The additive model
with different smoothness for the components

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Abstract We consider an additive regression model consisting of two components f^0 and g^0, where the first component f^0 is in some sense "smoother" than the second g^0. Smoothness is here described in terms of a semi-norm on the class of regression functions. We use a penalized least squares estimator $(\hat{f}, \hat{g})$ of (f^0, g^0) and show that the rate of convergence for $\hat{f}$ is faster than the rate of convergence for $\hat{g}$. In fact, both rates are generally as fast as in the case where one of the two components is known. The theory is illustrated by a simulation study. Our proofs rely on recent results from empirical process theory.

Keywords and phrases. Additive model, oracle rate, penalized least squares, smoothness.

Subject classification. 62G08, 62G05.

1 Introduction

Additive modelling has a long history (Stone [1985], Hastie and Tibshirani [1990]) and is very useful for dealing with the curse of dimensionality. Important estimation methods for such models are for example spline smoothing (Wahba [1990]) or iterative back fitting (Mammen et al. [1999]). Our contribution in this paper is to show that standard spline smoothing or more generally penalized least squares can estimate "smoother" components at a faster rate than "rough" components. In fact, we show an oracle rate for the smoother component, which is as fast as in the case where the rough component is known. Similarly (but perhaps less surprisingly) the rougher component can be estimated as fast as in the case where the smooth component is known. These results are in the same spirit as results for semi-parametric models (Bickel et al. [1998]) saying that the parametric part (the parameter of interest) is estimated with parametric rate despite the presence of an infinite-dimensional nuisance parameter. We make use of recent empirical process theory to deal with an infinite-dimensional parameter of interest.

For simplicity we consider the additive model with two components (extensions to more components can be derived essentially along the same lines). Let $(X_i, Z_i)_{i=1}^n$ be i.i.d. input variables and $\{Y_i\}_{i=1}^n$ be i.i.d. real-valued output variables. The model is

$$Y_i = f^0(X_i) + g^0(Z_i) + \epsilon_i, \ i = 1, \ldots, n,$$

where $f^0 \in \mathcal{F}$, $g^0 \in \mathcal{G}$ with $\mathcal{F}$ and $\mathcal{G}$ linear function spaces. Moreover, $\epsilon := (\epsilon_1, \ldots, \epsilon_n)^T$ is a vector of i.i.d. centered noise variables, independent of

$\{(X_i, Z_i)\}_{i=1}^n$. For a vector $v \in \mathbb{R}^n$ we write $\|v\|_n^2 := v^T v / n$. We study the estimator

$$(\hat{f}, \hat{g}) := \arg\min_{f,g} \left\{ \|Y - f - g\|_n^2 + \lambda^2 I^2(f) + \mu^2 J^q(g) \right\},$$

where I is a semi-norm on $\mathcal{F}$, J is a semi-norm on $\mathcal{G}$ and λ and μ are tuning parameters. Moreover, $1 \leq q \leq 2$ is some fixed constant. We consider the case where the "smoothness" induced by I is larger than the "smoothness" induced by J. For example, when both X and Z are bounded real-valued random variables, one may think of I as being some Sobolev norm, J being the total variation norm and $q = 1$. Note that we restrict ourselves to a squared norm in the penalty for the smoother part. A generalization here is straightforward but technical. Also a generalization to values of $q > 2$ is not difficult but is omitted to avoid complicated expressions.

We show that with an appropriate choice of the regularization parameters λ and μ the rate of convergence for the smoother function f^0 is faster than the rate for the less smooth function g^0. For each component we obtain the rate of convergence corresponding to the situation were the other component is known. This result is established assuming an incoherence condition between X_1 and Z_1 (see Condition 2.4).

The results in this paper are related to Wahl [2014]. The latter studies an additive two-component model and applies restricted least squares instead of the penalized least squares used here. Another important paper on the topic is Efromovich [2013] where adaptive rates are derived using a method including blockwise shrinkage. Related is also the paper Müller and van de Geer [2013] where a partial linear model is studied with the linear part being high-dimensional. The method used there is penalized least squares with ℓ_1-penalty on the linear part.

1.1 Organization of the paper

In the next section we outline the conditions used. Main condition is an entropy condition (Condition 2.1) which describes the assumed roughness of the functions f^0 and g^0. Section 3 contains the main theoretical result in Theorem 3.1. Section 4 presents a simulation study. All proofs are in Section 5.

2 Conditions

Let P be the distribution of (X, Z) and $\|\cdot\|$ be the $L_2(P)$-norm. For arbitrary positive constants R and M we let $\mathcal{F}(R, M) := \{f \in \mathcal{F} : \|f\| \leq R, \ I(f) \leq M\}$ and $\mathcal{G}(R, M) := \{g \in \mathcal{G} : \|g\| \leq R, \ J(g) \leq M\}$.

Let $\|\cdot\|_\infty$ be the supremum norm. The entropy of $(\mathcal{F}(R,M), \|\cdot\|_\infty)$ is denoted by $\mathcal{H}_\infty(\cdot, \mathcal{F}(R,M))$. The entropy integral $\mathcal{J}_\infty(\cdot, \mathcal{F}(R,M))$ is defined as

$$\mathcal{J}_\infty(z, \mathcal{F}(R,M)) := z \int_0^1 \sqrt{\mathcal{H}_\infty(uz, \mathcal{F}(R,M), \|\cdot\|_\infty)} du, \ z > 0$$

which we assume to exist.

For the class $\mathcal{G}$ the entropy $\mathcal{H}_\infty(\cdot, \mathcal{G}(R,M))$ and entropy integral $\mathcal{J}_\infty(\cdot, \mathcal{G}(R,M))$ are defined similarly. We shall however use a somewhat relaxed version of entropy and entropy integral for $\mathcal{G}$. Let $\mathcal{A}_n$ be the set of all subsets of cardinality n within the support of Z_1 (equal points are allowed). For $A_n \in \mathcal{A}_n$ and g a real-valued function on this support we let

$$\|g\|_{A_n,\infty} := \max_{z \in A_n} |g(z)|.$$

The entropy of the class $(\mathcal{G}(R,M), \|\cdot\|_{A_n})$ endowed with $\|\cdot\|_{A_n}$-norm is denoted by $\mathcal{H}_{A_n}(\cdot, \mathcal{G}(R,M))$. The uniform entropy is

$$\mathcal{H}_n(\cdot, \mathcal{G}(R,M)) := \sup_{A_n \in \mathcal{A}_n} \mathcal{H}_{A_n}(\cdot, \mathcal{G}(R,M)).$$

We furthermore define the entropy integral

$$\mathcal{J}_n(z, \mathcal{G}(R,M)) := z \int_0^1 \sqrt{\mathcal{H}_n(uz, \mathcal{G}(R,M))} du, \ z > 0 \tag{1}$$

assuming again it exists. Note that $\mathcal{H}_n(\cdot, \mathcal{G}(R,M)) \leq \mathcal{H}_\infty(\cdot, \mathcal{G}(R,M))$ and consequently $\mathcal{J}_n(\cdot, \mathcal{G}(R,M)) \leq \mathcal{J}_\infty(\cdot, \mathcal{G}(R,M))$.

We fix the "roughness indices" $0 < \alpha < \beta < 1$ and assume the following bounds on the entropy integrals for $\mathcal{F}(R,M)$ and $\mathcal{G}(R,M)$. The reason for the more stringent version of entropy (or entropy integral) for $\mathcal{F}(R,M)$ is apparent from Lemma 5.3 where we consider for $f \in \mathcal{F}(R,M)$ conditional versions of $f(X_1)$ given Z_1.

Condition 2.1. *For $R \leq M$ and some constants $A_I \geq 1$ and $A_J \geq 1$, it holds that*

$$\mathcal{J}_\infty(z, \mathcal{F}(R,M)) \leq A_I M^\alpha z^{1-\alpha}, \ z > 0,$$

and

$$\mathcal{J}_n(z, \mathcal{G}(R,M)) \leq A_J M^\beta z^{1-\beta}, \ z > 0.$$

As an illustration, suppose that $X_1 \in [0,1]$ and $I^2(f) = \int |f^{(k)}(x)|^2 dx$, where $f^{(k)}$ denotes the k-th derivative of f. Then $\alpha = 1/(2k)$ and the constant A_I depends only on the smallest eigenvalue of the matrix $\mathbb{E}\psi^T(X_1)\psi(X_1)$ where $\psi(X_1) = (1, X_1, \ldots, X_1^{k-1})$ (see e.g. Birman and Solomjak [1967]). Similar bounds hold for a general class of Besov spaces, see Birgé and Massart [2000].

We assume $\sup\{\|f\|_\infty : f \in \mathcal{F}(R,M)\}$ is bounded by a constant proportional to M and similarly for $\mathcal{G}(R,M)$. Without loss of generality we assume the proportionality constant to be equal to 1.

Condition 2.2. *For some constant $B \geq 1$ and all $M > 0$ and any $R \leq M/B$ it holds that*

$$\sup_{f \in \mathcal{F}(R,M)} \|f\|_\infty \leq M,$$

and

$$\sup_{g \in \mathcal{G}(R,M)} \|g\|_\infty \leq M.$$

For a sub-Gaussian random variable $Z \in \mathbb{R}$ and $\Psi(z) := \exp[|z|^2] - 1$, we define the Orlicz norm

$$\|Z\|_\Psi := \inf\{L > 0 : \mathbb{E}\Psi(Z/L) < 1\}.$$

We will assume that the noise is sub-Gaussian. Extension to sub-exponential noise is straightforward but omitted to avoid technical digressions.

Condition 2.3. *The error ϵ_1 is independent of (X_1, Z_1) and satisfies for some constant $K_\epsilon \geq 1$*

$$\|\epsilon_1\|_\Psi \leq K_\epsilon.$$

Recall that P denotes the distribution of (X, Z). Let $p := dP/d\nu$ be the density of P with respect to a dominating product measure $\nu := \nu_1 \times \nu_2$ with marginal densities p_1 and p_2. We define

$$r(x, z) := \frac{p(x, z)}{p_1(x)p_2(z)}.$$

We let

$$\gamma^2 := \int (r - 1)^2 p_1 p_2 d\nu$$

(assumed to exist). Note that γ is the χ^2-"distance" between the densities p and $p_1 p_2$.

We impose the following incoherence condition.

Condition 2.4. *It holds that $\gamma < 1$.*

Define

$$f_{\mathrm{P}} := E(f(X_1)|Z_1 = \cdot), \quad f_{\mathrm{A}} := f - f_{\mathrm{P}}.$$

The subscript "P" stands for "projection", and "A" stands for "anti-projection". Note that f_{P} is a function with the support of Z_1 as domain. We assume this function to be smooth.

Condition 2.5. *For some constant Γ it holds that*

$$J(f_{\mathrm{P}}) \leq \Gamma \|f\|.$$

To illustrate this condition, suppose that Z_1 is real-valued and $J(g) = \int |g^{(m)}(z)| dz$. Suppose moreover that

$$\sup_x \int |p^{(m)}(x|z)| dz \leq \Gamma,$$

where $p^{(m)}(x|z) := \frac{d^m}{dz^m}(p(x,z)/p_2(z))$. Then, interchanging differentiation and integration (and assuming this is allowed)

$$J(f_{\mathrm{P}}) = \int \left| \int f(x)p^{(m)}(x|z)d\nu_1(x) \right| dz \leq \Gamma \int |f(x)|d\nu_1(x) \leq \|f\|.$$

3   Main result

We define
$$\tau_R(f,g) := \|f+g\| + \lambda I(f) + (\mu/R)^{\frac{2-q}{q}}\mu J(g). \tag{2}$$

We moreover let
$$\tau_I^2(f) := \|f\|^2 + \lambda^2 I^2(f). \tag{3}$$

Theorem 3.1. *Assume Conditions 2.1, 2.2, 2.3, 2.4 and 2.5. Suppose that for some $0 < \delta < 1$, $\max\{A_I^2, A_J^2\}/n \leq n^{-\delta}$ and $(A_I^2/n)^{\frac{1}{1+\alpha}} \leq (A_J^2/n)^{\frac{1}{1+\beta}}n^{-\delta}$. There exist a universal constant C and constants c, c_0, c_1, c_2 depending on α, β, γ, δ, B, Γ, q and K_ϵ as well as on $I(f^0)$ and $J(g^0)$ such that for $n \geq c_0$ and*

$$\sqrt{n}\lambda^{1+\alpha} = c_1 A_I, \ \sqrt{n}\mu^{1+\beta} = c_1 A_J,$$

$$R = c_2\mu, \ R_I = c_2\lambda,$$

one has

$$\mathbb{P}\left(\tau_R(\hat{f} - f^0, \hat{g} - g^0) \leq R, \ \tau_I(\hat{f} - f^0) \leq R_I\right) \geq 1 - C\exp[-n\lambda^2/c].$$

The proof is given in Section 5.

Theorem 3.1 does not provide the explicit dependence on the constants. This dependence can in principle be deduced from Lemmas 5.6 and 5.7 albeit that the expressions are somewhat complicated. In an asymptotic formulation, considering α, β, γ, δ, B, Γ, q, K_ϵ as well as $I(f^0)$ and $J(g^0)$, as fixed, we get for $\lambda^2 \asymp A_I^{\frac{2}{1+\alpha}} n^{-\frac{1}{1+\alpha}}$ and $\mu^2 \asymp A_J^{\frac{2}{1+\beta}} n^{-\frac{1}{1+\beta}}$, the rates

$$\|\hat{f} - f^0\|^2 = \mathcal{O}_{\mathbf{P}}(A_I^{\frac{2}{1+\alpha}} n^{-\frac{1}{1+\alpha}}), \ \|\hat{g} - g^0\|^2 = \mathcal{O}_{\mathbf{P}}(A_J^{\frac{2}{1+\beta}} n^{-\frac{1}{1+\beta}}),$$

$$I(\hat{f}) = \mathcal{O}_{\mathbf{P}}(1), \ J(\hat{g}) = \mathcal{O}_{\mathbf{P}}(1).$$

Example 3.1. *Suppose that X_1 and Z_1 take values in the interval $[0,1]$ and that $I^2(f) = \int |f^{(k)}(x)|^2 dx$ and $J^2(g) = \int |g^{(m)}(z)|^2 dz$ with $m < k$. Then with $q = 2$ the estimator is a spline and easy to calculate as the loss function as well as the penalties are quadratic forms. The rates of convergence are $\|\hat{f} - f^0\| = \mathcal{O}_{\mathbf{P}}(n^{-\frac{k}{2k+1}})$ and $\|\hat{g} - g^0\| = \mathcal{O}_{\mathbf{P}}(n^{-\frac{m}{2m+1}})$. See Section 4 for some numerical results.*

Example 3.2. *Suppose that X_1 takes its values in $[0,1]$ and Z_1 is real-valued. Let $I^2(f) := \int |f^{(k)}(x)|^2 dx$ with $k > 1$ and $J(g) := \mathrm{TV}(g)$ be the total variation of g. Then with $q = 1$ the estimator is again easy to calculate (the problem being formally equivalent to a Lasso problem). The rates of convergence are $\|\hat{f} - f^0\| = \mathcal{O}_{\mathbf{P}}(n^{-\frac{k}{2k+1}})$ and $\|\hat{g} - g^0\| = \mathcal{O}_{\mathbf{P}}(n^{-\frac{1}{3}} \log^{\frac{1}{3}} n)$. Indeed, Condition 2.1 for the class $\mathcal{G}$ now holds with $\beta = 1/2$ and $A_J \asymp \sqrt{\log n}$. This follows e.g. from Lemma 2.2 in van de Geer [2000]. We note that once we have this fast rate for $\|\hat{f} - f^0\|$, the $(\log n)$-term in the rate for $\|\hat{g} - g^0\|$ can be easily removed using instead of the uniform entropy $\mathcal{H}_n$ the $\|\cdot\|_n$-entropy bound from Birman and Solomjak [1967] with $\|\cdot\|_n$-being the empirical L_2-norm (i.e. for a real-valued function m on the support of (X_1, Z_1), $\|m\|_n := (\sum_{i=1}^{n} m^2(X_i, Z_i)/n)^{1/2}$).*

4   Simulation results

In this simulation study, we show that the results of Theorem 3.1 also (approximately) hold empirically. We consider Example 3.1. We estimate each of the "true" functions f^0 and g^0 in the cases where neither functions are known and the cases where one of them is known. We will see that, for each function, the rate of convergence of the estimator when neither of the "true" functions is known is of the same order than that when one of the components is known. For this, we will show the plots of the MSE of the four estimators in four different scenarios (see Figure 1). However, we will only show the plots of the estimators when correlation$(X, Z) = 0.8$, SNR $= 7$ since analogous results hold for the other scenarios.

Let X and U be independent uniformly distributed random variables with values in $(0, 1)$. Define $Z = a X + (1 - a) U$ with a an appropriate constant such that the correlation between X and Z is equal to ρ (which we will define later).

We use B-splines of order 6 (piecewise polynomials of degree 5) to represent each of the functions f and g (see de Boor [2001]). We write

$$f(x) = \sum_{i=1}^{K} \gamma_{f,i} b_{f,i}(x), \qquad g(z) = \sum_{j=1}^{K} \gamma_{g,j} b_{g,j}(z),$$

where $b_{f,i}, b_{g,j}, \ i, j = 1, ..., K$ are the basis functions of the B-spline parametrization, $\gamma_f = (\gamma_{f,1}, ..., \gamma_{f,K}), \gamma_g = (\gamma_{g,1}, ..., \gamma_{g,K})$ are the parameters vectors of f and g, respectively, and $K + 6$ is the number of knots, which we choose to be $\lceil 3\sqrt{n}/5 \rceil + 6$ where n represents the number of observations. Denote by $(x_1, ..., x_n)$ and $(z_1, ..., z_n)$ realizations of the dependent random variables X and Z and let $x_{(r)}$ be the r-th order statistic of the sample from X ($r = 1, \ldots, n$). For estimating the function f (and analogously for the function g), we place the first and last 6 knots (corresponding to the order of the B-spline) in $x_{(1)}$ and $x_{(n)}$, respectively, and position the remaining knots uniformly in $\{x_{(2)}, ..., x_{(n-1)}\}$.

We define the penalizations as

$$I^2(f) := \int_0^1 |f'''(x)|^2 \mathrm{d}x \ + \ \int_0^1 |f(x)|^2 \mathrm{d}x,$$

$$J^2(g) := \int_0^1 |g''(z)|^2 \mathrm{d}z \ + \ \int_0^1 |g(z)|^2 \mathrm{d}z$$

and the $(i,j) - th$ components of the matrices $\Omega_f, \Omega_g \in \mathbb{R}^{K \times K}$ as

$$(\Omega_f)_{i,j} := \int_0^1 b'''_{f,i}(x) b'''_{f,j}(x) \mathrm{d}x + \int_0^1 b_{f,i}(x) b_{f,j}(x) \mathrm{d}x$$

and

$$(\Omega_g)_{i,j} := \int_0^1 b''_{g,i}(z) b''_{g,j}(z) \mathrm{d}z + \int_0^1 b_{g,i}(z) b_{g,j}(z) \mathrm{d}z$$

Then, we can write $I^2(f) = \gamma_f^T \Omega_f \gamma_f$ and $J^2(g) = \gamma_g^T \Omega_g \gamma_g$. Moreover, using Cholesky, we can find matrices $H_f, H_g \in \mathbb{R}^{K \times K}$ such that $\Omega_f = H_f^T H_f$ and $\Omega_g = H_g^T H_g$.

The case where both f^0 and g^0 are unknown:
Consider the two-components model:

$$Y_i = f^0(X_i) + g^0(Z_i) + \epsilon_i, \qquad i = 1, ..., n, \tag{4}$$

where $\epsilon_i, \ i = 1, ..., n$ are i.i.d. centered Gaussian random variables with variance σ^2. The estimator is

$$(\hat{f}, \hat{g}) := \arg\min_{f,g} \left\{ \|Y - f - g\|_n^2 + \lambda^2 I^2(f) + \mu^2 J^2(g) \right\}.$$

We took $\lambda = 14 \, n^{-3/7}$ and $\mu = 0.3 \, n^{-2/5}$. The constants of both tuning parameters are chosen by minimizing the mean square error[1] of the estimators for the case $n = 5000$. Candidates for the constants were taken from the grid $(\{1, 2, 3, ..., 20\} \times \{0.1, 0.2, 0.3, ..., 1\})$, where the first set corresponds to the constant of λ and the second to the constant of μ.

The case where f^0 or g^0 is known:
If g^0 is known we re-write equation (4) as

$$Y_i^f = f^0(X_i) + \epsilon_i,$$

with $Y_i^f = Y_i - g^0(Z_i)$, $i = 1, \ldots, n$.

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[1]Estimated using 100 simulations.

We then use the estimator

$$\hat{f}^s := \arg\min_f \left\{ \|Y^f - f\|_n^2 + \lambda^2 I^2(f) \right\}.$$

The tuning parameter is taken to be $\lambda = 14\, n^{-3/7}$.

Similarly, if f^0 is known we let $Y^g := Y - f^0$ and

$$\hat{g}^s := \arg\min_g \left\{ \|Y^g - g\|_n^2 + \mu^2 J^2(g) \right\}$$

with $\mu = 0.3\, n^{-2/5}$.

Simulations:
Define the Signal-to-Noise ratio as $\text{SNR} := \text{var}(f^0(X) + g^0(Z))/\sigma^2$. For our simulations, we consider the following scenarios:

- $f^0(x) = -10\sin(1.9x + 0.2\pi) + \mathbb{E}[10\sin(1.9x + 0.2\pi)]$.
- $g^0(z) = 3e^{-500(z-0.1)^2} - \mathbb{E}[3e^{-500(z-0.1)^2}]$.
- $\text{SNR} \in \{0.5, 7\}$.
- $\rho \in \{0.2, 0.8\}^2$.
- $n \in \{100, 150, 200, ..., 5000\}$.

The error variance σ^2 was chosen in each scenario to match the above given Signal-to-Noise ratios. For each n the average of 100 simulations is used to estimate the mean square error. In Figure 1, we see that the rate of convergence of $\hat{f}$ and of $\hat{f}^s$ are of similar order and that the same applies to $\hat{g}^s$ and $\hat{g}$. In other words, for each function f^0 and g^0, the rate of convergence of the estimators when both functions are unknown (approximately) corresponds to the case when one of them is known. These results agree with Theorem 3.1 and hold in the four simulation scenarios. Moreover, we see that the convergence of $\hat{f}^s$ and $\hat{f}$ to f^0 is faster than that of $\hat{g}^s$ and $\hat{g}$ to g^0, which is also established in Theorem 3.1.

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[2]The value $\rho = 0.2$ corresponds to $a = 0.169$ and the value $\rho = 0.8$ to $a = 0.571$.

(a) $\rho = 0.2$, SNR $= 0.5$

(b) $\rho = 0.2$, SNR $= 7$

(c) $\rho = 0.8$, SNR $= 0.5$
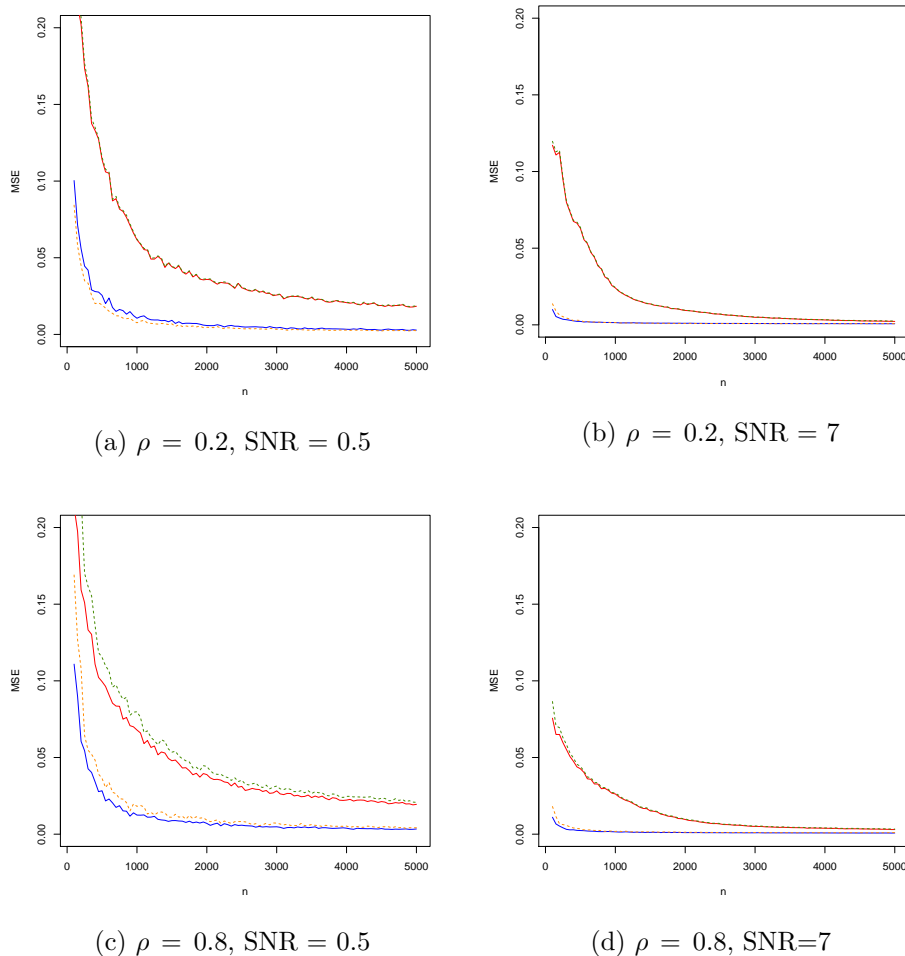
(d) $\rho = 0.8$, SNR$=7$

Figure 1: Estimated MSE for each of the computed estimators: $\hat{f}^s$ (blue line), $\hat{g}^s$ (red line), $\hat{f}$ (orange dotted line), and $\hat{g}$ (green dotted line) for the four simulation scenarios.

The log-transformed data from Figure 1 for the scenario $\rho = 0.8$ and SNR $= 7$ is plotted in Figure 2. Here, we fit a linear regression on each curve considering only those observations corresponding to $n \geq 1000$ and print the slope of these and the theoretical slope[3] in the legend of the plot. With SNR$=7$ it is not clear whether the slopes of the regression line of the estimators agree with their theoretical counterpart. For lower SNR however the agreement is remarkably good (not shown here).

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[3]Recall that by Theorem 3.1 we have $\log ||\hat{f} - f^0||_2^2 = \log(c_1) - (6/7)\log(n)$ and $\log ||\hat{g} - g^0||_2^2 = \log(c_2) - (4/5)\log(n)$, where c_1 and c_2 are constants depending on those of the tuning parameters.
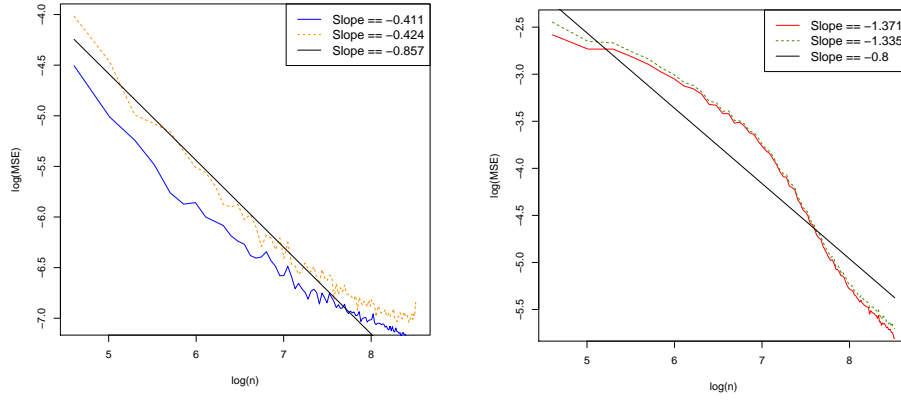
(a) Log transformed data for $\hat{f}^s$ (blue line) and $\hat{f}$ (orange dotted line)

(b) Log transformed data for $\hat{g}^s$ (red line) and $\hat{g}$ (green dotted line)

Figure 2: Log-transformed data for the case $\rho = 0.8$ and SNR $= 7$. A black line using the theoretical slope and an arbitrary intercept was drawn for graphical comparison.

The plots of both f^0 and g^0 and their corresponding estimators for the scenario $\rho = 0.8$ and SNR $= 7$ are displayed in Figure 3. We can see that, as the number of observations increases, the functions $\hat{f}$ and $\hat{g}$ converge to $\hat{f}^s$ and $\hat{g}^s$, respectively. This happens while all of them improve their estimation of the true functions f^0 and g^0 appropriately. We note that $\hat{f}$ and $\hat{f}^s$ are almost identical to f^0 when the number of observations is large. However, $\hat{g}$ and $\hat{g}^s$ can only resemble but not describe perfectly g^0. This is probably due to the highly variable second and third derivatives of g^0 in comparison with those of f^0, as can be seen in Figure 4.
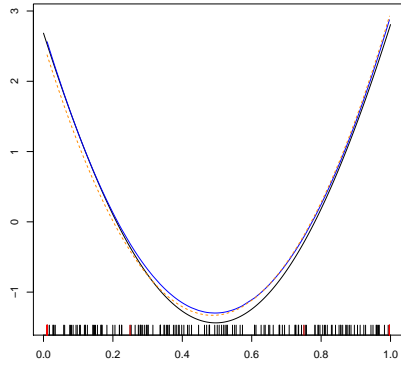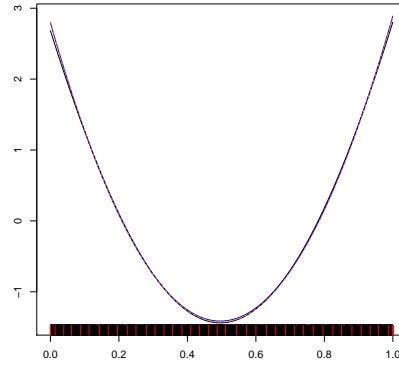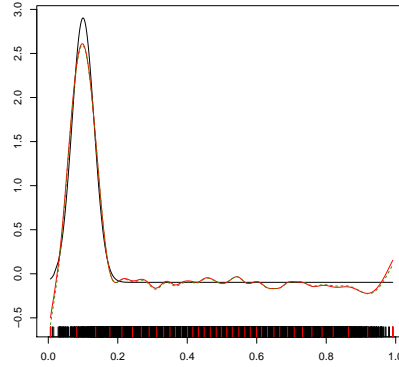
(a) $n = 150$ (b) $n = 5000$
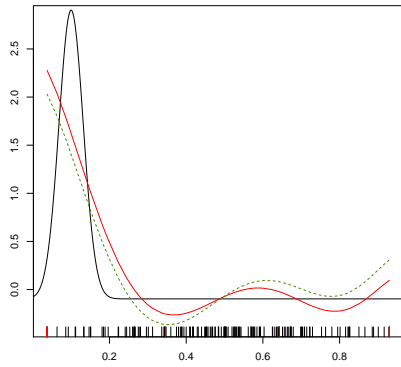
(c) $n = 150$ (d) $n = 5000$

Figure 3: Plots of the true functions f^0 and g^0 (black lines) with their corresponding estimators $\hat{f}^s$ (blue lines), $\hat{f}$ (orange dotted lines), $\hat{g}^s$ (red lines), $\hat{g}$ (green dotted lines), for $\rho = 0.8$ and SNR $= 7$. The data are represented with black small vertical lines and knots positions with red small vertical lines. For each $n \in \{150, 5000\}$, we use a single simulation (not 100 simulations).
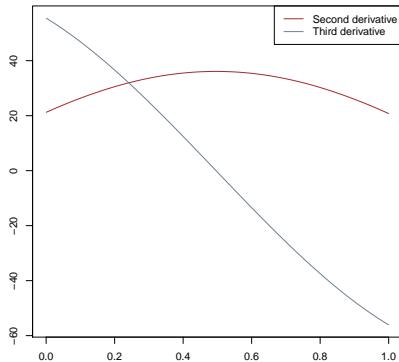
(a) Derivatives of f^0.
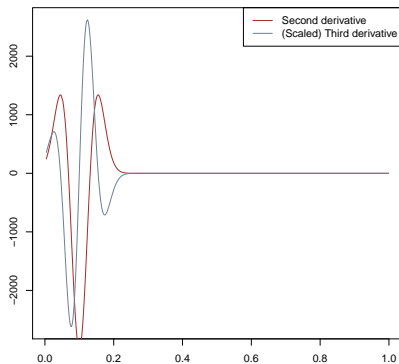
(b) Derivatives of g^0.

Figure 4: Second and third derivatives of the true functions f^0 and g^0 in $[0, 1]$. The third derivative of g^0 was multiplied by 0.02 for an easier comparison.

5 Proofs

We use the notation $P_n := \sum_{i=1}^{n} \delta_{(X_i, Z_i)}/n$ for the empirical measure based on $\{(X_i, Z_i)\}_{i=1}^{n}$.

The proof is organized as follows. We first present some preliminary results needed for the proof of the faster rate for $\hat{f}$. Then we look in Subsection 5.2 at the global rate for both components. We use here the convexity of the least squares loss function and the penalties to localize the problem to the set $\mathcal{M}(R) := \{(f, g) : \tau_R(f, g) \leq R\}$, and then show that indeed $(\hat{f} - f^0, \hat{g} - g^0) \in \mathcal{M}(R)$ provided that the random part of the problem is under control. In Subsection 5.5 we show the random part is indeed under control with large probability. For this result, we need recent findings from empirical process theory, in particular the convergence of empirical norms and inner products. Here,

we apply some results from van de Geer [2014]. The application is somewhat elaborate: for an additive model with p components there are $\binom{p+1}{2} - 1$ terms to consider. If there is only one component, say f, one needs to consider the behaviour of $\|f\|_n^2 - \|f\|^2$ and $\epsilon^T f/n$ uniformly over some collection of functions f. If there are two components f and g the number of terms to consider is five: namely uniform convergence $\|f\|_n^2$, $\|g\|_n^2$, $P_n fg$, $\epsilon^T f/n$ and $\epsilon^T g/n$ to their theoretical counterparts. This is done in Subsection 5.4. Subsection 5.2 takes such uniform convergence for granted. The same is true in Subsection 5.3 where we show the faster rate for the estimator $\hat{f}$ of the smoother component: the results are on a random event which is shown to have large probability in Subsection 5.6 using results from empirical process theory given in Subsection 5.4. We finally collect all pieces in Subsection 5.7 to finish the proof of the main result in Theorem 3.1.

5.1 Preliminaries

Lemma 5.1. *Assume Condition 2.4 and suppose $\int f p_1 d\nu_1 = 0$. Then*

$$\|f + g\|^2 \geq (1 - \gamma)(\|f\| + \|g\|)^2.$$

Proof. We have

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2 + 2 \int (fg) p d\nu.$$

Moreover, since $\int f p_1 = 0$,

$$\left| \int fg p d\nu \right| = \left| \int fg(r-1)p_1 p_2 d\nu \right| \leq \gamma \left(\int f^2 g^2 p_1 p_2 d\nu \right)^{1/2} = \gamma \|f\| \|g\|.$$

Hence,

$$\|f + g\|^2 \geq \|f\|^2 + \|g\|^2 - 2\gamma \|f\| \|g\|$$
$$= (1 - \gamma)(\|f\|^2 + \|g\|^2) + \gamma(\|f\| - \|g\|)^2 \geq (1 - \gamma)(\|f\| + \|g\|)^2.$$

$\square$

Lemma 5.2. *Assume Condition 2.4 and suppose $\int f p_1 d\nu_1 = 0$. We have that*

$$\|f_{\mathrm{P}}\|^2 \leq \gamma \|f\|^2$$

and

$$\|f_{\mathrm{A}}\|^2 \geq (1 - \gamma^2)\|f\|^2.$$

Proof. We have

$$f_{\mathrm{P}} = \int f(r-1) p_1 d\nu_1.$$

Hence

$$\|f_{\mathrm{P}}\| \leq \|r - 1\| \|f\| = \gamma \|f\|,$$

and

$$\|f_{\mathrm{A}}\|^2 = \|f\|^2 - \|f_{\mathrm{P}}\|^2 \geq (1 - \gamma^2)\|f\|^2.$$

$\square$.

Lemma 5.3. *Assume Conditions 2.1 and 2.2. Then*

$$\mathcal{J}_\infty(z, \{f_A : \ f \in \mathcal{F}(R, M)\}) \leq 2\mathcal{J}_\infty(z, \mathcal{F}(R, M)), \ z > 0$$

and for $R \leq M/B$

$$\sup_{f \in \mathcal{F}(R,M)} \|f_A\|_\infty \leq 2M.$$

Proof. Let $u > 0$ and $f, \ \tilde{f} \in \mathcal{F}(R, M)$ be arbitrary, satisfying $\|f - \tilde{f}\|_\infty \leq u$. Then clearly also

$$\|f_P - \tilde{f}_P\|_\infty = \|E(f(X_1) - \tilde{f}(X_1)|Z = \cdot)\|_\infty \leq u.$$

So then

$$\|f_A - \tilde{f}_A\|_\infty \leq \|f_P - \tilde{f}_P\|_\infty + \|f - \tilde{f}\|_\infty \leq 2u.$$

Similarly, for $f \in \mathcal{F}(R, M)$, we have

$$\|f_A\|_\infty \leq \|f_P\|_\infty + \|f\|_\infty \leq 2M.$$

$\square$

5.2 A global bound

We define

$$\mathcal{M}(R) := \{(f, g) : \ \tau_R(f, g) \leq R\} \tag{5}$$

and for a sufficiently small value $\delta_0 > 0$, to be chosen later the sets

$$\mathcal{T}_1(R) := \left\{ \sup_{(f,g)\in\mathcal{M}(R)} \left| \|f + g\|_n^2 - \|f + g\|^2 \right| \leq \delta_0^2 R^2 \right\},$$

$$\mathcal{T}_2(R) := \left\{ \sup_{(f,g)\in\mathcal{M}(R)} |\epsilon^T(f + g)|/n \leq \delta_0^2 R^2 \right\},$$

and

$$\mathcal{T}(R) := \mathcal{T}_1(R) \cap \mathcal{T}_2(R). \tag{6}$$

Lemma 5.4. *Take $\delta_0 \leq \frac{1}{20}$ and suppose that*

$$\lambda^2 I^2(f^0) + \mu^2 J^q(g^0) \leq \delta_0^2 R^2. \tag{7}$$

Then on $\mathcal{T}(R)$, we have $\|\hat{m} - m^0\|^2 + \lambda^2 I^2(\hat{f}) + \mu^2 J^q(\hat{g}) \leq 4\delta_0^2 R^2$ and $\tau_R(\hat{f} - f^0, \hat{g} - g^0) \leq R$.

Proof. Define

$$\tilde{f} := t\hat{f} + (1 - t)f^0, \ \tilde{g} := t\hat{g} + (1 - t)g^0$$

with

$$t := \frac{R}{R + \tau_R(\hat{f} - f^0, \hat{g} - g^0)}.$$

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Then $\tau_R(\tilde{f} - f^0, \tilde{g} - g^0) \leq R$. Let $\tilde{m} := \tilde{f} + \tilde{g}$ and $m^0 := f^0 + g^0$. By convexity

$$\|\tilde{m} - m^0\|_n^2 + \lambda^2 I^q(\tilde{f}) + \mu^2 J^q(\tilde{g}) \leq 2\epsilon^T(\tilde{m} - m^0) + \lambda^2 I^q(f^0) + \mu^2 J^q(g^0).$$

On $\mathcal{T}(R)$ we find

$$\|\tilde{m} - m^0\|^2 + \lambda^2 I^q(\tilde{f}) + \mu^2 J^q(\tilde{g}) \leq 4\delta_0^2 R^2.$$

Hence

$$I(\tilde{f}) \leq (2\delta_0)R/\lambda,$$

and

$$J(\tilde{g}) \leq (2\delta_0)^{2/q}(R/\mu)^{2/q} \leq 2\delta_0(R/\mu)^{2/q}$$

where in the last step we used $2\delta_0 < 1$ and $2/q \geq 1$. Since by (7) it holds that $I(f^0) \leq 2\delta_0 R/\lambda$ we get

$$I(\tilde{f} - f^0) \leq 4\delta_0 R/\lambda.$$

Also, by (7) we have $J(g^0) \leq (2\delta_0)^{2/q}(R/\mu)^{2/q} \leq 2\delta_0(R/\mu)^{2/q}$ so that

$$J(\tilde{g} - g^0) \leq 4\delta_0(R/\mu)^{2/q}.$$

We find

$$\lambda I(\tilde{f} - f^0) \leq 4\delta_0 R$$

as well as

$$(\mu/R)^{\frac{2-q}{q}}\mu J(\tilde{g} - g^0) \leq 4\delta_0 R.$$

But then

$$\tau_R(\tilde{f} - f^0, \tilde{g} - g^0) = \|\tilde{m} - m^0\| + \lambda I(\tilde{f} - f^0) + (\mu/R)^{\frac{2-q}{q}}\mu J(\tilde{g} - g^0)$$

$$\leq 10\delta_0 R \leq R/2$$

where we used $\delta_0 \leq \frac{1}{20}$. This implies $\tau_R(\hat{f} - f^0, \hat{g} - g^0) \leq R$. Repeating the argument completes the proof. $\qquad\square$

5.3 A tighter bound for the smoother part

Let $\mathcal{F}(R_I) := \left\{ f : \tau_I(f) \leq R_I \right\}.$

For δ_1 sufficiently small we define

$$\mathcal{T}_{1,I}(R_I) := \left\{ \sup_{f \in \mathcal{F}(R_I)} \left| \|f_A\|_n^2 - \|f_A\|^2 \right| \leq \delta_1^2 R_I^2 \right\},$$

$$\mathcal{T}_{I,2}(R_I) := \left\{ \sup_{f \in \mathcal{F}(R_I)} |\epsilon^T f_A|/n \leq \delta_1^2 R_I^2 \right\},$$

$$\mathcal{T}_{I,3}(R_I, R) := \left\{ \sup_{(f,g) \in \mathcal{M}(R): \ f \in \mathcal{F}(R_I)} |P_n f_A(g + f_P)| \leq \delta_1^2 R_I^2 \right\}$$

and we let

$$\mathcal{T}_I(R_I, R) := \mathcal{T}_{I,1}(R_I) \cap \mathcal{T}_{I,2}(R_I) \cap \mathcal{T}_{I,3}(R_I, R). \tag{8}$$

Lemma 5.5. *Assume Condition 2.4 and 2.5. Suppose the condition (7)*

$$\lambda^2 I^2(f^0) + \mu^2 J^2(g^0) \leq \delta_0^2 R^2$$

of Lemma 5.4 holds, with $\delta_0 \leq \frac{1}{20}$ given as there. Suppose moreover that

$$\lambda^2 I^2(f^0) \leq \delta_1^2 R_I^2, \tag{9}$$

$$2\mu^2 \Gamma(2\delta_0 R/\mu)^{\frac{2(q-1)}{q}} \leq \delta_1^2 R_I^2, \ 2\mu^2 \Gamma^q / R_I^{2-q} \leq \delta_1^2 \tag{10}$$

and $\delta_1^2 \leq \frac{(1-\gamma^2)}{36}$. Then on $\mathcal{T}(R) \cap \mathcal{T}_I(R_I)$ it holds that $\tau_I(\hat{f} - f^0) \leq R_I$.

Proof. We use the Basic Inequality

$$\|Y - \hat{f} - \hat{g}\|_n^2 + \lambda^2 I^2(\hat{f}) + \mu^2 J^q(\hat{g}) \leq \|Y - f^0 - (\hat{g} + \hat{f}_{\mathrm{P}} - f_{\mathrm{P}}^0)\|_n^2$$

$$+ \lambda^2 I^2(f^0) + \mu^2 J^q(\hat{g} + \hat{f}_{\mathrm{P}} - f_{\mathrm{P}}^0).$$

This gives that

$$\|\hat{f}_{\mathrm{A}} - f_{\mathrm{A}}^0\|^2 + \lambda^2 I^2(\hat{f}) + \mu^2 J^q(\hat{g})$$

$$\leq 2\epsilon^T (\hat{f}_{\mathrm{A}} - f_{\mathrm{A}}^0)/n - 2(\hat{f}_{\mathrm{A}} - f_{\mathrm{A}}^0)^T (\hat{g} - g^0 + \hat{f}_{\mathrm{P}} - f_{\mathrm{P}}^0)/n$$

$$+ \|\hat{f}_{\mathrm{A}} - f_{\mathrm{A}}^0\|^2 - \|\hat{f}_{\mathrm{A}} - f_{\mathrm{A}}^0\|_n^2 + \lambda^2 I^2(f^0) + \mu^2 J^q(\hat{g} + \hat{f}_{\mathrm{P}} - f_{\mathrm{P}}^0).$$

By convexity the inequality also holds if we replace $\hat{f}$ by $\tilde{f} := t\hat{f} + (1-t)\hat{f}^0$ with

$$t := \frac{R_I}{R_I + \tau_I(\hat{f} - f^0)}.$$

Before exploiting this, we derive a bound for $J^q(\hat{g} + \tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0)$. We use that for positive a and b,

$$(a + b)^q - a^q \leq 2(a + b)^{q-1} b \leq 2(a^{q-1} + b^{q-1})b = 2a^{q-1}b + 2b^q$$

Hence

$$J^q(\hat{g} + \tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0) - J^q(\hat{g}) \leq 2J^{q-1}(\hat{g})J(\tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0) + 2J^q(\tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0)$$

$$\leq 2\Gamma J^{q-1}(\hat{g})\|\tilde{f} - f^0\| + 2\Gamma^q \|\tilde{f} - f^0\|^q$$

where in the last step we used Condition 2.5. On $\mathcal{T}(R)$ we have $J(\hat{g}) \leq (2\delta_0 R/\mu)^{2/q}$ by Lemma 5.4. We also have $\|\tilde{f} - f^0\| \leq R_I$. Hence

$$J^q(\hat{g} + \tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0) - J^q(\hat{g}) \leq 2\Gamma(2\delta_0 R/\mu)^{\frac{2(q-1)}{q}} R_I + 2\Gamma^q R_I^q.$$

But then by condition (10)

$$\mu^2 J^q(\hat{g} + \tilde{f}_{\mathrm{P}} - f_{\mathrm{P}}^0) \leq 2\delta_1^2 R_I^2.$$

We insert this result in the Basic Inequality with $\hat{f}$ replaced by $\tilde{f}$:

$$\|\tilde{f}_{\mathrm{A}} - f_{\mathrm{A}}^0\|^2 + \lambda^2 I^2(\tilde{f}) \leq 6\delta_1^2 R_I^2 + \lambda^2 I^2(f_0).$$

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Invoking (9) we get

$$\|\tilde{f}_{\mathrm{A}} - f_{\mathrm{A}}^0\|^2 + \lambda^2 I^2(\tilde{f} - f_0) \leq 6\delta_1^2 R_I^2 + 3\lambda^2 I^2(f^0) \leq 9\delta_1^2 R_I^2.$$

Since by Lemma 5.2 $\|\tilde{f} - f^0\|^2 \geq \|\tilde{f}_A - f_A^0\|^2/(1 - \gamma^2)$, this implies

$$\tau_I^2(\tilde{f} - f^0) \leq 9\delta_1^2 R_I^2/(1 - \gamma^2) \leq R_I^2/4$$

using $\delta_1^2 \leq \frac{(1-\gamma^2)}{36}$. This implies $\tau_I(\hat{f} - f^0) \leq R_I$.

$\square$

5.4 Results from empirical process theory

We use Theorem 2.1 in van de Geer [2014] which is a consequence of results in Guédon et al. [2007] and combine this with Theorem 3.1. in van de Geer [2014]. We recall definition (1) of the entropy integral $\mathcal{J}_n$. Throughout, C_0 and C_1 are universal constants.

Theorem 5.1. *Fix some R_1, M_1, R_2 and M_2 and let*

$$K_1 := \sup_{f \in \mathcal{F}(R_1, M_1)} \|f\|_\infty, \ \ K_2 := \sup_{g \in \mathcal{G}(R_2, M_2)} \|g\|_\infty.$$

Define for all t and n

$$B_{1,1}(t, n) := \frac{R_1 \mathcal{J}_n(K_1, \mathcal{F}(R_1, M_1)) + R_1 K_1 \sqrt{t}}{\sqrt{n}} + \frac{\mathcal{J}_n^2(K_1, \mathcal{F}(R_1, M_1)) + K_1^2 t}{n},$$

$$B_{2,2}(t, n) := \frac{R_2 \mathcal{J}_n(K_2, \mathcal{G}(R_2, M_2)) + R_2 K_2 \sqrt{t}}{\sqrt{n}} + \frac{\mathcal{J}_n^2(K_2, \mathcal{G}(R_2, M_2)) + K_2^2 t}{n}$$

and

$$B_{1,2}(t, n) := \frac{R_1 \mathcal{J}_n(K_2, \mathcal{G}(R_2, M_2)) + R_2 \mathcal{J}_n(R_1 K_2/R_2, \mathcal{F}(R_1, M_1))}{\sqrt{n}}$$

$$+ \frac{R_1 K_2 \sqrt{t}}{\sqrt{n}} + \frac{K_1 K_2 t}{n}.$$

We have for all $t > 0$ with probability at least $1 - C_0 \exp[-t]$

$$\sup_{f \in \mathcal{F}(R_1, M_1)} \left| \|f\|_n^2 - \|f\|^2 \right| \leq C_1 B_{1,1}(t, n), \quad \sup_{g \in \mathcal{G}(R_2, M_2)} \left| \|g\|_n^2 - \|g\|^2 \right| \leq C_1 B_{2,2}(t, n).$$

Moreover, for $R_1 K_2 \leq R_2 K_1$ and all values of t and n satisfying

$$C_1 B_{1,1}(t, n) \leq R_1^2, \ \ C_1 B_{2,2}(t, n) \leq R_2^2$$

we have with probability at least $1 - C_0 \exp[-t]$

$$\sup_{f \in \mathcal{F}(R_1, M_1), \ g \in \mathcal{G}(R_2, M_2)} \left| (P_n - P) f g \right| \leq C_1 B_{1,2}(t, n).$$

The next result follows from standard arguments using Dudley's results (Dudley [1967]), see e.g. van der Vaart and Wellner [1996].

Theorem 5.2. *Assume Condition 2.3 on the noise. Fix some R_1, M_1, R_2 and M_2 and let*

$$K_1 := \sup_{f \in \mathcal{F}(R_1, M_1)} \|f\|_\infty, \ \ K_2 := \sup_{g \in \mathcal{G}(R_2, M_2)} \|g\|_\infty.$$

Consider values of t and n such that

$$C_1 B_{1,1}(t, n) \le R_1^2, \ \ C_1 B_{2,2}(t, n) \le R_2^2$$

with $B_{1,1}(t, n)$ and $B_{2,2}(t, n)$ given in Theorem 5.1. For these values, with probability at least $1 - C_0 \exp[-t]$, one has

$$\sup_{f \in \mathcal{F}(R_1, M_1)} |\epsilon^T f|/n \le C_1 B_{1,\epsilon}(t, n), \ \ \ \sup_{g \in \mathcal{G}(R_2, M_2)} |\epsilon^T g|/n \le C_1 B_{2,\epsilon}(t, n),$$

where

$$B_{1,\epsilon}(t, n) := \frac{K_\epsilon \mathcal{J}(R_1, \mathcal{F}(R_1, M_1)) + K_\epsilon R_1 \sqrt{t}}{\sqrt{n}}$$

and

$$B_{2,\epsilon}(t, n) := \frac{K_\epsilon \mathcal{J}(R_2, \mathcal{G}(R_2, M_2)) + K_\epsilon R_2 \sqrt{t}}{\sqrt{n}}.$$

Corollary 5.1. *Suppose Conditions 2.1, 2.2 and 2.3. Assume $R_1 \le M_1/B$ and $R_2 \le M_2/B$ where the constant B is from Condition 2.2. Let $B_{1,1}$, $B_{2,2}$, $B_{1,2}$ be defined as in Theorem 5.1 and $B_{1,\epsilon}$ and $B_{2,\epsilon}$ be defined as in Theorem 5.2. Then*

$$B_{1,1}(t, n) \le \frac{R_1 M_1 (A_I + \sqrt{t})}{\sqrt{n}} + \frac{M_1^2 (A_I^2 + t)}{n},$$

$$B_{2,2}(t, n) \le \frac{R_2 M_2 (A_J + \sqrt{t})}{\sqrt{n}} + \frac{M_2^2 (A_J^2 + t)}{n}.$$

$$B_{1,2}(t, n) \le \frac{A_J R_1 M_2 + A_I R_2^\alpha R_1^{1-\alpha} M_1^\alpha M_2^{1-\alpha}}{\sqrt{n}} + \frac{R_1 M_2 \sqrt{t}}{\sqrt{n}} + \frac{M_1 M_2 t}{n}$$

and

$$B_{1,\epsilon}(t, n) \le \frac{A_I K_\epsilon M_1^\alpha R_1^{1-\alpha} + K_\epsilon R_1 \sqrt{t}}{\sqrt{n}}, \ \ B_{2,\epsilon}(t, n) \le \frac{A_J K_\epsilon M_2^\beta R_2^{1-\beta} + K_\epsilon R_2 \sqrt{t}}{\sqrt{n}}.$$

The constants A_I and A_J are from Condition 2.1 and the constant K_ϵ from Condition 2.3.

Theorem 5.3. *Assume Conditions 2.1, 2.2 and 2.3. Let $\lambda \le R_I \le \mu \le R \le 1$ be constants and $L_I := R_I/\lambda$ and $L_J := (R/\mu)^{2/q}$.*

Case 1. *Assume $\lambda^2 \le 1/B^2$ and $\mu^2 \le R^{2-q}/B^q$. Suppose that for some $L \ge 4C_1$,*

$$\sqrt{n}\lambda^{1+\alpha} \ge LA_I, \ \ \sqrt{n}\mu^{1+\beta} \ge LA_J, \tag{11}$$

$$R \ge LL_J A_J/\sqrt{n}, \ \ R \ge K_\epsilon \lambda, \ \ R \ge L_J \lambda, \ \ R \ge K_\epsilon^{\frac{q}{q-(2-q)\beta}} \mu \tag{12}$$

and

$$\lambda^\alpha \le 1/L. \tag{13}$$

Then with probability at least $1 - 3C_0 \exp[-n\lambda^2/L^2]$ *it holds that*

$$\sup_{f \in \mathcal{F}(R,R/\lambda)} \left| \|f\|_n^2 - \|f\|^2 \right| \le \frac{4R^2}{L}, \quad \sup_{g \in \mathcal{G}(R,L_J)} \left| \|g\|_n^2 - \|g\|^2 \right| \le \frac{4R^2}{L},$$

$$\sup_{f \in \mathcal{F}(R,R/\lambda), \ g \in \mathcal{G}(R,L_J)} \left| (P_n - P)fg \right| \le \frac{4R^2}{L}$$

and

$$\sup_{f \in \mathcal{F}(R,R/\lambda)} |\epsilon^T f|/n \le \frac{2R^2}{L}, \quad \sup_{g \in \mathcal{G}(R,L_J)} |\epsilon^T g|/n \le \frac{2R^2}{L}.$$

Case 2. *Assume moreover that*

$$R_I \ge LL_J A_J/\sqrt{n}, \ R_I \ge K_\epsilon \lambda \tag{14}$$

Then with probability at least $1 - 3C_0 \exp[-n\lambda^2/L^2]$,

$$\sup_{f \in \mathcal{F}(R_I,L_I)} \left| \|f\|_n^2 - \|f\|^2 \right| \le \frac{4R_I^2}{L}, \quad \sup_{f \in \mathcal{F}(R_I,L_I), \ g \in \mathcal{G}(R,L_J)} \left| (P_n - P)fg \right| \le \frac{4R_I^2}{L}$$

and

$$\sup_{f \in \mathcal{F}(R_I,L_I)} |\epsilon^T f|/n \le \frac{2R_I^2}{L}.$$

Proof of Theorem 5.3.

Case 1. We first apply Corollary 5.1 with $R_1 = R_2 = R$ and $M_1 = R/\lambda$, $M_2 = L_J := (R/\mu)^{2/q}$. The condition $\lambda \le 1/B$ ensures $R_1 \le M_1/B$ and the condition $\mu \le R^{\frac{2-q}{q}}/B$ ensures that $R_2 \le M_2/B$. We let $B_{1,1}$, $B_{2,2}$, $B_{1,2}$ be defined as in Theorem 5.1 and $B_{1,\epsilon}$ and $B_{2,\epsilon}$ be defined as in Theorem 5.2 and insert the value $t = n\lambda^2/L^2$.

Case 1a for $\|f\|_n^2$.

$$B_{1,1}(t,n) \le \frac{R^2(A_I + \sqrt{t})}{\sqrt{n}\lambda} + \frac{R^2(A_I^2 + t)}{n\lambda^2} = \left(\frac{A_I + \sqrt{t}}{\sqrt{n}\lambda} + \frac{A_I^2 + t}{n\lambda^2} \right) R^2.$$

Now use that by (11) $\sqrt{n}\lambda \ge LA_I$ and $t = n\lambda^2/L^2$ to get

$$B_{1,1}(t,n) \le \left(\frac{2}{L} + \frac{2}{L^2} \right) \le \frac{4R^2}{L}$$

Case 1b For $\|g\|_n^2$.

$$B_{2,2}(t,n) \le \frac{RL_J(A_J + \sqrt{t})}{\sqrt{n}} + \frac{L_J^2(A_J^2 + t)}{n} = \left(\frac{L_J(A_J + \sqrt{t})}{\sqrt{n}R} + \frac{L_J^2(A_J^2 + t)}{nR^2} \right) R^2$$

$$\le \left(\frac{1}{L} + \frac{\sqrt{t}}{\sqrt{n}R} + \frac{1}{L^2} + \frac{t}{nR^2}\right)R^2,$$

where we used that $R \ge LL_J A_J/\sqrt{n}$ by (12). Insert $R \ge \lambda$ and $t = n\lambda^2/L^2$ to get

$$B_{2,2}(t,n) \le \left(\frac{2}{L} + \frac{2}{L^2}\right)R^2 \le \frac{4R^2}{L}.$$

Case 1c for $f^T g/n$**.** We already know by Cases 1a and 1b that $C_1 B_{1,1}(t,n) \le R^2$ and $C_1 B_{2,2}(t,n) \le R^2$ with probability at least $1 - C_0 \exp[-n\lambda^2/L^2]$. Moreover

$$B_{1,2}(t,n) \le \frac{RL_J A_J + R^{1+\alpha}L_J^{1-\alpha}A_I/\lambda^\alpha + RL_J\sqrt{t}}{\sqrt{n}} + \frac{RL_J t}{n\lambda}$$

$$= \left(\frac{L_J A_J}{\sqrt{n}R} + \frac{L_J^{1-\alpha}A_I \lambda}{\sqrt{n}R^{1-\alpha}\lambda^{1+\alpha}} + \frac{L_J\sqrt{t}}{\sqrt{n}R} + \frac{L_J t}{nR\lambda}\right)R^2.$$

Use $R \ge \lambda L_J$, $R \ge LL_J A_J/\sqrt{n}$ from (12) and $\sqrt{n}\lambda^{1+\alpha} \ge LA_I$ from (11) to find that

$$B_{1,2}(t,n) \le \left(\frac{1}{L} + \lambda^\alpha + \frac{\sqrt{t}}{\sqrt{n}\lambda} + \frac{t}{n\lambda^2}\right)R^2.$$

Apply now that by (13) $\lambda^\alpha \le 1/L$ and $t = n\lambda^2/L^2$ to get

$$B_{1,2}(t,n) \le \left(\frac{3}{L} + \frac{1}{L^2}\right) \le \frac{4R^2}{L}.$$

Case 1d for $\epsilon^T f/n$**.** We already know by Cases 1a and 1b that $C_1 B_{1,1}(t,n) \le R^2$ and $C_1 B_{2,2}(t,n) \le R^2$ with probability at least $1 - C_0 \exp[-n\lambda^2/L^2]$. Moreover

$$B_{1,\epsilon}(t,n) \le \frac{K_\epsilon A_I \lambda R}{\sqrt{n}\lambda^{1+\alpha}} + \frac{K_\epsilon R\sqrt{t}}{\sqrt{n}} = \left(\frac{K_\epsilon A_I \lambda}{\sqrt{n}\lambda^{1+\alpha}R} + \frac{K_\epsilon \sqrt{t}}{\sqrt{n}R}\right)R^2.$$

Invoke $\sqrt{n}\lambda^{1+\alpha} \ge LA_I$ from (11) and $R \ge K_\epsilon \lambda$ from (12) to obtain

$$B_{1,\epsilon}(t,n) \le \left(\frac{1}{L} + \frac{\sqrt{t}}{\sqrt{n}\lambda}\right)R^2.$$

With $t = n\lambda^2/L^2$ this gives

$$B_{1,\epsilon}(t,n) \le \frac{2R^2}{L}.$$

Case 1e for $\epsilon^T g/n$**.** We gave

$$B_{2,\epsilon}(t,n) \le \left(\frac{K_\epsilon A_J L_J^\beta}{\sqrt{n}R^{1+\beta}} + \frac{K_\epsilon \sqrt{t}}{\sqrt{n}R}\right)R^2.$$

Use $\sqrt{n}\mu^{1+\beta} \ge LA_J$ from (11) to find

$$B_{2,\epsilon}(t,n) \le \left(\frac{L_J^\beta (\mu/R)^{1+\beta}K_\epsilon}{L} + \frac{K_\epsilon \sqrt{t}}{\sqrt{n}R}\right)R^2.$$

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Next, we see that $L_J^\beta(\mu/R)^{1+\beta} \le 1/K_\epsilon$ since $R \ge K_\epsilon^{\frac{q}{q-(2-q)\beta}}\mu$ by (12). Moreover, also by (12) $R \ge K_\epsilon\lambda$. So with $t = n\lambda^2/L^2$

$$B_{2,\epsilon}(t,n) \le \left(\frac{1}{L} + \frac{\sqrt{t}}{\sqrt{n}\lambda}\right)R^2 = \frac{2R^2}{L}.$$

Case 2. Take $R_1 = R_I$, $R_2 = R$ and $M_1 = L_I$, $M_2 = L_J$. Then again $R_1 \le M_1/B$ and $R_2 \le M_2/B$. Also With these new values, we let $B_{1,1}$, $B_{2,2}$, $B_{1,2}$ be defined as in Theorem 5.1 and $B_{1,\epsilon}$ and $B_{2,\epsilon}$ be defined as in Theorem 5.2 and insert the value $t = n\lambda^2/L^2$.

Case 2a for $\|f\|_n^2$.

$$B_{1,1}(t,n) \le \frac{R_I^2(A_I + \sqrt{t})}{\sqrt{n}\lambda} + R_I^2\frac{A_I^2 + t}{n\lambda^2} = \left(\frac{A_I + \sqrt{t}}{\sqrt{n}R_I} + \frac{A_I + t}{nR_I^2}\right)R_I^2 \le 4R_I^2/L$$

since $\sqrt{n}R_I \ge \sqrt{n}\lambda \ge \sqrt{n}\lambda^{1+\alpha} \ge LA_I$ by (11) and $t = n\lambda^2/L^2$.

Case 2b for f^Tg/n. By similar arguments as in Case 1a (see also Case 2a) and 1b that $C_1B_{1,1}(t,n) \le R_I^2$ and $C_2B_{2,2}(t,n) \le R^2$ with probability at least $1 - C_0\exp[-n\lambda^2/L^2]$. Moreover

$$B_{1,2}(t,n) \le \frac{R_IL_JA_J}{\sqrt{n}} + \frac{R^\alpha R_I\lambda L_J^{1-\alpha}A_I}{\sqrt{n}\lambda^{1+\alpha}} + \frac{R_IL_J\sqrt{t}}{\sqrt{n}} + \frac{tR_IL_J}{n\lambda}$$

$$= \left(\frac{L_JA_J}{\sqrt{n}R_I} + \frac{R^\alpha\lambda L_J^{1-\alpha}A_I}{\sqrt{n}\lambda^{1+\alpha}R_I} + \frac{L_J\sqrt{t}}{\sqrt{n}R_I} + \frac{tL_J}{n\lambda R_I}\right)R_I^2.$$

Use that $R_I \ge LL_JA_J/\sqrt{n}$ (see (14)), $\sqrt{n}\lambda^{1+\alpha} \ge LA_I$ (see (11)) and $R \ge \lambda L_J$ (see (12)). We then get

$$B_{1,2}(t,n) \le \left(\frac{1}{L} + \lambda^\alpha + \frac{\sqrt{t}}{\sqrt{n}\lambda} + \frac{t}{n\lambda^2}\right)R_I^2.$$

With $t = n\lambda^2/L^2$ and $\lambda^\alpha \le 1/L$ (see (13)) this gives again

$$B_{1,2}(t,n) \le \frac{4R_I^2}{L}.$$

Case 2c for ϵ^Tf/n. By Case 2a, it holds that $C_1B_{1,1}(t,n) \le R_I^2$ with probability at least $1 - C_0\exp[-n\lambda^2/L^2]$. Moreover

$$B_{1,\epsilon}(t,n) \le \left(\frac{K_\epsilon A_I\lambda}{\sqrt{n}\lambda^{1+\alpha}R_I} + \frac{K_\epsilon\sqrt{t}}{\sqrt{n}R_I}\right)R_I^2.$$

From (11) we know $\sqrt{n}\lambda^{1+\alpha} \ge LA_I$ and from (14) $R_I \ge K_\epsilon\lambda$. With $t = n\lambda^2/L^2$ we find

$$B_{1,\epsilon}(t,n) \le \frac{2R_I^2}{L}.$$

The result now follows from the same arguments as in Case 2 of Theorem 5.3.

$\square$

Remark 5.1. *If we assume condition (11), then condition (12) is met for*

$$K_\epsilon \frac{\lambda}{\mu} \leq K_\epsilon^{\frac{q}{q-(2-q)\beta}} \leq \frac{R}{\mu} \leq \min\left\{ \left(\frac{\sqrt{n}}{LA_J} \right)^{\frac{q\beta}{(1+\beta)(2-q)}}, \left(\frac{\mu}{\lambda} \right)^{\frac{q}{2-q}} \right\}.$$

Under general conditions, the left hand side tends to zero and the right hand side tends to infinity as $n \to \infty$.

5.5 Application to $\mathcal{T}(R)$

Recall the definition (6) of the set $\mathcal{T}(R)$.

Lemma 5.6. *Let $\lambda \leq \mu \leq R \leq 1$. Assume Conditions 2.1, 2.2, 2.3 and 2.4. Assume that $\lambda^2 \leq (1-\gamma)/B^2$ and $\mu^2 \leq (1-\gamma)^q R^{2-q}/B^q$. Let*

$$L \geq \max\left\{ 4C_1(1-\gamma)^{1/2}, 16/((1-\gamma)^{1/2}\delta_0^2) \right\}$$

and

$$\sqrt{n}\lambda^{1+\alpha} \geq LA_I, \ \sqrt{n}\mu^{1+\beta} \geq LA_J,$$

$$\lambda^\alpha \leq (1-\gamma)^{\frac{1+\alpha}{2}}/L$$

and

$$K_\epsilon \frac{\lambda}{\mu} \leq K_\epsilon^{\frac{q}{q-(2-q)\beta}} \leq \frac{R}{\mu} \leq \min\left\{ \left(\frac{\sqrt{n}(1-\gamma)^{1/2}}{LA_J} \right)^{\frac{q\beta}{(1+\beta)(2-q)}}, \left(\frac{\mu}{\lambda} \right)^{\frac{q}{2-q}} \right\}.$$

Then

$$\mathbb{P}(\mathcal{T}(R)) \geq 1 - \exp[-n\lambda^2/L^2].$$

Proof. Recall the definition of $\mathcal{M}(R)$ given in (5) with $\tau_R(\cdot, \cdot)$ given in (2). Define $\tilde{\lambda}^2 := \lambda^2/(1-\gamma)$, $\tilde{\mu}^2 := \mu^2/(1-\gamma)$ and $\tilde{R}^2 := R^2/(1-\gamma)$. By Lemma 5.1

$$\mathcal{M}(R) \subset \left\{ (f, g) : \ \|f\| \leq \tilde{R}, \ \|g\| \leq \tilde{R}, \ I(f) \leq R/\lambda, \ J(g) \leq (R/\mu)^{\frac{2}{q}} \right\}$$

$$= \left\{ (f, g) : \ f \in \mathcal{F}(\tilde{R}, \tilde{R}/\tilde{\lambda}), \ g \in \mathcal{G}(\tilde{R}, (\tilde{R}/\tilde{\mu})^{\frac{2}{q}} \right\}.$$

We apply Case 1 of Theorem 5.3 with (λ, μ, R) replaced by $(\tilde{\lambda}, \tilde{\mu}, \tilde{R})$. We also replace L by $\tilde{L}^2 := L^2/(1-\gamma)$. Then

$$\sqrt{n}\tilde{\lambda}^{1+\alpha} = \sqrt{n}\lambda^{1+\alpha}/(1-\gamma)^{\frac{1+\alpha}{2}}$$

$$\geq LA_I/(1-\gamma)^{\frac{1+\alpha}{2}} = \tilde{L}A_I/(1-\gamma)^{\frac{\alpha}{2}} \geq \tilde{L}A_I.$$

Similarly

$$\sqrt{n}\tilde{\mu}^{1+\beta} \geq \tilde{L}A_J.$$

The condition $\lambda^\alpha \le (1-\gamma)^{\frac{1+\alpha}{2}}/L$ gives

$$\tilde{\lambda}^\alpha = \frac{\lambda^\alpha}{(1-\gamma)^{\alpha/2}} \le \frac{(1-\gamma)^{\frac{1+\alpha}{2}}}{L(1-\gamma)^{\frac{\alpha}{2}}} = \frac{1}{\tilde{L}}.$$

Furthermore

$$K_\epsilon \frac{\tilde{\lambda}}{\tilde{\mu}} \le K_\epsilon^{\frac{q}{q-(2-q)\beta}} \le \frac{\tilde{R}}{\tilde{\mu}} \le \min\left\{ \left(\frac{\sqrt{n}}{\tilde{L}A_J}\right)^{\frac{q\beta}{(1+\beta)(2-q)}}, \left(\frac{\tilde{\mu}}{\tilde{\lambda}}\right)^{\frac{q}{2-q}} \right\}.$$

By Remark 5.1 we conclude that the conditions for Case 1 of Theorem 5.3 are met. Clearly, for any f and g

$$\left| \|f+g\|_n^2 - \|f+g\|^2 \right| \le \left| \|f\|_n^2 - \|f\|_n^2 \right| + \left| \|g\|_n^2 - \|g\|^2 \right| + \left| 2(P_n - P)fg \right|.$$

By Case 1 of Theorem 5.3, for $\tilde{L} = L/(1-\gamma)^{1/2} \ge 4C_1$ and for $16\tilde{R}^2/\tilde{L} \le \delta_0^2 R^2$

$$\mathbb{P}(\mathcal{T}(R)) \ge 1 - \exp[-n\tilde{\lambda}^2/\tilde{L}^2].$$

The proof if finished by noting that $\tilde{R}^2/\tilde{L} = R^2/(L(1-\gamma)^{1/2})$ and $\tilde{\lambda}^2/\tilde{L}^2 = \lambda^2/L^2$.

$\square$

5.6 Application to $\mathcal{T}_I(R_I)$

Recall the definition (8) of the set $\mathcal{T}_I(R_I, R)$.

Lemma 5.7. *Assume Conditions 2.1, 2.2, 2.3, 2.4 and 2.5. Let $\lambda \le R_I \le \mu \le R \le 1$. Assume that $\lambda^2 \le (1-\gamma)/(2B)^2$ and $\mu^2 \le (1-\gamma)^q R^{2-q}/(2B)^q$. Let*

$$L \ge \max\left\{ 2C_1(1-\gamma)^{1/2}, 32/((1-\gamma)^{1/2}\delta_0^2), 32/(\delta_1^2) \right\}.$$

Take

$$\sqrt{n}\lambda^{1+\alpha} \ge LA_I, \quad \sqrt{n}\mu^{1+\beta} \ge LA_J,$$
$$\lambda^\alpha \le (2(1-\gamma))^{\frac{1+\alpha}{2}}/L$$

and

$$K_\epsilon \frac{\lambda}{\mu} \le K_\epsilon^{\frac{q}{q-(2-q)\beta}} \le \frac{R}{\mu} \le \min\left\{ \left(\frac{\sqrt{n}(1-\gamma)^{1/2}}{2LA_J}\right)^{\frac{q\beta}{(1+\beta)(2-q)}}, \left(\frac{\mu}{\lambda}\right)^{\frac{q}{2-q}} \right\}.$$

Also take

$$R_I \ge L(R/\mu)^{\frac{2}{q}}A_J/\sqrt{n}, \quad R_I \ge K_\epsilon\lambda$$
$$\Gamma R_I \le (2R/\mu)^{\frac{2}{q}}$$

Then

$$\mathbb{P}(\mathcal{T}_I(R_I, R)) \ge 1 - 3C_0 \exp[-n\lambda^2/L^2].$$

Proof. By Lemma 5.3

$$\mathcal{J}_\infty(z, \{f_A : \ f \in \mathcal{F}(R, M)\}) \leq 2\mathcal{J}_\infty(z, \mathcal{F}(R, M)), \ z > 0$$

and for $R \leq M/B$

$$\sup_{f \in \mathcal{F}(R,M)} \|f_A\|_\infty \leq 2M.$$

We can therefore apply similar arguments as for Case 2 of Theorem 5.3. We know that for $f \in \mathcal{F}(R_I)$, $\|f_A\| \leq \|f\| \leq R_I$. So

$$\mathcal{J}_\infty(z, \{f_A : \ f \in \mathcal{F}(R_I)\}) \leq 2\mathcal{J}_\infty(z, \mathcal{F}(R_I, R_I/\lambda)), \ z > 0$$

and

$$\sup_{f \in \mathcal{F}(R_I)} \|f_A\|_\infty \leq 2R_I/\lambda.$$

Moreover, for $f \in \mathcal{F}(R_I)$ and $g \in \mathcal{G}(R)$ we have

$$J(g + f_P) \leq J(g) + J(f_P) \leq (R/\mu)^{2/q} + \Gamma\|f\| \leq (R/\mu)^{2/q} + \Gamma R_I \leq (2R/\mu)^{2/q},$$

and

$$\|g + f_P\| \leq \|g\| + \|f_P\| \leq R/(1-\gamma)^{1/2} + R_I \leq 2R/(1-\gamma)^{1/2}.$$

It follows that

$$\{g + f_P : \ f \in \mathcal{F}(R_I), \ g \in \mathcal{G}(R)\} \subset \mathcal{G}(2R/(1-\gamma)^{1/2}, (2R/\mu)^{2/q}).$$

It is also clear that for any f and g

$$Pf_A g = Ef(X_1)g(Z_1) - E\left[E(f(X_1)|Z)g(Z_1)\right] = 0$$

and similarly $Pf_A f_P = 0$. By an appropriate replacements of the constants in Case 2 of Theorem 5.3 (as in the proof of Lemma (5.6) now using $(1-\gamma)^{1/2}/2$ instead of $(1-\gamma)^{1/2}$) the results follows.

$\square$

5.7 Finishing the proof of Theorem 3.1

We first note that since $\max\{A_I, A_J\} \leq n^{\frac{1-\delta}{2}}$ we $\lambda^{1+\alpha} = c_1 A_I/\sqrt{n} \leq n^{-\delta/2}$. So for n large λ will be small. The same is true for μ and for the ratio λ/μ.

In view of Lemma 5.4 we need $\lambda^2 I^2(f^0) + \mu^2 J^q(g^0) \leq \delta_0^2 R^2$. We take

$$R^2 = \max\left\{\mu^2 J^q(g^0)/(4\delta_0^2), K_\epsilon^{\frac{2q}{2-(2-q)\beta}}\right\}$$

and n sufficiently large such that

$$\lambda^2 I^2(f^0) \leq \mu^2 J^q(g^0).$$

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Take
$$L = \max\left\{2C_1(1-\gamma)^{1/2}, 32/((1-\gamma)^{1/2}\delta_0^2), 32/(\delta_1^2)\right\}.$$

Since
$$\max\left\{\frac{J^{q/2}(g^0)}{2\delta_0}, K_\epsilon^{\frac{q}{2-(2-q)\beta}}\right\} \leq \min\left\{\left(\frac{\sqrt{n}(1-\gamma)^{1/2}}{2LA_J}\right)^{\frac{q\beta}{(1+\beta)(2-q)}}, \left(\frac{\mu}{\lambda}\right)^{\frac{q}{2-q}}\right\}$$

for n sufficiently large as $A_J \leq n^{\frac{1-\delta}{2}}$ we know from Remark 5.1 that the conditions for Lemma 5.6 are met for n sufficiently large. By Lemma 5.5 we also need $\lambda^2 I^2(f_0) \leq R_I^2/\delta_1^2$. For $R_I/\lambda = \max\{I(f^0)/\delta_1, K_\epsilon\}$.

$$R_I \geq L(R/\mu)^{\frac{2}{q}} A_J/\sqrt{n}$$

for n sufficiently large so we can also apply Lemma 5.7.

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