ON BAHADUR-KIEFER TYPE PROCESSES FOR SUMS AND RENEWALS IN DEPENDENT CASES

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Abstract We study the asymptotic behaviour of Bahadur-Kiefer processes that are generated by summing partial sums of (weakly or strongly dependent) random variables and their renewals. Known results for i.i.d. case will be extended to dependent cases.

Key words: partial sums; renewals; Bahadur-Kiefer type processes; Wiener process; fractional Brownian motion; strong approximations.

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1 Introduction

In this work we intend to deal with Bahadur-Kiefer type processes that are based on partial sums and their renewals of weakly, as well as strongly, dependent sequences of random variables. In order to initiate our approach, let $\{Y_0, Y_1, Y_2, \ldots\}$ be random variables which have the same marginal distribution and, to begin with, satisfy the following assumptions:

(i) $\mathrm{E}Y_0 = \mu > 0$;
(ii) $\mathrm{E}(Y_0^2) < \infty$.

In terms of the generic sequence $\{Y_j,\ j = 0,1,2,\ldots\}$, with $t \geq 0$, we define

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$$S(t) := \sum_{i=1}^{[t]} Y_i, \tag{1.1}$$

$$N(t) := \inf\{s \geq 1 : S(s) > t\}, \tag{1.2}$$

$$Q(t) := S(t) + \mu N(\mu t) - 2\mu t, \tag{1.3}$$

whose respective appropriately normalized versions will be used in studying partial sums, their renewals, Bahadur-Kiefer type processes when the random variables in the sequence Y_i, $i = 0, 1, \ldots$ are weakly or strongly dependent.

The research area of what has become known as Bahadur-Kiefer processes was initiated by Bahadur [1] (cf. also Kiefer [22]) who established an almost sure representation of i.i.d. random variables based sample quantiles in terms of their empiricals. Kiefer [23] substantiated this work via studying the deviations between the sample quantile and its empirical processes. These three seminal papers have since been followed by many related further investigations (cf., e.g., Csörgő and Révész [10], [12, Chapter 5], Shorack [26], Csörgő [6], Deheuvels and Mason [17], [18], Deheuvels [15], [16], Csörgő and Horváth [7, Chapters 3-6], Csörgő and Szyszkowicz [13], and references in these works).

It follows from the results of Kiefer [23], and also from Vervaat [27], [28] as spelled out in Csáki *et al.* [5], that the original i.i.d. based Bahadur-Kiefer process cannot converge weakly to any non-degenerate random element of the $D[0,1]$ function space. On the other hand, Csörgő *et al.* [14] showed the opposite conclusion to hold true for long-range dependence based Bahadur-Kiefer processes. For an illustration and discussion of this conclusion, we refer to the Introduction and Corollary 1.2 of Csáki *et al.* [4]. For further results along these lines, we refer to Csörgő and Kulik [8], [9].

The study of the almost sure asymptotic behaviour of Bahadur-Kiefer type processes for sums and their renewals in the i.i.d. case was initiated by Horváth [21], Deheuvels and Mason [17], and augmented by further references and results as in Csörgő and Horváth [7, Chapter 2].

Vervaat [27], [28] initiated the study of limit theorems in general for processes with a positive drift and their inverses. For results on the asymptotic behaviour of integrals of Bahadur-Kiefer type processes for sums and their renewals, the so-called Vervaat processes, we refer to [5] in the i.i.d. case, [3] in the weakly dependent case, and [4] in the strongly dependent case.

Back to the topics of this paper on Bahadur-Kiefer type processes for sums and their renewals, the forthcoming Section 2 is concerned with the weakly dependent case, and Section 3 concludes results in terms of long-range dependent sequences of random variables. Both of these sections contain the relevant proofs as well.

2 Weakly dependent case

In this section we deal with weakly dependent random variables based Bahadur-Kiefer type processes. First we summarize the main results in the case when Y_i are i.i.d. random variables with finite 4-th moment.

Theorem A *Assume that* $\{Y_i, i = 0,1,\dots\}$ *are i.i.d. random variables with* $\mathrm{E}Y_0 = \mu > 0$, $\mathrm{E}(Y_0-\mu)^2 = \sigma^2 > 0$, *and* $\mathrm{E}Y_0^4 < \infty$. *Then we have*

$$Q(T) = \sigma\left(W(T) - W\left(T - \frac{\sigma}{\mu}W(T)\right)\right) + o_{\text{a.s.}}(T^{1/4}), \quad \text{as } T \to \infty, \tag{2.1}$$

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T}|Q(t/\mu)|}{(T\log\log T)^{1/4}(\log T)^{1/2}} = \frac{2^{1/4}\sigma^{3/2}}{\mu^{3/4}}, \quad \text{a.s.}, \tag{2.2}$$

$$\lim_{T\to\infty} \frac{\sup_{0\le t\le T}|Q(t/\mu)|}{(\log T)^{1/2}(\sup_{0\le t\le T}|\mu N(t)-t|)^{1/2}} = \frac{\sigma}{\mu^{1/2}}, \quad \text{a.s.}, \tag{2.3}$$

$$\lim_{T\to\infty} \mathrm{P}(T^{-1/4}|Q(T/\mu)| \le y) = 2\int_{-\infty}^{\infty} \Phi(y\mu^{3/4}\sigma^{-3/2}|x|^{-1/2})\varphi(x)\,dx - 1, \tag{2.4}$$

where Φ *is the standard normal distribution function and* φ *is its density.*

We note that (2.1) and (2.4) are due to Csörgő and Horváth [7], (2.2) is due to Horváth [21] and (2.3) is due to Deheuvels and Mason [17]. All these results can be found in [7].

For the case of i.i.d. random variables when the 4-th moment does not exist, we refer to Deheuvels and Steinebach [19].

In this section we assume that $S(t)$ can be approximated by a standard Wiener process as follows.

Assumption A *On the same probability space there exist a sequence* $\{Y_i,\ i = 0,1,2,\dots\}$ *of random variables, with the same marginal distribution, satisfying assumptions* (i) *and* (ii), *and a standard Wiener process* $W(t),\ t \ge 0$, *such that*

$$\sup_{0\le t\le T}|S(t) - \mu t - \sigma W(t)| = O_{\text{a.s.}}(T^{\beta}) \tag{2.5}$$

almost surely, as $T \to \infty$, *with* $\sigma > 0$, *where* $S(t)$ *is defined by* (1.1) *and* $\beta < 1/4$.

In the case of $1/4 \le \beta < 1/2$, there is a huge literature on strong approximation of the form (2.5) for weakly dependent random variables $\{Y_i\}$. The case $\beta < 1/4$ is treated in Berkes *et al.* [2], where Komlós-Major-Tusnády [24] type strong approximations as in (2.5) are proved under fairly general assumptions of dependence. For exact statements of, and conditions for, strong approximations that yield (2.5) to hold true for the partial sums as in Assumption A, we refer to [2].

Theorem 2.1 *Under* Assumption A *all the results* (2.1), (2.2), (2.3) and (2.4) *in* Theorem A *remain true.*

Proof. In fact, we only have to prove (2.1), for the other results follow from the latter. It follows from [7], Theorem 1.3 on p. 37, that under Assumption A we have

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T}\left|\frac{t}{\mu}-N(t)-\frac{\sigma}{\mu}W(t/\mu)\right|}{(T\log\log T)^{1/4}(\log T)^{1/2}} = 2^{1/4}\sigma^{3/2}\mu^{-7/4} \quad \text{a.s.}$$

and also

$$\sup_{0\le t\le T}|\mu t-\mu S(N(\mu t))| = O_{\text{a.s.}}(T^{\beta})$$

as $T\to\infty$. Hence, as $T\to\infty$, we arrive at

$$Q(T)=S(T)+\mu N(\mu T)-2\mu T = S(T)-\mu T-(S(N(\mu T))-\mu N(\mu T))+O_{\text{a.s.}}(T^{\beta})$$

$$=\sigma(W(t)-W(N(\mu T)))+O_{\text{a.s.}}(T^{\beta})=$$

$$\sigma\left(W(T)-W\left(T-\frac{\sigma}{\mu}W(T)\right)\right)+o_{\text{a.s.}}(T^{1/4}),$$

i.e., having (2.1) as desired. □

3 Strongly dependent case

In this section we deal with long range (strongly) dependent sequences, based on moving averages as defined by

$$\eta_j=\sum_{k=0}^{\infty}\psi_k\xi_{j-k}, \quad j=0,1,2,\ldots, \tag{3.1}$$

where $\{\xi_k,-\infty<k<\infty\}$ is a double sequence of independent standard normal random variables, and the sequence of weights $\{\psi_k,k=0,1,2,\ldots\}$ is square summable. Then $\mathrm{E}(\eta_0)=0$, $\mathrm{E}(\eta_0^2)=\sum_{k=0}^{\infty}\psi_k^2=:\sigma^2$ and, on putting $\tilde{\eta}_j=\eta_j/\sigma$, $\{\tilde{\eta}_j,j=0,1,2,\ldots\}$ is a stationary Gaussian sequence with $\mathrm{E}(\tilde{\eta}_0)=0$ and $\mathrm{E}(\tilde{\eta}_0^2)=1$. If $\psi_k\sim k^{-(1+\alpha)/2}\ell(k)$ with a slowly varying function, $\ell(k)$, at infinity, then $\mathrm{E}(\eta_j\eta_{j+n})\sim b_\alpha n^{-\alpha}\ell^2(n)$, where the constant b_α is defined by

$$b_\alpha=\int_0^{\infty}x^{-(1+\alpha)/2}(1+x)^{-(1+\alpha)/2}dx.$$

Now let $G(\cdot)$ be a real valued Borel measurable function, and define the subordinated sequence $Y_j=G(\tilde{\eta}_j)$, $j=0,1,2,\ldots$. We assume throughout that $J_1:=\mathrm{E}(G(\tilde{\eta}_0)\tilde{\eta}_0)\neq 0$. We say in this case that the Hermite rank of the function $G(\cdot)$ is equal to 1 (cf. Introduction of [4]).

For $1/2<H<1$ let $\{W_H(t),t\ge 0\}$ be a fractional Brownian motion (fbm), i.e., a mean-zero Gaussian process with covariance

$$\mathrm{E}W_H(s)W_H(t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}). \tag{3.2}$$

Based on a strong approximation result of Wang *et al.* [29], what follows next, was proved in Section 2 of Csáki *et al.* [4].

Theorem B *Let η_j be defined by* (3.1) *with $\psi_k \sim k^{-(1+\alpha)/2}$, $0 < \alpha < 1$, and put $\tilde{\eta}_j = \eta_j/\sigma$ with $\sigma^2 := \mathrm{E}(\eta_0^2) = \sum_{k=0}^{\infty} \psi_k^2$. Let $G(\cdot)$ be a function whose Hermite rank is* 1, *and put $Y_j = G(\tilde{\eta}_j)$, $j = 0, 1, 2, \ldots$. Furthermore, let $\{S(t), t \geq 0\}$ be as in* (1.1) *and assume condition* (ii). *Then, on an appropriate probability space for the sequence $\{Y_j = G(\tilde{\eta}_j),\ j = 0, 1, \ldots\}$, one can construct a fractional Brownian motion $W_{1-\alpha/2}(\cdot)$ such that, as $T \to \infty$, we have*

$$\sup_{0 \leq t \leq T} \left| S(t) - \mu t - \frac{J_1 \kappa_\alpha}{\sigma} W_{1-\alpha/2}(t) \right| = o_{\text{a.s.}}(T^{\gamma/2+\delta}), \tag{3.3}$$

where $\mu = \mathrm{E}(Y_0)$,

$$\kappa_\alpha^2 = 2\frac{\int_0^\infty x^{-(\alpha+1)/2}(1+x)^{-(\alpha+1)/2}\,dx}{(1-\alpha)(2-\alpha)}, \tag{3.4}$$

$\gamma = 2 - 2\alpha$ for $\alpha < 1/2$, $\gamma = 1$ for $\alpha \geq 1/2$ and $\delta > 0$ is arbitrary.

Moreover, if we also assume condition (i), *then, as $T \to \infty$,*

$$\sup_{0 \leq t \leq T} \left| \mu N(\mu t) - \mu t + \frac{J_1 \kappa_\alpha}{\sigma} W_{1-\alpha/2}(t) \right| = o_{\text{a.s.}}(T^{\gamma/2+\delta} + T^{(1-\alpha/2)^2+\delta}), \tag{3.5}$$

with γ as right above, and arbitrary $\delta > 0$.

Now, for use in the sequel, we state iterated logarithm results for fractional Brownian motion and its increments, which follows from Ortega's extension in [25] of Csörgő and Révész [11], [12, Section 1.2].

Theorem C *For $T > 0$ let a_T be a nondecreasing function of T such that $0 < a_T \leq T$ and a_T/T is nonincreasing. Then*

$$\limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |W_{1-\alpha/2}(t+s) - W_{1-\alpha/2}(t)|}{a_T^{1-\alpha/2}(2(\log T/a_T + \log\log T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.6}$$

If $\lim_{T\to\infty}(\log(T/a_T))/(\log\log T) = \infty$, then we have lim *instead of* lim sup *in* (3.6).

First we give an invariance principle for $Q(T)$ defined by (1.3) if $\gamma/2 < (1 - \alpha/2)^2$, which corresponds to the i.i.d. case when the forth moment exists. Equivalently, we assume that

$$0 < \alpha < 2 - \sqrt{2}. \tag{3.7}$$

Note that in (3.8) below, the random time argument of $W_{1-\alpha/2}$ is strictly positive for large enough T with probability 1. So, without loss of generality, we may define $W_{1-\alpha/2}(T-u) = 0$ if $u > T$.

Theorem 3.1 *Under the conditions of* Theorem B*, including* (i) *and* (ii)*, assuming* (3.7)*, as* $T \to \infty$*, we have*

$$Q(T) = \frac{J_1\kappa_\alpha}{\sigma}(W_{1-\alpha/2}(T) - W_{1-\alpha/2}(N(\mu T)) + o_{\text{a.s.}}(T^{\gamma/2+\delta})$$

$$= \frac{J_1\kappa_\alpha}{\sigma}\left(W_{1-\alpha/2}(T) - W_{1-\alpha/2}\left(T - \frac{J_1\kappa_\alpha}{\sigma\mu}W_{1-\alpha/2}(T)\right)\right) + o_{\text{a.s.}}(T^{\gamma/2+\delta}). \quad (3.8)$$

Proof. Put $c = J_1\kappa_\alpha/\sigma$. Then

$$Q(T) = S(T) - \mu T + \mu N(\mu T) - \mu T$$

$$= cW_{1-\alpha/2}(T) + o_{\text{a.s.}}(T^{\gamma/2+\delta}) + \mu(N(\mu T) - T).$$

But

$$\mu(T - N(\mu T)) = S(N(\mu T)) - \mu N(\mu T) + \mu T - S(N(\mu T))$$

$$= cW_{1-\alpha/2}(N(\mu T)) + o_{\text{a.s.}}((N(\mu T))^{\gamma/2+\delta}) + \mu T - S(N(\mu T)),$$

and using (3.5) and Theorem C, we have

$$cW_{1-\alpha/2}(N(\mu T)) = cW_{1-\alpha/2}\left(T - \frac{c}{\mu}W_{1-\alpha/2}(T) + o_{\text{a.s.}}(T^{\gamma/2+\delta} + T^{(1-\alpha/2)^2+\delta})\right)$$

$$= cW_{1-\alpha/2}\left(T - \frac{c}{\mu}W_{1-\alpha/2}(T)\right) + o_{\text{a.s.}}(T^{(\gamma/2+\delta)(1-\alpha/2)} + T^{(1-\alpha/2)^3}).$$

On the other hand (cf. [4]), $N(\mu T) = O_{\text{a.s.}}(T)$ and

$$\mu T - S(N(\mu T)) = o_{\text{a.s.}}(T^{\gamma/2+\delta}).$$

Since $(1-\alpha/2)^3 \le \gamma/2 < (1-\alpha/2)^2$, this dominates all the other remainder terms in the proof. Thus the proof of Theorem 3.1 is now complete. □

The proof of Theorem 3.1 also yields the following result.

Proposition 1. *As* $T \to \infty$*,*

$$\mu T - \mu N(\mu T)) = \frac{J_1\kappa_\alpha}{\sigma}W_{1-\alpha/2}\left(T - \frac{J_1\kappa_\alpha}{\sigma\mu}W_{1-\alpha/2}(T)\right) + o_{\text{a.s.}}(T^{\gamma/2+\delta}).$$

Now we are to give a limsup result for $Q(\cdot)$. For this we need a Strassen-type functional law of the iterated logarithm for fbm, due to Goodman and Kuelbs [20].

Theorem D *Let*

$$\mathbf{K} = \{T_H g(t), 0 \le t \le 1, \int_{-\infty}^{1} g^2(u)\,du \le 1\},$$

where

$$T_H g(t) = \frac{1}{k_H}\int_0^t (t-u)^{H-1/2} g(u)\,du + \frac{1}{k_H}\int_{-\infty}^0 (t-u)^{H-1/2} - (-u)^{H-1/2}) g(u)\,du,$$

and

$$k_H^2 = \int_{-\infty}^0 ((1-s)^{H-1/2} - (-s)^{H-1/2})^2\,ds + \int_0^1 (1-s)^{2H-1}\,ds.$$

Then, almost surely, $\mathbf{K}$ *is the set of limit points of the net of stochastic processes*

$$\frac{W_H(nt)}{(2n^{2H}\log\log n)^{1/2}}, 0 \le t \le 1, \tag{3.9}$$

as $n \to \infty$.

Theorem 3.2 *Under the conditions of* Theorem 3.1, *we have*

$$\limsup_{T\to\infty} \frac{|Q(T)|}{T^{(1-\alpha/2)^2}(\log\log T)^{1/2-\alpha/4}(\log T)^{1/2}} = \frac{2^{1-\alpha/4}(J_1\kappa_\alpha)^{2-\alpha/2}}{\sigma^{2-\alpha/2}\mu^{1-\alpha/2}} \quad \text{a.s.} \tag{3.10}$$

Proof. It follows from Theorem C that

$$|W_{1-\alpha/2}(T)| \le (1+\delta)T^{1-\alpha/2}(2\log\log T)^{1/2}$$

with probability 1 for any $\delta > 0$ if T is large enough. Hence, applying Theorem C with $a_T = (1+\delta)c/\mu T^{1-\alpha/2}(2\log\log T)^{1/2}$, $c = J_1\kappa_\alpha/\sigma$, we obtain

$$c \sup_{|s|\le a_T} |W_{1-\alpha/2}(T) - W_{1-\alpha/2}(T-s)| \le c(1+\delta)a_T^{1-\alpha/2}(2\log T)^{1/2},$$

almost surely for large enough T. Since $\delta > 0$ is arbitrary, we obtain the upper bound in (3.10).

To obtain the lower bound, we follow the proof in the i.i.d. case, given in Csörgő and Horváth [7]. On choosing

$$g(s) = \begin{cases} \frac{1}{k_H}((1-s)^{H-1/2} - (-s)^{H-1/2}), & s \le 0, \\ \frac{1}{k_H}(1-s)^{H-1/2}, & 0 < s \le 1, \end{cases}$$

in Theorem D, we have

$$f(t) = \frac{1}{k_H}\int_{-\infty}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) g(s)\,ds + \frac{1}{k_H}\int_0^t (t-s)^{H-1/2} g(s)\,ds.$$

It can be seen that $\int_{-\infty}^{1} g^2(s)\,ds = 1$, and $\{f(t),\ 0 \le t \le 1\}$ is a continuous increasing function with $f(0) = 0$, $f(1) = 1$, and hence by Theorem D it is in **K**. For $0 < \delta < 1$, on considering the function

$$g_\delta(s) = \begin{cases} g(s), & 0 \le s \le 1-\delta, \\ 0, & 1-\delta \le s \le 1, \end{cases}$$

we define

$$f_\delta(t) = \begin{cases} f(t), & 0 \le t \le 1-\delta, \\ f(1-\delta), & 1-\delta \le t \le 1. \end{cases}$$

Then it can be seen that the latter function is in **K**, and hence it is a limit function of the net of stochastic processes as in (3.9). It follows that there is a sequence T_k of random variables such that, in our context,

$$\lim_{k\to\infty} \sup_{0\le t\le 1} \left| \frac{W_{1-\alpha/2}(T_k t)}{T_k^{1-\alpha/2}(2\log\log T_k)^{1/2}} - f_\delta(t) \right| = 0.$$

Using Theorem C with $a_T = f(1-\delta)c/\mu T^{1-\alpha/2}(2\log\log T)^{1/2}$, we get

$$\lim_{T\to\infty} \frac{\sup_{T(1-\delta)\le t\le T} c|W_{1-\alpha/2}(t+a_T) - W(t)|}{c a_T^{1-\alpha/2}(2\log T)^{1/2}} = 1 \qquad \text{a.s.}$$

Since δ is arbitrary, and $\lim_{\delta\to 0} f_\delta(t) = f(t)$, the lower bound follows as in Csörgő and Horváth [7], p. 28. This completes the proof of Theorem 3.2. □

Next we give the limiting distribution of $Q(T)$.

Theorem 3.3 *Under the conditions of* Theorem 3.1, *we have*

$$\lim_{T\to\infty} \mathrm{P}\left(Q(T) T^{-(1-\alpha/2)^2} \le y \right) = \int_{-\infty}^{\infty} \varphi(x) \Phi\left(\frac{y\sigma^{2-\alpha/2}\mu^{1-\alpha/2}}{|x|^{1-\alpha/2}(J_1\kappa_\alpha)^{2-\alpha/2}} \right) dx. \quad (3.11)$$

Proof. According to Theorem 3.1 we have to determine the limiting distribution of

$$c\left(W_{1-\alpha/2}(T) - W_{1-\alpha/2}\left(T - \frac{c}{\mu} W_{1-\alpha/2}(T) \right) \right),$$

where $c = J_1\kappa_\alpha/\sigma$. Via the scaling property of fbm, i.e.,

$$\widetilde{W}(v) := T^{-1+\alpha/2} W_{1-\alpha/2}(Tv), \quad v \ge 0,$$

is also an fbm with parameter $1-\alpha/2$. So we have to determine the limiting distribution of

$$c\left(\widetilde{W}(1) - \widetilde{W}(1 - c_1 T^{-\alpha/2}\widetilde{W}(1)) \right),$$

as $T \to \infty$, where $c_1 = J_1\kappa_\alpha/(\sigma\mu)$.

For $u > 0$, the joint distribution of $\widetilde{W}(1)$, $\widetilde{W}(u)$ is bivariate normal with density

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\left(-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-2r\frac{xy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right),$$

where $\sigma_1^2=\mathrm{E}(W_{1-\alpha/2}^2(1))=1$, $\sigma_2^2=\mathrm{E}(W_{1-\alpha/2}^2(u))=u^{2-\alpha}$ and

$$r=\frac{1+u^{2-\alpha}-|1-u|^{2-\alpha}}{2\sigma_1\sigma_2}.$$

Now consider the conditional density

$$\mathrm{P}(\widetilde{W}(u)\in dz|\widetilde{W}(1)=x)=\frac{1}{\sigma_2\sqrt{1-r^2}}\varphi\left(\frac{z-r\sigma_2 x}{\sigma_2\sqrt{1-r^2}}\right)dz,$$

where $u=1-c_1xT^{-\alpha/2}$.

So the density function of $\widetilde{W}(1)-\widetilde{W}(u)$ is equal to

$$\mathrm{P}(\widetilde{W}(1)-\widetilde{W}(u)\in dY)=\int_{-\infty}^{T^{\alpha/2}/c_1}\frac{1}{\sigma_2\sqrt{1-r^2}}\varphi(x)\varphi\left(\frac{x-Y-r\sigma_2 x}{\sigma_2\sqrt{1-r^2}}\right)dx\,dY$$

and hence its distribution function is

$$\mathrm{P}(\widetilde{W}(1)-\widetilde{W}(u)\le Z)=\int_{-\infty}^{T^{\alpha/2}/c_1}\varphi(x)\Phi\left(\frac{Z-x+r\sigma_2 x}{\sigma_2\sqrt{1-r^2}}\right)dx,\quad -\infty<Z<\infty.$$

It can be seen that, as $T\to\infty$,

$$\sigma_2\sqrt{1-r^2}\sim\frac{|c_1x|^{1-\alpha/2}}{T^{\alpha/2-\alpha^2/4}},$$

$$\frac{x-xr\sigma_2}{\sigma_2\sqrt{1-r^2}}=O(T^{-\alpha/2+\alpha^2/4}).$$

Hence, as $T\to\infty$,

$$\mathrm{P}(\widetilde{W}(1)-\widetilde{W}(u)\le Z)\sim\int_{-\infty}^{T^{\alpha/2}/c_1}\varphi(x)\Phi\left(\frac{ZT^{\alpha/2-\alpha^2/4}}{|c_1x|^{1-\alpha/2}}\right)dx.$$

Putting $Z=yT^{\alpha^2/4-\alpha/2}/c$, and taking the limit $T\to\infty$, we finally obtain (3.11). □

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