EXTENSIONS OF TWO q-GOSPER IDENTITIES WITH AN EXTRA INTEGER PARAMETER

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Abstract. According to the method of series rearrangement, we establish the extensions of two q-Gosper identities with an extra integer parameter. The limiting cases of them produce the generalizations of Gosper's two ${}_3F_2(\frac{3}{4})$-series identities with an additional integer parameter. Meanwhile, several related results are also given.

1. Introduction

For a complex variable x, define the shifted-factorial by

$$(x)_n=\begin{cases}\prod_{k=0}^{n-1}(x+k), & n>0;\\ 1, & n=0;\\ \frac{(-1)^n}{\prod_{k=1}^{-n}(k-x)}, & n<0.\end{cases}$$

For simplifying the expressions, we shall use the abbreviated symbol:

$$\begin{bmatrix}a, & b, & \cdots, & c\\ \alpha, & \beta, & \cdots, & \gamma\end{bmatrix}_n=\frac{(a)_n(b)_n\cdots(c)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

Following Bailey [1], define the hypergeometric series by

$${}_{1+r}F_s\left[\begin{matrix}a_0, & a_1, & \cdots, & a_r\\ & b_1, & \cdots, & b_s\end{matrix}\,\Big|\, z\right]=\sum_{k=0}^{\infty}\begin{bmatrix}a_0, & a_1, & \cdots, & a_r\\ 1, & b_1, & \cdots, & b_s\end{bmatrix}_n z^k,$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then two ${}_3F_2(\frac{3}{4})$-series identities due to Gosper (cf. [7, Equations (1.4) and (1.5)]) read, respectively, as

$${}_3F_2\left[\begin{matrix}3x, & 1-3x, & -n\\ & \frac{1}{2}, & -3n\end{matrix}\,\Big|\,\frac{3}{4}\right]=\begin{bmatrix}\frac{1}{3}+x, & \frac{2}{3}-x\\ \frac{1}{3}, & \frac{2}{3}\end{bmatrix}_n, \tag{1}$$

$${}_3F_2\left[\begin{matrix}3x, & 2-3x, & -n\\ & \frac{3}{2}, & -1-3n\end{matrix}\,\Big|\,\frac{3}{4}\right]=\begin{bmatrix}\frac{2}{3}+x, & \frac{4}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{bmatrix}_n. \tag{2}$$

For two complex numbers x and q, define the q-shifted factorial by

$$(x;q)_n=\begin{cases}\prod_{i=0}^{n-1}(1-xq^i), & n>0;\\ 1, & n=0;\\ \frac{1}{\prod_{j=n}^{-1}(1-xq^j)}, & n<0.\end{cases}$$

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The fraction form of it reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix} q \Bigg]_n = \frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman [6], define the q-series by

$${}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\;\Big|\; q; z\right] = \sum_{k=0}^{\infty}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix}\;\Big|\; q\right]_k \left\{(-1)^k q^{\binom{k}{2}}\right\}^{s-r} z^k,$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then the q-analogues of (1) and (2) due to Chu [2, Equations (3.9a) and (3.9b)] can respectively be stated as

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix} x, q/x \\ q, -q, q^{1/2}, -q^{1/2}, q^{-3n} \end{matrix}\;\Big|\; q\right]_k q^k = \left[\begin{matrix} qx, q^2/x \\ q, q^2 \end{matrix}\;\Big|\; q^3\right]_n, \tag{3}$$

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix} x, q^2/x \\ q, -q, q^{3/2}, -q^{3/2}, q^{-1-3n} \end{matrix}\;\Big|\; q\right]_k q^k = \left[\begin{matrix} q^2x, q^4/x \\ q^2, q^4 \end{matrix}\;\Big|\; q^3\right]_n. \tag{4}$$

Inspired by the work of [3]-[5], we shall establish, in terms of the method of series rearrangement, the extensions of (3) and (4) with an extra integer parameter which involve the generalizations of (1) and (2) with an additional integer parameter in Section 2. Meanwhile, several related results are also offered in Section 3.

2. Extensions of two q-Gosper identities

Theorem 1. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix} x, q^{1-\ell}/x \\ q, -q^{1-\ell}, q^{1/2}, -q^{1/2}, q^{-3n} \end{matrix}\;\Big|\; q\right]_k q^k = \left[\begin{matrix} x, -x \\ x^2, -1 \end{matrix}\;\Big|\; q\right]_\ell$$
$$\times\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix} q^{-\ell}, x^2q^{-1} \\ q, x^2q^{\ell} \end{matrix}\;\Big|\; q\right]_i\left[\begin{matrix} q^{1+i}x, q^{2-i}/x \\ q, q^2 \end{matrix}\;\Big|\; q^3\right]_n.$$

Proof. Setting $a = x^2q^{-1}$, $b = xq^k$ and $c = -x$ in the terminating ${}_6\phi_5$-series identity (cf. [6, p. 42]):

$${}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-\ell} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, q^{1+\ell}a \end{matrix}\;\Big|\; q; \frac{q^{1+\ell}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\;\Big|\; q\right]_\ell, \tag{5}$$

we get the equation:

$$\left[\begin{matrix} q^{1-\ell}/x, -q^{1-\ell}/x \\ q^{1-\ell}/x^2, -q^{1-\ell+k} \end{matrix}\;\Big|\; q\right]_\ell \sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix} q^{-\ell}, xq^k, x^2q^{-1} \\ q, x, x^2q^{\ell} \end{matrix}\;\Big|\; q\right]_i$$
$$\times\left[\begin{matrix} q^{1-\ell+k}/x \\ q^{1-\ell}/x \end{matrix}\;\Big|\; q\right]_{\ell-i} = 1. \tag{6}$$

Then there is the following relation:

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix} x, q^{1-\ell}/x \\ q, -q^{1-\ell}, q^{1/2}, -q^{1/2}, q^{-3n} \end{matrix}\;\Big|\; q\right]_k q^k$$
$$= \sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix} x, q^{1-\ell}/x \\ q, -q^{1-\ell}, q^{1/2}, -q^{1/2}, q^{-3n} \end{matrix}\;\Big|\; q\right]_k q^k\left[\begin{matrix} q^{1-\ell}/x, -q^{1-\ell}/x \\ q^{1-\ell}/x^2, -q^{1-\ell+k} \end{matrix}\;\Big|\; q\right]_\ell$$
$$\times\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix} q^{-\ell}, xq^k, x^2q^{-1} \\ q, x, x^2q^{\ell} \end{matrix}\;\Big|\; q\right]_i\left[\begin{matrix} q^{1-\ell+k}/x \\ q^{1-\ell}/x \end{matrix}\;\Big|\; q\right]_{\ell-i}.$$

Interchange the summation order for the last double sum to achieve

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{1-\ell}/x\\ q,-q^{1-\ell},q^{1/2},-q^{1/2},q^{-3n}\end{matrix}\Big|\,q\right]_k q^k\\ &=\left[\begin{matrix}q^{1-\ell}/x,-q^{1-\ell}/x\\ q^{1-\ell}/x^2,-q^{1-\ell}\end{matrix}\Big|\,q\right]_\ell\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix}q^{-\ell},x^2q^{-1}\\ q,x^2q^{\ell}\end{matrix}\Big|\,q\right]_i\\ &\times\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}xq^i,q^{1-i}/x\\ q,-q,q^{1/2},-q^{1/2},q^{-3n}\end{matrix}\Big|\,q\right]_k q^k.\end{aligned}$$

Calculating the series on the last line by (3), we attain Theorem 1 to complete the proof. □

When $\ell=0$, Theorem 1 reduces to (3). Another concrete formula is displayed as follows.

Corollary 2 ($\ell=1$ in Theorem 1)**.**

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,1/x\\ q,-1,q^{1/2},-q^{1/2},q^{-3n}\end{matrix}\Big|\,q\right]_k q^k\\ &=\frac{1}{2}\left[\begin{matrix}qx,q^2/x\\ q,q^2\end{matrix}\Big|\,q^3\right]_n+\frac{1}{2}\left[\begin{matrix}q^2x,q/x\\ q,q^2\end{matrix}\Big|\,q^3\right]_n.\end{aligned}$$

Performing the replacement $a\to q^{3x}$ in Theorem 1 and then letting $q\to 1$, we obtain the following equation.

Proposition 3. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}3x, & 1-\ell-3x, & -n\\ & \frac{1}{2}, & -3n\end{matrix}\Big|\,\frac{3}{4}\right]=\left[\begin{matrix}3x\\ 6x\end{matrix}\right]_\ell\\ &=\sum_{i=0}^{\ell}(-1)^i\frac{6x+2i-1}{6x-1}\left[\begin{matrix}-\ell,6x-1\\ 1,6x+\ell\end{matrix}\right]_i\left[\begin{matrix}\frac{1+i}{3}+x, & \frac{2-i}{3}-x\\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\right]_n.\end{aligned}$$

When $\ell=0$, Proposition 3 reduces to (1). Another concrete formula is expressed as follows.

Corollary 4 ($\ell=1$ in Proposition 3)**.**

$${}_3F_2\left[\begin{matrix}3x, & -3x, & -n\\ & \frac{1}{2}, & -3n\end{matrix}\Big|\,\frac{3}{4}\right]=\frac{1}{2}\left[\begin{matrix}\frac{1}{3}+x, & \frac{2}{3}-x\\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\right]_n+\frac{1}{2}\left[\begin{matrix}\frac{2}{3}+x, & \frac{1}{3}-x\\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\right]_n.$$

Theorem 5. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{2-\ell}/x\\ q,-q,q^{3/2},-q^{3/2-\ell},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k=\left[\begin{matrix}xq^{-1},-xq^{-1/2}\\ x^2q^{-1},-q^{-1/2}\end{matrix}\Big|\,q\right]_\ell\\ &\times\sum_{i=0}^{\ell}(-1)^i q^{(\ell-\frac{1}{2})i}\frac{1-x^2q^{2i-2}}{1-x^2q^{-2}}\left[\begin{matrix}q^{-\ell},x,x^2q^{-2}\\ q,xq^{-1},x^2q^{\ell-1}\end{matrix}\Big|\,q\right]_i\left[\begin{matrix}q^{2+i}x,q^{4-i}/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n.\end{aligned}$$

Proof. Taking $a=x^2q^{-2}$, $b=xq^k$ and $c=-xq^{-1/2}$ in (5), we gain the equation:

$$\begin{aligned}&\left[\begin{matrix}q^{2-\ell}/x,-q^{3/2-\ell}/x\\ q^{2-\ell}/x^2,-q^{3/2-\ell+k}\end{matrix}\Big|\,q\right]_\ell\sum_{i=0}^{\ell}(-1)^i q^{(\ell-\frac{1}{2})i}\frac{1-x^2q^{2i-2}}{1-x^2q^{-2}}\left[\begin{matrix}q^{-\ell},xq^k,x^2q^{-2}\\ q,xq^{-1},x^2q^{\ell-1}\end{matrix}\Big|\,q\right]_i\\ &\times\left[\begin{matrix}q^{2-\ell+k}/x\\ q^{2-\ell}/x\end{matrix}\Big|\,q\right]_{\ell-i}=1.\end{aligned}$$

Then we can proceed as follows:

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix}x,q^{2-\ell}/x\\q,-q,q^{3/2},-q^{3/2-\ell},q^{-1-3n}\end{bmatrix}_k q^k\\&=\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix}x,q^{2-\ell}/x\\q,-q,q^{3/2},-q^{3/2-\ell},q^{-1-3n}\end{bmatrix}_k q^k\begin{bmatrix}q^{2-\ell}/x,-q^{3/2-\ell}/x\\q^{2-\ell}/x^2,-q^{3/2-\ell+k}\end{bmatrix}_\ell\\&\times\sum_{i=0}^{\ell}(-1)^iq^{(\ell-\frac{1}{2})i}\frac{1-x^2q^{2i-2}}{1-x^2q^{-2}}\begin{bmatrix}q^{-\ell},xq^k,x^2q^{-2}\\q,xq^{-1},x^2q^{\ell-1}\end{bmatrix}_i\begin{bmatrix}q^{2-\ell+k}/x\\q^{2-\ell}/x\end{bmatrix}_{\ell-i}.\end{aligned}$$

Interchange the summation order for the last double sum to achieve

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix}x,q^{2-\ell}/x\\q,-q,q^{3/2},-q^{3/2-\ell},q^{-1-3n}\end{bmatrix}_k q^k\\&=\begin{bmatrix}q^{2-\ell}/x,-q^{3/2-\ell}/x\\q^{2-\ell}/x^2,-q^{3/2-\ell}\end{bmatrix}_\ell\sum_{i=0}^{\ell}(-1)^iq^{(\ell-\frac{1}{2})i}\frac{1-x^2q^{2i-2}}{1-x^2q^{-2}}\begin{bmatrix}q^{-\ell},x,x^2q^{-2}\\q,xq^{-1},x^2q^{\ell-1}\end{bmatrix}_i\\&\times\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix}xq^i,q^{2-i}/x\\q,-q,q^{3/2},-q^{3/2},q^{-1-3n}\end{bmatrix}_k q^k.\end{aligned}$$

Evaluating the series on the last line by (4), we attain Theorem 5 to finish the proof. □

When $\ell=0$, Theorem 5 reduces to (4). Another concrete formula is displayed as follows.

Corollary 6 ($\ell=1$ in Theorem 5)**.**

$$\begin{aligned}&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix}x,q/x\\q,-q,q^{3/2},-q^{1/2},q^{-1-3n}\end{bmatrix}_k q^k\\&=\frac{1-x}{(1+q^{1/2})(1-xq^{-1/2})}\begin{bmatrix}q^3x,q^3/x\\q^2,q^4\end{bmatrix}_n\\&+\frac{1-xq^{-1}}{(1+q^{-1/2})(1-xq^{-1/2})}\begin{bmatrix}q^2x,q^4/x\\q^2,q^4\end{bmatrix}_n.\end{aligned}$$

Employing the substitution $a\to q^{3x}$ in Theorem 5 and then letting $q\to1$, we obtain the following equation.

Proposition 7. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\begin{aligned}&{}_3F_2\begin{bmatrix}3x, & 2-\ell-3x, & -n\\ & \frac{3}{2}, & -1-3n\end{bmatrix}\frac{3}{4}\end{bmatrix}=\begin{bmatrix}3x-1\\6x-1\end{bmatrix}_\ell\\&\times\sum_{i=0}^{\ell}(-1)^i\Big(\frac{3x+i-1}{3x-1}\Big)^2\begin{bmatrix}-\ell,6x-2\\1,6x+\ell-1\end{bmatrix}_i\begin{bmatrix}\frac{2+i}{3}+x, & \frac{4-i}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{bmatrix}_n.\end{aligned}$$

When $\ell=0$, Proposition 7 reduces to (2). Another concrete formula is expressed as follows.

Corollary 8 ($\ell=1$ in Proposition 7)**.**

$$\begin{aligned}&3F_2\begin{bmatrix}3x, & 1-3x, & -n\\ & \frac{3}{2}, & -1-3n\end{bmatrix}\frac{3}{4}\end{bmatrix}\\&=\frac{3x}{6x-1}\begin{bmatrix}1+x, & 1-x\\ \frac{2}{3}, & \frac{4}{3}\end{bmatrix}_n+\frac{3x-1}{6x-1}\begin{bmatrix}\frac{2}{3}+x, & \frac{4}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{bmatrix}_n.\end{aligned}$$

Proposition 9. *For a nonnegative integer ℓ and a complex number x, there holds*

$$ {}_3F_2\left[\begin{matrix}3x, & 2-\ell-3x, & -n\\ & \frac{3}{2}-\ell, & -1-3n\end{matrix}\ \Big|\ \frac{3}{4}\right]=\left[\begin{matrix}3x-1,3x-\frac{1}{2}\\ 6x-1,-\frac{1}{2}\end{matrix}\right]_\ell \times\sum_{i=0}^{\ell}\left(\frac{3x+i-1}{3x-1}\right)^2\left[\begin{matrix}-\ell,6x-2\\ 1,6x+\ell-1\end{matrix}\right]_i\left[\begin{matrix}\frac{2+i}{3}+x, & \frac{4-i}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n. $$

Proof. Performing the replacement $q\to q^2$ in Theorem 5, we have

$$ \sum_{k=0}^{n}(q^{-6n};q^6)_k\left[\begin{matrix}x,q^{4-2\ell}/x\\ q^2,-q^2,q^3,-q^{3-2\ell},q^{-2-6n}\end{matrix}\ \Big|\ q^2\right]_k q^{2k}=\left[\begin{matrix}xq^{-2},-xq^{-1}\\ x^2q^{-2},-q^{-1}\end{matrix}\ \Big|\ q^2\right]_\ell \times\sum_{i=0}^{\ell}(-1)^i q^{(2\ell-1)i}\frac{1-x^2q^{4i-4}}{1-x^2q^{-4}}\left[\begin{matrix}q^{-2\ell},x,x^2q^{-4}\\ q^2,xq^{-2},x^2q^{2\ell-2}\end{matrix}\ \Big|\ q^2\right]_i\left[\begin{matrix}q^{4+2i}x,q^{8-2i}/x\\ q^4,q^8\end{matrix}\ \Big|\ q^6\right]_n. $$

Replace q by $-q^{1/2}$ to gain

$$ \sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{2-\ell}/x\\ q,-q,q^{3/2-\ell},-q^{3/2},q^{-1-3n}\end{matrix}\ \Big|\ q\right]_k q^{k}=\left[\begin{matrix}xq^{-1},xq^{-1/2}\\ x^2q^{-1},q^{-1/2}\end{matrix}\ \Big|\ q\right]_\ell \times\sum_{i=0}^{\ell}q^{(\ell-\frac{1}{2})i}\frac{1-x^2q^{2i-2}}{1-x^2q^{-2}}\left[\begin{matrix}q^{-\ell},x,x^2q^{-2}\\ q,xq^{-1},x^2q^{\ell-1}\end{matrix}\ \Big|\ q\right]_i\left[\begin{matrix}q^{2+i}x,q^{4-i}/x\\ q^2,q^4\end{matrix}\ \Big|\ q^3\right]_n. $$

Employing the substitution $a\to q^{3x}$ in the last equation and then letting $q\to 1$, we acquire Proposition 9. □

When $\ell=0$, Proposition 9 also reduces to (2). Another concrete formula is expressed as follows.

Corollary 10 ($\ell=1$ in Proposition 9)**.**

$$ {}_3F_2\left[\begin{matrix}3x, & 1-3x, & -n\\ & \frac{1}{2}, & -1-3n\end{matrix}\ \Big|\ \frac{3}{4}\right] = 3x\left[\begin{matrix}1+x, & 1-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n+(1-3x)\left[\begin{matrix}\frac{2}{3}+x, & \frac{4}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n. $$

3. Several related results

Theorem 11. *For a nonnegative integer ℓ and a complex number x, there holds*

$$ \sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{-\ell}/x\\ q,-1,q^{1/2-\ell},-q^{1/2},q^{-3n}\end{matrix}\ \Big|\ q\right]_k q^{k} = \left[\begin{matrix}qx,q^{1/2}x\\ qx^2,q^{1/2}\end{matrix}\ \Big|\ q\right]_\ell\sum_{i=0}^{\ell}q^{(\ell+\frac{1}{2})i}\frac{1+xq^i}{2(1+x)}\left[\begin{matrix}q^{-\ell},x^2\\ q,x^2q^{\ell+1}\end{matrix}\ \Big|\ q\right]_i \times\left\{\left[\begin{matrix}q^{1+i}x,q^{2-i}/x\\ q,q^2\end{matrix}\ \Big|\ q^3\right]_n+\left[\begin{matrix}q^{2+i}x,q^{1-i}/x\\ q,q^2\end{matrix}\ \Big|\ q^3\right]_n\right\}. $$

Proof. Setting $a=x^2$, $b=xq^k$ and $c=q^{1/2}x$ in (5), we get the equation:

$$ \left[\begin{matrix}q^{-\ell}/x,q^{1/2-\ell}/x\\ q^{-\ell}/x^2,q^{1/2-\ell+k}\end{matrix}\ \Big|\ q\right]_\ell\sum_{i=0}^{\ell}q^{(\ell+\frac{1}{2})i}\frac{1-x^2q^{2i}}{1-x^2}\left[\begin{matrix}q^{-\ell},xq^k,x^2\\ q,qx,x^2q^{\ell+1}\end{matrix}\ \Big|\ q\right]_i \times\left[\begin{matrix}q^{k-\ell}/x\\ q^{-\ell}/x\end{matrix}\ \Big|\ q\right]_{\ell-i}=1. $$

Then there is the following relation:

$$\begin{aligned}
&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix} x,q^{-\ell}/x \\ q,-1,q^{1/2-\ell},-q^{1/2},q^{-3n} \end{bmatrix} q\Bigg]_k q^k \\
&= \sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix} x,q^{-\ell}/x \\ q,-1,q^{1/2-\ell},-q^{1/2},q^{-3n} \end{bmatrix} q\Bigg]_k q^k \begin{bmatrix} q^{-\ell}/x,q^{1/2-\ell}/x \\ q^{-\ell}/x^2,q^{1/2-\ell+k} \end{bmatrix} q\Bigg]_\ell \\
&\times \sum_{i=0}^{\ell} q^{(\ell+\frac{1}{2})i}\frac{1-x^2q^{2i}}{1-x^2}\begin{bmatrix} q^{-\ell},xq^k,x^2 \\ q,qx,x^2q^{\ell+1} \end{bmatrix} q\Bigg]_i \begin{bmatrix} q^{k-\ell}/x \\ q^{-\ell}/x \end{bmatrix} q\Bigg]_{\ell-i}.
\end{aligned}$$

Interchange the summation order for the last double sum to achieve

$$\begin{aligned}
&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix} x,q^{-\ell}/x \\ q,-1,q^{1/2-\ell},-q^{1/2},q^{-3n} \end{bmatrix} q\Bigg]_k q^k \\
&= \begin{bmatrix} q^{-\ell}/x,q^{1/2-\ell}/x \\ q^{-\ell}/x^2,q^{1/2-\ell} \end{bmatrix} q\Bigg]_\ell \sum_{i=0}^{\ell} q^{(\ell+\frac{1}{2})i}\frac{1+xq^i}{1+x}\begin{bmatrix} q^{-\ell},x^2 \\ q,x^2q^{\ell+1} \end{bmatrix} q\Bigg]_i \\
&\times \sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix} xq^i,q^{-i}/x \\ q,-1,q^{1/2},-q^{1/2},q^{-3n} \end{bmatrix} q\Bigg]_k q^k.
\end{aligned}$$

Calculating the series on the last line by Corollary 2, we attain Theorem 11 to complete the proof. □

Corollary 12 ($\ell = 1$ in Theorem 11)**.**

$$\begin{aligned}
&\sum_{k=0}^{n}(q^{-3n};q^3)_k\begin{bmatrix} x,1/qx \\ q,-1,q^{-1/2},-q^{1/2},q^{-3n} \end{bmatrix} q\Bigg]_k q^k \\
&= \frac{1-qx}{2(1-q^{1/2})(1+xq^{1/2})}\begin{bmatrix} qx,q^2/x \\ q,q^2 \end{bmatrix} q^3\Bigg]_n \\
&+ \frac{1}{2}\begin{bmatrix} q^2x,q/x \\ q,q^2 \end{bmatrix} q^3\Bigg]_n - \frac{q^{1/2}(1-x)}{2(1-q^{1/2})(1+xq^{1/2})}\begin{bmatrix} q^3x,1/x \\ q,q^2 \end{bmatrix} q^3\Bigg]_n.
\end{aligned}$$

Performing the replacement $a \to q^{3x}$ in Theorem 11 and then letting $q \to 1$, we obtain the following equation.

Proposition 13. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\begin{aligned}
&{}_3F_2\begin{bmatrix} 3x, & -\ell-3x, & -n \\ & \frac{1}{2}-\ell, & -3n \end{bmatrix} \frac{3}{4}\Bigg] = \frac{1}{2}\begin{bmatrix} 1+3x,\frac{1}{2}+3x \\ 1+6x,\frac{1}{2} \end{bmatrix}_\ell \\
&\times \sum_{i=0}^{\ell}\begin{bmatrix} -\ell,6x \\ 1,1+6x+\ell \end{bmatrix}_i \left\{\begin{bmatrix} \frac{1+i}{3}+x, & \frac{2-i}{3}-x \\ \frac{1}{3}, & \frac{2}{3} \end{bmatrix}_n + \begin{bmatrix} \frac{2+i}{3}+x, & \frac{1-i}{3}-x \\ \frac{1}{3}, & \frac{2}{3} \end{bmatrix}_n\right\}.
\end{aligned}$$

Corollary 14 ($\ell = 1$ in Proposition 13)**.**

$$\begin{aligned}
&{}_3F_2\begin{bmatrix} 3x, & -1-3x, & -n \\ & -\frac{1}{2}, & -3n \end{bmatrix} \frac{3}{4}\Bigg] \\
&= \frac{1+3x}{2}\begin{bmatrix} \frac{1}{3}+x, & \frac{2}{3}-x \\ \frac{1}{3}, & \frac{2}{3} \end{bmatrix}_n + \frac{1}{2}\begin{bmatrix} \frac{2}{3}+x, & \frac{1}{3}-x \\ \frac{1}{3}, & \frac{2}{3} \end{bmatrix}_n - \frac{3x}{2}\begin{bmatrix} 1+x, & -x \\ \frac{1}{3}, & \frac{2}{3} \end{bmatrix}_n.
\end{aligned}$$

Theorem 15. *For a nonnegative integer ℓ and a complex number x, there holds*

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{1-\ell}/x\\ q,-q^{1-\ell},q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k$$
$$=\left[\begin{matrix}x,-x\\ x^2,-1\end{matrix}\Big|\,q\right]_\ell\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{(1-x^2q^{-1})(1-q^{1/2})}\left[\begin{matrix}q^{-\ell},x^2q^{-1}\\ q,x^2q^{\ell}\end{matrix}\Big|\,q\right]_i$$
$$\times\left\{\frac{1-xq^i}{1+xq^{i-1/2}}\left[\begin{matrix}q^{3+i}x,q^{3-i}/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n-\frac{q^{1/2}(1-xq^{i-1})}{1+xq^{i-1/2}}\left[\begin{matrix}q^{2+i}x,q^{4-i}/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n\right\}.$$

Proof. In accordance with (6), we gain the following relation:

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{1-\ell}/x\\ q,-q^{1-\ell},q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k$$
$$=\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{1-\ell}/x\\ q,-q^{1-\ell},q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k\left[\begin{matrix}q^{1-\ell}/x,-q^{1-\ell}/x\\ q^{1-\ell}/x^2,-q^{1-\ell+k}\end{matrix}\Big|\,q\right]_\ell$$
$$\times\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix}q^{-\ell},xq^k,x^2q^{-1}\\ q,x,x^2q^{\ell}\end{matrix}\Big|\,q\right]_i\left[\begin{matrix}q^{1-\ell+k}/x\\ q^{1-\ell}/x\end{matrix}\Big|\,q\right]_{\ell-i}.$$

Interchange the summation order for the last double sum to achieve

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q^{1-\ell}/x\\ q,-q^{1-\ell},q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k$$
$$=\left[\begin{matrix}q^{1-\ell}/x,-q^{1-\ell}/x\\ q^{1-\ell}/x^2,-q^{1-\ell}\end{matrix}\Big|\,q\right]_\ell\sum_{i=0}^{\ell}(-1)^i q^{\ell i}\frac{1-x^2q^{2i-1}}{1-x^2q^{-1}}\left[\begin{matrix}q^{-\ell},x^2q^{-1}\\ q,x^2q^{\ell}\end{matrix}\Big|\,q\right]_i$$
$$\times\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}xq^i,q^{1-i}/x\\ q,-q,q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k.$$

Evaluating the series on the last line by the equivalent form of Corollary 6:

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,q/x\\ q,-q,q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k$$
$$=\frac{1-x}{(1-q^{1/2})(1+xq^{-1/2})}\left[\begin{matrix}q^3x,q^3/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n$$
$$+\frac{1-xq^{-1}}{(1-q^{-1/2})(1+xq^{-1/2})}\left[\begin{matrix}q^2x,q^4/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n,$$

we attain Theorem 15 to finish the proof. □

Corollary 16 ($\ell=1$ in Theorem 15)**.**

$$\sum_{k=0}^{n}(q^{-3n};q^3)_k\left[\begin{matrix}x,1/x\\ q,-1,q^{1/2},-q^{3/2},q^{-1-3n}\end{matrix}\Big|\,q\right]_k q^k$$
$$=\frac{1-qx}{2(1-q^{1/2})(1+xq^{1/2})}\left[\begin{matrix}q^4x,q^2/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n$$
$$+\frac{(1-x)^2}{2(1+xq^{-1/2})(1+xq^{1/2})}\left[\begin{matrix}q^3x,q^3/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n$$
$$-\frac{q^{1/2}(1-xq^{-1})}{2(1-q^{1/2})(1+xq^{-1/2})}\left[\begin{matrix}q^2x,q^4/x\\ q^2,q^4\end{matrix}\Big|\,q^3\right]_n.$$

Employing the substitution $a\to q^{3x}$ in Theorem 15 and then letting $q\to1$, we obtain the following equation.

Proposition 17. *For a nonnegative integer ℓ and a complex number x, there holds*

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}3x, & 1-\ell-3x, & -n\\ & \frac{1}{2}, & -1-3n\end{matrix}\,\Big|\,\frac{3}{4}\right]\\
&= \left[\begin{matrix}3x\\6x\end{matrix}\right]_\ell \sum_{i=0}^{\ell}(-1)^i\frac{6x+2i-1}{6x-1}\left[\begin{matrix}-\ell, 6x-1\\1, 6x+\ell\end{matrix}\right]_i\\
&\times\left\{(3x+i)\left[\begin{matrix}\frac{3+i}{3}+x, & \frac{3-i}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n-(3x+i-1)\left[\begin{matrix}\frac{2+i}{3}+x, & \frac{4-i}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n\right\}.
\end{aligned}
$$

Corollary 18 ($\ell=1$ in Proposition 13)**.**

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}3x, & -3x, & -n\\ & \frac{1}{2}, & -1-3n\end{matrix}\,\Big|\,\frac{3}{4}\right]\\
&= \frac{1+3x}{2}\left[\begin{matrix}\frac{4}{3}+x, & \frac{2}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n+\frac{1-3x}{2}\left[\begin{matrix}\frac{2}{3}+x, & \frac{4}{3}-x\\ \frac{2}{3}, & \frac{4}{3}\end{matrix}\right]_n.
\end{aligned}
$$

With the change of the parameter ℓ, these theorems and propositions can create more concrete formulas. They will not be laid out in this paper.

The case $n\to\infty$ of (2) reads as

$$
{}_2F_1\left[\begin{matrix}3x, & 2-3x\\ & \frac{3}{2}\end{matrix}\,\Big|\,\frac{1}{4}\right]=\frac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2/3+x)\Gamma(4/3-x)}.
$$

Letting $x=1/3$ in this equation, we recover the beautiful series for π (cf. [8, Equation (27)]):

$$
\sum_{k=0}^{\infty}\frac{k!}{(3/2)_k}\frac{1}{4^k}=\frac{2\pi}{3\sqrt{3}}.
$$

Unfortunately, we don't find new series for π from the family of summation formulas for ${}_3F_2(\frac{3}{4})$-series.

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