RATIONAL REPRESENTATIONS AND PERMUTATION REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT. We investigate the question which \mathbb{Q} -valued characters and characters of \mathbb{Q} -representations of finite groups are \mathbb{Z} -linear combinations of permutation characters. This question is known to reduce to that for quasi-elementary groups, and we give a solution in that case. As one of the applications, we exhibit a family of simple groups with rational representations whose smallest multiple that is a permutation representation can be arbitrarily large.

1. INTRODUCTION

Many rational invariants of a finite group G are encoded in the ring $\operatorname{Char}_{\mathbb{Q}}(G)$ of rationally-valued characters, the ring $R_{\mathbb{Q}}(G)$ of rational representations, and the ring $\operatorname{Perm}(G)$ of virtual permutation representations. All three have the same \mathbb{Z} -rank, and there are natural inclusions with finite cokernels

$$\operatorname{Perm}(G) \longrightarrow R_{\mathbb{Q}}(G) \longrightarrow \operatorname{Char}_{\mathbb{Q}}(G).$$

The quotient $\operatorname{Char}_{\mathbb{Q}}(G)/R_{\mathbb{Q}}(G)$ is studied by the theory of Schur indices, and the purpose of this paper is to investigate the other two,

$$C(G) = \frac{R_{\mathbb{Q}}(G)}{\operatorname{Perm}(G)}$$
 and $\hat{C}(G) = \frac{\operatorname{Char}_{\mathbb{Q}}(G)}{\operatorname{Perm}(G)}$.

They have exponent dividing |G| by Artin's induction theorem, and Serre remarked that C(G) need not be trivial ([14] Exc. 13.4). It is trivial for *p*-groups [6, 12, 13], and it is known for nilpotent groups [11] (see also §2), for Weyl groups of Lie groups [15, 9] and in other special cases [1, 7]. It follows from the general results of Dress, Kletzing, and Hambleton-Taylor-Williams [4, 5, 9, 8], that the study of C(G) for a group *G* reduces, in principle, to that of its quasi-elementary subgroups, or of its 'basic' quasi-elementary subquotients. Specifically, for subgroups the statement is that of the two maps

$$\prod_{\substack{Q \leq G \\ \text{quasi-elem.}}} \mathcal{C}(Q) \xrightarrow{\text{Ind}} \mathcal{C}(G) \xrightarrow{\text{Res}} \prod_{\substack{Q \leq G \\ \text{quasi-elem.}}} \mathcal{C}(Q),$$

the first one is surjective and the second one injective, and similarly for \hat{C} . This is also an immediate consequence of Solomon's induction theorem, see §3.

Our first observation is that the composite map allows us to describe C(G) and $\hat{C}(G)$ explicitly, in a way that bypasses the representation theory of G — purely in terms of quasi-elementary subgroups and the 'Res Ind' maps between them; in fact, it is enough to consider maximal quasi-elementary subgroups, i.e. *p*-normalisers of cyclic subgroups of G. In §3 we give a simple formula for the Res Ind maps, and in §4 we prove one of the main results of the paper, which describes C(Q) and $\hat{C}(Q)$ for a *p*-quasi-elementary group $Q = C \rtimes P$.

Its simplest formulation is:

Theorem 1.1 (=Theorem 4.6). Let ρ be an irreducible rational representation of a *p*-quasi-elementary group $Q = C \rtimes P$. (So *C* is cyclic, *P* a *p*-group, and $p \nmid |C|$.) The order of ρ in C(Q) is $\frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}$, where $\hat{\psi}$ is the (unique) rational irreducible constituent of $\operatorname{Res}_C \rho$ and $\hat{\pi}$ a rational irreducible constituent of $\operatorname{Res}_P \rho$ of minimal dimension.

Together with the aforementioned 'Res Ind' formula, it gives a way to compute C(G) and $\hat{C}(G)$ efficiently in a given finite group G. Incidentally, it also gives an algorithm to find $Perm(G) \subset R_{\mathbb{Q}}(G)$ without computing the subgroup lattice, which is now implemented in Magma [2]. In §5 and §6 we illustrate applications of this approach to proving both triviality and non-triviality of C(G), as we shall now describe.

In general, C(G) remains somewhat mysterious, especially in non-soluble groups. Already Frobenius knew that $C(A_n)$ is trivial for all n. It was announced by Solomon in [15] that $C(PSL_2(\mathbb{F}_q))$ is trivial for all prime powers q. In §5 we explain how this, and the same statement for $GL_2(\mathbb{F}_q)$ and $PGL_2(\mathbb{F}_q)$, follow from the results of §3 and §4.

There is, to our knowledge, no example in the literature of a simple group with non-trivial C(G). In §6 we show:

Theorem 1.2 (=Theorem 6.1 and Corollary 6.6). The exponent of the 2-part of C(G) is unbounded in the families $G = PSL_k(\mathbb{F}_p)$ and $G = SL_k(\mathbb{F}_p)$. Moreover, $\hat{C}(PSL_k(\mathbb{F}_p)) \neq \{1\}$ for all even $k \geq 4$ and all odd primes p.

Notation. Throughout the paper, G denotes a finite group. We write

$\operatorname{Char}(G)$	=	the character ring G ,		
$\operatorname{Char}_{\mathbb{Q}}(G)$	=	the ring of \mathbb{Q} -valued characters,		
$R_{\mathbb{Q}}(G)$	=	the ring of characters of virtual $\mathbb{Q}G$ -representations,		
$\operatorname{Perm}(G)$	=	the ring of characters of virtual permutation representations,		
C(G)	=	$R_{\mathbb{Q}}(G)/\operatorname{Perm}(G),$		
$\hat{\mathrm{C}}(G)$	=	$\operatorname{Char}_{\mathbb{Q}}(G)/\operatorname{Perm}(G),$		
$\mathbb{Q}(\chi)$	=	the field of character values of a complex character χ ,		
$m(\chi)$	=	the Schur index of an irreducible complex character χ over $\mathbb{Q}(\chi)$.		
x a complex character x of C define its trace and when x is irreducible its				

For a complex character χ of G, define its *trace* and, when χ is irreducible, its *rational hull* as

$$\operatorname{Tr} \chi = \sum_{\substack{\sigma \in \operatorname{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \\ \hat{\chi} = m(\chi) \operatorname{Tr} \chi} \chi^{\sigma} \in \operatorname{Char}_{\mathbb{Q}}(G),$$

If χ is irreducible, then Tr χ is a \mathbb{Q} -*irreducible character* and $\hat{\chi}$ is the character of an *irreducible rational representation*. We write

 $\begin{array}{lll} \operatorname{Irr}(G) &=& \operatorname{the set of (complex) irreducible characters of } G, \\ \operatorname{Irr}_{\mathbb{Q}}(G) &=& \operatorname{the set of } \mathbb{Q}\text{-irreducible characters of } G, \\ \mu(\alpha,\beta) &=& \frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} = \operatorname{multiplicity of } \alpha \text{ in } \beta, \\ && \operatorname{used for characters } \alpha \in \operatorname{Irr}_{\mathbb{Q}}(G), \beta \in \operatorname{Char}_{\mathbb{Q}}(G), \text{ and} \\ && \operatorname{also for rational representations } \alpha, \beta \text{ with } \alpha \text{ irreducible.} \end{array}$

We write $x \sim y$ for conjugate elements. A *p*-quasi-elementary group is one of the form $G = C \rtimes P$ with *C* cyclic, and *P* a *p*-group; throughout the paper we adopt the convention that $p \nmid |C|$.

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2. Basic facts

Lemma 2.1. An inclusion $N \triangleleft G$ induces injections $C(G/N) \hookrightarrow C(G)$, $\hat{C}(G/N) \hookrightarrow \hat{C}(G)$.

Proof. Suppose $\bar{\rho}$ is a representation of G/N, which lifts to $\rho \in \operatorname{Perm} G$. Write

$$\rho = \bigoplus \mathbb{C}[G/H_i]^{\oplus n_i}, \qquad n_i \in \mathbb{Z}.$$

For a subgroup H < G recall that $\mathbb{C}[G/H]^N \cong \mathbb{C}[G/NH]$, as G-representations (see e.g. [3], proof of Thm. 2.8(5)). Therefore,

$$\bar{\rho} = \rho^N = \bigoplus \mathbb{C}[G/NH_i]^{\oplus n_i} \in \operatorname{Perm}(G/N),$$

as required.

Lemma 2.2. Let ρ be an irreducible rational representation and $\tau \in \operatorname{Irr} G$ its constituent, so $\operatorname{Tr} \tau \in \operatorname{Irr}_{\mathbb{Q}}(G)$ and $\rho = m(\tau) \operatorname{Tr} \tau$. The order of $\operatorname{Tr} \tau$ in $\hat{C}(G)$ is $m(\tau)$ times the order of ρ in C(G).

Proof. Clear from the definitions of C(G) and C(G).

This allows us to immediately deduce results about $\hat{C}(G)$ from those about C(G), and conversely.

Nilpotent groups. Some statements seem to have a cleaner formulation for C(G) than for $\hat{C}(G)$, and for some it is the other way around. Let us briefly illustrate this with an example of nilpotent groups:

Theorem 2.3 (Rasmussen [11] Thm 5.2). Let $G = G_2 \times G_{2'}$ be a nilpotent group, where G_2 is its Sylow 2-subgroup. Then C(G) is trivial, unless $G_{2'} \neq \{1\}$ and there exists an irreducible character χ of G_2 with $m(\chi) = 2$ and such that one of the following holds:

- (1) $\mathbb{Q}(\chi) \neq \mathbb{Q}$, or
- (2) $\mathbb{Q}(\chi) = \mathbb{Q}$ and there exists a prime divisor q of |G| such that the order of 2 (mod q) is even.

The conditions turn out to be much simpler if one transforms this into a result about $\hat{C}(G)$. The following follows easily from [11, Thm. 4.2] and standard facts about Schur indices:

Theorem 2.4. Let $\chi = \chi_2 \chi_{2'}$ be an irreducible character of a nilpotent group $G = G_2 \times G_{2'}$ as above. Then the order of $\operatorname{Tr} \chi$ in $\hat{C}(G)$ is $m(\chi_2)$ (which is 1 or 2).

Metabelian and supersolvable groups. The following theorem will be of central importance in what follows. It implies that knowing the order of every \mathbb{Q} irreducible representation in $\hat{C}(G)$ determines the structure of $\hat{C}(G)$ completely when G is metabelian or supersoluble (e.g. nilpotent or quasi-elementary). It does not hold in arbitrary groups, as first noted by Berz [1]; the smallest counterexample is $G = C_3 \times SL_2(\mathbb{F}_3)$.

Theorem 2.5 (Berz [1]). If G is metabelian or supersoluble, then $Perm(G) \subseteq R_{\mathbb{Q}}(G)$ is freely generated by $n_{\rho}\rho$, as ρ ranges over irreducible rational representations of G, and

$$n_{\rho} = \gcd_{H \le G} \mu(\rho, \mathbb{Q}[G/H]).$$

Lemma 2.6. If $G = A \rtimes V$ with A abelian and V an elementary abelian p-group, then $\hat{C}(G) = \{1\}$.

Proof. By Theorem 2.5, it is enough to show that every complex irreducible character τ of G occurs exactly once in $\mathbb{C}[G/H]$ for a suitable H < G. This is clear when dim $\tau = 1$. Otherwise $\tau = \operatorname{Ind}_{AU}^G \chi$, for some subgroup U of V and a 1-dimensional character χ of AU (see [14, Part II, §8.2]). Let H be a subgroup of V that is complementary to U, i.e. HU = V and $H \cap U = \{1\}$. By Mackey's formula, we have

$$\langle \tau, \mathbb{C}[G/H] \rangle = \langle \chi, \operatorname{Res}_{AU} \operatorname{Ind}_{H}^{G} \mathbf{1} \rangle = \langle \chi, \operatorname{Ind}_{AU\cap H}^{AU} \mathbf{1} \rangle = \langle \chi, \mathbb{C}[AU] \rangle = 1. \qquad \Box$$

Recall that a *p*-quasi-elementary group $G = C \rtimes P$ is *basic* if the kernel K of $P \to \operatorname{Aut}(C)$ is trivial or isomorphic to D_8 or has normal *p*-rank one.

Proposition 2.7 ([7], Proposition 5.2). Let $G = C \rtimes P$ be basic *p*-quasi-elementary. Let A_p be a maximal cyclic subgroup of $K = \ker(P \to \operatorname{Aut}(C))$ that is normal in P (it is all of K if K is cyclic, and has index 2 in K otherwise), let $A = CA_p$, and let χ be a faithful one-dimensional character of A. Then $\rho = \operatorname{Tr} \operatorname{Ind}_A^G \chi$ is a \mathbb{Q} -irreducible character, and

order of
$$\rho$$
 in $\hat{\mathcal{C}}(G) = \frac{|P|}{|A_p| \cdot \max_{\substack{H \leq P \\ H \cap A_p = 1}} |H|}.$

3. $\hat{C}(G)$ as a Mackey functor

Let \mathcal{R} be a Mackey subfunctor of the character ring Mackey functor $\operatorname{Char}(G)$. This simply means that for any finite group G, $\mathcal{R}(G)$ is a subgroup of $\operatorname{Char}(G)$ such that if $H \leq G$ are finite groups, then

- for all $\rho \in \mathcal{R}(H)$, $\operatorname{Ind}_{H}^{G} \rho \in \mathcal{R}(G)$,
- for all $\tau \in \mathcal{R}(G)$, $\operatorname{Res}_H \tau \in \mathcal{R}(H)$,
- for all $\rho \in \mathcal{R}(H)$ and $g \in G$, $\rho^g \in \mathcal{R}(H^g)$.

Here are some examples:

- $R_K(G)$, the representation ring of G over a fixed subfield K of \mathbb{C} ,
- Char_K(G), the ring generated by K-valued characters, with fixed $K \subset \mathbb{C}$,
- Perm(G), the ring generated by permutation characters,
- the subgroup of Char(G) generated by characters of degree divisible by a fixed integer n.

If p is a prime number, write $\mathcal{R}(G)_p$ for $\mathcal{R}(G) \otimes \mathbb{Z}_p$, where \mathbb{Z}_p is the ring of p-adic integers.

Proposition 3.1. Let G be a finite group, fix a prime number p, and let \mathcal{F}_p be a family of subgroups of G such that every p-quasi-elementary subgroup of G is conjugate to a subgroup of some $Q \in \mathcal{F}_p$. Then

$$\prod_{Q\in\mathcal{F}_p}\operatorname{Res}_Q:\mathcal{R}(G)_p\longrightarrow\prod_{Q\in\mathcal{F}_p}\mathcal{R}(Q)_p$$

is injective. Dually,

$$\sum_{Q} \operatorname{Ind}_{Q}^{G} : \prod_{Q \in \mathcal{F}_{p}} \mathcal{R}(Q)_{p} \longrightarrow \mathcal{R}(G)_{p}$$

is surjective.

Proof. By Solomon's induction theorem, a prime-to-p multiple d of the trivial representation can be written as

$$d\mathbf{1}_G = \sum_i n_i \operatorname{Ind}_{H_i}^G \mathbf{1}_{H_i}$$

for some *p*-quasi-elementary subgroups H_i and integers n_i . Because $\operatorname{Ind}_{H_i^g}^G \mathbf{1}_{H_i^g} \cong$ $\operatorname{Ind}_{H_i}^G \mathbf{1}_{H_i}$, we may assume that each H_i is contained in some $Q_i \in \mathcal{F}_p$. Taking tensor products with any $\rho \in \mathcal{R}(G)$ yields

$$d\rho = \sum_{i} n_i \operatorname{Ind}_{H_i}^G \operatorname{Res}_{H_i} \rho.$$

If all $\operatorname{Res}_{H_i} \rho$ were 0, then so would be $d\rho$, and therefore also ρ . This proves injectivity. Also, the equation shows that $d\rho \in \operatorname{Im}\left(\sum_Q \operatorname{Ind}_Q^G \mathcal{R}(Q)\right)$, which proves surjectivity, since d is invertible in \mathbb{Z}_p .

Corollary 3.2. For $S, T \in \mathcal{F}_p$ write $\alpha_{S,T} = \operatorname{Res}_T \operatorname{Ind}_S^G : \hat{C}(S) \longrightarrow \hat{C}(T)$. Then

$$\hat{\mathcal{C}}(G)_p \cong Image \Big(\prod_T \sum_S \alpha_{S,T} : \prod_{S \in \mathcal{F}_p} \hat{\mathcal{C}}(S) \longrightarrow \prod_{T \in \mathcal{F}_p} \hat{\mathcal{C}}(T) \Big).$$

In particular, $\hat{C}(G)_p = 1$ if and only if for all pairs $S, T \in \mathcal{F}_p$ and all $\rho \in R_{\mathbb{Q}}(S)$ (equivalently, for those ρ whose class in C(S) is nontrivial), we have $\operatorname{Res}_T^G \operatorname{Ind}_S^G \rho \in \operatorname{Perm}(T)$. The same also holds for C(G).

Proof. Apply Proposition 3.1 to \mathcal{R} being Perm, $R_{\mathbb{Q}}$, and $\text{Char}_{\mathbb{Q}}$.

Corollary 3.3. Let \mathcal{F} be a family of subgroups of G such that every quasi-elementary subgroup is conjugate to a subgroup of some $Q \in \mathcal{F}$. Then

$$\hat{\mathcal{C}}(G) \longleftrightarrow \prod_{Q \in \mathcal{F}} \hat{\mathcal{C}}(Q)$$

via the (product of) restriction maps. Consequently, the kernel of the composition

$$R_{\mathbb{Q}}(G) \xrightarrow{\prod \operatorname{Res}} \prod_{Q \in \mathcal{F}} R_{\mathbb{Q}}(Q) \longrightarrow \prod_{Q \in \mathcal{F}} \hat{C}(Q)$$

is Perm(G). Dually, the composition

$$\prod_{Q \in \mathcal{F}} R_{\mathbb{Q}}(Q) \xrightarrow{\text{Ind}} R_{\mathbb{Q}}(G) \to \hat{\mathcal{C}}(G)$$

is onto. The same holds with $R_{\mathbb{O}}$ replaced by $\operatorname{Char}_{\mathbb{O}}$ and \hat{C} by C.

Remark 3.4. The theorem and the two corollaries give a very efficient way of computing $\hat{C}(G)_p, \hat{C}(G), C(G)_p, C(G)$ and of finding $\operatorname{Perm}(G)$ as a subring of $R_{\mathbb{Q}}(G) \leq \operatorname{Char}_{\mathbb{Q}}(G)$, without computing the full lattice of subgroups of G.

Remark 3.5. One possible family \mathcal{F}_p is the set of maximal *p*-quasi-elementary subgroups of *G*. These are of the form

$$Q = C \rtimes \operatorname{Syl}_n(N_G(C)),$$

where C is cyclic of order prime to p. Possible families \mathcal{F} in Corollary 3.3 are $\mathcal{F} = \bigcup_p \mathcal{F}_p$, as p ranges over prime divisors of |G|, or alternatively $\mathcal{F} = \{N_G(C)\}$ as C ranges over (representatives of conjugacy classes of) cyclic subgroups of G.

Notation 3.6. For the remainder of this section we use the following notation:

CC(G)	=	the set of conjugacy classes of G ,
$CC_{\rm cyc}(G)$	=	the set of conjugacy classes of cyclic subgroups of G ,
[x]	=	the conjugacy class of x , when x is either an element of G
		or a cyclic subgroup,
$\operatorname{Tr}^* \chi$	=	the normalised trace $\operatorname{Tr}^* \chi = \frac{1}{[\mathbb{Q}(\chi):\mathbb{Q}]} \operatorname{Tr} \chi$ of a character χ ,
au(D)	=	$\tau(y)$, where $D \leq G$ is a cyclic subgroup, y is any generator
		of D , and $\tau \in \operatorname{Char}_{\mathbb{Q}}(G) \otimes \mathbb{Q}$. The rationality of τ ensures
		that $\tau(y)$ only depends on D and not on the generator y.

Note in particular, that for any character χ of G and any cyclic subgroup D of G, $\operatorname{Tr}^* \chi(D)$ is the average value of χ on the generators of D.

Lemma 3.7. Let H_1 , H_2 be two subgroups of G. Let τ_i be a character of H_i , i = 1, 2, and assume that τ_1 is \mathbb{Q} -valued. Then

$$\langle \operatorname{Ind}_{H_1}^G \tau_1, \operatorname{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{\operatorname{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \sum_{D_1 \leq H_1 \atop D_1 \sim C} \tau_1(D_1) \cdot \sum_{D_2 \leq H_2 \atop D_2 \sim C} \operatorname{Tr}^* \tau_2(D_2).$$

Proof. First, note that by definition of inner products and of induced class functions,

$$\langle \operatorname{Ind}_{H_1}^G \tau_1, \operatorname{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \overline{\left(\sum_{y \in [x] \cap H_1} \tau_1(y)\right)} \left(\sum_{y \in [x] \cap H_2} \tau_2(y)\right).$$

The idea of the proof is to partition the set of conjugacy classes of elements of G according to conjugacy classes of cyclic subgroups they generate, and to use the fact that for a rational character τ , $\tau(x) = \tau(x')$ whenever x and x' generate conjugate cyclic subgroups. We get

where

$$\begin{split} f(C) &= |Z_G(C)| \cdot \sum_{\substack{[x] \in CC(G) \\ \langle x \rangle = C}} \left(\sum_{y \in [x] \cap H_1} \tau_1(y) \right) \left(\sum_{y \in [x] \cap H_2} \tau_2(y) \right) \\ &= |Z_G(C)| \cdot \#\{k : x \sim x^k\} \cdot \\ \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{[x] \in CC(G) \\ [x] \sim C}} \sum_{y \in [x] \cap H_2} \tau_2(y) \\ &= |N_G(C)| \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \sum_{\substack{generators \\ y \text{ of } D_2}} \tau_2(y) \\ &= |N_G(C)| \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \phi(|C|) \operatorname{Tr}_{\mathbb{Q}/\mathbb{Q}}^* \tau_2(y), \end{split}$$

as claimed.

Corollary 3.8. Suppose $H_1 < Q_1 < G$, $H_2 < Q_2 < G$, and let χ_i be irreducible characters of H_i . Set $\tau_i = \operatorname{Ind}_{H_i}^{Q_i} \chi_i$, and $\rho_i = \operatorname{Tr} \tau_i$. Assume that τ_2 is irreducible. Then

$$\mu(\rho_{2}, \operatorname{Res}_{Q_{2}} \operatorname{Ind}_{Q_{1}}^{G} \rho_{1}) = \frac{[\mathbb{Q}(\tau_{1}) : \mathbb{Q}]}{|H_{1}| \cdot |H_{2}|} \sum_{\substack{[C] \in CC_{\operatorname{cyc}}(G) \\ \sum_{\substack{D_{1} \leq H_{1} \\ D_{1} \sim C}} \operatorname{Tr}^{*} \chi_{1}(D_{1}) \cdot \sum_{\substack{D_{2} \leq H_{2} \\ D_{2} \sim C}} \operatorname{Tr}^{*} \chi_{2}(D_{2})$$

Proof.

$$\mu(\rho_2, \operatorname{Res}_{Q_2} \operatorname{Ind}_{Q_1}^G \rho_1) = \langle \tau_2, \operatorname{Res}_{Q_2} \operatorname{Ind}_{H_1}^G (\sum_{\sigma \in \operatorname{Gal}(\mathbb{Q}(\tau_1)/\mathbb{Q})} (\chi_1)^{\sigma}) \rangle$$

$$= \langle \operatorname{Ind}_{H_2}^G \chi_2, \operatorname{Ind}_{H_1}^G (\sum_{\sigma \in \operatorname{Gal}(\mathbb{Q}(\tau_1)/\mathbb{Q})} (\chi_1)^{\sigma}) \rangle$$

$$= \frac{1}{[\mathbb{Q}(\chi_1):\mathbb{Q}(\tau_1)]} \langle \operatorname{Ind}_{H_2}^G \chi_2, \operatorname{Ind}_{H_1}^G (\operatorname{Tr} \chi_1) \rangle$$

$$= \frac{1}{[\mathbb{Q}(\chi_1):\mathbb{Q}(\tau_1)]|H_1|\cdot|H_2|} \sum_{[C]\in CC_{\operatorname{cyc}}(G)} |N_G(C)|\phi(|C|) \cdot \\ \cdot \sum_{D_1 \sim C} \sum_{D_1 \sim C} \operatorname{Tr} \chi_1(D_1) \cdot \sum_{D_2 \sim C} \sum_{D_2 \sim C} \operatorname{Tr}^* \chi_2(y)$$

$$= \frac{[\mathbb{Q}(\tau_1):\mathbb{Q}]}{|H_1|\cdot|H_2|} \sum_{D_1 \sim C} [C]\in CC_{\operatorname{cyc}}(G)} |N_G(C)|\phi(|C|) \cdot \\ \cdot \sum_{D_1 \sim C} \sum_{D_1 \sim C} \operatorname{Tr}^* \chi_1(D_1) \cdot \sum_{D_2 \leq H_2} \operatorname{Tr}^* \chi_2(D_2).$$

Lemma 3.9. If C is a cyclic group, and χ is a 1-dimensional character of C, then $(\operatorname{Tr}^* \chi)(C) = \mu(\operatorname{ord}(\chi))/\phi(\operatorname{ord}(\chi))$, where μ is the Moebius mu function, and $\operatorname{ord}(\chi)$ is the smallest natural number n such that $\chi^n = \mathbf{1}$.

Proof. It is enough to prove the lemma for faithful characters χ , since we may, without loss of generality, replace C by $C/\ker \chi$. Let g be a generator of C. Then

$$(\operatorname{Tr}^* \chi)(C) = \frac{1}{[\mathbb{Q}(\chi) : \mathbb{Q}])} \operatorname{Tr} \chi(g) = \frac{1}{\phi(\operatorname{ord}(\chi))} \operatorname{Tr} \chi(g).$$

If |C| = n, then $\chi(g)$ is a primitive *n*-th root of unity, and the fact that its trace is $\mu(n)$ is classical.

Corollary 3.10. Let G be a group and p^r a prime power. Then $\hat{C}(G)$ has an element of order p^r if and only if there exist two p-quasi-elementary subgroups Q_1 , Q_2 of G, irreducible monomial characters $\tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i$ of Q_i , and an integer k, such that

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- the rational character $\operatorname{Tr} \tau_2$ has order divisible by p^{k+r} in $\hat{C}(Q_2)$, and
- the rational number

$$\frac{|\mathbb{Q}(\tau_{1}):\mathbb{Q}|}{|H_{1}||H_{2}|} \cdot \sum_{\substack{|U|\in CC_{cyc}(G)\\|U|=CC_{cyc}(G)\\|U|=CC_{cyc}(G)}} |N_{G}(U)|\phi(|U|) \cdot \sum_{\substack{|D_{1}\leq H_{1}\\|D_{1}\sim U}} \frac{\mu([D_{1}:D_{1}\cap \ker\chi_{1}])}{\phi([D_{1}:D_{1}\cap \ker\chi_{1}])} \cdot \sum_{\substack{|D_{2}\leq H_{2}\\|D_{2}\sim U}} \frac{\mu([D_{2}:D_{2}\cap \ker\chi_{2}])}{\phi([D_{2}:D_{2}\cap \ker\chi_{2}])}$$

has p-adic valuation at most k.

In this case, $\operatorname{Ind}_{Q_1}^G \operatorname{Tr} \tau_1$ has order divisible by p^r in $\widehat{\mathrm{C}}(G)$.

Remark 3.11.

- Note that it is enough to take the last two sums in the above formula only over those D_i for which $D_i \cap \ker \chi_i$ has square-free index in D_i , since for the others $\mu(\operatorname{ord}(\operatorname{Res}_{D_i}\chi_i)) = 0$. For example if χ_i are faithful, then the outer sum may be taken over U of square free order.
- If, say, H_1 is cyclic, the sum $\sum_{\substack{D_1 \leq H_1 \\ D_1 \sim U}} has at most one term for every U.$
- If Q_1 , Q_2 are basic and H_1 , H_2 are cyclic, then Proposition 2.7 gives a simple expression for the order of Tr τ_2 in $\hat{C}(Q_2)$.

Proof of Corollary 3.10. By Corollary 3.2, $\hat{C}(G)_p$ has an element of order p^r if and only if there exist *p*-quasi-elementary subgroups Q_1 , Q_2 , and characters $\rho_i \in$ $\operatorname{Irr}_{\mathbb{Q}}(Q_i)$, such that ρ_2 has order p^{k+r} in $\hat{C}(Q_2)$ for some *k*, and $\mu(\rho_2, \operatorname{Res}_{Q_2} \operatorname{Ind}^G \rho_1)$ has *p*-adic valuation at most *k*. Quasi-elementary groups are M-groups, so if τ_i is a complex irreducible constituent of ρ_i , then there exist subgroups $H_i \leq Q_i$ such that $\tau_i = \operatorname{Ind}_{H_i}^{Q_i} \chi_i$ for 1-dimensional characters $\chi_i \in \operatorname{Irr}(H_i)$. The result therefore follows from Corollary 3.8 in combination with Lemma 3.9.

4. QUASI-ELEMENTARY GROUPS

The aim of this section is to provide several formulae of theoretical and algorithmic interest for the orders of characters in $\hat{C}(G)$ and C(G) when G is quasielementary. Let $G = C \rtimes P$ with P a p-group and C cyclic of order coprime to p; we identify P with a Sylow subgroup of G.

Lemma 4.1. Let N be a normal subgroup of a finite group G, let η be an irreducible character of N, and let θ be a complex irreducible constituent of $\operatorname{Ind}_N^G \eta$. Write $\mathcal{G}_{\eta} = \operatorname{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})$, and similarly for \mathcal{G}_{θ} . Then

$$\frac{[\mathbb{Q}(\eta):\mathbb{Q}]}{[\mathbb{Q}(\theta):\mathbb{Q}]} = \frac{\#\{\gamma \in \mathcal{G}_{\eta} \mid \langle \eta^{\gamma}, \operatorname{Res}_{N} \theta \rangle \neq 0\}}{\#\{\gamma \in \mathcal{G}_{\theta} \mid \langle \operatorname{Ind}_{N}^{G} \eta, \theta^{\gamma} \rangle \neq 0\}}.$$

In particular, if $\operatorname{Ind}_N^G \eta$ is irreducible, then

$$\frac{[\mathbb{Q}(\eta):\mathbb{Q}]}{[\mathbb{Q}(\theta):\mathbb{Q}]} = \#\{\gamma \in \mathcal{G}_{\eta} \mid \langle \eta^{\gamma}, \operatorname{Res}_{N} \theta \rangle \neq 0\}.$$

Proof. The *G*-action on the characters of *N* commutes with the Galois action. Every Galois conjugate of θ is a constituent of $\operatorname{Ind}^G \eta^{\gamma}$ for some $\gamma \in \mathcal{G}_{\eta}$, and moreover the number of distinct Galois conjugates of θ in η^{γ} is independent of γ . Also, the number of Galois conjugates of η in $\operatorname{Res}_N \theta^{\gamma}$ is independent of $\gamma \in \mathcal{G}_{\theta}$. So an inclusion–exclusion count gives

$$#\mathcal{G}_{\theta} = #\mathcal{G}_{\eta} \cdot \frac{\#\{\gamma \in \mathcal{G}_{\theta} \mid \langle \operatorname{Ind}_{N}^{G} \eta, \theta^{\gamma} \rangle \neq 0\}}{\#\{\gamma \in \mathcal{G}_{\eta} \mid \langle \eta^{\gamma}, \operatorname{Res}_{N} \theta \rangle \neq 0\}}.$$

Lemma 4.2. Let η be an irreducible complex representation of G, with rational hull $\hat{\eta}$. Then

$$\dim \hat{\eta} = \dim \eta \cdot m(\eta) \cdot [\mathbb{Q}(\eta) : \mathbb{Q}].$$

Proof. The rational hull of η is given by

$$\hat{\eta} = m(\eta) \sum_{\gamma \in \operatorname{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})} \eta^{\gamma},$$

whence the claim follows.

Theorem 4.3. Let $G = C \rtimes X$ with C cyclic of order coprime to |X|. Let τ be a complex irreducible character of G with rational hull $\rho = \hat{\tau}$, let π be a complex irreducible constituent of $\operatorname{Res}_X \tau$ with rational hull $\hat{\pi}, \psi$ an irreducible constituent of $\operatorname{Res}_C \tau$ with rational hull $\hat{\psi}, K_{\psi}$ the stabiliser of ψ under the X-action on $\operatorname{Irr}(C)$, and let ξ be a complex irreducible constituent of $\operatorname{Res}_{K_{\psi}} \pi$. Then

$$\begin{split} \mu(\rho, \operatorname{Ind}_X^G \hat{\pi}) &= \\ &= \frac{m(\pi)}{m(\tau)} \langle \xi, \operatorname{Res}_{K_{\psi}} \pi \rangle \cdot \# \{ \text{Galois conjugates } \pi' \text{ of } \pi \mid \langle \operatorname{Res}_{K_{\psi}} \pi', \xi \rangle \neq 0 \} \\ &= \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}. \end{split}$$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$. So $K = K_{\psi}$ is assumed to be the kernel of the X-action on C. Recall that ψ denotes a complex constituent of $\tau|_C$. In particular, $\tau = \operatorname{Ind}_{CK}^G \psi \xi$, as explained in [14, Part II, §8.2]. We have

$$\rho = m(\tau) \sum_{\substack{\gamma \in \operatorname{Gal}(\mathbb{Q}(\tau)/\mathbb{Q}) \\ \gamma \in \operatorname{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})}} \tau^{\gamma}; \quad \dim \rho = m(\tau)[\mathbb{Q}(\tau) : \mathbb{Q}] \dim \tau, \\
\hat{\pi} = m(\pi) \sum_{\substack{\gamma \in \operatorname{Gal}(\mathbb{Q}(\pi)/\mathbb{Q}) \\ \gamma \in \operatorname{Gal}(\mathbb{Q}(\pi)/\mathbb{Q})}} \pi^{\gamma}; \quad \dim \hat{\pi} = m(\pi)[\mathbb{Q}(\pi) : \mathbb{Q}] \dim \pi.$$

Thus

$$\mu(\rho, \operatorname{Ind}_X^G \hat{\pi}) = \frac{1}{m(\tau)} \langle \tau, \operatorname{Ind}_X^G \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \operatorname{Ind}_{CK}^G \psi \xi, \operatorname{Ind}_X^G \hat{\pi} \rangle$$
$$= \frac{1}{m(\tau)} \langle \operatorname{Res}_X \operatorname{Ind}_{CK}^G \psi \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \operatorname{Ind}_K^X \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \xi, \operatorname{Res}_K \hat{\pi} \rangle,$$

where the last line follows from Mackey's formula, noting that $CK \setminus G/X$ consists of one double coset, and that $CK \cap X = K$.

Next, X acts on the representations of K by conjugation, and there is a Clifford theory decomposition

(4.4)
$$\operatorname{Res}_{K} \pi = e \sum_{g \in X/\operatorname{Stab}_{X} \xi} \xi^{g}.$$

Recall that the constituents of $\hat{\pi}$ are Galois conjugates of π , and we select those whose restriction to K contains ξ :

$$\Omega = \big\{ \gamma \in \operatorname{Gal}(\mathbb{Q}(\pi) : \mathbb{Q}) \ \big| \ \langle \operatorname{Res}_K \pi^{\gamma}, \xi \rangle \neq 0 \big\}.$$

The inner product $\langle \operatorname{Res}_K \pi^{\gamma}, \xi \rangle = \langle \operatorname{Res}_K \pi, \xi^{\gamma^{-1}} \rangle$ is the same (and equals *e*) for every $\gamma \in \Omega$, since $\xi^{\gamma^{-1}}$ is irreducible and so must be one of ξ^g in (4.4). So we have

$$\frac{1}{m(\tau)}\langle \xi, \operatorname{Res}_K \hat{\pi} \rangle = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \operatorname{Res}_K \pi \rangle,$$

which proves the first equality.

It remains to show that

(4.5)
$$\frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \operatorname{Res}_K \pi \rangle = \frac{\dim \psi \dim \hat{\pi}}{\dim \rho}.$$

By comparing dimensions in (4.4), and since $\tau = \operatorname{Ind}_{CK}^G \psi \xi$, we see that

$$\langle \xi, \operatorname{Res}_K \pi \rangle = e = \frac{\dim \pi}{[X : \operatorname{Stab}_X \xi] \dim \xi} = \frac{[X : K] \dim \pi}{[X : \operatorname{Stab}_X \xi] \dim \tau} = \frac{[\operatorname{Stab}_X \xi : K] \dim \pi}{\dim \tau}$$

 \mathbf{so}

$$\mu(\rho, \operatorname{Ind}^G \hat{\pi}) = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \operatorname{Res}_K \pi \rangle = |\Omega| \cdot [\operatorname{Stab}_X \xi : K] \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau}.$$

Consider the two groups

$$H_1 = \{ \gamma \in \operatorname{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q}) \mid \langle (\psi\xi)^{\gamma}, \operatorname{Res}_{CK} \tau \rangle \neq 0 \}, H_2 = \{ \gamma \in \operatorname{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \mid \langle \xi^{\gamma}, \operatorname{Res}_K \pi \rangle \neq 0 \}.$$

There is a natural projection $H_1 \to H_2$ given by the restriction of Galois action to $\mathbb{Q}(\xi)$, whose kernel consists of precisely those elements of $\operatorname{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q})$ that act trivially on ξ , and through the action of some $g \in X$ on ψ (this last condition is equivalent to the Galois element being in H_1). Thus, the kernel is isomorphic to the subgroup of G/CK that acts trivially on ξ , i.e. to $\operatorname{Stab}_X \xi/K$. We deduce that

$$\mu(\rho, \operatorname{Ind}^G \hat{\pi}) = |\Omega| \frac{|H_1|}{|H_2|} \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau}.$$

Now, by applying Lemma 4.1 first to $CK \triangleleft G$ with $\theta = \tau$, $\eta = \psi \xi$, and then to $K \triangleleft X$ with $\theta = \pi$, $\eta = \xi$, we find that

$$|H_1| = \frac{[\mathbb{Q}(\xi) : \mathbb{Q}][\mathbb{Q}(\psi) : Q]}{[\mathbb{Q}(\tau) : \mathbb{Q}]} \quad \text{and} \quad |H_2| = |\Omega| \frac{[\mathbb{Q}(\xi) : \mathbb{Q}]}{[\mathbb{Q}(\pi) : \mathbb{Q}]},$$

so that

$$\begin{split} \mu(\rho, \operatorname{Ind}^{G} \hat{\pi}) &= |\Omega| \cdot \frac{|H_{1}|}{|H_{2}|} \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau} \\ &= |\Omega| \cdot \frac{[\mathbb{Q}(\xi) : \mathbb{Q}] \cdot [\mathbb{Q}(\psi) : \mathbb{Q}] / [\mathbb{Q}(\tau) : \mathbb{Q}]}{|\Omega| [\mathbb{Q}(\xi) : \mathbb{Q}] / [\mathbb{Q}(\pi) : \mathbb{Q}]} \cdot \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau} \\ &= \frac{[\mathbb{Q}(\psi) : \mathbb{Q}] \cdot [\mathbb{Q}(\pi) : \mathbb{Q}] m(\pi) \dim \pi}{[\mathbb{Q}(\tau) : \mathbb{Q}] m(\tau) \dim \tau} = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}, \end{split}$$

where the last equality follows from Lemma 4.2.

Theorem 4.6. Let $G = C \rtimes P$ be p-quasi-elementary, let ρ be an irreducible rational representation of G. Let ψ be a complex irreducible constituent of $\operatorname{Res}_C \rho$ with rational hull $\hat{\psi}$, and let $\hat{\pi}$ be a rational irreducible constituent of $\operatorname{Res}_P \rho$ of minimal dimension. Denote by π a complex irreducible constituent of $\hat{\pi}$, by ξ a complex irreducible constituent of $\pi|_{K_{\psi}}$, where $K_{\psi} \leq P$ is the stabiliser in P of ψ , and by τ a complex irreducible constituent of ρ such that $\operatorname{Res}_P \tau$ contains π . Then

order of
$$\rho$$
 in $C(G) = \mu(\rho, \operatorname{Ind}_P^G \hat{\pi}) = \frac{\dim \psi \dim \hat{\pi}}{\dim \rho}$
= $\frac{m(\pi)}{m(\tau)} \langle \xi, \operatorname{Res}_{K_{\psi}} \pi \rangle \cdot \# \{ Galois \ conjugates \ \pi' \ of \ \pi \mid \xi \subset \operatorname{Res}_{K_{\psi}} \pi' \}.$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$ (see Lemma 2.1). Thus, $K = K_{\psi}$ is assumed to be the kernel of the *P*-action on *C*. Under this assumption, if $H \leq G$ intersects *C* non-trivially, then

$$\langle \rho, \mathbb{C}[G/H] \rangle_G = \langle \operatorname{Res}_H \rho, \mathbf{1} \rangle_H = 0.$$

Write o for the order of ρ in C(G). By Theorem 2.5, we have

$$\begin{array}{lll} \rho \cdot \langle \rho, \rho \rangle &=& \gcd_{H \leq G} \langle \rho, \mathbb{C}[G/H] \rangle_G = \gcd_{H \leq P} \langle \rho, \mathbb{C}[G/H] \rangle_G \\ &=& \gcd_{H < P} \langle \rho|_P, \mathbb{C}[P/H] \rangle_P. \end{array}$$

Because C(P) = 1 by the Ritter-Segal theorem [12, 13], we can replace the permutation representations $\mathbb{C}[P/H]$ by all rational representations of P in the last term. This is clearly the same as just taking the rational irreducible constituents $\hat{\pi}_1, ..., \hat{\pi}_k$ of $\rho|_P$, so

(4.7)
$$o = \frac{1}{\langle \rho, \rho \rangle} \gcd_{j} \langle \rho |_{P}, \hat{\pi}_{j} \rangle = \gcd_{j} \frac{\langle \rho, \operatorname{Ind}_{P}^{G} \hat{\pi}_{j} \rangle}{\langle \rho, \rho \rangle} = \gcd_{j} \mu(\rho, \operatorname{Ind}_{P}^{G} \hat{\pi}_{j}).$$

The theorem will therefore follow from Theorem 4.3, once we show that the gcd may be replaced by the term corresponding to any $\hat{\pi}$ of minimal dimension. Now, by Theorem 4.3 and by Lemma 4.2,

$$\mu(\rho, \operatorname{Ind}_P^G \hat{\pi}_j) = \frac{\dim \hat{\psi} \dim \hat{\pi}_j}{\dim \rho} = \frac{\dim \hat{\psi} m(\pi_j) \dim \pi_j[\mathbb{Q}(\pi_j) : \mathbb{Q}]}{\dim \rho}$$

where π_j is a complex irreducible constituent of $\hat{\pi}_j$. We argue as in [17, §2]: if p = 2, then all the terms $m(\pi_j)$, dim π_j , $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are powers of 2, so gcd and minimum are the same. If p is odd, then $m(\pi_j) = 1$, and moreover, either some $\pi_j = \mathbf{1}$, in which case the claim is clear, or else all dim π_j are powers of p, while all $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are (p-1) times powers of p ([17, Lemma 2.1]), so again gcd and minimum are the same.

5. EXAMPLES:
$$\operatorname{GL}_2(\mathbb{F}_q)$$
, $\operatorname{PGL}_2(\mathbb{F}_q)$, $\operatorname{SL}_2(\mathbb{F}_q)$ and $\operatorname{PSL}_2(\mathbb{F}_q)$

Theorem 5.1. For every prime power $q = p^n$, the group $G = \operatorname{GL}_2(\mathbb{F}_q)$ has $\hat{C}(G) = \{1\}$.

Proof. By Corollary 3.3, it suffices to show that every maximal quasi-elementary subgroup $Q = C \rtimes P$ of $G = \operatorname{GL}_2(\mathbb{F}_q)$ is contained in some $\overline{Q} < G$ with $\widehat{C}(\overline{Q}) = 1$. Pick $C = \langle g \rangle$ cyclic, and let $P = \operatorname{Syl}_l(N_G(C))$ for some prime number l. Write f(t) for the characteristic polynomial of g.

Case 1 (split Cartan). Suppose f(t) has distinct roots $a, b \in \mathbb{F}_q^{\times}$. Then g is conjugate to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and its centraliser is the split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times}, \qquad N_G(C) < (\mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times}) \rtimes C_2,$$

with $C_2 = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$. Here $\bar{Q} = (\mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times}) \rtimes C_2$ has trivial $\hat{C}(\bar{Q})$ by Corollary 2.6. **Case 2** (non-split Cartan). Suppose f(t) is irreducible over \mathbb{F}_q . Then the cen-

traliser of C is the non-split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q[g]^{\times} \cong \mathbb{F}_{q^2}^{\times}, \qquad N_G(C) < \mathbb{F}_{q^2}^{\times} \rtimes \operatorname{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \cong \mathbb{F}_{q^2}^{\times} \rtimes C_2.$$

Again $\bar{Q} = \mathbb{F}_{q^2}^{\times} \rtimes C_2$ has trivial \hat{C} by Corollary 2.6.

Case 3 (scalars). Suppose $g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a scalar matrix. Then $Q = C \rtimes P$ can be embedded into one of the following:

- if l = p: $\bar{Q} = C \times U = C \times \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$; in this case U is an elementary abelian *p*-group; or
- if either l is odd and l|(q-1), or l=2 and $q\equiv 1 \mod 4$: $\overline{Q}=H \rtimes C_2$ with H = split Cartan; or
- if either l is odd and l|(q+1), or l=2 and $q\equiv 3 \mod 4$: $\overline{Q}=H \rtimes C_2$ with H = non-split Cartan.

In all these cases, $\hat{C}(\bar{Q})$ is trivial by Corollary 2.6.

Case 4 (non-semisimple). Finally suppose that g is not semisimple, say $g = g_s g_u$ with g_s central and $g_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ non-trivial unipotent. Then

$$N_G C = N_G \langle g_u \rangle = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| ac^{-1} \in \mathbb{F}_p^{\times} \right\} \\ = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{F}_q^{\times} \right\} \cdot \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{F}_q \right\} \cdot \left\{ \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_p^{\times} \right\} \\ \cong \mathbb{F}_q^{\times} \times (\mathbb{F}_q \rtimes \mathbb{F}_p^{\times}).$$

If l = p, then Q can be embedded into $\overline{Q} = \langle g_s \rangle \times U$, where $U = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \} \cong \mathbb{F}_q$ is an elementary abelian p-group. In this case $\hat{C}(\bar{Q})$ is trivial by Corollary 2.6. Otherwise, Q can be embedded into $\bar{Q} \cong (\mathbb{F}_q^{\times} \times \langle g_u \rangle) \rtimes \mathbb{F}_p^{\times}$, where the action in the semi-direct product is faithful. If τ is an irreducible character of \bar{Q} such that $\operatorname{Res}_{\langle q_n \rangle} \tau$ is faithful, then $\overline{Q}/\ker \tau$ satisfies the assumptions of Proposition 2.7 with $K = \{1\}$, so Tr $\tau \in \text{Perm}(\bar{Q})$. Otherwise, $\text{Res}_{\langle g_u \rangle} \tau = \dim \tau \cdot \mathbf{1}$, so τ factors through an abelian quotient, and $\operatorname{Tr} \tau \in \operatorname{Perm}(\overline{Q})$ e.g. by Corollary 2.6.

Remark 5.2. It is also not hard to deduce the structure of \hat{C} for the related classical groups:

- $G = PGL_2(\mathbb{F}_q)$. Combined with Lemma 2.1, the theorem implies C(G) = 1.
- $G = \operatorname{SL}_2(\mathbb{F}_q)$. In general, $\hat{C}(G) \neq 1$. For example, $\operatorname{SL}_2(\mathbb{F}_3)$ has C = 1 and $\hat{C} \cong \mathbb{Z}/2\mathbb{Z}$ (it has a 2-dimensional irreducible symplectic representation), and $SL_2(\mathbb{F}_{17})$ has $C \cong \mathbb{Z}/4\mathbb{Z}$.
- $G = \text{PSL}_2(\mathbb{F}_q)$. It is a result of Solomon, announced in [15], that $\hat{C}(G) = 1$. This can also be seen following the argument for GL_2 in Theorem 5.1: the analogues of Q are the images of $Q \cap \mathrm{SL}_2(\mathbb{F}_q)$ in $\mathrm{PSL}_2(\mathbb{F}_q)$, and they are dihedral in Cases 1 and 2 of the theorem, elementary abelian or dihedral (p=2) in Case 3 and isomorphic to $\mathbb{F}_p \rtimes \mathbb{F}_p^{\times}$ in Case 4. Again, all these groups have $\hat{C} = 1$, so $\hat{C}(G) = 1$.

6.
$$\operatorname{PSL}_n(\mathbb{F}_p)$$

Let ord₂ denote the 2-adic valuation of a rational number, $\operatorname{ord}_2\left(2^x \cdot \frac{a}{b}\right) = x$, where $2 \nmid ab$.

Theorem 6.1. Let $k \geq 4$ be an integer, and p a prime. The groups $PSL_k(\mathbb{F}_n)$, and therefore also $SL_k(\mathbb{F}_p)$, have $\hat{C}(G)$ of exponent divisible by $2^{\min(\operatorname{ord}_2(k), \operatorname{ord}_2(p-1))}$.

In the remainder of the section we prove the theorem using Corollary 3.10. We will construct a 2-quasi-elementary subgroup $Q = C \rtimes P$ of $G = PSL_k(\mathbb{F}_p)$ and a rational character ρ of Q such that $\operatorname{Ind}_Q^G \rho$ has order divisible by $2^{\min(\operatorname{ord}_2(k), \operatorname{ord}_2(p-1))}$ in $\hat{\mathcal{C}}(G)$.

Lemma 6.2. Let p be an odd prime and $k \ge 4$ an integer. If k = 4, assume that $p \equiv 1 \pmod{4}$. Then there exists a prime number l that divides $p^{k-2} - 1$ but does not divide $p^s - 1$ for any s < k - 2.

Proof. This is a special case of Zsigmondy's Theorem [18].

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Write Q_{2^N} for the generalised quaternion group of order 2^N .

Lemma 6.3. The group $SL_2(\mathbb{F}_q)$, $q = p^k$ has a 2-Sylow subgroup of the form

- S = {(¹₀*)} ≅ C^k_p if p = 2;
 S = ⟨c,h⟩ ≅ Q_{2^N}, c = (^α_{0 α⁻¹}), h = (⁰_{-1 0}) with α ∈ ℝ[×]_q of exact order 2^{N-1}||q − 1, if q ≡ 1 mod 4;
 S = ⟨c,h⟩ ≅ Q_{2^N}, c = (^{α − β}_{β α}), h = (^{γ δ}_{δ −γ}) with α + β√−1 ∈ ℝ[×]_q of exact order 2^{N-1}||q + 1 and any choice of γ, δ ∈ ℝ_q with γ² + δ² = −1, if q ≡ 3

Conjugation by the matrix $\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is an automorphism of S, acting as -1 in the first case, as $c \mapsto c, h \mapsto h^{-1}$ in the second case, and as $c \mapsto c^{-1}, h \mapsto hc^{2m+1}$ for some m in the last case.

Proof. Direct computation.

From now on, G will denote $\mathrm{PSL}_k(\mathbb{F}_p)$. The theorem only has content when k is even and p is odd, so we will assume this. Write

$$n = \operatorname{ord}_2(p-1) \ge 1,$$
 $N = \operatorname{ord}_2(p^{k-2}-1) \ge 3,$ $m = \operatorname{ord}_2(k-2) \ge 1.$

Case A: Either k > 4 or $p \equiv 1 \mod 4$. Let A be a generator of a nonsplit Cartan subgroup $\mathbb{F}_{p^{k-2}}^{\times} = \operatorname{GL}_1(\mathbb{F}_{p^{k-2}}) < \operatorname{GL}_{k-2}(\mathbb{F}_p)$, and l a prime divisor of $p^{k-2} - 1$ as in Lemma 6.2. The conditions on l imply that the normaliser of $\langle A^{\frac{p^{k-2}-1}{l}} \rangle \cong C_l$ in $\operatorname{GL}_{k-2}(\mathbb{F}_p)$ is generated by A and by the Frobenius automorphism $F \in \operatorname{Gal}(\mathbb{F}_{p^{k-2}}/\mathbb{F}_p)$ of order k-2. Note that F has determinant -1, since it is an odd permutation on a normal basis of $\mathbb{F}_{p^{k-2}}/\mathbb{F}_p$. Define

$$c_p = \begin{pmatrix} 1 & 1 \\ & 1 \\ & & I_{k-2} \end{pmatrix}, \qquad c_l = \begin{pmatrix} 1 & 1 \\ & & A^{\frac{p^{k-2}-1}{l}} \end{pmatrix},$$
$$x = \begin{pmatrix} d^{-1} & \\ & & U \end{pmatrix}, \qquad f = \begin{pmatrix} -1 & 1 \\ & & F^{(k-2)/2^m} \end{pmatrix},$$

where $U = A^{\frac{p^{k-2}-1}{2^N}}$ and $d = \det U$. We view these matrices as representing elements of $G = \text{PSL}_k(\mathbb{F}_p)$. Write

$$C = \langle c_p c_l \rangle \cong C_{pl}, \quad P = \langle x, f \rangle \cong C_{2^N} \rtimes C_{2^m}, \quad Q = CP \cong (C_p \times C_l) \rtimes (C_{2^N} \rtimes C_{2^m}).$$

Note that C_{2^N} acts trivially on C_l , and through a C_{2^n} quotient on C_p , while C_{2^m} acts through a C_2 quotient on C_p and faithfully on C_l .

Case B: $p \equiv 3 \mod 4$ and k = 4. We take the same c_p as in Case A, and $C = \langle c_p \rangle$. A 2-Sylow of the centraliser of C in G is isomorphic to $\{1\} \times \text{Syl}_2(\text{SL}_2(\mathbb{F}_p))$, which is isomorphic to Q_{2^N} by the last case of Lemma 6.3. A 2-Sylow of the normaliser is

$$P = \operatorname{Syl}_2 N_G(C) = \operatorname{Syl}_2 Z_G(C) \rtimes \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 & \\ & & & 1 \end{pmatrix} \cong Q_{2^N} \rtimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is in fact isomorphic to the semi-dihedral group $SD_{2^{N+1}}$. Again, we let Q = CP.

In both cases, write K for the centraliser of C in P. Thus, $K \cong C_{2^{N-n}}$ in case A, and $K \cong Q_{2^N}$ in case B, where the isomorphism is that of Lemma 6.3. Let A_p be K in case A, and a cyclic subgroup of index 2 in K that is normal in Q in case B. Let χ be faithful irreducible characters of CA_p , $\tau = \operatorname{Ind}_{CA_p}^Q \chi$ and $\rho = \operatorname{Tr} \tau \in \operatorname{Irr}_{\mathbb{Q}}(Q)$.

Lemma 6.4. The character ρ has order 2^n in $\hat{C}(Q)$.

Proof. We will use Proposition 2.7. The biggest subgroup of P that intersects CA_p trivially is of order 2 in case B, and of order 2^m in case A. So the order of ρ in $\hat{C}(G)$ is $2^{N+1-(N-1)-1} = 2$ in case B, and $2^{N+m-(N-n)-m} = 2^n$ in case A.

Finally, we show that $\operatorname{Ind}_Q^G \rho$ has order divisible by $2^{\min(\operatorname{ord}_2(k),\operatorname{ord}_2(p-1))}$ in $\hat{C}(G)$. We will use Corollary 3.10 with $Q_1 = Q_2 = Q$ and $\chi_1 = \chi_2 = \chi$. In view of Lemma 6.4, it suffices to show that

(6.5)
$$\sum_{[U] \in CC_{cyc}(G)} S(U)$$

has 2-adic valuation at most $n - \min(\operatorname{ord}_2(k), \operatorname{ord}_2(p-1))$, where for $U \leq CA_p$,

$$S(U) = \frac{[\mathbb{Q}(\tau):\mathbb{Q}]}{|CA_p|^2} |N_G(U)|\phi(|U|) \cdot \left(\sum_{\substack{D \le CA_p \\ D \sim U}} \frac{\mu([D:D \cap \ker \chi])}{\phi([D:D \cap \ker \chi])}\right)^2$$

Note that since CA_p is cyclic and χ is faithful, this simplifies to

$$S(U) = \frac{[\mathbb{Q}(\tau):\mathbb{Q}]}{|CA_p|^2\phi(|U|)} |N_G(U)|\mu(|U|)^2,$$

see Remark 3.11. In particular, S(U) = 0 if U has non-square-free order. Case A.

The subgroups of CK of square-free order are C_{2lp} , C_{lp} , C_{2l} , C_l , C_{2p} , C_p , C_2 , and C_1 . We will show that $S(C_{lp}) + S(C_{2lp})$ has a strictly lower 2-adic valuation than the rest of the sum, and that this valuation is $n - \min(\operatorname{ord}_2(k), n)$. A summary of the calculations that follow is:

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$$\begin{aligned} \operatorname{ord}_{2}[\mathbb{Q}(\tau):\mathbb{Q}] &= \operatorname{ord}_{2}(l-1) + N - n - 1 - m, \\ \operatorname{ord}_{2}|CK|^{2} &= 2(N-n), \\ \phi(|C_{lp}|) &= \phi(|C_{2lp}|) &= (l-1)(p-1), \\ |N_{G}(C_{lp})| &= |N_{G}(C_{2lp})| &= \frac{(k-2)p(p^{k-2}-1)(p-1)}{\gcd(k,p-1)}, \\ \operatorname{ord}_{2}(S(C_{lp}) + S(C_{2lp})) &= \operatorname{ord}_{2}(2S(C_{lp})) \\ &= 1 + \operatorname{ord}_{2}(l-1) + N - n - 1 - m - 2(N-2) + N + \\ n + m - \min(\operatorname{ord}_{2}(k), n) - \operatorname{ord}_{2}(l-1) + n \\ &= n - \min(\operatorname{ord}_{2}(k), n). \end{aligned}$$

The assertions concerning |CK| and $\phi(|C_{lp}|)$ are clear.

Since the conjugation action of P on Irr(CK) is through Galois automorphisms, and $ker(P \to Aut(CK))$ has index 2^{n+m} in P, we have

$$[\mathbb{Q}(\tau):\mathbb{Q}] = 2^{-n-m}[\mathbb{Q}(\chi):\mathbb{Q}] = \frac{p-1}{2^n} \frac{(l-1)2^{N-n-1}}{2^m},$$

with 2-adic valuation $\operatorname{ord}_2(l-1) + N - n - 1 - m$.

The normaliser $N_{\mathrm{GL}_k(p)}$ of the preimage of C_{lp} under $\mathrm{SL} \to \mathrm{PSL}$ consists of block diagonal matrices, with the normaliser of non-split Cartan in the lower right corner (order $(k-2)(p^{k-2}-1)$), and a Borel subgroup in the top left (order $p(p-1)^2$). The determinant is surjective on $N_{\mathrm{GL}_k(p)}$, and $N_{\mathrm{GL}_k(p)}$ contains $Z(\mathrm{GL}_k(p))$, so the normaliser of C_{lp} in PSL has order $\frac{(k-2)(p^{k-2}-1)p(p-1)}{\gcd(k,p-1)}$, with 2-adic valuation $N+n+m-\min(\operatorname{ord}_2(k),n)$. This is also the normaliser of C_{2lp} .

It remains to show that the rest of the sum in equation (6.5) has strictly greater 2adic valuation than $\operatorname{ord}_2(S(C_{lp})+S(C_{2lp}))$. If $U \leq C$, then $|N_G(U)|$ and $|N_G(UC_2)|$ agree up to a power of p, $\phi(|U|) = \phi(|UC_2|)$, while $\mu(|U|) = -\mu(|UC_2|)$. It follows that the 2-adic valuation of $S(U) + S(UC_2)$ is at least 1 greater than that of S(U).

Moreover, for any $U \leq C_{lp}$, the normaliser of U in G contains that of C_{lp} , while $1/\phi(|U|)$ has strictly greater 2-adic valuation than $1/\phi(|C_{lp}|)$ whenever $U \neq C_{lp}$. This establishes the claim.

Case B. The subgroups of CA_p of square-free order are C_1 , C_2 , C_p , and C_{2p} . We will show that $\operatorname{ord}_2(\sum S(U)) = \operatorname{ord}_2(S(C_p) + S(C_{2p})) = 0$. Again, we summarise the calculations as follows:

$$\begin{aligned} \operatorname{ord}_2[\mathbb{Q}(\tau):\mathbb{Q}] &= N-3, \\ \operatorname{ord}_2|CA_p|^2 &= 2N-2, \\ \phi(|C_p|) &= \phi(|C_{2p}|) &= p-1, \\ |N_G(C_p)| &= p^4 |N_G(C_{2p})| &= p^4 \cdot \frac{(p-1)^3 p^2 (p+1)}{2}, \\ \operatorname{ord}_2(S(C_p) + S(C_{2p})) &= \operatorname{ord}_2((1+p^4)S(C_{2p})) \\ &= 1+N-3-2N+2-1+N+1=0. \end{aligned}$$

The assertions concerning $|CA_p|$ and ϕ are clear.

It follows from the description of the *P*-action on $\operatorname{Irr}(CK)$ that $[\mathbb{Q}(\tau) : \mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(\chi) : \mathbb{Q}]$, and has 2-adic valuation 2^{N-3} .

The normaliser of C_{2p} in GL₄ is block diagonal, with all invertible matrices in the bottom right corner, and Borel in the top left. So its order in PSL is $\frac{(p-1)^3p^2(p+1)}{2}$ with 2-adic valuation N + 1. Finally, $|N(C_p)| = p^4 |N(C_{2p})|$, e.g. see Murray [10] §4.

It remains to show that the 2-adic valuation of $S(C_1) + S(C_2)$ is positive. The normaliser of C_2 in GL_4 is $GL_2 \times GL_2$, so the order of the normaliser in PSL is $\frac{(p-1)^3 p^2 (p+1)^2}{2}$, with 2-adic valuation 2N, and the normaliser of C_1 is even bigger. So the 2-adic valuations of $S(C_1)$ and of $S(C_2)$ are positive.

Corollary 6.6. As G ranges over the simple groups $PSL_k(\mathbb{F}_p)$, and therefore also over $SL_k(\mathbb{F}_p)$, the exponent of $C(G)_2$ is unbounded.

Proof. If $\operatorname{ord}_2(k) > \operatorname{ord}_2(p-1)$, then by [16, Lemma 5.6(1)] all Schur indices in $\operatorname{PSL}_k(\mathbb{F}_p)$ are trivial. So the assertion follows from Theorem 6.1.

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