        {
55            const ulong & aj = A[j];
56            const ulong & jmp = Jmp[j];
57
58            if ( r == 0 ) // all-zero word
59            {
60                A[j] = 0;
61                return;
62            }
63
64            r -= jmp;
65            r /= nine;
66
67            A[j] = r % nine;
68            r /= nine;
69
70        }
71    }
72
73    ulong rank(ulong r)
74    {
75        ulong e = 0;
76
77        for (ulong j=0; j<n; ++j)
78            if ( r >= m1[j] ) e = j;
79
80        return e;
81    }
82
83    void unrank(ulong r)
84    {
85        ulong e = rank(r);
86
87        unrank(r, A);
88
89        A[e] = 0;
90    }
91
92    ulong rank()
93    {
94        return unrank();
95    }
96
97    void unrank()
98    {
99        unrank();
100    }
101
102    ulong rank(ulong r, ulong *A)
103    {
104        const ulong & nine = m1[0];
105
106        if ( r == 0 ) // all-zero word
107        {
108            A[0] = 0;
109            return 0;
110        }
111
112        const ulong & jmp = Jmp[0];
113
114        ulong d = r % nine;
115        r /= nine;
116
117        A[0] = d;
118
119        for (ulong j=1; j<n; ++j)
120        {
121            const ulong & aj = A[j];
122            const ulong & jmp = Jmp[j];
123
124            if ( r == 0 ) // all-zero word
125            {
126                A[j] = 0;
127                return j;
128            }
129
130            r -= jmp;
131            r /= nine;
132
133            A[j] = r % nine;
134            r /= nine;
135
136        }
137
138        return n;
139    }
140
141    void unrank(ulong r, ulong *A)
142    {
143        const ulong & nine = m1[0];
144
145        if ( r == 0 ) // all-zero word
146        {
147            A[0] = 0;
148            return;
149        }
150
151        const ulong & jmp = Jmp[0];
152
153        ulong d = r % nine;
154        r /= nine;
155
156        A[0] = d;
157
158        for (ulong j=1; j<n; ++j)
159        {
160            const ulong & aj = A[j];
161            const ulong & jmp = Jmp[j];
162
163            if ( r == 0 ) // all-zero word
164            {
165                A[j] = 0;
166                return;
167            }
168
169            r -= jmp;
170            r /= nine;
171
172            A[j] = r % nine;
173            r /= nine;
174
175        }
176
177        A[n] = 0;
178    }
179
180    ulong rank(ulong r, ulong *A, ulong *e)
181    {
182        const ulong & nine = m1[0];
183
184        if ( r == 0 ) // all-zero word
185        {
186            A[0] = 0;
187            *e = 0;
188            return 0;
189        }
190
191        const ulong & jmp = Jmp[0];
192
193        ulong d = r % nine;
194        r /= nine;
195
196        A[0] = d;
197
198        for (ulong j=1; j<n; ++j)
199        {
200            const ulong & aj = A[j];
201            const ulong & jmp = Jmp[j];
202
203            if ( r == 0 ) // all-zero word
204            {
205                A[j] = 0;
206                *e = j;
207                return j;
208            }
209
210            r -= jmp;
211            r /= nine;
212
213            A[j] = r % nine;
214            r /= nine;
215
216        }
217
218        A[n] = 0;
219
220        *e = n;
221
222        return n;
223    }
224
225    void unrank(ulong r, ulong *A, ulong *e)
226    {
227        const ulong & nine = m1[0];
228
229        if ( r == 0 ) // all-zero word
230        {
231            A[0] = 0;
232            *e = 0;
233            return;
234        }
235
236        const ulong & jmp = Jmp[0];
237
238        ulong d = r % nine;
239        r /= nine;
240
241        A[0] = d;
242
243        for (ulong j=1; j<n; ++j)
244        {
245            const ulong & aj = A[j];
246            const ulong & jmp = Jmp[j];
247
248            if ( r == 0 ) // all-zero word
249            {
250                A[j] = 0;
251                *e = j;
252                return;
253            }
254
255            r -= jmp;
256            r /= nine;
257
258            A[j] = r % nine;
259            r /= nine;
260
261        }
262
263        A[n] = 0;
264
265        *e = n;
266
267        return;
268    }
269
270    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr)
271    {
272        const ulong & nine = m1[0];
273
274        if ( r == 0 ) // all-zero word
275        {
276            A[0] = 0;
277            *e = 0;
278            *tr = 0;
279            return 0;
280        }
281
282        const ulong & jmp = Jmp[0];
283
284        ulong d = r % nine;
285        r /= nine;
286
287        A[0] = d;
288
289        for (ulong j=1; j<n; ++j)
290        {
291            const ulong & aj = A[j];
292            const ulong & jmp = Jmp[j];
293
294            if ( r == 0 ) // all-zero word
295            {
296                A[j] = 0;
297                *e = j;
298                *tr = j;
299                return j;
300            }
301
302            r -= jmp;
303            r /= nine;
304
305            A[j] = r % nine;
306            r /= nine;
307
308        }
309
310        A[n] = 0;
311
312        *e = n;
313
314        *tr = n;
315
316        return n;
317    }
318
319    void unrank(ulong r, ulong *A, ulong *e, ulong *tr)
320    {
321        const ulong & nine = m1[0];
322
323        if ( r == 0 ) // all-zero word
324        {
325            A[0] = 0;
326            *e = 0;
327            *tr = 0;
328            return;
329        }
330
331        const ulong & jmp = Jmp[0];
332
333        ulong d = r % nine;
334        r /= nine;
335
336        A[0] = d;
337
338        for (ulong j=1; j<n; ++j)
339        {
340            const ulong & aj = A[j];
341            const ulong & jmp = Jmp[j];
342
343            if ( r == 0 ) // all-zero word
344            {
345                A[j] = 0;
346                *e = j;
347                *tr = j;
348                return;
349            }
350
351            r -= jmp;
352            r /= nine;
353
354            A[j] = r % nine;
355            r /= nine;
356
357        }
358
359        A[n] = 0;
360
361        *e = n;
362
363        *tr = n;
364
365        return;
366    }
367
368    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n)
369    {
370        const ulong & nine = m1[0];
371
372        if ( r == 0 ) // all-zero word
373        {
374            A[0] = 0;
375            *e = 0;
376            *tr = 0;
377            *n = 0;
378            return 0;
379        }
380
381        const ulong & jmp = Jmp[0];
382
383        ulong d = r % nine;
384        r /= nine;
385
386        A[0] = d;
387
388        for (ulong j=1; j<n; ++j)
389        {
389            const ulong & aj = A[j];
390            const ulong & jmp = Jmp[j];
391
392            if ( r == 0 ) // all-zero word
393            {
394                A[j] = 0;
395                *e = j;
396                *tr = j;
397                *n = j;
398                return j;
399            }
400
401            r -= jmp;
402            r /= nine;
403
404            A[j] = r % nine;
405            r /= nine;
406
407        }
408
409        A[n] = 0;
410
411        *e = n;
412
413        *tr = n;
414
415        *n = n;
416
417        return n;
418    }
419
420    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n)
421    {
422        const ulong & nine = m1[0];
423
424        if ( r == 0 ) // all-zero word
425        {
426            A[0] = 0;
427            *e = 0;
428            *tr = 0;
429            *n = 0;
430            return;
431        }
432
433        const ulong & jmp = Jmp[0];
434
435        ulong d = r % nine;
436        r /= nine;
437
438        A[0] = d;
439
440        for (ulong j=1; j<n; ++j)
441        {
442            const ulong & aj = A[j];
443            const ulong & jmp = Jmp[j];
444
445            if ( r == 0 ) // all-zero word
446            {
447                A[j] = 0;
448                *e = j;
449                *tr = j;
450                *n = j;
451                return;
452            }
453
454            r -= jmp;
455            r /= nine;
456
457            A[j] = r % nine;
458            r /= nine;
459
460        }
461
462        A[n] = 0;
463
464        *e = n;
465
466        *tr = n;
467
468        *n = n;
469
470        return;
471    }
472
473    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m)
474    {
475        const ulong & nine = m1[0];
476
477        if ( r == 0 ) // all-zero word
478        {
479            A[0] = 0;
480            *e = 0;
481            *tr = 0;
482            *n = 0;
483            *m = 0;
484            return 0;
485        }
486
487        const ulong & jmp = Jmp[0];
488
489        ulong d = r % nine;
490        r /= nine;
491
492        A[0] = d;
493
494        for (ulong j=1; j<n; ++j)
495        {
496            const ulong & aj = A[j];
497            const ulong & jmp = Jmp[j];
498
499            if ( r == 0 ) // all-zero word
500            {
501                A[j] = 0;
502                *e = j;
503                *tr = j;
504                *n = j;
505                *m = j;
506                return j;
507            }
508
509            r -= jmp;
510            r /= nine;
511
512            A[j] = r % nine;
513            r /= nine;
514
515        }
516
517        A[n] = 0;
518
519        *e = n;
520
521        *tr = n;
522
523        *n = n;
524
525        *m = n;
526
527        return n;
528    }
529
530    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m)
531    {
532        const ulong & nine = m1[0];
533
534        if ( r == 0 ) // all-zero word
535        {
536            A[0] = 0;
537            *e = 0;
538            *tr = 0;
539            *n = 0;
540            *m = 0;
541            return;
542        }
543
544        const ulong & jmp = Jmp[0];
545
546        ulong d = r % nine;
547        r /= nine;
548
549        A[0] = d;
550
551        for (ulong j=1; j<n; ++j)
552        {
553            const ulong & aj = A[j];
554            const ulong & jmp = Jmp[j];
555
556            if ( r == 0 ) // all-zero word
557            {
558                A[j] = 0;
559                *e = j;
560                *tr = j;
561                *n = j;
562                *m = j;
563                return;
564            }
565
566            r -= jmp;
567            r /= nine;
568
569            A[j] = r % nine;
570            r /= nine;
571
572        }
573
574        A[n] = 0;
575
576        *e = n;
577
578        *tr = n;
579
580        *n = n;
581
582        *m = n;
583
584        return;
585    }
586
587    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d)
588    {
589        const ulong & nine = m1[0];
590
591        if ( r == 0 ) // all-zero word
592        {
593            A[0] = 0;
594            *e = 0;
595            *tr = 0;
596            *n = 0;
597            *m = 0;
598            *d = 0;
599            return 0;
600        }
601
602        const ulong & jmp = Jmp[0];
603
604        ulong d = r % nine;
605        r /= nine;
606
607        A[0] = d;
608
609        for (ulong j=1; j<n; ++j)
610        {
611            const ulong & aj = A[j];
612            const ulong & jmp = Jmp[j];
613
614            if ( r == 0 ) // all-zero word
615            {
616                A[j] = 0;
617                *e = j;
618                *tr = j;
619                *n = j;
620                *m = j;
621                *d = j;
622                return j;
623            }
624
625            r -= jmp;
626            r /= nine;
627
628            A[j] = r % nine;
629            r /= nine;
630
631        }
632
633        A[n] = 0;
634
635        *e = n;
636
637        *tr = n;
638
639        *n = n;
640
641        *m = n;
642
643        *d = n;
644
645        return n;
646    }
647
648    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d)
649    {
650        const ulong & nine = m1[0];
651
652        if ( r == 0 ) // all-zero word
653        {
654            A[0] = 0;
655            *e = 0;
656            *tr = 0;
657            *n = 0;
658            *m = 0;
659            *d = 0;
660            return;
661        }
662
663        const ulong & jmp = Jmp[0];
664
665        ulong d = r % nine;
666        r /= nine;
667
668        A[0] = d;
669
670        for (ulong j=1; j<n; ++j)
671        {
672            const ulong & aj = A[j];
673            const ulong & jmp = Jmp[j];
674
675            if ( r == 0 ) // all-zero word
676            {
677                A[j] = 0;
678                *e = j;
679                *tr = j;
680                *n = j;
681                *m = j;
682                *d = j;
683                return;
684            }
685
686            r -= jmp;
687            r /= nine;
688
689            A[j] = r % nine;
690            r /= nine;
691
692        }
693
694        A[n] = 0;
695
696        *e = n;
697
698        *tr = n;
699
700        *n = n;
701
702        *m = n;
703
704        *d = n;
705
706        return;
707    }
708
709    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2)
710    {
711        const ulong & nine = m1[0];
712
713        if ( r == 0 ) // all-zero word
714        {
715            A[0] = 0;
716            *e = 0;
717            *tr = 0;
718            *n = 0;
719            *m = 0;
720            *d = 0;
721            *e2 = 0;
722            return 0;
723        }
724
725        const ulong & jmp = Jmp[0];
726
727        ulong d = r % nine;
728        r /= nine;
729
730        A[0] = d;
731
732        for (ulong j=1; j<n; ++j)
733        {
734            const ulong & aj = A[j];
735            const ulong & jmp = Jmp[j];
736
737            if ( r == 0 ) // all-zero word
738            {
739                A[j] = 0;
740                *e = j;
741                *tr = j;
742                *n = j;
743                *m = j;
744                *d = j;
745                *e2 = j;
746                return j;
747            }
748
749            r -= jmp;
750            r /= nine;
751
752            A[j] = r % nine;
753            r /= nine;
754
755        }
756
757        A[n] = 0;
758
759        *e = n;
760
760        *tr = n;
761
762        *n = n;
763
764        *m = n;
765
766        *d = n;
767
768        *e2 = n;
769
770        return n;
771    }
772
773    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2)
774    {
775        const ulong & nine = m1[0];
776
777        if ( r == 0 ) // all-zero word
778        {
779            A[0] = 0;
780            *e = 0;
781            *tr = 0;
782            *n = 0;
783            *m = 0;
784            *d = 0;
785            *e2 = 0;
786            return;
787        }
788
789        const ulong & jmp = Jmp[0];
790
791        ulong d = r % nine;
792        r /= nine;
793
794        A[0] = d;
795
796        for (ulong j=1; j<n; ++j)
797        {
798            const ulong & aj = A[j];
799            const ulong & jmp = Jmp[j];
800
801            if ( r == 0 ) // all-zero word
802            {
803                A[j] = 0;
804                *e = j;
805                *tr = j;
806                *n = j;
807                *m = j;
808                *d = j;
809                *e2 = j;
810                return;
811            }
812
813            r -= jmp;
814            r /= nine;
815
816            A[j] = r % nine;
817            r /= nine;
818
819        }
820
821        A[n] = 0;
822
823        *e = n;
824
824        *tr = n;
825
826        *n = n;
827
828        *m = n;
829
830        *d = n;
831
832        *e2 = n;
833
834        return;
835    }
836
837    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2)
838    {
839        const ulong & nine = m1[0];
840
841        if ( r == 0 ) // all-zero word
842        {
843            A[0] = 0;
844            *e = 0;
845            *tr = 0;
846            *n = 0;
847            *m = 0;
848            *d = 0;
849            *e2 = 0;
850            *d2 = 0;
851            return 0;
852        }
853
854        const ulong & jmp = Jmp[0];
855
856        ulong d = r % nine;
857        r /= nine;
858
859        A[0] = d;
860
860        for (ulong j=1; j<n; ++j)
861        {
862            const ulong & aj = A[j];
863            const ulong & jmp = Jmp[j];
864
865            if ( r == 0 ) // all-zero word
866            {
867                A[j] = 0;
868                *e = j;
869                *tr = j;
870                *n = j;
871                *m = j;
872                *d = j;
873                *e2 = j;
874                *d2 = j;
875                return j;
876            }
877
878            r -= jmp;
879            r /= nine;
880
881            A[j] = r % nine;
882            r /= nine;
883
884        }
885
886        A[n] = 0;
887
887        *e = n;
888
888        *tr = n;
889
889        *n = n;
890
890        *m = n;
891
891        *d = n;
892
892        *e2 = n;
893
893        *d2 = n;
894
895        return n;
896    }
897
898    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2)
899    {
900        const ulong & nine = m1[0];
901
902        if ( r == 0 ) // all-zero word
903        {
904            A[0] = 0;
905            *e = 0;
906            *tr = 0;
907            *n = 0;
908            *m = 0;
909            *d = 0;
910            *e2 = 0;
911            *d2 = 0;
912            return;
913        }
914
915        const ulong & jmp = Jmp[0];
916
917        ulong d = r % nine;
918        r /= nine;
919
920        A[0] = d;
921
921        for (ulong j=1; j<n; ++j)
922        {
923            const ulong & aj = A[j];
924            const ulong & jmp = Jmp[j];
925
926            if ( r == 0 ) // all-zero word
927            {
928                A[j] = 0;
929                *e = j;
930                *tr = j;
931                *n = j;
932                *m = j;
933                *d = j;
934                *e2 = j;
935                *d2 = j;
936                return;
937            }
938
939            r -= jmp;
940            r /= nine;
941
942            A[j] = r % nine;
943            r /= nine;
944
945        }
946
947        A[n] = 0;
948
948        *e = n;
949
949        *tr = n;
950
950        *n = n;
951
951        *m = n;
952
952        *d = n;
953
953        *e2 = n;
954
954        *d2 = n;
955
956        return;
957    }
958
959    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2, ulong *e3)
960    {
961        const ulong & nine = m1[0];
962
963        if ( r == 0 ) // all-zero word
964        {
965            A[0] = 0;
966            *e = 0;
967            *tr = 0;
968            *n = 0;
969            *m = 0;
970            *d = 0;
971            *e2 = 0;
972            *d2 = 0;
973            *e3 = 0;
974            return 0;
975        }
976
977        const ulong & jmp = Jmp[0];
978
979        ulong d = r % nine;
980        r /= nine;
981
982        A[0] = d;
983
983        for (ulong j=1; j<n; ++j)
984        {
985            const ulong & aj = A[j];
986            const ulong & jmp = Jmp[j];
987
988            if ( r == 0 ) // all-zero word
989            {
990                A[j] = 0;
991                *e = j;
992                *tr = j;
993                *n = j;
994                *m = j;
995                *d = j;
996                *e2 = j;
997                *d2 = j;
998                *e3 = j;
999                return j;
1000            }
1001
1002            r -= jmp;
1003            r /= nine;
1004
1005            A[j] = r % nine;
1006            r /= nine;
1007
1008        }
1009
1010        A[n] = 0;
1011
1011        *e = n;
1012
1012        *tr = n;
1013
1013        *n = n;
1014
1014        *m = n;
1015
1015        *d = n;
1016
1016        *e2 = n;
1017
1017        *d2 = n;
1018
1018        *e3 = n;
1019
1020        return n;
1021    }
1022
1023    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2, ulong *e3)
1024    {
1025        const ulong & nine = m1[0];
1026
1027        if ( r == 0 ) // all-zero word
1028        {
1029            A[0] = 0;
1030            *e = 0;
1031            *tr = 0;
1032            *n = 0;
1033            *m = 0;
1034            *d = 0;
1035            *e2 = 0;
1036            *d2 = 0;
1037            *e3 = 0;
1038            return;
1039        }
1040
1041        const ulong & jmp = Jmp[0];
1042
1043        ulong d = r % nine;
1044        r /= nine;
1045
1046        A[0] = d;
1047
1047        for (ulong j=1; j<n; ++j)
1048        {
1049            const ulong & aj = A[j];
1050            const ulong & jmp = Jmp[j];
1051
1052            if ( r == 0 ) // all-zero word
1053            {
1054                A[j] = 0;
1055                *e = j;
1056                *tr = j;
1057                *n = j;
1058                *m = j;
1059                *d = j;
1060                *e2 = j;
1061                *d2 = j;
1062                *e3 = j;
1063                return;
1064            }
1065
1066            r -= jmp;
1067            r /= nine;
1068
1069            A[j] = r % nine;
1070            r /= nine;
1071
1072        }
1073
1074        A[n] = 0;
1075
1075        *e = n;
1076
1076        *tr = n;
1077
1077        *n = n;
1078
1078        *m = n;
1079
1079        *d = n;
1080
1080        *e2 = n;
1081
1081        *d2 = n;
1082
1082        *e3 = n;
1083
1084        return;
1085    }
1086
1087    ulong rank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2, ulong *e3, ulong *d3)
1088    {
1089        const ulong & nine = m1[0];
1090
1091        if ( r == 0 ) // all-zero word
1092        {
1093            A[0] = 0;
1094            *e = 0;
1095            *tr = 0;
1096            *n = 0;
1097            *m = 0;
1098            *d = 0;
1099            *e2 = 0;
1100            *d2 = 0;
1101            *e3 = 0;
1102            *d3 = 0;
1103            return 0;
1104        }
1105
1106        const ulong & jmp = Jmp[0];
1107
1108        ulong d = r % nine;
1109        r /= nine;
1110
1111        A[0] = d;
1112
1112        for (ulong j=1; j<n; ++j)
1113        {
1114            const ulong & aj = A[j];
1115            const ulong & jmp = Jmp[j];
1116
1117            if ( r == 0 ) // all-zero word
1118            {
1119                A[j] = 0;
1120                *e = j;
1121                *tr = j;
1122                *n = j;
1123                *m = j;
1124                *d = j;
1125                *e2 = j;
1126                *d2 = j;
1127                *e3 = j;
1128                *d3 = j;
1129                return j;
1130            }
1131
1132            r -= jmp;
1133            r /= nine;
1134
1135            A[j] = r % nine;
1136            r /= nine;
1137
1138        }
1139
1140        A[n] = 0;
1141
1141        *e = n;
1142
1142        *tr = n;
1143
1143        *n = n;
1144
1144        *m = n;
1145
1145        *d = n;
1146
1146        *e2 = n;
1147
1147        *d2 = n;
1148
1148        *e3 = n;
1149
1149        *d3 = n;
1150
1150        return n;
1151    }
1152
1153    void unrank(ulong r, ulong *A, ulong *e, ulong *tr, ulong *n, ulong *m, ulong *d, ulong *e2, ulong *d2, ulong *e3, ulong *d3)
1154    {
1155        const ulong & nine = m1[0];
1156
1157        if ( r == 0 ) // all-zero word
1158        {
1159            A[0] = 0;
1160            *e = 0;
1161            *tr = 0;
1162            *n = 0;
1163            *m = 0;
1164            *d = 0;
1165            *e2 = 0;
1166            *d2 = 0;
1167            *e3 = 0;
1168            *d3 = 0;
1169            return;
1170        }
1171
1172        const ulong & jmp = Jmp[0];
1173
1174        ulong d = r % nine;
1175        r /= nine;
1176
1177        A[0] = d;
1178
1178        for (ulong j=1; j<n; ++j)
1179        {
1180            const ulong & aj = A[j];
1181            const ulong & jmp = Jmp[j];
1182
1183            if ( r == 0 ) // all-zero word
1184            {
1185                A[j] = 0;
1186                *e = j;
1187                *tr = j;
1188                *n = j;
1189                *m = j;
1190                *d = j;
1191                *e2 = j;
1192                *d2 = j;
1193                *e3 = j;
1194                *d3 = j;
1195                return;
1196            }
1197
1198            r -= jmp;
1199            r /= nine;
1200
1201            A[j] = r % nine;
1202            r /= nine;
1203
1204        }
1205
1206        A[n] = 0;
1207
1207        *e = n;
1208
1208        *tr = n;
1209
1209        *n = n;
1210
1210        *m = n;
1211
1211        *d = n;
1212
1212        *e2 = n;
1213
1213        *d2 = n;
1214
1214        *e3 = n;
1215
1215        *d3 = n;
12
```

```

24
25 }           // backward (digits decreasing, fast)
26 {
27     const ulong ten = nine + 1;
28     r += jmp * ten;
29     r += ( ten - aj );
30 }
31
32     break; // and return r
33 }
34 else // not on last track
35 {
36     if ( fwq ) // forward (digits decreasing, slow)
37 {
38         r += nine;
39         r += jmp * ( nine - aj );
40     }
41     else // backward (digits increasing, slow)
42 {
43         r += jmp * aj;
44     }
45 }
46
47 // may switch direction:
48 const ulong ap = aj & 1UL;
49 const ulong mp = nine & 1UL;
50 if ( ap != mp ) fwq = ! fwq;
51
52     ++j;
53 }
54
55     return r;
56 }

1 ulong unrank(ulong r, ulong *B) const
2 // Write mixed radix number with rank r into B[].
3 // Return index of last non-zero digit, 0 also for the all-zero word.
4 {
5     bool fwq = true; // whether going forward
6
7     ulong j = 0; // track
8     while ( j < n )
9     {
10         const ulong & nine = m1[j];
11         const ulong & jmp = Jmp[j];
12
13         if ( fwq ) // forward (same as for subset-lex order)
14 {
15             if ( r <= nine ) // forward (digits increasing, fast)
16 {
17                 // on last non-zero digit
18                 B[j] = r;
19                 break; // done
20             }
21             else // forward (digits decreasing, slow)
22 {
23                 r -= nine;
24                 ulong d = nine; // new digit
25                 while ( r > jmp )
26 {
27                     r -= jmp;
28                     d -= 1;
29                 }
30                 B[j] = d;
31             }
32         }
33     else // backward
34 {
35         const ulong ten = nine + 1;
36         const ulong jj = jmp * ten;
37         if ( r > jj ) // backward (digits decreasing, fast)
38 {
39             // on last non-zero digit
40             r -= jj;

```

```

41          B[j] = ten - r;
42          break; // done
43      }
44      else           // backward (digits increasing, slow)
45      {
46          ulong d = 0; // new digit
47          while ( r > jmp )
48          {
49              r -= jmp;
50              d += 1;
51          }
52          B[j] = d;
53      }
54
55      const ulong ap = B[j] & 1UL;
56      const ulong mp = nine & 1UL;
57      if ( ap != mp ) fwq = ! fwq; // switch direction
58
59      ++j;
60  }
61
62  const ulong tr = j;
63  while ( ++j < n ) { B[j] = 0; }
64
65  return tr;
66 }

```