# TWO FLOOR BUILDING NEEDING EIGHT COLORS

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ABSTRACT. Motivated by frequency assignment in office blocks, we study the chromatic number of the adjacency graph of 3-dimensional parallelepiped arrangements. In the case each parallelepiped is within one floor, a direct application of the Four-Colour Theorem yields that the adjacency graph has chromatic number at most 8. We provide an example of such an arrangement needing exactly 8 colours. We also discuss bounds on the chromatic number of the adjacency graph of general arrangements of 3-dimensional parallelepipeds according to geometrical measures of the parallelepipeds (side length, total surface or volume).

## 1. INTRODUCTION

The Graph Colouring Problem for Office Blocks was raised by BAE Systems at the 53rd European Study Group with Industry in 2005 [1]. Consider an office complex with space rented by several independent organisations. It is likely that each organisation uses its own wireless network (WLAN) and ask for a safe utilisation of it. A practical way to deal with this issue is to use a so-called "stealthy wallpaper" in the walls and ceilings shared between different organisations, which would attenuate the relevant frequencies. Yet, the degree of screening produced will not be sufficient if two distinct organisations have adjacent offices, that is, two offices in face-to-face contact on opposite sides of just one wall or floor-ceiling. In this case, the WLANs of the two organisations need to be using two different channels (the reader is referred to the report by Allwright *et al.* [1] for the precise technical motivations).

This problem can be modeled as a graph coloring problem by building a *conflict graph* corresponding to the office complex: to each organisation corresponds a vertex, and two vertices are adjacent if the corresponding territories share a wall, floor, or ceiling area. The goal is to assign a color (frequency) to each vertex (organisation) such that adjacent vertices are assigned distinct colors. In addition, not every graph may occur as the conflict graph of an existing office complex. However, the structure of such conflict graphs is not clear and various fundamental questions related to the problem at hands were asked. Arguably, one of the most natural questions concerns the existence of bounds on the chromatic number of such conflict graphs. More specifically, which additional constraints one should add to the model to ensure "good" upper bounds on the chromatic number of conflict graphs? These additional constraints should be meaningful regarding the practical problem, reflecting as much as possible real-world situations. Indeed, as noted by Tietze [7], complete graphs of arbitrary size are conflict graphs, that is, for every integer $n$, there can be $n$ organisations whose territories all are in face-to-face contact with

each other. The reader is referred to the paper by Reed and Allwright [6] for a description of Tietze's construction. Besicovitch [4] and Tietze [8] proved that this is still the case if the territories are asked to be convex polyhedra.

An interesting condition is when the territories are required to be *rectangular parallelepipeds* (sometimes called *cuboids*), that is, a 3-dimensional solid figure bounded by six rectangles aligned with a fixed set of Cartesian axes. For convenience, we shall call *box* a rectangular parallelepiped. When all territories are boxes, the *clique number* of any conflict graph, that is, the maximum size of a complete subgraph, is at most 4. However, Reed and Allwright [6] and also later Magnant and Martin [5] designed arrangements of boxes that yield conflict graphs requiring an arbitrarily high number of colours.

On the other hand, if the building is assumed to have floors (in the usual way) and each box is 1-*floor*, i.e. restricted to be within one floor, then the chromatic number is bounded by 8: on each floor, the obtained conflict graph is planar and hence can be coloured using 4 colours [2, 3]. It is natural to ask whether this bound is tight. As noted during working sessions in Oxford (see the acknowledgments), it can be shown that up to 6 colours can be needed, by using an arrangement of boxes spanning three floors. Such a construction is shown in Figure 1.
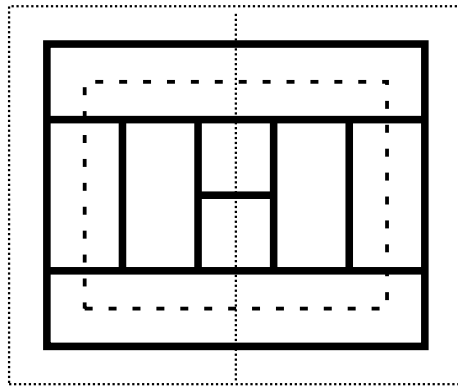


FIGURE 1. An arrangement of 1-floor boxes spanning three floors and requiring six colours. The solid, dotted, and dashed lines indicate the middle, top, and bottom floors, respectively.

The purpose of this note is to show that the upper bound is actually tight. More precisely, we shall build an arrangement of 1-floor boxes that spans two floors and yields a conflict graph requiring 8 colours. From now on, we shall identify a box arrangement with its conflict graph for convenience. In particular, we assign colors directly to the boxes and define the *chromatic number of an arrangement* as that of the associated conflict graph.

**Theorem 1.** *There exists an arrangement of* 1-*floor boxes spanning two floors with chromatic number* 8.

The boxes considered in Theorem 1 have one of their geometrical measures bounded: their height is at most one floor. We also discuss bounds on the chromatic number of box arrangements with respect to some other geometrical measures: the

side lengths, the surface area and the volume. More precisely, assuming that boxes have integer coordinates, we obtain the following.

**Theorem 2.** *We consider a box arrangement $A$ with integer coordinates.*

(1) *If there exists one fixed dimension such that every box in $A$ has length at most $\ell$ in this dimension, then $A$ has chromatic number at most $4(\ell + 1)$.*

(2) *If for each box, there is one dimension such that the length of this box in this dimension if at most $\ell$, then $A$ has chromatic number at most $12(\ell+1)$.*

(3) *If the total surface area of each box in $A$ is at most $s$, then $A$ has chromatic number at most $9\sqrt[3]{4s} + 13$.*

(4) *If the volume of each box in $A$ is at most $v$, then $A$ has chromatic number at most $24\sqrt[4]{6v} + 13$.*

In the next section, we give the proof of Theorem 1 and in the last section we indicate how to obtain the bounds given in Theorem 2.

## 2. PROOF OF THEOREM 1

We shall construct an arrangement of 1-floor boxes that is not 7-colorable. Before that, we need the following definition. Consider an arrangement $A$ and let $A_1$, $A_2$ and $A_3$ be (non-necessarily disjoint) subsets of the boxes in $A$. Given a proper coloring $c$ of $A$, let $C_i$ be the set of colors used for the boxes in $A_i$, for $i \in \{1, 2, 3\}$. The *signature $\sigma_A(c)$ of $A$ with respect to $c$* is defined to be $a_x b_y c$, where $a := |C_1|$, $x := |C_1 \cup C_2|$, $b := |C_2|$, $y := |C_2 \cup C_3|$, and $c := |C_3|$. The collection of all signatures can be endowed with a partial order: if $s = a_x b_y c$ and $s' = a'_{x'} b'_{y'} c'$ are two signatures, then $s \leqslant s'$ if $a \leqslant a'$, $x \leqslant x'$, $b \leqslant b'$, $y \leqslant y'$ and $c \leqslant c'$.

To build the desired arrangement, we use the arrangement $X$ of 1-floor boxes described in Figure 2 as a building brick. The arrangement $X$ has three specific *regions*, $X_1$, $X_2$ and $X_3$. We also abusively write $X_1$, $X_2$ and $X_3$ to mean the subsets of boxes of $X$ respectively intersecting the regions $X_1$, $X_2$ and $X_3$ (note that some boxes may belong to several subsets). We start by giving some properties of the signatures of $X$ with respect to proper colorings and according to the three subsets $X_1$, $X_2$ and $X_3$.
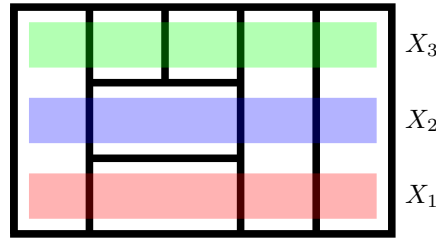


FIGURE 2. The gadget $X$ with the regions $X_1$, $X_2$ and $X_3$.

**Assertion 3.** *For every proper coloring $c$ of $X$,*

(1) $\sigma_X(c) \geqslant 3_3 2_4 4$,

(2) $\sigma_X(c) \geqslant 3_4 3_3 2$, *or*

(3) $\sigma_X(c) \geqslant 2_3 3_4 3$.

The proof of this assertion does not need any insight, we thus omit it. However, the reader interested in checking its accuracy should first note that one can restrict to the cases where the three vertical (in Figure 2) boxes are respectively colored either 1, 2 and 1; or 1, 1 and 2; or 1, 2 and 3.
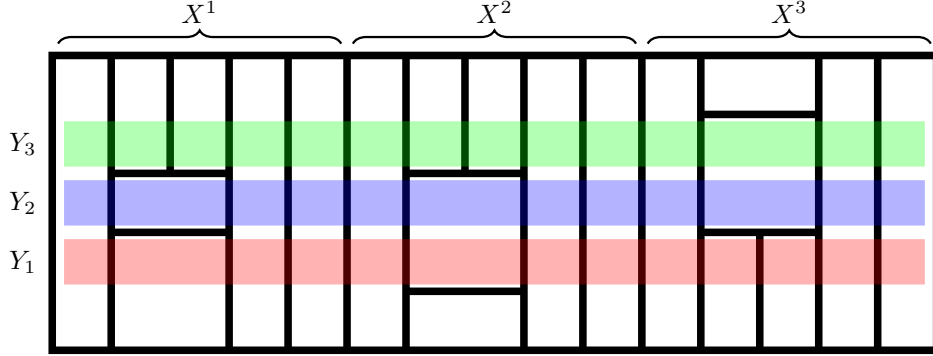


FIGURE 3. The gadget $Y$ with the regions $Y_1$, $Y_2$ and $Y_3$.

Now, the arrangement $Y$ is obtained from three copies $X^1$, $X^2$ and $X^3$ of the arrangement $X$. We define three regions $Y_1$, $Y_2$ and $Y_3$ on $Y$ as depicted in Figure 3. As previously, we also write $Y_1$, $Y_2$ and $Y_3$ for the subsets of boxes intersecting the region $Y_1$, $Y_2$ and $Y_3$, respectively. We set $X_i^j := Y_i \cap X^j$ for $(i, j) \in \{1, 2, 3\}^2$.

**Assertion 4.** *In any proper coloring of $Y$, at least four colors are used in one of the three regions.*

*Proof.* Suppose on the contrary that there is a proper coloring $c$ of $Y$ with at most three colors in each of $Y_1$, $Y_2$ and $Y_3$. For $i \in \{1, 2, 3\}$, the restriction of $c$ to $X^i$ is a proper coloring of $X^i$, which we identify to $c$. The condition on $c$ implies that none of $\sigma_{X^1}(c)$, $\sigma_{X^2}(c)$ and $\sigma_{X^3}(c)$ fulfills inequality (1) of Assertion 3. In particular, note that exactly 3 different colours appear on $X_2^1$, and they also appear on $X_2^2$ and on $X_2^3$. Since $X_1^2 = X_2^2$, these three colours appear on $Y_1$. Similarly, since $X_3^3 = X_2^3$, these three colours appear on $Y_3$.

Assume now that $\sigma_{X^1}(c)$ satisfies (2). Then exactly three colours appear on $X_1^1$, one of which does not appear on $X_2^1$ as $\left| c(X_1^1 \cup X_2^1) \right| \geqslant 4$. Thus in total at least four colours appear on $X_1^1 \cup X_1^2 \subset Y_1$, which contradicts our assumption on $c$. It remains to deal with the case where $\sigma_{X^1}(c)$ fulfills (3) of Assertion 3. Thanks to the symmetry of (2) and (3) with respect to the regions $X_1$ and $X_3$, the same reasoning as above applied to $X_3^3$ instead of $X_1^2$ yields that four colours appear on $Y_3$, a contradiction. $\qquad\square$

To finish the construction, we need the following definition. Consider two copies $Y^1$ and $Y^2$ of $Y$. The regions $Y_i^1$ and $Y_j^2$ *fully overlap* if every box in $Y_i^1$ is in face-to-face contact with every box in $Y_j^2$. Observe that for every pair $(i, j) \in \{1, 2, 3\}$, there exists a 2-floor arrangement of $Y^1$ and $Y^2$ such that $Y_i^1$ and $Y_j^2$ fully overlap: it is obtained by rotating $Y^2$ ninety degrees, adequately scaling it (i.e. stretching it horizontally) and placing it on top of $Y^1$.
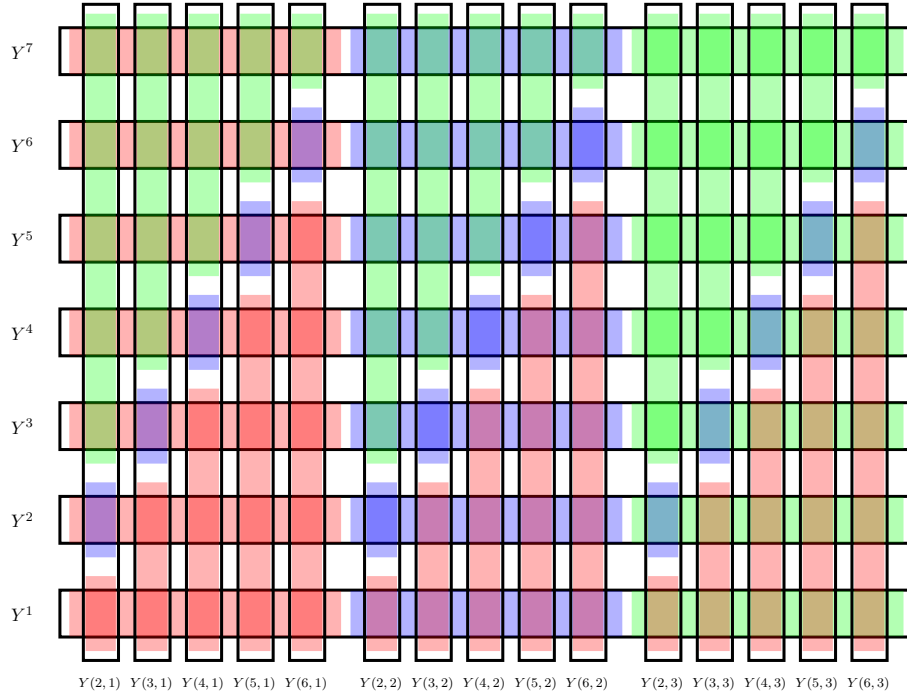
FIGURE 4. Schematic view of the arrangement $Z$.

We are now in a position to build the desired arrangement $Z$ spanning two floors. To this end, we use several copies of $Y$. The first floor of $Z$ is composed of seven parallel copies $Y^1, \ldots, Y^7$ of $Y$ (drawn horizontally in Figure 4). The second floor of $Z$ is composed of fifteen parallel copies of $Y$ (drawn vertically in Figure 4): for each $j \in \{1, 2, 3\}$ and each $i \in \{2, \ldots, 6\}$, a copy $Y(i, j)$ of $Y$ is placed such that the first region of $Y(i, j)$ fully overlap the regions $Y_j^1, \ldots, Y_j^{i-1}$, the second region of $Y(i, j)$ fully overlaps the region $Y_j^i$, and the third region of $Y(i, j)$ fully overlaps the regions $Y_j^{i+1}, \ldots, Y_j^7$.

Consider a proper coloring of $Z$. Assertion 4 ensures that each copy of $Y$ in $Z$ has a region for which at least four different colours are used. In particular, there exists $j \in \{1, 2, 3\}$ such that three regions among $Y_j^1, \ldots, Y_j^7$ are colored using four colours. Let these regions be $Y_j^{i_1}$, $Y_j^{i_2}$ and $Y_j^{i_3}$ with $1 \leqslant i_1 < i_2 < i_3 \leqslant 7$. Now, consider the arrangement $Y(i_2, j)$. By Assertion 4, there exists $k \in \{1, 2, 3\}$ such that the $k$-th region of $Y(i_2, j)$ is also colored using at least four different colors. Consequently, as this region and the region $Y_j^{i_k}$ fully overlap, they are colored using at least eight different colors. This concludes the proof.

## 3. BOUNDS WITH RESPECT TO GEOMETRICAL MEASURES

In this part, we provide bounds on the chromatic number of boxes arrangements provided that the boxes satisfy some geometrical constraints. Namely, we prove Theorem 2, which is recalled here for the reader's ease.

**Theorem 2.** *We consider a box arrangements $A$ with integer coordinates.*

(1) *If there exists one fixed dimension such that every box in $A$ has length at most $\ell$ in this dimension, then $A$ has chromatic number at most $4(\ell + 1)$.*

(2) *If for each box, there is one dimension such the length of this box in this dimension if at most $\ell$, then $A$ has chromatic number at most $12(\ell + 1)$.*

(3) *If the total surface area of each box in $A$ is at most $s$, then $A$ has chromatic number at most $9\sqrt[3]{4s} + 13$.*

(4) *If the volume of each box in $A$ is at most $v$, then $A$ has chromatic number at most $24\sqrt[4]{6v} + 13$.*

*Proof.*

(1) The conflict graph corresponding to an arrangement where the boxes have height at most $\ell$ can be vertex partitioned into $\ell + 1$ planar graphs $P_0, \dots, P_\ell$. Indeed if the distance between the levels of two boxes is at least $\ell + 1$, then these two boxes are not adjacent. So the planar graphs are obtained by assigning, for each $x$, all the boxes that have their floor at level $x$ to be in the graph $P_k$ where $k := x \mod (\ell+1)$. Consequently, the whole conflict graph has chromatic number at most $4(\ell + 1)$.

(2) The boxes can be partitioned into three sets according to the dimension in which the length is bounded. In other words, $A$ is partitioned into $U_1$, $U_2$ and $U_3$ such that for each $i \in \{1, 2, 3\}$, all boxes in $U_i$ have length at most $\ell$ in dimension $i$. Consequently, (1) ensures that each of $U_1$, $U_2$ and $U_3$ has chromatic number at most $4(\ell+1)$ and, therefore, $A$ has chromatic number at most $3 \cdot 4(\ell+1) = 12(\ell+1)$.

(3) For each box, the minimum length taken over all three dimensions is at most $\sqrt{s}$, and thus (2) implies that the chromatic number of $A$ is $O(\sqrt{s})$. However, one can be more careful. Let us fix a positive integer $\ell$, to be made precise later. The set of boxes is partitioned as follows. Let $U$ be the set of boxes with lengths in every dimension at least $\ell$ and let $R$ be the set all remaining boxes, that is, $R := A \setminus U$. By (2), the arrangement $R$ has chromatic number at most $12\ell$. Now consider a box $B$ in $U$ with dimensions $x$, $y$ and $z$, each being at least $\ell$. We shall give an upper bound on the number of boxes of $U$ that can be adjacent to $B$. The surface of a face of a box in $U$ is at least $\ell^2$. So in $U$ there are at most $s/\ell^2$ that have a face totally adjacent to a face of $B$. Some boxes of $U$ could also be adjacent to $B$ without having a face totally adjacent to a face of $B$. In this case, such a box is adjacent to an edge of $B$. For an edge of length $w$, there are at most $w/\ell + 1$ such boxes. So the number of boxes of $U$ adjacent to $B$ but having no face totally adjacent to a face of $B$ is at most $4(x + y + z)/\ell + 12$. Since $\ell \leqslant \min\{x, y, z\}$, we deduce that $2\ell(x+y+z) \leqslant 2xy+2yz+2xz \leqslant s$. Hence the total number of boxes in $U$ that are adjacent to $B$ is at most $s/\ell^2 + 2s/\ell^2 + 12 = 3s/\ell^2 + 12$. Consequently, by degeneracy, $U$ has chromatic number at most $3s/\ell^2 + 13$. Therefore, $A$ has chromatic number at most $12\ell + 3s/\ell^2 + 13$. Setting $\ell := \sqrt[3]{s/2}$ yields the upper bound $9\sqrt[3]{4.s} + 13$.

(4) Once again, for a fixed parameter $\ell$ to be made precise later, the set of boxes is partitioned into two parts: the part $U$, composed of all the boxes with lengths in every dimension at least $\ell$ and the part $R$, composed of all the remaining boxes. By (2), we know that $R$ has chromatic number at most $12\ell$. Let $B$ be a box in $U$ with dimensions $x$, $y$ and $z$. Since $\ell \leqslant \min\{x, y, z\}$, the volume $v_B$ of $B$ satisfies that $6v \geqslant 6v_B = 6xyz \geqslant 2(\ell xy + \ell xz + \ell yz) = \ell s_B$, where $s_B$ is the total surface area of $B$. So every box in $U$ has total surface area at most $6v/\ell$ and thus (3) implies that $U$ has chromatic number at most $9\sqrt[3]{4.6v/\ell} + 13$. Therefore, $A$ has

chromatic number at most $9\sqrt[3]{24v/\ell} + 12\ell + 13$. Setting $\ell$ to be $\sqrt[4]{3v/8}$ yields the upper bound $24\sqrt[4]{6v} + 13$. $\qquad\square$

In the previous theorem, we are mainly concerned with the order of magnitude of the functions of the different parameters. However, even in this context, we do not have any non trivial lower bound on the corresponding chromatic numbers.

## Acknowledgments.

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