

Exponential speed of uniform convergence of the cell density toward equilibrium for subcritical mass in a Patlak-Keller-Segel model

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Abstract

This paper is concerned with a chemotaxis aggregation model for cells, more precisely with a parabolic-elliptic semilinear Patlak-Keller-Segel system in a ball of \mathbb{R}^N for $N \geq 2$. For $N = 2$, this system is well known for its critical mass 8π . It has been proved in [24] that it also exhibits a critical mass phenomenon for $N \geq 3$. The main result of this paper is the exponential speed of uniform convergence of radial solutions toward the unique steady state in the subcritical case for $N \geq 2$. We stress that this covers in particular the classical Keller-Segel system with $N = 2$, and that the result improves on the known results even for this most studied problem. A key tool is an associated one-dimensional degenerate parabolic problem (PDE_m) where m is proportional to the total mass of cells. The proof exploits its formal gradient flow structure $u_t = -\nabla \mathcal{F}[u(t)]$ on an "infinite dimensional Riemannian manifold". In particular, we show a new Hardy type inequality, equivalent to the strict convexity of \mathcal{F} at any steady state of subcritical mass, which heuristically explains the exponential speed of convergence.

Contents

1	Introduction	2
1.1	Origin of the problem	2
1.1.1	Case of dimension $N \geq 3$	3
1.1.2	Case of dimension $N = 2$	4
1.2	Main result	5
1.3	A Hardy type inequality and exponential convergence in a L^2 weighted space	6
1.4	Heuristics	8
2	Preliminary results for dimension $N = 2$	12
2.1	Local wellposedness and regularity for problem (PDE_m)	13
2.2	Subcritical case : Lyapunov functional and convergence in $C^1([0, 1])$	14
3	A Hardy type inequality	14
4	Convergence with exponential speed in $L^2((0, 1), \frac{dx}{x^{2-q}})$	17
5	Convergence with exponential speed in $C^1([0, 1])$	20
6	Appendix : proofs of the preliminary results for dimension $N = 2$	25
6.1	Reminder on continuous dynamical systems and Lyapunov functionals	25
6.2	Wellposedness and regularity for problem (PDE_m)	26
6.3	Subcritical case : Lyapunov functional and convergence in $C^1([0, 1])$	31

1 Introduction

1.1 Origin of the problem

In this paper, we are interested in the speed of convergence toward steady states of solutions of the following problem, called (PDE_m) :

$$\boxed{u_t = x^{2-\frac{2}{N}} u_{xx} + u u_x^q} \quad t > 0 \quad 0 < x \leq 1 \quad (1)$$

$$u(t, 0) = 0 \quad t \geq 0 \quad (2)$$

$$u(t, 1) = m \quad t \geq 0 \quad (3)$$

$$u_x(t, x) \geq 0 \quad t > 0 \quad 0 \leq x \leq 1, \quad (4)$$

where $m \geq 0$, N is an integer greater or equal to 2 and q is the critical exponent, i.e.

$$\boxed{q = \frac{2}{N}}.$$

Note that this parabolic problem has degenerate diffusion since $x^{2-\frac{2}{N}}$ vanishes at $x = 0$ and that its nonlinearity involves the gradient and is moreover non Lipschitz when $N \geq 3$ since $0 < q < 1$.

Problem (PDE_m) arose for $N = 2$ in the articles [3] of P. Biler, G. Karch, P. Laurençot and T. Nadzieja and [20] of N. Kavallaris and P. Souplet and then in [23, 24] for $N \geq 3$ as a key tool in the study of radial solutions of the following chemotaxis system (PKS_q) , supposed to describe a collection of cells diffusing in the open unit ball $D \subset \mathbb{R}^N$ and emitting a chemical which attracts themselves :

$$\rho_t = \Delta \rho - \nabla[\rho^q \nabla c] \quad t > 0 \quad \text{on } D \quad (5)$$

$$-\Delta c = \rho \quad t > 0 \quad \text{on } D, \quad (6)$$

with the following boundary conditions :

$$\frac{\partial \rho}{\partial \nu} - \rho^q \frac{\partial c}{\partial \nu} = 0 \quad \text{on } \partial D \quad (7)$$

$$c = 0 \quad \text{on } \partial D, \quad (8)$$

where ρ is the cell density and c the chemoattractant concentration.

Note that this model relies on the following assumptions :

- Cells diffuse much more slowly than the chemoattractant.
- The cell flux \vec{F} due to the chemoattractant is here described by $\vec{F} = \chi \nabla c$ where

$$\chi(\rho) = \rho^q$$

is the sensitivity of cells to the chemoattractant.

- On the boundary ∂D , there is a no flux condition for ρ and a Dirichlet conditions for c .

This system (PKS_q) is a particular case of the Patlak-Keller-Segel model. To know more about the latter, the reader can refer to the original works [26] of C.S. Patlak and [21] of E.F. Keller and L.A. Segel. For a review on mathematics of chemotaxis, see the chapter written by M.A. Herrero in [14] and the article [16] of T. Hillen and K. J. Painter. For a review on the Patlak-Keller-Segel model, see both articles of D. Horstmann [17, 18].

We also would like to very briefly recall some important results for the case $N = 2$ and $q = 1$:
- It is known thanks to the works [15] of M.A. Herrero and J.L. Velazquez and [3] of P. Biler, G. Karch, P. Laurençot and T. Nadzieja that 8π is a critical mass for radial solutions in a ball.

- In the case of the whole plane \mathbb{R}^2 , this system has a similar behaviour. See [9] by J. Dolbeault and B. Perthame, [5] by A. Blanchet, J.A. Carrillo and N. Masmoudi, [4] by P. Biler, G. Karch, P. Laurençot and T. Nadzieja and [6] by A. Blanchet, J. Dolbeault and B. Perthame.
- For general solutions in a bounded domain of \mathbb{R}^2 , the results are slightly different since for a mass 4π blow-up at a point of the boundary of the domain can occur (see the book [29] of T. Suzuki).

We now want to recall what is essential to know about the relation between problems (PKS_q) and (PDE_m) (much more can be found in [24]):

- m is proportional to the total mass of cells $\int_B \rho$ which is a conserved quantity in time.
- The derivative of u is the quantity with physical interest since u_x is proportional to the cell density ρ , up to a rescaling in time and a change of variable.
More precisely, denoting $\rho(t, y) = \tilde{\rho}(t, |y|)$ for $t \geq 0$ and $y \in \overline{D}$, we have

$$\tilde{\rho}(t, x) = N^{\frac{2}{q}} u_x(N^2 t, x^N) \quad \text{for all } x \in [0, 1].$$

- The power $q = \frac{2}{N}$ is critical. Indeed, as a particular case of [19] by D. Horstmann and M. Winkler, we know that the solutions are global in time when $q < \frac{2}{N}$ and can blow up if $q > \frac{2}{N}$

From now on, we will only focus on problem (PDE_m) , which becomes our chemotaxis model. We will now list some facts that we have obtained in [23, 24] for $N \geq 3$ and will later establish some similar results that we need for the case $N = 2$.

1.1.1 Case of dimension $N \geq 3$

In [23], we have proved the existence of a unique maximal classical solution u of problem (PDE_m) with initial condition $u_0 \in Y_m$ and existence time $T_{max} = T_{max}(u_0) > 0$, where we denote

$$Y_m = \{u \in C([0; 1]), u \text{ nondecreasing}, u'(0) \text{ exists}, u(0) = 0, u(1) = m\}$$

and "classical" means here that

$$u \in C([0, T_{max}) \times [0, 1]) \cap C^1((0, T_{max}) \times [0, 1]) \cap C^{1,2}((0, T_{max}) \times (0, 1)).$$

Actually, we obtained more information about the regularity of the solutions and will refer to [23] when necessary.

In [24], we showed that the stationary solutions of (PDE_m) are the restrictions to $[0, 1]$ of a family of functions $(U_a)_{a \geq 0}$ on $[0, +\infty)$ with the following simple structure :

- $U_1 \in C^1([0, 1]) \cap C^2((0, 1])$, $U_1(0) = 0$, $U_1'(0) = 1$, U_1 is increasing on $[0, A]$ for some $A > 0$ and reaches its maximum M at $x = A$ after which U_1 is flat.
- All $(U_a)_{a \geq 0}$ are obtained by dilation of U_1 , i.e. $U_a(x) = U_1(ax)$ for all $x \geq 0$.

An easy consequence of this description is that

- If $0 \leq m < M$, then there exists a unique stationary solution. The latter is given by $U_a|_{[0,1]}$, where $a = a(m) \in [0, A]$ is uniquely determined by m .
- If $m = M$, there exists a continuum of steady states : $(U_a|_{[0,1]})_{a \geq A}$.
Note that the corresponding cell densities have their support strictly inside D when $a > A$.

- If $m > M$, there is no stationary solution.

We call M the critical mass of problem (PDE_m) , which is justified by the following result proved in [24, Theorems 1.2 and 1.3], valid for any $u_0 \in Y_m$:

- If $m \leq M$, then

$$T_{max}(u_0) = +\infty$$

and there exists $a \geq 0$ such that

$$u(t) \xrightarrow[t \rightarrow +\infty]{} U_a \quad \text{in} \quad C^1([0, 1]).$$

More precisely, $a = a(m) \in [0, A)$ if $0 \leq m < M$ and $a \geq A$ if $m = M$.

- If $m > M$, then

$$T_{max}(u_0) < \infty.$$

1.1.2 Case of dimension $N = 2$

For $N = 2$, there is also such a critical mass phenomenon, well studied, with critical mass $M = 2$ corresponding to 8π in the original Patlak-Keller-Segel model (PKS_1) (see [3, 15]).

Problem (PDE_m) then reads

$$\boxed{u_t = x u_{xx} + u u_x} \quad t > 0 \quad 0 < x \leq 1 \quad (9)$$

$$u(t, 0) = 0 \quad t \geq 0 \quad (10)$$

$$u(t, 1) = m \quad t \geq 0 \quad (11)$$

$$u_x(t, x) \geq 0 \quad t > 0 \quad 0 \leq x \leq 1, \quad (12)$$

where $m \geq 0$.

It is easy to see that its stationary solutions are all

$$(U_a|_{[0,1]})_{a \geq 0}$$

where

$$U_a(x) = U_1(ax)$$

and

$$U_1(x) = \frac{x}{1 + \frac{x}{2}}$$

for all $x \in [0, 1]$, $a \geq 0$.

The description of the set of steady states easily gives :

- If $m < 2$, there exists a unique classical steady state of problem (PDE_m) , namely $U_a|_{[0,1]}$ where

$$a = a(m) = \frac{m}{1 - \frac{m}{2}} \in [0, +\infty).$$

- If $m \geq 2$, there is no classical stationary solution of problem (PDE_m) but only a singular one $\bar{U} = m$ (singular in the sense that the boundary condition at $x = 0$ is lost).

Remark 1.1. *A deep difference with the case $N \geq 3$ is that the steady states here do not reach their upper bound 2 and that the critical value switches from the regular to the singular regime. Actually, for all $a > 0$, $\dot{U}_a > 0$ on $[0, 1]$. We will see in Theorem 2.1 vii) that this property is shared with the solution u at any time $t > 0$, which means, coming back to the cell density interpretation, that cells are present in the whole ball D . This is in contrast with the case $N \geq 3$, where, at least in the critical mass case, the cells are sometimes present only in a ball strictly inside D .*

It is possible to show a similar result as for $N \geq 3$, i.e. that if $0 \leq m < 2$, for any $u_0 \in Y_m$, then

$$T_{max}(u_0) = +\infty$$

and

$$u(t) \xrightarrow[t \rightarrow +\infty]{} U_a \quad \text{in} \quad C^1([0, 1])$$

where

$$a = a(m) = \frac{m}{1 - \frac{m}{2}} \in [0, +\infty).$$

In [3], for the subcritical case $0 \leq m < 2$, the exponential speed of convergence of $u(t)$ toward the unique stationary solution $U_{a(m)}$ as $t \rightarrow +\infty$ was proved for all L^p norms with $1 \leq p < \infty$ when the initial condition u_0 is continuous and nondecreasing with $u_0(0) = 0$ and $u_0(1) = m$ (a larger class than Y_m) and also in L^∞ norm for some initial conditions for which global in time $W^{1,\infty}$ bound is known (the result then following by interpolation between L^1 and $W^{1,\infty}$).

As far as we know, the mere convergence in C^1 norm was unknown, and a stronger result (the exponential convergence in C^1 norm) will actually be obtained below, by a very different technique from that in [3]. See section 2 for more details.

1.2 Main result

The main goal of this paper is to study the speed of convergence of solutions of (PDE_m) toward the unique stationary solution U_a for the subcritical case $0 < m < M$ ($m = 0$ being obvious since $u = 0$ because $u_0 \in Y_0 = \{0\}$) when

$$\boxed{N \geq 2}. \tag{13}$$

From now on, we fix

$$\boxed{0 < m < M} \tag{14}$$

and

$$\boxed{u_0 \in Y_m}. \tag{15}$$

We denote u the global solution of (PDE_m) with initial condition u_0 . We know that

$$u(t) \xrightarrow[t \rightarrow +\infty]{} U_a \quad \text{in} \quad C^1([0, 1]),$$

where $U_a = U_{a(m)}$ is the unique stationary state of problem (PDE_m) .

Building on this qualitative information, we shall obtain a stronger quantitative one, namely the exponential speed of convergence in $C^1([0, 1])$.

Theorem 1.1. *Assume (13)(14)(15).*

Let $U_a = U_{a(m)}$ be the unique stationary solution of (PDE_m) , i.e. problem (1)-(4), and let $\lambda_1 = \lambda_1(a) > 1$ be the best constant of the Hardy type inequality in Proposition 1.1 below.

Let $\lambda \in (0, \lambda_1 - 1)$.

Then there exists $C = C(u_0, \lambda) > 0$ such that for all $t \geq 1$,

$$\|u(t) - U_a\|_{C^1([0,1])} \leq C \exp(-\lambda \dot{U}_a(1)^q t).$$

Remark 1.2. *We recall that the derivative of u is, up to a multiplicative constant and a change of variables, the radial part of the cell density ρ in the original Patlak-Keller-Segel model (PKS_q) . Hence, this result is equivalent to the exponential speed of uniform convergence of $\rho(t)$ toward ρ_a where ρ_a is the cell density corresponding to U_a .*

The proof of Theorem 1.1 consists of two steps:

- We first establish exponential convergence in an appropriate weighted L^2 norm, by means of a linearization procedure and a suitable Hardy type inequality.
- We then deduce exponential C^1 convergence by using a smoothing effect after a suitable transformation of the equation.

In the next subsection, we describe the first step of the proof.

1.3 A Hardy type inequality and exponential convergence in a L^2 weighted space

The following result, which is a Hardy type inequality, requires as a natural framework the two Hilbert spaces $L \supset Y_m$ and H , where

$$L = L^2 \left((0, 1), \frac{dx}{x^{2-q}} \right)$$

is equipped with the norm

$$\|h\|_L = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}}}$$

and

$$H = \left\{ h \in L, \int_0^1 \dot{h}^2 < \infty, h(0) = h(1) = 0 \right\}$$

with the norm

$$\|h\|_H = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2}.$$

Note that, actually, $H = H_0^1 \subset C^{\frac{1}{2}}([0, 1])$ and the norms on H and H_0^1 are equivalent (see Remark 3.1).

Remark 1.3. *It is very natural to introduce L from the viewpoint of the evolution equation (PDE_m). This will be justified in the following heuristics subsection.*

Proposition 1.1. *Let $a \in (0, A)$.*

There exists $\lambda_1 = \lambda_1(a) > 1$ such that for all $h \in H$,

$$\int_0^1 \frac{\dot{h}^2}{U_a^q} \geq \lambda_1 \int_0^1 \frac{h^2}{x^{2-q}}. \quad (16)$$

Moreover, there exists $\phi_1 \in H$ such that there is equality if and only if $h = c\phi_1$ for some $c \in \mathbb{R}$.

As will be explained with much more details in subsection 1.4, the evolution problem (PDE_m) can formally be seen as a gradient flow equation

$$u_t = -\nabla \mathcal{F}[u(t)]$$

on some “infinite dimensional Riemannian manifold” (\mathcal{M}, g) where

$$\mathcal{M} = \{u \in Y_m^1, \dot{u} > 0 \text{ on } [0, 1]\}$$

is an open set of the affine space

$$Y_m^1 = Y_m \cap C^1([0, 1])$$

and the metric g is defined by

$$g_u(h, h) = \int_0^1 \frac{h^2}{x^{2-q} \dot{u}^q}$$

for all $u \in \mathcal{M}$ and $h \in T_u \mathcal{M}$, $T_u \mathcal{M}$ denoting the tangent space to \mathcal{M} at u .

The previous result is actually equivalent to the strict convexity of the Lyapunov functional \mathcal{F} at U_a , which makes us expect an exponential speed of convergence toward U_a , measured with the Riemannian distance $d_{\mathcal{M}}(U_a, \cdot)$ defined by the metric g (which is equivalent to $\|\cdot\|_L$ near U_a).

Its proof relies on the theory of compact self-adjoint operators on a separable Hilbert space and on a technique used in the article [1] of P.R Beesack about extensions of Hardy’s inequality.

We enjoy the opportunity to thank Philippe Souplet for suggesting this reading.

The following result shows rigorously the expected exponential speed of convergence in L :

Lemma 1.1. *Under the assumptions of Theorem 1.1, there exists $C = C(u_0, \lambda) > 0$ such that*

$$\|u(t) - U_a\|_L \leq C \exp(-\lambda \dot{U}_a(1)^q t)$$

for all $t \geq 0$.

This result, though not the strongest, is the core of our paper. Its proof is inspired by both the gradient flow structure of problem (PDE_m) and the fact that \mathcal{M} is an open set of an affine space, which allows us to consider all the situation from the viewpoint of U_a . More precisely, if we define

$$h(t) = u(t) - U_a$$

and consider

$$\gamma(t) = g_{U_a}(h(t), \dot{h}(t)),$$

we want to get a differential inequality on the latter. Since h satisfies

$$h_t = L_{U_a} h + F(x, h, \dot{h}) \tag{17}$$

where

$$L_{U_a} = x^{2-q} \dot{U}_a^q \frac{d}{dx} \left[\frac{\dot{h}}{\dot{U}_a^q} \right] + \dot{U}_a^q h$$

is the linearized operator at $u = U_a$ and F is some remainder term, we will have two parts to deal with in the derivative of γ . The first term can be managed thanks to the Hardy type inequality in Proposition 1.1. The second imposes to sacrifice a bit of the first one, but without any serious damage since there was anyhow no hope to reach the limit case $\lambda = \lambda_1$, at least by this way.

Remark 1.4.

- i) *We can show that the degenerate parabolic equation (17) satisfied by h is regularizing in time from L to $C^1([0, 1])$, at least for large time (see Lemma 5.1). This will be enough to deduce the exponential speed of convergence toward steady states in $C^1([0, 1])$, i.e. Theorem 1.1, as an easy consequence of Lemma 1.1.*
- ii) *The constant C we get is unbounded as $\lambda \rightarrow \lambda_1$ so that we cannot get the same result with $\lambda = \lambda_1 - 1$.*
- iii) *We think that the upper rate $\lambda_1 \dot{U}_a(1)^q$ is not optimal. We believe λ_1 is but not $\dot{U}_a(1)^q$ because it follows from the following rough inequality : $\int_0^1 \frac{h^2}{x^{2-q}} \geq \dot{U}_a(1)^q \int_0^1 \frac{h^2}{x^{2-q} \dot{U}_a^q}$ for any $h \in L$ because U_a is concave.*
- iv) *In dimension $N \geq 3$, an interesting question is to know whether the exponential speed of convergence degenerates or not for $a = A$. Indeed, we can see that $\lambda_1(A) = 1$ since if we set $w_A = \frac{d}{da} \Big|_{a=A} U_a$, we remark that $-\frac{d}{dx} \frac{w_A}{\dot{U}_A^q} = \frac{w_A}{x^{2-q}}$. Hence we can guess that $\lambda_1(a) \rightarrow 1$ as $a \rightarrow A$. But, since the center manifold seems to be made of the steady states $(U_a)_{a \geq A}$, it is not clear that the exponential speed of convergence should disappear. It would then be very different for the critical mass for $N = 2$ and $q = 1$ since the speed of convergence degenerates and is no longer exponential. This has been done in [20]. It was known that infinite time blow-up of u_x occurs. Of course, uniform convergence toward the constant singular steady state $\bar{U} = 2$ cannot hold in this case since $u(t, 0) = 0$. However, the authors proved that $|u(t) - 2|_1 \sim C \sqrt{t} e^{-\sqrt{2}t}$ as $t \rightarrow \infty$.*

1.4 Heuristics

Although the proof of Lemma 1.1 (cf. sections 3-4) can be read without any reference to the following heuristic arguments, we think that they shed some light on the underlying ideas and on the intuition that led to the rigorous proof. Indeed, the latter is inspired by a gradient flow approach, in the spirit of the seminal work of F. Otto [25], a strategy which has already been used successfully for the Patlak-Keller-Segel model. For instance, applying these ideas to system (PKS_1) in \mathbb{R}^2 for the subcritical mass case, A. Blanchet, V. Calvez and J.A. Carrillo recovered in [2] the global in time existence of weak solutions and V. Calvez and J.A. Carrillo proved in [8] the exponential speed of convergence of radial solutions toward equilibrium, but measured with the Wasserstein distance \mathcal{W}_2 .

First, we would like to recall a basic fact about gradient flows in a Euclidean space which provides a sufficient condition to have an exponential speed of convergence to the stationary point. We will give its rigorous proof, even though it is very simple, because it is the scheme for proofs in a more general infinite dimensional setting, as we will then see on a well-known instance in an infinite dimensional Hilbert space. Finally, we will see, without searching to be rigorous, that these ideas are inspiring in the case of problem (PDE_m) which turns out to define a gradient flow on an "infinite dimensional Riemannian manifold".

For basic knowledge about strict Lyapunov functional and Lasalle's invariance principle, we refer the reader to [10, Chapter 9] or to [28, Appendix G]. We also recall some useful properties in subsection 6.1.

We consider the following differential equation in the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ having a gradient flow structure, i.e.

$$\dot{x}(t) = -\nabla F(x(t))$$

with $F : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth.

Lemma 1.2. *Let $x_0 \in \mathbb{R}^d$.*

If the trajectory starting from x_0 is relatively compact in \mathbb{R}^d (then global), if there exists a unique stationary point x_∞ and if F is strictly convex at $x = x_\infty$, i.e. F satisfies for some $\alpha_1 > 0$,

$$d^2 F(x_\infty)(\dot{x}, \dot{x}) \geq \alpha_1 |\dot{x}|^2 \text{ for all } \dot{x} \in \mathbb{R}^n,$$

then for any $\alpha \in (0, \alpha_1)$, there exists $C = C(x_0, \alpha) > 0$ such that for all $t \geq 0$

$$|x(t) - x_\infty| \leq C \exp(-\alpha t).$$

Proof of Lemma 1.2. First, we observe that F is a strict Lyapunov function since

$$\frac{d}{dt} F(x(t)) = -|\nabla F(x(t))|^2.$$

Since the trajectory $(x(t))_{t \geq 0}$ starting from x_0 is relatively compact, i.e. bounded in the context of an Euclidean space, then from Lasalle's invariance principle, the ω -limit set is made of stationary points. But since there is only one stationary point x_∞ , i.e. verifying

$$\nabla F(x_\infty) = 0,$$

we can deduce the convergence of $x(t)$ toward x_∞ .

This implies in particular that x_∞ is the minimum of F , so that moreover

$$d^2 F(x_\infty) \geq 0.$$

It is then not surprising that the strict convexity assumption on F will give information about the speed of convergence of $x(t)$ toward x_∞ . Indeed, if we denote

$$h(t) = x(t) - x_\infty$$

and

$$\gamma(t) = |h(t)|^2,$$

we have

$$\dot{\gamma}(t) = -2\langle \nabla F(x(t)), h(t) \rangle.$$

But $\nabla F(x_\infty) = 0$, so

$$\nabla F(x(t)) = d(\nabla F)(x_\infty).h(t) + \epsilon(h(t))h(t)$$

where $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$. Hence,

$$\dot{\gamma}(t) = -2d^2 F(x_\infty).(h(t), h(t)) + \epsilon(h(t))|h(t)|^2$$

Now, let $\alpha < \alpha_1$.

Since $h(t) \xrightarrow{h \rightarrow 0} 0$, there exists $t_0 > 0$ such that for all $t \geq t_0$, $\epsilon(h(t)) \leq 2(\alpha_1 - \alpha)$. Then, for all $t \geq t_0$,

$$\dot{\gamma}(t) \leq -2\alpha \gamma(t),$$

which implies

$$\gamma(t) \leq \gamma(t_0) \exp(-2\alpha t)$$

for all $t \geq t_0$ and finally we have for all $t \geq 0$,

$$\gamma(t) \leq C \exp(-2\alpha t)$$

where $C = C(x_0, \alpha)$ because γ is bounded. Whence the result. \square

As said before, this scheme can also be used in an infinite dimensional setting, like a Hilbert space. For example, let us consider the heat equation with Dirichlet condition on an bounded domain Ω

$$u_t = \Delta u.$$

This equation defines a continuous dynamical system on $L^2(\Omega)$ endowed with its standard scalar product (\cdot, \cdot) . and is moreover regularizing so that, for $t > 0$, $u(t) \in H_0^1(\Omega)$. If we define

$$F(u) = \int_{\Omega} \frac{|\nabla u|^2}{2},$$

then for $t > 0$,

$$u_t = -\nabla F(u(t))$$

since

$$(\nabla F(u), h) = dF(u).h = \int_{\Omega} \nabla u \nabla h = - \int_{\Omega} \Delta u h = (-u_t, h)$$

for all $h \in H_0^1(\Omega)$.

It is easy to see that F is a strict Lyapunov function and that 0 is the only stationary solution since the only harmonic function in Ω vanishing on the boundary is the zero function.

Moreover, since F is quadratic,

$$d^2 F(u).(h, h) = 2F(h) = \int_{\Omega} |\nabla h|^2 \geq \lambda_1(\Omega) \|h\|_{L^2(\Omega)}^2$$

by Poincaré inequality, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in Ω with Dirichlet condition. The same computation as above shows that for any $u_0 \in L^2(\Omega)$, for any $\lambda < \lambda_1(\Omega)$, there exists $C = C(u_0, \lambda) > 0$ such that for all $t \geq 0$,

$$\|u(t)\|_{L^2(\Omega)} \leq C \exp(-\lambda t).$$

Note that, actually, the proof also works for $\lambda = \lambda_1$ in this particular instance because F is quadratic so that $u \mapsto \nabla F(u)$ is linear hence there is no $o(h)$ to deal with.

Another more general setting where this method can be applied is that of "infinite dimensional Riemannian manifolds". This idea has been deeply exploited in the very nice paper [25] concerning the porous medium equation.

It turns out that problem (PDE_m) has this kind of gradient flow structure and we will try to take advantage of it. In what follows, we will consider the case of dimension $N \geq 3$ but all this discussion can be made for the case $N = 2$.

If we denote the "infinite dimensional manifold" (actually an open set of the affine space Y_m^1)

$$\mathcal{M} = \{u \in Y_m^1, \dot{u} > 0 \text{ on } [0, 1]\}$$

where we recall that

$$Y_m^1 = Y_m \cap C^1([0, 1]),$$

we know that for $t > 0$, $u(t) \in Y_m^1$ and then for t large enough,

$$u(t) \in \mathcal{M}$$

since $u(t) \xrightarrow[t \rightarrow +\infty]{} U_a$ in $C^1([0, 1])$ and $\dot{U}_a > 0$ on $[0, 1]$.

We can define the "Riemannian metric" g on \mathcal{M} by

$$g_u(h, k) = \int_0^1 \frac{hk}{x^{2-q}\dot{u}^q} \quad (18)$$

for any $u \in \mathcal{M}$ and any $(h, k) \in T_u\mathcal{M}^2$, where actually, for any $u \in \mathcal{M}$

$$T_u\mathcal{M} = \mathcal{T}$$

with

$$\mathcal{T} = \{h \in C^1([0, 1]), h(0) = h(1) = 0\}$$

since \mathcal{M} is an open set of the affine space Y_m^1 which has \mathcal{T} as direction (actually, $Y_m^1 = mId_{[0,1]} + \mathcal{T}$).

Now, we recall the strict Lyapunov functional \mathcal{F} used in [24] to prove convergence toward steady states in the critical and subcritical mass cases :

$$\mathcal{F}[u] = \int_0^1 \frac{\dot{u}^{2-q}}{(2-q)(1-q)} - \frac{u^2}{2x^{2-q}}.$$

\mathcal{F} can be guessed by the following equivalent formulation of (1)

$$u_t = x^{2-q}\dot{u}^q \left[\frac{d}{dx} \frac{\dot{u}^{1-q}}{1-q} + \frac{u}{x^{2-q}} \right]. \quad (19)$$

It is easy to see formally that

$$u_t = -\nabla\mathcal{F}[u(t)],$$

which explains intuitively why \mathcal{F} is a strict Lyapunov functional for (PDE_m) .

Indeed, for any $h \in T_u\mathcal{M}$, we have by definition

$$g_u(\nabla\mathcal{F}[u], h) = d\mathcal{F}(u).h$$

and moreover, by formal computation and integration by parts, we get

$$d\mathcal{F}(u).h = \int_0^1 \frac{\dot{u}^{1-q}\dot{h}}{1-q} - \frac{uh}{x^{2-q}} = - \int_0^1 \left[\frac{\ddot{u}}{\dot{u}^q} + \frac{u}{x^{2-q}} \right] h = -g_u(u_t, h).$$

Since we study the subcritical mass case, there exists a unique steady state U_a so we have to compute the second derivative of \mathcal{F} at this point. Formally, we get

$$\boxed{d^2\mathcal{F}[U_a].(h, h) = \int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}}}.$$

As explained before, since $u(t) \xrightarrow{t \rightarrow +\infty} U_a$ in $C^1([0, 1])$, then U_a is the minimum of \mathcal{F} so that we can naturally expect that, for any $h \in T_{U_a}\mathcal{M}$,

$$d^2F(U_a).(h, h) \geq 0.$$

If we can prove the stronger result that for some $\alpha_1 > 0$, we have for all $h \in T_{U_a}\mathcal{M}$,

$$d^2\mathcal{F}[U_a].(h, h) \geq \alpha_1 g_{U_a}(h, h)$$

or equivalently that for some $\lambda_1 > 1$, for all $h \in T_{U_a}\mathcal{M}$,

$$\int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} \geq \lambda_1 \int_0^1 \frac{h^2}{x^{2-q}}, \quad (20)$$

then we can hope to prove that the speed of convergence is exponential as before.

Remark 1.5. We thank Philippe Souplet for pointing out the following intuitive explanation of the fact that $\lambda_1 > 1$ in the present context. Indeed, for the subcritical case, the steady states of (19) form an increasing family $(U_a)_{a \in (0, A)}$ of solutions of

$$\frac{d}{dx}f(\dot{u}) + V(x)u = 0$$

where f is the increasing function on $[0, +\infty)$ defined for all $v \geq 0$ by

$$f(v) = \frac{v^{1-q}}{1-q}$$

and

$$V(x) = \frac{1}{x^{2-q}} > 0.$$

Hence, for any $a \in (0, A)$, $w_a = \frac{d}{da}U_a > 0$ and w_a formally satisfies

$$\frac{d}{dx}[f'(\dot{U}_a)\dot{w}_a] + V(x)w_a = 0.$$

If $\phi_1 > 0$ is an eigenvector for the first eigenvalue λ_1 , i.e. satisfies

$$\frac{d}{dx}[f'(\dot{U}_a)\dot{\phi}_1] + \lambda_1 V(x)\phi_1 = 0,$$

then it is easy to see by integration by parts that

$$(\lambda_1 - 1) \int_0^1 V w_a \phi_1 = [f'(\dot{U}_a)w_a \dot{\phi}_1]_0^1 > 0$$

by Hopf maximum principle on the boundary. Therefore, $\lambda_1 > 1$.

But here, there is an additional difficulty since we have a "Riemannian structure". Indeed, the metric g here depends on the point u , so that if we set

$$\gamma_0(t) = g_{u(t)}(u(t) - U_a, u(t) - U_a)$$

and differentiate it, there will be an extra term. This strategy is in some sense very natural since it takes into account the gradient flow structure. Nevertheless, because of this extra term,

we preferred to also take advantage of the fact that \mathcal{M} is an open set of an affine space by considering

$$\gamma(t) = g_{U_a}(u(t) - U_a, u(t) - U_a),$$

i.e. we fixed the point U_a and consider the difference $u(t) - U_a$ belonging to the tangent space $T_{U_a}\mathcal{M}$. Hence, this strategy of linearization somehow uses both the gradient flow structure via the good relation between g and the flow, and the "affine structure" because we can fix U_a and consider the situation from its viewpoint.

Finally, we also remark that if $U \in \mathcal{M}$ is near of U_a , then all measures by the metrics $g_u(h, h)$ are comparable to

$$\|h\|_L = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}}}.$$

Hence, recalling that the Riemannian metric $d_{\mathcal{M}}$ on \mathcal{M} between U_a and U is defined by

$$d_{\mathcal{M}}(U_a, U)^2 = \inf_{\{u \in C^1([0,1], \mathcal{M}), u(0)=U_a, u(1)=U\}} \int_0^1 g_{u(t)}(u_t, u_t) dt,$$

it is clear that $d_{\mathcal{M}}(U_a, U)$ is equivalent to $\|U - U_a\|_L$ for U near of U_a . This consideration naturally leads us to introduce the Hilbert space $L \supset Y_m$, where

$$L = L^2 \left((0, 1), \frac{dx}{x^{2-q}} \right).$$

It is also very natural to make the proof of the Hardy type inequality (20) in a larger space than $T_{U_a}\mathcal{M}$, namely for all $h \in H$, where H is the Hilbert space

$$H = \left\{ h \in L, \int_0^1 \dot{h}^2 < \infty, h(0) = h(1) = 0 \right\}$$

equipped with the norm

$$\|h\|_H = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2}.$$

Outline of the rest of the paper. In section 2, we state some preliminary results for dimension $N = 2$ which will be proved in the appendix.

The next sections are devoted to proofs. In section 3, we will get the strict convexity of \mathcal{F} (or \mathcal{G} if $N = 2$) at U_a by showing its equivalent form expressed in the Hardy type inequality of Proposition 1.1.

In section 4, we show Lemma 1.1 which establishes the exponential speed of convergence toward the steady state in L .

In section 5, we prove that the degenerate parabolic equation satisfied by $h = u - U_a$ is regularizing for large time from L to $C^1([0, 1])$, i.e. Lemma 5.1 which therefore easily implies Theorem 1.1. In the appendix, we also recall some basic facts about continuous dynamical systems and Lyapunov functionals.

2 Preliminary results for dimension $N = 2$

In this section, we focus on the most studied case of dimension 2, well-known for its critical mass 8π if we come back to the original Keller-Segel system (5). Our aim is to state the results that lead us to Lemma 2.5, i.e. to the C^1 convergence of $u(t)$ toward the unique steady state U_a that we mentioned in the introduction.

We would like to remark that problem (PDE_m) is simpler for $N = 2$ (see (9)) than for $N \geq 3$ (see (1)) since its nonlinearity is then locally Lipschitz (even bilinear) in (u, u_x) . Accordingly, the convergence results for $N = 2$, as well as the required wellposedness and regularity properties, can be proved by similar ideas as in [23, 24] which treat the case $N \geq 3$. We point out that some of the wellposedness issues for $N = 2$ have been addressed in [20, 3], but that they do not provide all the necessary properties that we need. Therefore, and also for the sake of completeness, we chose to give all the proofs in Appendix, trying to be reasonably self-contained.

2.1 Local wellposedness and regularity for problem (PDE_m)

We first give a wellposedness and regularity theorem which requires the introduction of the following "norm" \mathcal{N} and some notation.

Definition 2.1. For any real function u defined on $(0, 1]$, we set

$$\mathcal{N}[u] = \sup_{x \in (0,1]} \frac{u(x)}{x}.$$

Notation 2.1. Let $m \geq 0$ and $\gamma > 0$.

- $Y_m = \{u \in C([0, 1]) \text{ nondecreasing, } u'(0) \text{ exists, } u(0) = 0, u(1) = m\}$.
- $Y_m^1 = Y_m \cap C^1([0, 1])$.
- $Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0, 1]), \sup_{x \in (0,1]} \frac{|u'(x) - u'(0)|}{x^\gamma} < \infty\}$.

Theorem 2.1. Let $K > 0$ and $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$.

- i) There exists $T_{max} = T_{max}(u_0) > 0$ and a unique maximal classical solution of (PDE_m) with initial condition u_0 , i.e.

$$u \in C([0, T_{max}) \times [0, 1]) \cap C^1((0, T_{max}) \times [0, 1]) \cap C^{1,2}((0, T_{max}) \times (0, 1))$$

verifying (9)(10)(11)(12) and $u(0) = u_0$.

Moreover, u satisfies the following condition :

$$\sup_{t \in (0, T]} \sqrt{t} \|u(t)\|_{C^1([0,1])} < \infty \text{ for any } T \in (0, T_{max}). \quad (21)$$

- ii) There exists $\tau = \tau(K) > 0$ such that $T_{max} \geq \tau$.

- iii) Blow up alternative : $T_{max} = +\infty$ or $\lim_{t \rightarrow T_{max}} \mathcal{N}[u(t)] = +\infty$

- iv) $u \in C^\infty((0, T_{max}) \times (0, 1])$.

- v) If $u_0 \in Y_m^{1,\gamma}$ with $\frac{1}{2} < \gamma \leq 1$ then $u \in C([0, T_{max}), C^1([0, 1]))$.

- vi) For all $t \in (0, T_{max})$, $u(t) \in Y_m^{1,1}$.

- vii) $u_x(t, x) > 0$ for all $(t, x) \in (0, T_{max}) \times [0, 1]$.

At least the four first points were known explicitly or implicitly (see [20]). Concerning point vii), to our knowledge, it was only proved that for all $t \in (0, T_{max})$,

$$u_x(t, 0) > 0.$$

Although vii) is expected, its proof is rather technical and moreover this fact will turn out to be essential in the proof of Lemma 2.4.

2.2 Subcritical case : Lyapunov functional and convergence in $C^1([0, 1])$

From now on, we only focus on the subcritical case

$$\boxed{m < 2}$$

which corresponds to mass lower than 8π for the original Keller-Segel system (5).

Then, the classical solutions of (PDE_m) are globally defined. More precisely :

Lemma 2.1. *Let $m < 2$ and $u_0 \in Y_m$. Then*

$$T_{max}(u_0) = +\infty.$$

The next lemma, stating in particular the relative compactness of the trajectory $\{u(t), t \geq 1\}$ in Y_m^1 for any initial condition $u_0 \in Y_m^1$, will also be useful to check that $(T(t))_{t \geq 0}$ defined below is a continuous dynamical system on Y_m^1 .

Definition 2.2. *Let $u_0 \in Y_m^1$ and $t \geq 0$.*

We define $T(t)u_0 = u(t)$ where u is the classical solution of problem (PDE_m) with initial condition u_0 .

Lemma 2.2. *Let $m < 2$, $t_0 > 0$ and $K > 0$.*

Then, $\{T(t)u_0, \mathcal{N}[u_0] \leq K, t \geq t_0\}$ is relatively compact in Y_m^1 .

Lemma 2.3. *$(T(t))_{t \geq 0}$ is a continuous dynamical system on Y_m^1 .*

We now introduce a functional which is an analogue of \mathcal{F} in the case $q = 1$.

Definition 2.3. *Let $\mathcal{M} = \{u \in Y_m^1, u_x > 0 \text{ on } [0, 1]\}$.*

We define for all $u \in \mathcal{M}$,

$$\mathcal{G}[u] = \int_0^1 u_x [\ln u_x - 1] - \frac{u^2}{2x}.$$

Indeed, we have the following result.

Lemma 2.4. *\mathcal{G} is a strict Lyapunov functional for $(T(t))_{t \geq 0}$.*

As a consequence, we finally get :

Lemma 2.5. *Let $0 \leq m < 2$ and $u_0 \in Y_m$. Then*

$$u(t) \xrightarrow[t \rightarrow +\infty]{} U_a \quad \text{in} \quad C^1([0, 1])$$

where

$$a = \frac{m}{1 - \frac{m}{2}}.$$

3 A Hardy type inequality

The aim of this section is to prove Proposition 1.1. First, we will need to establish some intermediate lemmas. For reader's convenience, we recall that

$$L = L^2 \left((0, 1), \frac{dx}{x^{2-q}} \right) \quad \text{and} \quad H = \left\{ h \in L, \int_0^1 \dot{h}^2 < \infty, h(0) = h(1) = 0 \right\}$$

and that L and H are equipped with the following norms

$$\|h\|_L^2 = \int_0^1 \frac{h^2}{x^{2-q}} \quad \text{and} \quad \|h\|_H^2 = \int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2.$$

Remark 3.1. *Actually, we can see that*

$$H = H_0^1$$

and that $\|\cdot\|_{H_0^1}$ and $\|\cdot\|_H$ are equivalent.

Indeed, $H \subset H_0^1$ with continuous embedding is obvious and the reverse is also true by the standard Hardy inequality

$$\int_0^1 \frac{h^2}{x^2} \leq 4 \int_0^1 \dot{h}^2. \quad (22)$$

valid for any $h \in H_0^1$. Note also that L and H are separable Hilbert spaces.

We will need the following compactness result.

Lemma 3.1. *The imbedding $H \subset L$ is compact.*

Proof. For any $\alpha \in (0, 1]$, we denote

$$C_0^\alpha = \{h \in C^\alpha([0, 1]), h(0) = 0\}$$

the Banach space equipped with the norm

$$\|h\|_{C_0^\alpha} = \sup_{\substack{(x,y) \in [0,1]^2 \\ x \neq y}} \frac{|h(x) - h(y)|}{|x - y|^\alpha}.$$

It is clear that $H \subset C_0^{\frac{1}{2}}$ with continuous imbedding since if $h \in H$,

$$|h(x) - h(y)| = \left| \int_y^x \dot{h} \right| \leq \sqrt{|x - y|} \sqrt{\int_0^1 \dot{h}^2}.$$

Now, let $\gamma \in (\frac{1-q}{2}, \frac{1}{2})$. The imbedding $C_0^{\frac{1}{2}} \subset C_0^\gamma$ is compact and the imbedding $C_0^\gamma \subset L$ is continuous since for all $h \in C_0^\gamma$,

$$\|h\|_L^2 = \int_0^1 \frac{h^2}{x^{2-q}} \leq \|h\|_{C_0^\gamma}^2 \int_0^1 \frac{1}{x^{2-q-2\gamma}}$$

with $\int_0^1 \frac{1}{x^{2-q-2\gamma}} < \infty$ since $2 - q - 2\gamma < 1$. □

The following lemma, whose proof relies on a technique used in [1] to get extensions of Hardy's inequality, will be essential in the proof of Proposition 1.1.

Lemma 3.2. *Let $0 < a < A$. Then, for all $h \in H$*

$$\int_0^1 \frac{\dot{h}^2}{U_a^q} - \frac{h^2}{x^{2-q}} \geq 0 \quad (23)$$

with equality if and only if $h = 0$.

Before giving the proof, we recall some useful properties of U_a . For all $a \geq 0$,

$$x^{2-\frac{2}{N}} \ddot{U}_a + U_a \dot{U}_a^{\frac{2}{N}} = 0 \quad (24)$$

and

$$\dot{U}_a(0) = a.$$

This implies the concavity of U_a , so

$$\dot{U}_a(1) \leq \dot{U}_a \leq a \text{ on } [0, 1].$$

Moreover, for all $x \in [0, 1]$,

$$U_a(x) = U_1(ax).$$

Since U_1 is increasing on $[0, A]$ (and flat after $x = A$) for some $A > 0$, then

$$\text{for } 0 < a < A, \dot{U}_a > 0 \text{ on } [0, 1].$$

Proof. We denote for all $x \in [0, 1]$,

$$w_a(x) = \frac{d}{da}U_a(x) = x \dot{U}_1(ax).$$

We see that $w_a > 0$ on $(0, 1]$ since $0 < a < A$.

Moreover, for all $x \in [0, 1]$, noting first that

$$\dot{w}_a(x) = \dot{U}_1(ax) + ax\ddot{U}_1(ax),$$

we have

$$\begin{aligned} \frac{\dot{w}_a(x)}{\dot{U}_a^q(x)} &= \frac{\dot{U}_1(ax)^{1-q}}{a^q} + a^{1-q}x \frac{\ddot{U}_1(ax)}{\dot{U}_1(ax)^q} \\ &= \frac{\dot{U}_1(ax)^{1-q}}{a^q} - \frac{U_1(ax)}{ax^{1-q}} \quad \text{by (24)} \end{aligned}$$

then, we obtain

$$\begin{aligned} \frac{d}{dx} \left[\frac{\dot{w}_a}{\dot{U}_a^q} \right] &= (1-q)a^{1-q} \frac{\ddot{U}_1(ax)}{\dot{U}_1(ax)^q} + (1-q) \frac{U_1(ax)}{ax^{2-q}} - \frac{\dot{U}_1(ax)}{x^{1-q}} \\ &= -(1-q)a^{1-q} \frac{U_1(ax)}{(ax)^{2-q}} + (1-q) \frac{U_1(ax)}{ax^{2-q}} - \frac{w_a(x)}{x^{2-q}} = -\frac{w_a(x)}{x^{2-q}} \end{aligned}$$

again by (24), so we see that w_a satisfies

$$\frac{d}{dx} \left[\frac{\dot{w}_a}{\dot{U}_a^q} \right] + \frac{w_a}{x^{2-q}} = 0. \quad (25)$$

We note that this equation could also be obtained by differentiating (24) with respect to a .

The proof of Lemma 3.2 will be made by density. Therefore, we first make the following computation for any $h \in C_c^\infty((0, 1))$.

$$\begin{aligned} \int_0^1 \frac{[\dot{h} - \frac{\dot{w}_a}{w_a}h]^2}{\dot{U}_a^q} &= \int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} + \int_0^1 \left[\frac{\dot{w}_a}{w_a} \right]^2 \frac{h^2}{\dot{U}_a^q} - \int_0^1 2h\dot{h} \left[\frac{\dot{w}_a}{w_a} \frac{1}{\dot{U}_a^q} \right] \\ &= \int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} + \int_0^1 \left[\frac{\dot{w}_a}{w_a} \right]^2 \frac{h^2}{\dot{U}_a^q} + \int_0^1 h^2 \frac{d}{dx} \left[\frac{\dot{w}_a}{w_a} \frac{1}{\dot{U}_a^q} \right] \\ &= \int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} + \int_0^1 h^2 \left(\left[\frac{\dot{w}_a}{w_a} \right]^2 \frac{1}{\dot{U}_a^q} + \frac{d}{dx} \left[\frac{\dot{w}_a}{w_a} \frac{1}{\dot{U}_a^q} \right] \right) \end{aligned}$$

where we used $h(0) = h(1) = 0$ in the integration by parts.

From (25), we deduce

$$\frac{d}{dx} \left[\frac{1}{w_a} \frac{\dot{w}_a}{\dot{U}_a^q} \right] + \frac{1}{\dot{U}_a^q} \left[\frac{\dot{w}_a}{w_a} \right]^2 = \frac{d}{dx} \left[\frac{\dot{w}_a}{\dot{U}_a^q} \right] \frac{1}{w_a} = -\frac{1}{x^{2-q}}$$

which, coming back to the previous computation, implies

$$\int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} = \int_0^1 \frac{[\dot{h} - \frac{\dot{w}_a}{w_a}h]^2}{\dot{U}_a^q} \geq 0.$$

Let $h \in H$. Since $H = H_0^1(0, 1)$ with equivalent norms, then $C_c^\infty((0, 1))$ is dense in H so there exists $h_n \in C_c^\infty((0, 1))$ such that $h_n \rightarrow h$ in L and $\dot{h}_n \rightarrow \dot{h}$ in $L^2((0, 1), dx)$ with convergence almost everywhere in $(0, 1)$ and domination by two functions respectively in L and $L^2(0, 1)$.

Hence, by Lebesgue's dominated convergence theorem, the previous equation is also valid for h .

Since $w_a \left(\frac{h}{w_a} \right)' = \dot{h} - \frac{\dot{w}_a}{w_a} h$, there is equality if and only if $h = c w_a$ for some $c \in \mathbb{R}$ almost everywhere on $(0, 1)$, but actually everywhere on $[0, 1]$ since h and w_a are both continuous. Now, we note that $h(1) = 0$ and $w_a(1) > 0$ since $a < A$, so $h = c w_a$ implies $c = 0$, i.e $h = 0$. \square

Proof of Proposition 1.1. The following procedure is standard. Considering the symmetric bilinear form Λ defined on H by

$$\Lambda(h, k) = \int_0^1 \frac{\dot{h}\dot{k}}{\dot{U}_a^q} \quad \text{for all } (h, k) \in H^2,$$

it is easy to see that Λ is continuous and coercive. Hence, we can apply the Lax-Milgram theorem and prove that for any $\varphi \in H'$, there exists a unique $h \in H$ such that

$$\Lambda(h, \cdot) = \varphi.$$

Thanks to Lemma 3.2, any $f \in L$ defines $\varphi_f \in H'$ by

$$\varphi_f(k) = \int_0^1 \frac{f k}{x^{2-q}} \quad \text{for all } k \in H.$$

We then define $T : L \rightarrow L$ by $Tf = h$ where $h \in H$ is such that $\Lambda(h, \cdot) = \varphi_f$. It is easy to see that T is self-adjoint, continuous (thanks to Lax-Milgram) and even compact, thanks to Lemma 3.1.

The end of the proof, which relies on the theory of compact self-adjoint operators on a separable Hilbert space, is completely similar to that of [11, Theorem 2, p.336]. Moreover, since the infimum in

$$\lambda_1 = \inf_{\substack{h \in H \\ h \neq 0}} \frac{\Lambda(h, h)}{\|h\|_L^2}$$

is reached, then Lemma 3.2 implies $\lambda_1 > 1$. \square

4 Convergence with exponential speed in $L^2 \left((0, 1), \frac{dx}{x^{2-q}} \right)$

Proof of Lemma 1.1. We let

$$u = U_a + h.$$

To get the result, it is equivalent to show the existence of $C > 0$ such that

$$\gamma(t) \leq C \exp(-2 \lambda \dot{U}_a(1)^q t)$$

where

$$\gamma(t) = g_{U_a}(h(t), h(t)) = \int_0^1 \frac{h(t)^2}{x^{2-q} \dot{U}_a^q}.$$

An easy computation shows that for any $t > 0$ and any $x \in (0, 1]$,

$$h_t = L_{U_a} h + F(x, h, \dot{h}) \tag{26}$$

where

$$L_{U_a} h = [x^{2-q}] \ddot{h} + \left[q \frac{U_a}{\dot{U}_a^{1-q}} \right] \dot{h} + [\dot{U}_a^q] h \tag{27}$$

$$= x^{2-q} \dot{U}_a^q \frac{d}{dx} \left[\frac{\dot{h}}{\dot{U}_a^q} \right] + \dot{U}_a^q h \tag{28}$$

and

$$F(x, h, \dot{h}) = \frac{q}{\dot{U}_a^{1-q}} h \dot{h} + \left[h \dot{U}_a^q + U_a \dot{U}_a^q \right] \left[\left(1 + \frac{\dot{h}}{\dot{U}_a} \right)^q - 1 - q \frac{\dot{h}}{\dot{U}_a} \right]. \quad (29)$$

We already know from [23, Proposition 2.1] that if $t_0 > 0$,

$$h_x(t, x) = \psi(t, x^{\frac{q}{2}}) \quad \text{for all } t \geq t_0, x \in [0, 1] \quad (30)$$

where

$$\psi \in C^{1,\infty}([t_0, \infty) \times [0, 1]) \quad (31)$$

(ψ having odd derivatives vanishing at $x = 0$). Formula (30) implies

$$\frac{\partial^2 h}{\partial x^2}(t, x) = \frac{q}{2} \frac{\partial \psi}{\partial x}(t, x^{\frac{q}{2}}) \quad \text{for all } t \geq t_0, x \in (0, 1]. \quad (32)$$

Moreover, from [24, Theorem 1.2], we know that

$$\|h(t)\|_{C^1([0,1])} \xrightarrow{t \rightarrow +\infty} 0. \quad (33)$$

We will need the following lemma, whose proof is postponed just after this one :

Lemma 4.1. *Let $t > 0$. Denoting $h = h(t)$, we have :*

$$\frac{h L_{U_a} h}{x^{2-q} \dot{U}_a^q} \in L^1(0, 1)$$

and

$$\int_0^1 \frac{h L_{U_a} h}{x^{2-q} \dot{U}_a^q} = - \int_0^1 \left[\frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} \right]. \quad (34)$$

By (33), there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$\left\| \frac{h(t)}{\dot{U}_a} \right\|_{\infty, [0,1]} \leq \frac{1}{2} \quad \text{and} \quad \|\dot{h}(t)\|_{\infty, [0,1]} \leq \min \left(1, \frac{2\epsilon}{K a^{1+q}}, \frac{\delta}{K} \right). \quad (35)$$

Recalling (29), there exists $K > 0$ such that

$$\left| \frac{h F(x, h, \dot{h})}{\dot{U}_a^q} \right| \leq K \left[h^2 |\dot{h}| + h^2 \dot{h}^2 + U_a |h| \dot{h}^2 \right] \quad (36)$$

for all $(x, t) \in [0, 1] \times [t_0, \infty)$ (h and \dot{h} depending on t).

Let $\delta > 0$ such that $\lambda + \delta \in (0, \lambda_1 - 1)$ and $\epsilon = \frac{\lambda_1 - 1 - \lambda - \delta}{\lambda_1} > 0$ which satisfies

$$(1 - \epsilon)(\lambda_1 - 1) - \epsilon = \lambda + \delta. \quad (37)$$

It is easy to see that for any $t \geq t_0 + 1$ there exists $M(t) > 0$ such that

$$\sup_{[t-1, t+1]} \|h_t\|_{\infty, [0,1]} \leq M(t).$$

This follows from (26)(27)(32)(29) since for all $t \geq t_0$, $\|\dot{h}(t)\|_{\infty, [0,1]} \leq 1$ by (35).

Let $t \geq t_0 + 1$. From now on, we denote $h = h(t)$. We want to differentiate $\gamma(t)$ under the integral sign by applying Lebesgue's dominated theorem. This is allowed since

$$\left| \frac{h_t h}{x^{2-q} \dot{U}_a^q} \right| \leq \|h_t\|_{\infty, [0,1]} \|\dot{h}\|_{\infty, [0,1]} \frac{1}{x^{1-q} \dot{U}_a^q(1)} \leq \frac{M(t)}{x^{1-q} \dot{U}_a^q(1)}.$$

Hence, we have :

$$\begin{aligned}
\dot{\gamma}(t) &= 2 \int_0^1 \frac{h_t h}{x^{2-q} \dot{U}_a^q} = 2 \int_0^1 \frac{L_{U_a} h h}{x^{2-q} \dot{U}_a^q} + 2 \int_0^1 \frac{F(x, h, \dot{h}) h}{x^{2-q} \dot{U}_a^q} \\
&= -2 \int_0^1 \left[\frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} \right] + 2 \int_0^1 \frac{F(x, h, \dot{h}) h}{x^{2-q} \dot{U}_a^q} \\
&= -2(1-\epsilon) \int_0^1 \left[\frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} \right] - 2\epsilon \int_0^1 \left[\frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} \right] + 2 \int_0^1 \frac{F(x, h, \dot{h}) h}{x^{2-q} \dot{U}_a^q} \\
&\leq -2[(1-\epsilon)(\lambda_1 - 1) - \epsilon] \int_0^1 \frac{h^2}{x^{2-q}} - 2\epsilon \int_0^1 \frac{\dot{h}^2}{\dot{U}_a^q} + 2 \int_0^1 \frac{F(x, h, \dot{h}) h}{x^{2-q} \dot{U}_a^q}
\end{aligned}$$

where we have used Lemma 4.1 and Proposition 1.1. Moreover, by (36), it is easy to see that

$$\begin{aligned}
\int_0^1 \frac{F(x, h, \dot{h}) h}{x^{2-q} \dot{U}_a^q} &\leq K \left[\|\dot{h}\|_{\infty, [0,1]} + \|\dot{h}\|_{\infty, [0,1]}^2 \right] \int_0^1 \frac{h^2}{x^{2-q}} + K \int_0^1 \frac{U_a |h|}{x} x^q \dot{h}^2 \\
&\leq 2K \|\dot{h}\|_{\infty, [0,1]} \int_0^1 \frac{h^2}{x^{2-q}} + Ka \|\dot{h}\|_{\infty, [0,1]} \int_0^1 \dot{h}^2.
\end{aligned}$$

Hence, coming back to the previous calculation and applying (37), we have

$$\begin{aligned}
\dot{\gamma}(t) &\leq -2(\lambda + \delta - K \|\dot{h}\|_{\infty, [0,1]}) \int_0^1 \frac{h^2}{x^{2-q}} + \left[Ka \|\dot{h}\|_{\infty, [0,1]} - \frac{2\epsilon}{a^q} \right] \int_0^1 \dot{h}^2 \\
&\leq -2\lambda \int_0^1 \frac{h^2}{x^{2-q}} \quad \text{because of (35)} \\
&\leq -2\lambda \dot{U}_a(1)^q \gamma(t)
\end{aligned}$$

Then for all $t \geq t_0$,

$$\gamma(t) \leq C_1 \exp(-2\lambda \dot{U}_a(1)^q t)$$

where

$$C_1 = \gamma(t_0) \exp(2\lambda \dot{U}_a(1)^q t_0)$$

depends on λ and u_0 .

Since u_0 has a derivative at $x = 0$, it is clear that for some $\bar{\alpha}$ large enough, $U_{\bar{\alpha}}$ is a supersolution (see the proof of [24, Lemma 4.1]). Hence by the comparison principle (see [23, Lemma 4.1]), we have $u(t, x) \leq U_{\bar{\alpha}}(x) \leq \bar{\alpha}x$ for all $x \in [0, 1]$ and $t \geq 0$, which implies that γ is bounded. So there exists $C_2 = C_2(t_0, u_0)$ such that for all $t \in [0, t_0]$,

$$\gamma(t) \leq C_2 \exp(-2\lambda \dot{U}_a(1)^q t),$$

whence the result with $C = \max(C_1, C_2)$ depending on u_0 and λ . \square

Remark : we see that t_0 depends on λ and that $t_0 \rightarrow +\infty$ as $\lambda \rightarrow \lambda_1$. Hence, since t_0 may possibly go to infinity, then we have no bound on C . So, we cannot get the result for $\lambda = \lambda_1$, at least by this way.

Proof of Lemma 4.1. Fixing $\epsilon \in (0, 1)$, since $h \in C^2((0, 1])$ and from formula (28), we see that :

$$\int_{\epsilon}^1 \frac{L_{U_a} h h}{x^{2-q} \dot{U}_a^q} = \int_{\epsilon}^1 \frac{d}{dx} \left[\frac{\dot{h}}{\dot{U}_a^q} \right] h + \frac{h^2}{x^{2-q}} = - \int_{\epsilon}^1 \left[\frac{\dot{h}^2}{\dot{U}_a^q} - \frac{h^2}{x^{2-q}} \right] - \frac{h(\epsilon) \dot{h}(\epsilon)}{\dot{U}_a^q(\epsilon)}$$

since $h(1) = 0$. Then, since $h(0) = 0$ and $h \in C^1([0, 1])$, we have

$$\frac{h^2}{x^{2-q}} \in L^1(0, 1)$$

and

$$\frac{h(\epsilon)\dot{h}(\epsilon)}{\dot{U}_a^q(\epsilon)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Moreover,

$$\frac{d}{dx} \left[\frac{\dot{h}}{\dot{U}_a^q} \right] \in L^1(0, 1)$$

since

$$\begin{aligned} \frac{d}{dx} \left[\frac{\dot{h}}{\dot{U}_a^q} \right] &= \frac{\ddot{h}}{\dot{U}_a^q} + q \frac{U_a}{x^{2-q}\dot{U}_a} \dot{h}, \\ \left| \frac{U_a}{x} \right| &\leq a \end{aligned}$$

and because of (32). Finally, we get the result by letting ϵ go to zero since the Lebesgue's dominated theorem can be applied. \square

5 Convergence with exponential speed in $C^1([0, 1])$

We first give a regularizing estimate from L to $C^1([0, 1])$ for problem (17).

Lemma 5.1. *Let*

$$\begin{aligned} N &\geq 2, \\ 0 &< m < M, \\ U_a &= U_{a(m)} \end{aligned}$$

the unique stationary solution of (PDE_m) (equations (1)-(4)) and

$$u_0 \in Y_m.$$

Then, there exists $\bar{t} = \bar{t}(u_0) > 0$, $T = T(N, u_0) > 0$, $C = C(N, u_0) > 0$ such that, for all $t_0 \geq \bar{t}$ and $t \in (0, T]$,

$$\|u(t_0 + t) - U_a\|_{C^1([0,1])} \leq \frac{C}{t^\beta} \|u(t_0) - U_a\|_L,$$

where

$$\beta = \beta(N) = 1 + \frac{N}{4}.$$

Before giving the proof of this lemma, we need to recall some notation and well-known properties of the Dirichlet heat semigroup on the open unit ball B of \mathbb{R}^{N+2} .

Properties 5.1. *We denote $(S(t))_{t \geq 0}$ the Dirichlet heat semigroup on $B = B(0, 1) \subset \mathbb{R}^{N+2}$,*

$$C_0(\bar{B}) = \{f \in C(\bar{B}), f = 0 \text{ on } \partial B\}$$

and

$$C_0^1(\bar{B}) = \{f \in C^1(\bar{B}), f = 0 \text{ on } \partial B\}.$$

For $p > 1$,

$$\|S(t)f\|_{W^{1,p}(B)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B). \quad (38)$$

Moreover, let $p > N + 2$. Since there exists $C > 0$ such that for all $t > 0$,

$$\|S(t)f\|_{C_0^1(\bar{B})} \leq \frac{C}{\sqrt{t}} \|f\|_{C(\bar{B})} \text{ for all } f \in C_0(\bar{B})$$

and

$$\|S(t)f\|_{C(\bar{B})} \leq \frac{C}{t^{\frac{N+2}{2p}}} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B)$$

then there exists $C > 0$ such that for all $t > 0$,

$$\|S(t)f\|_{C^1(\overline{B})} \leq \frac{C}{t^\gamma} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B) \quad (39)$$

where $\gamma = \frac{1}{2} + \frac{N+2}{2p} < 1$.

Proof of Lemma 5.1. As before, we denote $h(t) = u(t) - U_a$. Now, we set

$$\begin{aligned} w(t, y) &= \frac{u(N^2t, |y|^N)}{|y|^N} \\ W_a(y) &= \frac{U_a(|y|^N)}{|y|^N} \\ f(t, y) &= \frac{h(N^2t, |y|^N)}{|y|^N} \end{aligned}$$

for all $t \geq 0$ and $y \in \overline{B}$ where

$$B = B(0, 1) \subset \mathbb{R}^{N+2}$$

denotes the open unit ball in \mathbb{R}^{N+2} . Then w is a radial classical solution of the following transformed problem called $(tPDE_m)$:

$$\begin{aligned} w_t &= \Delta w + N^2 w \left(w + \frac{y \cdot \nabla w}{N} \right)^q && \text{on } (0, T] \times \overline{B} \\ w(0) &= w_0 && \text{on } (0, T] \times \overline{B} \\ w + \frac{y \cdot \nabla w}{N} &\geq 0 && \text{on } (0, T] \times \overline{B} \\ w &= m && \text{on } [0, T] \times \partial B \end{aligned} \quad (40)$$

Here, "classical" means that for any $T > 0$,

$$w \in C([0, T] \times \overline{B}) \cap C^{1,2}((0, T] \times \overline{B}).$$

Note also that W_a is a radial stationary solution of $(tPDE_m)$ and that

$$f = w - W_a$$

which implies obviously $f = 0$ on ∂B .

All these facts rely on the following calculations relating h to \tilde{f} (and also u to w and U_a to W_a), where

$$f(t, y) = \tilde{f}(t, |y|) \text{ for all } (t, y) \in [0, +\infty) \times \overline{B}.$$

We have for $0 < t \leq T$ and $0 < x \leq 1$:

$$h(t, x) = x \tilde{f} \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right). \quad (41)$$

$$h_t(t, x) = \frac{x}{N^2} \tilde{f}_t \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right).$$

$$\begin{aligned} h_x(t, x) &= \left[\tilde{f} + \frac{r \tilde{f}_r}{N} \right] \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right) \\ &= \left[f + \frac{y \cdot \nabla f}{N} \right] \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right). \end{aligned} \quad (42)$$

$$\begin{aligned} x^{2-\frac{2}{N}} h_{xx}(t, x) &= \frac{x}{N^2} \left[\tilde{f}_{rr} + \frac{N+1}{r} \tilde{f}_r \right] \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right) \\ &= \frac{x}{N^2} \Delta f \left(\frac{t}{N^2}, x^{\frac{1}{N}} \right). \end{aligned}$$

Remark: we would like to mention that problem (40), which exhibits simple Laplacian diffusion, was already used in [23] to prove existence of a solution for (PDE_m) and in [24] to get some estimates implying relative compactness of the trajectories in $C^1([0, 1])$. Actually, the solution u was obtained from w by the formula

$$u(t, x) = x w\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right)$$

and w was obtained as a limit of solutions w^ϵ of approximations of (40) (because the nonlinearity is non Lipschitz), the regularity of w following from that of w^ϵ since for $\alpha \in [0, \frac{2}{N})$, w^ϵ have a bound in $C^{1+\alpha/2, 2+\alpha}$ uniform in ϵ . See section 4.5 in [23] for more details.

Since $\dot{U}_a > 0$ and $W_a \in C^1(\overline{B})$, then we have

$$W_a + \frac{y \cdot \nabla W_a}{N} \text{ positive and bounded on } \overline{B}. \quad (43)$$

A simple computation shows that

$$f_t = \Delta f + \Phi(y, f, \nabla f) \quad (44)$$

where

$$\begin{aligned} \Phi(y, f, \nabla f) = & N^2 f \left[W_a + \frac{y \cdot \nabla W_a}{N} + f + \frac{y \cdot \nabla f}{N} \right]^q \\ & + N^2 W_a \left(W_a + \frac{y \cdot \nabla W_a}{N} \right)^q \left[\left(1 + \frac{f + \frac{y \cdot \nabla f}{N}}{W_a + \frac{y \cdot \nabla W_a}{N}} \right)^q - 1 \right]. \end{aligned} \quad (45)$$

We observe that, since $\tilde{f}(r) = \frac{h(r^N)}{r^N}$ and B is the unit ball in \mathbb{R}^{N+2} , then

$$\|f\|_{L^2(B)}^2 = \int_0^1 |S_{N+1}| r^{N+1} \frac{h(r^N)^2}{r^{2N}} dr = \frac{|S_{N+1}|}{N} \int_0^1 \frac{h^2}{x^{2-\frac{2}{N}}} dx = \frac{|S_{N+1}|}{N} \|h\|_L^2$$

hence

$$\|f\|_{L^2(B)} = \sqrt{\frac{|S_{N+1}|}{N}} \|h\|_L.$$

Other observation : by (33), we know that for t large enough $\|h(t)\|_{C^1([0,1])}$ is as small as desired. Hence, since $h(t, 0) = 0$, we deduce from (41) that $\|f(t)\|_{C(\overline{B})}$ can be made as small as we wish for large t and then $\|y \cdot \nabla f(t)\|_{C(\overline{B})}$ also from (42). Hence, there exists

$$\bar{t}_0 = \bar{t}_0(u_0) > 0$$

such that for all $t \geq \bar{t}_0$, for all $y \in \overline{B}$,

$$\left| \frac{f + \frac{y \cdot \nabla f}{N}}{W_a + \frac{y \cdot \nabla W_a}{N}} \right| \leq \frac{1}{2}.$$

This fact, the boundedness of $W_a + \frac{y \cdot \nabla W_a}{N}$ on \overline{B} (above and below by a positive constant) and (45) imply, for any $p \geq 2$, the existence of $C > 0$ such that for all $t \geq \bar{t}_0$,

$$\|\Phi(y, f(t), \nabla f(t))\|_{L^p(B)} \leq C \|f(t)\|_{W^{1,p}(B)} \quad (46)$$

and

$$\|\Phi(y, f(t), \nabla f(t))\|_{C(\overline{B})} \leq C \|f(t)\|_{C^1(\overline{B})} \quad (47)$$

We will now use the regularizing effect of the Dirichlet heat semigroup recalled in Properties 5.1

to show a similar property for (44).

First step. We show that for all $p \in (1, +\infty)$, there exists $T_0 > 0$ and $C > 0$ such that

$$(A) \quad \|f(t_0 + t)\|_{W^{1,p}} \leq Ct^{-1/2}\|f(t_0)\|_p, \quad \text{for all } t_0 \geq \bar{t}_0 \text{ et } t \in (0, T_0]$$

and

$$(A') \quad \|f(t_0 + t)\|_p \leq C\|f(t_0)\|_p, \quad \text{pour tout } t_0 \geq \bar{t}_0 \text{ et } t \in (0, T_0].$$

Let $t_0 \geq \bar{t}$ and $t \geq t_0$.

Since w is a classical solution of $(tPDE_m)$ for $t > 0$, then f is a classical solution of (44), hence also a mild solution. So,

$$f(t_0 + t) = S(t)f(t_0) + \int_0^t S(t-s)\Phi(y, f(t_0 + s), \nabla f(t_0 + s)) ds. \quad (48)$$

Then, by (46) and (38) we obtain : (C_1 being a positive constant which may vary from line to line)

$$\|f(t_0 + t)\|_{W^{1,p}(B)} \leq \frac{C_1}{\sqrt{t}}\|f(t_0)\|_{L^p(B)} + \int_0^t \frac{C_1}{\sqrt{t-s}}\|f(t_0 + s)\|_{W^{1,p}(B)} ds,$$

from which follows

$$\sqrt{t}\|f(t_0 + t)\|_{W^{1,p}(B)} \leq C_1\|f(t_0)\|_{L^p(B)} + \sqrt{t} \int_0^t \frac{C_1}{\sqrt{s(t-s)}}\sqrt{s}\|f(t_0 + s)\|_{W^{1,p}(B)} ds.$$

We notice that $\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ by the change of variable $x = \frac{s}{t}$.

Let $T_0 = \frac{1}{4C_1^2}$. Denoting

$$a(T_0) = \sup_{t \in (t_0, t_0 + T_0]} \sqrt{t}\|f(t)\|_{W^{1,p}(B)},$$

we get

$$a(T_0) \leq C_1\|f(t_0)\|_{L^p(B)} + C_1\sqrt{T_0} a_1(T_0)$$

which, by the choice of T_0 , gives

$$a(T_0) \leq 2C_1\|f(t_0)\|_{L^p(B)}.$$

Hence, for all $t \in (t_0, T_0]$,

$$\|f(t_0 + t)\|_{W^{1,p}(B)} \leq \frac{2C_1}{\sqrt{t}}\|f(t_0)\|_{L^p(B)},$$

which proves (A) and allows thanks to (48) and (46) again to get (A').

Second step. Let us set $n = N + 2$.

We show by iteration the existence of $p \in (n, +\infty)$ and $C > 0$ independent of the solution f such that

$$(B) \quad \|f(t_0 + t)\|_{W^{1,p}(B)} \leq Ct^{-1/2-(n/2)(1/2-1/p)}\|f(t_0)\|_2 \quad \text{for all } t_0 \geq \bar{t}_0 \text{ et } t \in (0, T_0].$$

et

$$(B') \quad \|f(t_0 + t)\|_{L^p(B)} \leq Ct^{-(n/2)(1/2-1/p)}\|f(t_0)\|_2 \quad \text{for all } t_0 \geq \bar{t}_0 \text{ et } t \in (0, T_0].$$

Indeed, this is true for $p = 2$ thanks to (A) and (A').

Assume that (B) and (B') are true for some $p \in [2, +\infty)$.

If $p < n$, then we prove that (B) and (B') are true for $p = p^*$, $p^* < +\infty$ being the optimal exponent such that we have the following Sobolev imbedding

$$W^{1,p}(B) \subset L^{p^*}(B).$$

Indeed, we have by (A) and Sobolev embedding that

$$\begin{aligned} \|f(t_0 + t)\|_{W^{1,p^*}} &\leq C(t/2)^{-1/2} \|f(t_0 + (t/2))\|_{p^*} \leq C(t/2)^{-1/2} \|f(t_0 + (t/2))\|_{W^{1,p}} \\ &\leq C(t/2)^{-1/2} (t/2)^{-1/2 - (n/2)(1/2 - 1/p)} \|f(t_0)\|_2 \\ &= C(t/2)^{-1/2 - (n/2)(1/2 - 1/p + 1/n)} \|f(t_0)\|_2 = C' t^{-1/2 - (n/2)(1/2 - 1/p^*)} \|f(t_0)\|_2, \end{aligned}$$

and, by Sobolev embedding and (B), we have

$$\|f(t_0 + t)\|_{p^*} \leq C \|f(t_0 + t)\|_{W^{1,p}} \leq C t^{-1/2 - (n/2)(1/2 - 1/p)} \|f(t_0)\|_2 = C t^{-(n/2)(1/2 - 1/p^*)} \|f(t_0)\|_2.$$

Iterating this process, we obtain after a finite number of steps some $p \in [n, +\infty)$ such that (B) and (B') are true.

- If $p > n$, this is the result we wanted.
- If $p = n$, since B (resp. (B')) is true for $p = 2$ and $p = n$, we can interpolate between $W^{1,2}(B)$ and $W^{1,n}(B)$ (resp. $L^2(B)$ and $L^n(B)$) and get (B) (resp. (B')) for some $p_0 \in (\frac{n}{2}, n)$, which, by an application of the previous process, shows (B) (resp. (B')) for $p = p_0^* > n$.

Last step. We can now prove the result by making a last iteration.

Coming back to (48), by (47) and (39) we obtain :

$$\|f(t_0 + t)\|_{C^1(\overline{B})} \leq \frac{C}{t^\gamma} \|f(t_0)\|_{L^p(B)} + \int_0^t \frac{C}{\sqrt{t-s}} \|f(t_0 + s)\|_{C^1(\overline{B})} ds,$$

from which follows

$$t^\gamma \|f(t_0 + t)\|_{C^1(\overline{B})} \leq C \|f(t_0)\|_{L^p(B)} + t^\gamma \int_0^t \frac{C}{s^\gamma \sqrt{t-s}} s^\gamma \|f(t_0 + s)\|_{C^1(\overline{B})} ds.$$

Note that $t^\gamma \int_0^t \frac{ds}{s^\gamma \sqrt{t-s}} = \int_0^1 \frac{dx}{x^\gamma \sqrt{1-x}} \sqrt{t}$ which is well defined since $\gamma = \frac{1}{2} + \frac{n}{2p} < 1$.

Let $T = \frac{1}{4C^2}$. Denoting

$$b(T) = \sup_{t \in (t_0, t_0 + T]} t^\gamma \|f(t)\|_{C^1(\overline{B})},$$

we get

$$b(T) \leq C \|f(t_0)\|_{L^p(B)} + C\sqrt{T} a(T)$$

which, by the choice of T, gives

$$b(T) \leq 2C \|f(t_0)\|_{L^p(B)}.$$

Hence, for all $t \in (t_0, T]$,

$$\|f(t_0 + t)\|_{C^1(\overline{B})} \leq \frac{2C}{t^\gamma} \|f(t_0)\|_{L^p(B)},$$

which implies by (B') that

$$\|f(t_0 + t)\|_{C^1(\overline{B})} \leq \frac{C'}{t^{\frac{1}{2} + \frac{n}{2p}}} \|f(t_0 + t/2)\|_{L^p(B)} \leq \frac{C'}{t^{\frac{1}{2} + \frac{n}{4}}} \|f(t_0)\|_{L^2(B)}.$$

This implies the result since

$$\|h(t_0 + t)\|_{C^1([0,1])} \leq C_1 \|f(t_0 + t)\|_{C^1(\overline{B})} \leq \frac{C_2}{t^{\frac{1}{2} + \frac{n}{4}}} \|f(t_0)\|_{L^2(B)} = \frac{C_3}{t^{\frac{1}{2} + \frac{n}{4}}} \|h(t_0)\|_L.$$

□

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. This follows from Lemma 1.1 and from Lemma 5.1, which is a regularizing in time estimate. Indeed, using notation of Lemma 5.1 (having fixed $p > N$), let $t \geq \bar{t} + T$. Then $t - T \geq \bar{t}$ so we obtain,

$$\|u(t) - U_a\|_{C^1([0,1])} \leq \frac{C}{T^\gamma} \|u(t - T) - U_a\|_L \leq C(u_0, p) \exp(-\lambda \dot{U}_a(1)^q (t - T)),$$

which gives the result. \square

6 Appendix : proofs of the preliminary results for dimension $N = 2$

In this section, we first recall some basic facts about continuous dynamical systems and Lyapunov functionals. In the next subsections are the proofs of all results of section 2.

6.1 Reminder on continuous dynamical systems and Lyapunov functionals

For reader's convenience, we fast recall some very basic facts on continuous dynamical systems, which are general but will be given in the context of

$$Y_m^1 = Y_m \cap C^1([0, 1])$$

endowed with the induced topology of $C^1([0, 1])$. For reference, see [10, chap. 9].

Here follow the definitions of a continuous dynamical system, its trajectories, stationary points and ω -limit sets.

Definition 6.1. *A continuous dynamical system on Y_m^1 is a one-parameter family of mappings $(T(t))_{t \geq 0}$ from Y_m^1 to Y_m^1 such that :*

- i) $T(0) = Id.$
- ii) $T(t + s) = T(t)T(s)$ for any $t, s \geq 0.$
- iii) For any $t \geq 0, T(t) \in C(Y_m^1, Y_m^1).$
- iv) For any $u_0 \in Y_m, t \mapsto T(t)u_0 \in C((0, \infty), Y_m^1).$

Definition 6.2. *Let $u_0 \in Y_m^1.$*

- u_0 is a stationary point if for all $t \geq 0, T(t)u_0 = u_0.$
- $\gamma_1(u_0) = \{T(t)u_0, t \geq 1\}$ is the trajectory of u_0 from $t = 1.$
- $\omega(u_0) = \{v \in Y_m^1, \exists t_n \rightarrow +\infty, t_n \geq 1, T(t_n)u_0 \xrightarrow[n \rightarrow +\infty]{} v \text{ in } Y_m^1\}$ is the ω -limit set of $u_0.$

Now we give the definition of a strict Lyapunov functional and Lasalle's invariance principle.

Definition 6.3.

- i) $\mathcal{F} \in C(Y_m^1, \mathbb{R})$ is a Lyapunov functional if for all $u_0 \in Y_m^1,$
 $t \mapsto \mathcal{F}[T(t)u_0]$ is nonincreasing on $[0, +\infty).$
- ii) A Lyapunov functional \mathcal{F} is a strict Lyapunov functional if

$$\mathcal{F}[T(t)u_0] = \mathcal{F}[u_0] \text{ for all } t \geq 0 \text{ implies that } u_0 \text{ is an equilibrium point.}$$

Proposition 6.1. *Lasalle's invariance principle.*

Let $u_0 \in Y_m^1.$ Assume that the dynamical system $(T(t))_{t \geq 0}$ admits a strict Lyapunov functional and that $\gamma_1(u_0)$ is relatively compact in $Y_m^1.$

Then the ω -limit set $\omega(u_0)$ is nonempty and consists of equilibria of the dynamical system.

See [10, p. 143] for a proof.

6.2 Wellposedness and regularity for problem (PDE_m)

We first remark that there is a classical comparison principle available for problem (PDE_m) , which will for instance imply the uniqueness of the maximal classical solution in Theorem 2.1.

Lemma 6.1. *Let $T > 0$. Assume that :*

- $u_1, u_2 \in C([0, T] \times [0, 1]) \cap C^1((0, T] \times [0, 1]) \cap C^{1,2}((0, T] \times (0, 1))$.
- For all $t \in (0, T]$, $u_1(t)$ and $u_2(t)$ are nondecreasing.
- There exists $i_0 \in \{1, 2\}$ and some $\gamma < 1$ such that

$$\sup_{t \in (0, T]} t^\gamma \|u_{i_0}(t)\|_{C^1([0, 1])} < \infty. \quad (49)$$

Suppose moreover that :

$$u_{1t} \leq x u_{1xx} + u_1 u_{1x} \quad \text{for all } (t, x) \in (0, T] \times (0, 1) \quad (50)$$

$$u_{2t} \geq x u_{2xx} + u_2 u_{2x} \quad \text{for all } (t, x) \in (0, T] \times (0, 1) \quad (51)$$

$$u_1(0, x) \leq u_2(0, x) \quad \text{for all } x \in [0, 1] \quad (52)$$

$$u_1(t, 0) \leq u_2(t, 0) \quad \text{for } t \geq 0 \quad (53)$$

$$u_1(t, 1) \leq u_2(t, 1) \quad \text{for } t \geq 0 \quad (54)$$

Then $u_1 \leq u_2$ on $[0, T] \times [0, 1]$.

The proof of this result was given in [20] under weaker assumptions. We give a different one in this simpler context.

Proof of Lemma 6.1. Let us set

$$z = (u_1 - u_2) e^{-\int_0^t (\|u_{i_0}(s)\|_{C^1} + 1) ds},$$

well defined thanks to (49). The hypotheses made show that

$$z \in C([0; T] \times [0; 1]) \cap C^1((0, T] \times [0, 1]) \cap C^{1,2}((0, T] \times (0, 1)).$$

Assume now by contradiction that $\max_{[0; T] \times [0; 1]} z > 0$.

By assumption, $z \leq 0$ on the parabolic boundary of $[0, T] \times [0, 1]$.

Hence, $\max_{[0; T] \times [0; 1]} z$ is reached at a point $(t_0, x_0) \in (0; T] \times (0; 1)$.

Then $z_x(t_0, x_0) = 0$ so $(u_1)_x(t_0, x_0) = (u_2)_x(t_0, x_0)$.

Moreover, $z_{xx}(t_0, x_0) \leq 0$ and $z_t(t_0, x_0) \geq 0$. But we have

$$z_t(t_0, x_0) \leq x z_{xx}(t_0, x_0) + [(u_{i_0})_x(t_0, x_0) - \|u_{i_0}(t_0)\|_{C^1} - 1] z(t_0, x_0).$$

The LHS of the inequality is nonnegative and the RHS is negative, whence the contradiction. \square

Before coming to the proof of Theorem 2.1, we need to fix some notation and recall some facts about the Dirichlet heat semigroup.

For reference, see for instance the book [22] of A. Lunardi.

Notation 6.1.

- B denotes the open unit ball in \mathbb{R}^4 .
- $Z_0 = \{W \in C(\overline{B}), W|_{\partial B} = 0\}$.

- $(S(t))_{t \geq 0}$ denotes the heat semigroup on Z_0 . It is the restriction on Z_0 of the Dirichlet heat semigroup on $L^2(B)$.
- $(X_\theta)_{\theta \in [0,1]}$ denotes the scale of interpolation spaces for $(S(t))_{t \geq 0}$, where $X_0 = Z_0$, $X_1 = D(-\Delta)$ and $X_\alpha \hookrightarrow X_\beta$ with dense continuous injection for any $\alpha > \beta$, $(\alpha, \beta) \in [0, 1]^2$.

Properties 6.1.

- $X_{\frac{1}{2}} = \{W \in C^1(\overline{B}), W|_{\partial B} = 0\}$.
- Let $\gamma_0 \in (0; \frac{1}{2}]$. For any $\gamma \in [0, 2\gamma_0)$,

$$X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$$

with continuous embedding.

- There exists $C_D \geq 1$ such that for any $\theta \in [0; 1]$, $W \in X_0$ and $t > 0$,

$$\|S(t)W\|_{X_\theta} \leq \frac{C_D}{t^\theta} \|W\|_\infty.$$

We just want to introduce some specific notation we are going to use.

Notation 6.2. Let $(a, b) \in (0, 1)^2$. We denote $I(a, b) = \int_0^1 \frac{ds}{(1-s)^a s^b}$. For all $t \geq 0$, $\int_0^t \frac{ds}{(t-s)^a s^b} = t^{1-a-b} I(a, b)$.

Notation 6.3. Let $m \geq 0$ and $\gamma > 0$.

- $Y_m = \{u \in C([0; 1]) \text{ nondecreasing, } u'(0) \text{ exists, } u(0) = 0, u(1) = m\}$.
- $Z_m = \{w \in C(\overline{B}), w|_{\partial B} = m\}$.
- $Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0, 1]), \sup_{x \in (0,1]} \frac{|u'(x) - u'(0)|}{x^\gamma} < \infty\}$.
- $Z_m^{1,\gamma} = \{w \in Z_m \cap C^1(\overline{B}), \sup_{y \in \overline{B} \setminus \{0\}} \frac{|\nabla w(y)|}{|y|^\gamma} < \infty\}$.

Proof of Theorem 2.1. We begin by giving a short proof of points i)ii)iii)iv).

We define the following transformation θ_0 , already remarked in [7, section 2.2] and [13, section 2], and also used in [20] :

$$\begin{aligned} \theta_0 : Y_m &\longrightarrow Z_m \\ u &\longrightarrow w \text{ where } w(y) = \frac{u(|y|^2)}{|y|^2} \text{ for all } y \in \overline{B} \setminus \{0\}. \end{aligned}$$

The next lemma has been proved in [23, Lemma 4.3].

Lemma 6.2. Let $m \geq 0$.

- i) θ_0 sends Y_m into Z_m .
- ii) If $\gamma > \frac{1}{2}$, then θ_0 sends $Y_m^{1,\gamma}$ into $Z_m^{1,2\gamma-1}$.

If $u_0 \in Y_m$, we set

$$w_0 = \theta_0(u_0) \in Z_m$$

and

$$w(t, y) = \frac{u(4t, |y|^2)}{|y|^2}$$

for all $y \in \overline{B}$.

Then we obtain a transformed problem called $(tPDE_m)$ with simple Laplacian diffusion in $B \subset \mathbb{R}^4$:

$$w_t = \Delta w + 4w \left(w + \frac{y \cdot \nabla w}{2} \right) \quad \text{on } (0, T] \times \overline{B} \quad (55)$$

$$w(0) = w_0 \quad (56)$$

$$w + \frac{y \cdot \nabla w}{N} \geq 0 \quad \text{on } (0, T] \times \overline{B} \quad (57)$$

$$w = m \quad \text{on } [0, T] \times \partial B \quad (58)$$

This relies on the following calculations relating u to \tilde{w} , where we denote

$$w(t, y) = \tilde{w}(t, |y|) \text{ for all } (t, y) \in [0, +\infty) \times \overline{B}.$$

We have for $0 < t \leq T$ and $0 < x \leq 1$:

$$u(t, x) = x \tilde{w} \left(\frac{t}{4}, \sqrt{x} \right). \quad (59)$$

$$u_t(t, x) = \frac{x}{4} \tilde{w}_t \left(\frac{t}{4}, \sqrt{x} \right).$$

$$\begin{aligned} u_x(t, x) &= \left[\tilde{w} + \frac{r \tilde{w}_r}{2} \right] \left(\frac{t}{4}, \sqrt{x} \right) \\ &= \left[w + \frac{y \cdot \nabla w}{2} \right] \left(\frac{t}{4}, \sqrt{x} \right). \end{aligned} \quad (60)$$

$$\begin{aligned} x u_{xx}(t, x) &= \frac{x}{4} \left[\tilde{w}_{rr} + \frac{3}{r} \tilde{w}_r \right] \left(\frac{t}{4}, \sqrt{x} \right) \\ &= \frac{x}{4} \Delta w \left(\frac{t}{4}, \sqrt{x} \right). \end{aligned}$$

The existence of a unique maximal classical solution w on $[0, T^*)$ of problem $(tPDE_m)$ with initial condition $w_0 \in Z_m$, i.e. a function

$$w \in C([0, T^*) \times \overline{B}) \cap C^{1,2}((0, T^*) \times \overline{B})$$

satisfying (55)(56)(57)(58) is standard.

Indeed, we can set $W = w - m$, get a corresponding equation for W , obtain by a fixed point argument a mild solution W on $[0, \tau^*]$ for some small $\tau^* > 0$ by use of the Dirichlet heat semigroup since the nonlinearity is locally Lipschitz in $(w, \nabla w)$ on $C^1(\overline{B})$, and finally exploit regularity results to prove that the solution is classical.

Moreover,

$$\sup_{t \in (0, \tau^*]} \sqrt{t} \|w(t)\|_{C^1(\overline{B})} < \infty \quad (61)$$

Again by iteration of regularity results on (55), it can also be proved that

$$w \in C^\infty((0, T^*) \times \overline{B}).$$

Since $\tau^* = \tau^*(\|w_0\|_{\infty, \overline{B}})$, we also get the blow-up alternative

$$T^* = +\infty \quad \text{or} \quad \lim_{t \rightarrow T^*} \|w(t)\|_{\infty, \overline{B}} = +\infty.$$

Now, we come back to problem (PDE_m) . Let $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$. By uniqueness, w is radial because w_0 is. Then, if we set

$$T_{max}(u_0) = 4T^*(w_0).$$

and for all $(t, x) \in [0, T_{max}) \times [0, 1]$,

$$u(t, x) = x \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right), \quad (62)$$

then we can check that u is the unique maximal classical solution of problem (PDE_m) with initial condition u_0 . The fact that u is nondecreasing on $[0, 1]$ will be shown in vii). It is easy to see that the small existence time $\tau^* = \tau^*(\|w_0\|_{\infty, \overline{B}})$ for w gives a small existence time $\tau = \tau(\mathcal{N}[u_0])$ for u , i.e. for each $K > 0$, there exists $\tau = \tau(K) > 0$ such that if $\mathcal{N}[u_0] \leq K$ then the solution u is at least defined on $[0, \tau]$. Moreover, the regularity of w implies the results of regularity on u . Hence, we have proved i)ii)iii) and iv).

v) If $u_0 \in Y_m^{1, \gamma}$ with $\gamma > 1/2$, then from Lemma 6.2 ii),

$$w_0 = \theta_0(u_0) \in Z_m^{1, 2\gamma-1} \subset C^1(\overline{B}).$$

We only have to check the continuity at $t = 0$ of $t \mapsto w(t) \in C^1(\overline{B})$. This is clear by the variation of constants formula since $t \mapsto S(t)\Phi \in C([0, +\infty), X_{\frac{1}{2}})$ for any $\Phi \in X_{\frac{1}{2}}$. Hence, we get a maximal classical solution

$$w \in C([0, T^*), C^1(\overline{B}))$$

which, thanks to formula (60), gives a maximal classical solution of (PDE_m)

$$u \in C([0, T_{max}), C^1([0, 1])).$$

vi) Let $(t, x) \in (0, T_{max}(u_0)) \times (0, 1]$. From formulas (59) and (60), we have

$$u(t, x) = x \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right)$$

and

$$u_x(t, x) = \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right) + \sqrt{x} \frac{\tilde{w}_r}{2}\left(\frac{t}{4}, \sqrt{x}\right).$$

These formulas allow to prove that $u(t) \in C^1([0, 1])$ with $u_x(t, 0) = \tilde{w}(\frac{t}{4}, 0)$. Since $w(\frac{t}{4})$ is radial, then $\tilde{w}_r(\frac{t}{4}, 0) = 0$. This implies that for any $y \in [0, 1]$,

$$\left| \tilde{w}\left(\frac{t}{4}, y\right) - \tilde{w}\left(\frac{t}{4}, 0\right) \right| \leq K \frac{y^2}{2}$$

and

$$\left| \tilde{w}_r\left(\frac{t}{4}, y\right) \right| \leq K y$$

where $K = \|\tilde{w}(\frac{t}{4})_{rr}\|_{\infty, [0, 1]}$. So, we obtain

$$|u_x(t, x) - u_x(t, 0)| \leq K x.$$

Hence, $u(t) \in Y_m^{1, 1}$.

vii) Let us now show that $u_x(t, x) > 0$ for all $(t, x) \in (0, T_{max}) \times [0, 1]$.

We prove the result in two steps. Let $T \in (0, T_{max})$.

First step : We now show that $v := u_x \geq 0$ on $(0, T] \times [0, 1]$.

We divide the proof in three parts.

- First part : We show the result for any $u_0 \in Y_m^{1, \gamma}$ where $\gamma > \frac{1}{2}$.
Since u satisfies on $(0, T] \times (0, 1]$

$$u_t = x u_{xx} + u u_x \quad (63)$$

and thanks to iv), we can now differentiate this equation with respect to x . We denote

$$b = 1 + u$$

and obtain the partial differential equation satisfied by v :

$$v_t = x v_{xx} + b v_x + v^2 \quad \text{on} \quad (0, T) \times (0, 1) \quad (64)$$

$$v(0, \cdot) = (u_0)' \quad (65)$$

$$v(t, 0) = u_x(t, 0) \quad \text{for} \quad t \in (0, T] \quad (66)$$

$$v(t, 1) = u_x(t, 1) \quad \text{for} \quad t \in (0, T] \quad (67)$$

By vi), we know that $u \in C([0, T], C^1([0, 1]))$, then $v \in C([0, T] \times [0, 1])$ and v reaches its minimum on $[0, T] \times [0, 1]$.

By comparison principle, we have

$$0 \leq u \leq m$$

so

$$u_x(t, 0) \geq 0$$

and

$$u_x(t, 1) \geq 0$$

for all $t \in (0, T]$. Then, from (65), (66) and (67), $v \geq 0$ on the parabolic boundary of $[0, T] \times [0, 1]$. From (64), we see that v cannot reach a negative minimum in $(0, T) \times (0, 1)$. So $v \geq 0$ on $[0, T] \times [0, 1]$.

- Second part : We show that if $u_0 \in Y_m$, there exists $\tau \in (0, T)$ such that for all $t \in [0, \tau]$, $u(t)$ is nondecreasing on $[0, 1]$.

Let $u_0 \in Y_m$. By Lemma 4.4 in [23], there exists a sequence $(u_{0,n})_{n \geq 1}$ of $Y_m^{1,1}$ such that

$$\|u_{0,n} - u_0\|_{\infty, [0,1]} \xrightarrow{n \rightarrow \infty} 0$$

and

$$\mathcal{N}[u_{0,n}] \leq \mathcal{N}[u_0].$$

Since $\mathcal{N}[u_{0,n}]$ is bounded, we know by ii) that there exists a common small existence time $\tau \in (0, T)$ for all solutions $(u_n(t))_{t \geq 0}$ of problem (PDE_m) with initial condition $u_{0,n}$. From first part, we know that for all $t \in [0, \tau]$ $u_n(t)$ is a nondecreasing function since $u_{0,n} \in Y_m^{1,1}$. To prove the result, it is sufficient to show that

$$\|u_n - u\|_{\infty, [0,1] \times [0, \tau]} \xrightarrow{n \rightarrow \infty} 0.$$

Let $\eta > 0$. By (21), there exists $C > 0$ such that for all $t \in [0, \tau]$, $\|u(t)_x\|_{\infty} \leq \frac{C}{\sqrt{t}}$. So we can choose $\eta' > 0$ such that

$$\eta' e^{\int_0^\tau [\|u(t)_x\|_{\infty} + 1] dt} \leq \eta$$

Let $n_0 \geq 1$ such that for all $n \geq n_0$, $\|u_{0,n} - u_0\|_{\infty, [0,1]} \leq \eta'$. Let $n \geq n_0$.

Let us set

$$z(t) = [u_n(t) - u(t)] e^{-\int_0^\tau [\|u(t)_x\|_{\infty} + 1] dt}$$

We see that z satisfies

$$z_t = x z_{xx} + b z_x + c z \quad (68)$$

where $b = u_n(t)$ and $c = [u_x - \|(u)_x\|_{\infty} - 1] < 0$.

Since $z \in C([0, \tau] \times [0, 1])$, z reaches its maximum and its minimum.

Assume that this maximum is greater than η' . Since $z = 0$ for $x = 0$ and $x = 1$ and $z \leq \eta'$ for $t = 0$, it can be reached only in $(0, \tau) \times (0, 1)$ but this is impossible because $c < 0$ and (68). We make the similar reasoning for the minimum. Hence, $|z| \leq \eta'$ on $[0, \tau] \times [0, 1]$.

Eventually, $\|u_n - u\|_{\infty, [0,1] \times [0, \tau]} \leq \eta' e^{\int_0^\tau [\|u(t)_x\|_{\infty} + 1] dt} \leq \eta$ for all $n \geq n_0$. Whence the result.

- Last part : Let $u_0 \in Y_m$. From the second part, there exists $\tau \in (0, T)$ such that for all $t \in [0, \tau]$, $u(t)$ is nondecreasing. Since $u \in C([\tau, T], C^1([0, 1]))$ and $u(\tau) \in Y_m^{1,1}$, we can apply the same argument as in the first part to deduce that for all $t \in [\tau, T]$, $u(t)$ is nondecreasing. This concludes the proof of the first step.

Second step : Let us show that $v > 0$ on $(0, T] \times [0, 1]$.

0 is clearly a subsolution of problem $(tPDE_m)$ so $w \geq 0$ on \bar{B} but by strong maximum principle we even have

$$w > 0$$

on B (see [12, Theorem 5 p.39]). Then, from formula (60) it follows that

$$v(t, 0) = u_x(t, 0) > 0$$

for $t \in (0, T]$.

Assume by contradiction that v is zero at some point in $(0, T) \times (0, 1)$.

Since v satisfies (64) and the underlying operator is parabolic on $(0, T) \times (0, 1]$, by the strong minimum principle (see [12, Theorem 5 p.39]), we deduce that $v = 0$ on $(0, T) \times (0, 1)$. Then, by continuity, $v(t, 0) = 0$ for $t \in (0, T)$ which contradicts the previous assertion.

Suppose eventually that $v(t, 1) = 0$ for some $t \in (0, T)$. From (63), we deduce that $u_{xx}(t, 1) = 0$, ie

$$v_x(t, 1) = 0.$$

Since $v^2 \geq 0$, we observe that v satisfies :

$$v_t \geq x v_{xx} + [1 + u]v_x \tag{69}$$

Since $v > 0$ on $(0, T) \times [\frac{1}{2}, 1)$ and the underlying operator in the above equation is uniformly parabolic on $(0, T) \times [\frac{1}{2}, 1]$, we can apply Hopf's minimum principle (cf. [27, Theorem 3, p.170]) to deduce that $v_x(t, 1) < 0$ what yields a contradiction. In conclusion, $u_x > 0$ on $(0, T] \times [0, 1]$ for all $T < T_{max}$, whence the result. □

6.3 Subcritical case : Lyapunov functional and convergence in $C^1([0, 1])$

Here are the proofs of results in subsection 2.2.

Proof of Lemma 2.1. $m = 0$ is trivial so we assume $0 < m < 2$.

Let $T_{max} = T_{max}(u_0)$.

From Theorem 2.1, in order to get $T_{max} = +\infty$, it is sufficient to prove that

$$\sup_{t \in [0, T_{max})} \mathcal{N}[u(t)] < \infty.$$

This fact easily follows from a comparison with a supersolution of problem (PDE_m) . The main idea is that since $m < 2$, if a_0 is large enough then

$$u_0 \leq U_{a_0}$$

and U_{a_0} is then a supersolution so for all $t \in [0, T_{max})$, $0 \leq u(t) \leq U_{a_0}$ hence

$$\mathcal{N}[u(t)] \leq a_0$$

since U_{a_0} is concave.

Now we give an explicit formula for a_0 which will end the proof. We denote

$$a = \frac{m}{1 - \frac{m}{2}}$$

which defines the unique steady state, i.e. satisfying $U_a(1) = m$.

First, since u_0 is differentiable at $x = 0$, $x \mapsto \frac{u_0(x)}{x}$ can be extended continuously to $[0; 1]$, so $m \leq \mathcal{N}[u_0] < +\infty$.

Let us set $x_0 = \frac{m}{\mathcal{N}[u_0]} \in (0, 1]$. We can check that for

$$a_0 = \frac{a}{x_0}$$

we have

$$U_{a_0}(x_0) = U_{a_0 x_0}(1) = m.$$

- For $x \in [0; x_0]$, $u_0(x) \leq \mathcal{N}[u_0]x \leq U_{a_0}(x)$ since by concavity, U_{a_0} is above its chord between $x = 0$ and $x = x_0$.

- For $x \in [x_0, 1]$, $u_0(x) \leq m = U_{a_0}(x_0) \leq U_{a_0}(x)$ since U_{a_0} is increasing.

Hence, $u_0 \leq U_{a_0}$ on $[0, 1]$ where

$$a_0 = \frac{\mathcal{N}[u_0]}{1 - \frac{m}{2}}$$

Remark : we actually proved the following stronger result, to be used in the next proof.

For each $K > 0$, for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, we have

$$\sup_{t \in [0, \infty)} \mathcal{N}[u(t)] \leq \frac{K}{1 - \frac{m}{2}}.$$

□

Proof of Lemma 2.2. Actually, we will prove the following stronger result :

If $m < 2$, $\gamma \in [0; 1)$, $t_0 > 0$ and $K > 0$, then there exists $D_K > 0$ such that for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, we have

$$\sup_{t \geq t_0} \|u(t)\|_{C^1, \frac{\gamma}{N}} \leq D_K.$$

Let $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. Let $w_0 = \theta_0(u_0)$.

First step : thanks to the final remark in the proof of Lemma 2.1, we have

$$\sup_{t \in [0, \infty)} \mathcal{N}[u(t)] \leq C_K := \frac{K}{1 - \frac{m}{2}}.$$

Since for $t \geq 0$, $\|w(t)\|_{\infty, \bar{B}} = \mathcal{N}[u(\frac{t}{4})]$, we deduce that w is global and that

$$\sup_{t \in [0, \infty)} \|w(t)\|_{\infty, \bar{B}} = \sup_{t \in [0, \infty)} \mathcal{N}[u(t)] \leq C_K.$$

Second step : Let

$$\tau = \frac{t_0}{4}$$

and $t \in [0, \tau]$.

Denoting $W_0 = w_0 - m$, then

$$w(t) - m = S(t)W_0 + 4 \int_0^t S(t-s)w \left(w + \frac{x \cdot \nabla w}{2} \right) ds, \quad (70)$$

so

$$\|w(t)\|_{C^1} \leq m + \frac{C_D}{\sqrt{t}}(C_K + m) + 4 \int_0^t \frac{C_D}{\sqrt{t-s}} C_K \|w(s)\|_{C^1} ds.$$

Setting $h(t) = \sup_{s \in (0, t]} \sqrt{s} \|w(s)\|_{C^1}$, we have $h(t) < \infty$ by (61) and

$$\sqrt{t} \|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(C_K + m) + 4C_K C_D \sqrt{t} \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} h(s) ds,$$

$$\sqrt{t}\|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(m + C_K) + 4C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right) \sqrt{t} h(t).$$

Let $T \in (0, \tau]$. Then,

$$h(T) \leq m\sqrt{\tau} + C_D(m + C_K) + 4C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right) \sqrt{T} h(T). \quad (71)$$

Setting $A = m\sqrt{\tau} + C_D(m + C_K)$ and $B = 8C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right)$, assume that there exists $T \in [0, \tau]$ such that

$$h(T) = 2A.$$

Then,

$$2A \leq A + \frac{B}{2}\sqrt{T} 2A \text{ which implies } T \geq \frac{1}{B^2}.$$

Let us set

$$\tau' = \min\left(\tau, \frac{1}{2B^2}\right).$$

Since $h \geq 0$ is nondecreasing, $h_0 = \lim_{t \rightarrow 0^+} h(t)$ exists and $h_0 \leq A$ by (71). So by continuity of h on $(0, \tau']$, $h(t) \leq 2A$ for all $t \in (0, \tau']$, that is to say :

$$\|w(t)\|_{C^1} \leq \frac{2A}{\sqrt{t}} \text{ for all } t \in (0, \tau'],$$

where A and τ' only depend on K . Then, setting $A_K = 2A$, we have

$$\sup_{t \in [0, \tau']} \sqrt{t}\|w(t)\|_{C^1} \leq A_K.$$

Third step : Let $\gamma_0 \in (\frac{\gamma}{2}, \frac{1}{2})$ and $t \in [0, \tau']$.

Setting $W = w - m$ and $W_0 = w_0 - m$, then for $t \geq 0$, due to (70), we get

$$\|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} \leq \frac{C_D}{t^{\frac{1}{2}+\gamma_0}}(C_K + m) + 4 \int_0^t \frac{C_D}{(t-s)^{\frac{1}{2}+\gamma_0}} C_K \frac{A_K}{\sqrt{s}} ds.$$

Then we deduce that :

$$\begin{aligned} t^{\frac{1}{2}+\gamma_0} \|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} &\leq C_D(C_K + m) + 4C_K C_D A_K t^{\frac{1}{2}+\gamma_0} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\gamma_0} \sqrt{s}} ds \\ &\leq C_D(m + C_K) + 4C_K C_D A_K I\left(\frac{1}{2} + \gamma_0, \frac{1}{2}\right) \sqrt{\tau'}. \end{aligned}$$

Hence, since $X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$, we deduce that there exists $A'_K > 0$ depending only on K such that

$$\|w(\tau')\|_{C^{1,\gamma}(\overline{B})} \leq \frac{A'_K}{\tau'^{\frac{1}{2}+\gamma_0}} =: A''_K.$$

Last step : Let $t' \geq \frac{t_0}{4}$. Since $\tau' \leq \frac{t_0}{4}$, we can apply the same arguments by taking $w_0(t' - \tau)$ as initial data instead of w_0 , so we obtain

$$\text{for all } t' \geq \frac{t_0}{4}, \|w(t')\|_{C^{1,\gamma}(\overline{B})} \leq A''_K.$$

Finally, coming back to $u(t)$, thanks to formula (62), we get an upper bound D_K for $\|u(t)\|_{C^{1,\frac{\gamma}{N}}}$ valid for any $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. \square

Proof of Lemma 2.3. Thanks to Lemma 2.1, we know that $T(t)$ is well defined for all $t \geq 0$ and by definition of a classical solution, $T(t)$ maps Y_m^1 into Y_m^1 .

ii) is clear by uniqueness of the global classical solution.

iv) comes from the fact that $u \in C((0, \infty), C^1([0, 1]))$.

iii) Let $t > 0$, $u_0 \in Y_m^1$ and $(u_n)_{n \geq 1} \in Y_m^1$.

Assume that $u_n \xrightarrow[n \rightarrow \infty]{C^1} u_0$. Let us show that $u_n(t) \xrightarrow[n \rightarrow \infty]{C^1} u(t)$.

We proceed in two steps.

First step : We want to show that if $u_n \xrightarrow[n \rightarrow \infty]{C^1} u_0$, then $u_n(t) \xrightarrow[n \rightarrow \infty]{C^0} u(t)$.

Actually, this has already been done in the proof of Theorem 2.1 vii) (in the first step, second part). Indeed, the argument there shows that if all the u_n exist on a common interval $[0, T_0]$, then we have

$$\|u_n - u\|_{\infty, [0,1] \times [0, T_0]} \xrightarrow[n \rightarrow \infty]{} 0.$$

But here, for all n , $T_{max}(u_n) = +\infty$, so this result can be applied to $T_0 = t$, which implies the result.

Second step : since $u_n \xrightarrow[n \rightarrow \infty]{C^1} u_0$, $\|u_n\|_{C^1}$ is bounded so there exists $K > 0$ such that for all $n \geq 1$, $\mathcal{N}[u_n] \leq K$. Then, from Lemma 2.2, since $t > 0$, $\{u_n(t), n \geq 1\}$ is relatively compact in Y_m^1 and has a single accumulation point $u(t)$ from first step. Whence the result. \square

Proof of Lemma 2.4. Let $t > 0$. We can differentiate the integral by applying Lebesgue's dominated theorem. Indeed, let $\eta > 0$ small enough so that

$$I = [t - \eta, t + \eta] \subset (0, T_{max}).$$

Note : here, for $0 \leq m < 2$, $T_{max} = +\infty$.

Since $u \in C(I, C^1([0, 1]))$, then $\frac{u}{x}$ is bounded on $I \times [0, 1]$. Since moreover, by Theorem 2.1 vii), for all $t \in I$, $u_x(t) > 0$ on $[0, 1]$, then $\ln(u_x)$ is bounded on $I \times [0, 1]$.

Let $(t, x) \in (0, T_{max}(u_0)) \times (0, 1]$. We recall that

$$u_x(t, x) = \tilde{w} \left(\frac{t}{4}, \sqrt{x} \right) + \sqrt{x} \frac{\tilde{w}_r}{2} \left(\frac{t}{4}, \sqrt{x} \right).$$

Hence,

$$u_{xx}(t, x) = \frac{3}{4\sqrt{x}} \tilde{w}_r \left(\frac{t}{4}, \sqrt{x} \right) + \frac{1}{4} \tilde{w}_{rr} \left(\frac{t}{4}, \sqrt{x} \right)$$

and

$$u_{xxx}(t, x) = -\frac{3}{8x^{\frac{3}{2}}} \tilde{w}_r \left(\frac{t}{4}, \sqrt{x} \right) + \left(\frac{3}{8x} + \frac{1}{4} \right) \tilde{w}_{rr} \left(\frac{t}{4}, \sqrt{x} \right) + \frac{1}{8\sqrt{x}} \tilde{w}_{rrr} \left(\frac{t}{4}, \sqrt{x} \right).$$

Since

$$u_t = x u_{xx} + u u_x$$

and

$$u_{t,x} = x u_{xxx} + [1 + u] u_{xx} + u_x^2,$$

it is now easy to see that u_t and $u_{t,x}$ are bounded on $I \times [0, 1]$.

Since $u \in C^2((0, T_{max}) \times (0, 1])$, then $u_{t,x} = u_{x,t}$.

Finally, $\ln(u_x) u_{t,x} - \frac{u u_t}{x}$ is bounded on $I \times [0, 1]$. Hence, by direct calculation,

$$\begin{aligned} \frac{d}{dt} \mathcal{G}[u(t)] &= \int_0^1 \ln(u_x) u_{t,x} - \frac{u u_t}{x} = - \int_0^1 \left[\frac{u_{xx}}{u_x} + \frac{u}{x} \right] u_t \\ &= - \int_0^1 \frac{u_t^2}{x u_x} \end{aligned}$$

where an integration by parts was made, using that $u_t(t, 0) = u_t(t, 1) = 0$.

It is easy to see that \mathcal{G} is continuous on \mathcal{M} , \mathcal{G} is nonincreasing on the trajectories, so we have proved that \mathcal{G} is a Lyapunov function. Now, assume that

$$\mathcal{G}[u(t)] = \mathcal{G}[u_0]$$

for all $t \geq 0$. This implies

$$\int_0^t \int_0^1 \frac{u_t^2}{xu_x} = 0$$

so

$$u_t = 0$$

on $[0, 1] \times [0, t]$ for all $t \geq 0$. Hence, by continuity, for all $t \geq 0$,

$$u(t) = u_0$$

i.e. u is a steady state of (PDE_m) . □

Proof of Lemma 2.5. Let $u_0 \in Y_m$ and u the global solution of (PDE_m) with initial condition u_0 . Let us set

$$u_1 = u(1) \in Y_m^1.$$

To get the result, we just have to study $\lim_{t \rightarrow +\infty} T(t)u_1$.

Thanks to Lemma 2.2, $\gamma_1(u_1)$ is relatively compact in Y_m^1 and since \mathcal{G} is a strict Lyapunov functional for $(T(t))_{t \geq 0}$, we know by Lasalle's invariance principle (Proposition 6.1) that the ω -limit set $\omega(u_1)$ is non empty and contains only stationary solutions. But since there exists only one steady state U_a where

$$a = \frac{m}{1 - \frac{m}{2}},$$

then

$$\omega(u_1) = \{U_a\}$$

so

$$T(t)u_1 \xrightarrow[t \rightarrow +\infty]{} U_a.$$

□

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References

- [1] Beesack, Paul R. Hardy's inequality and its extensions. *Pacific J. Math.* 11 1961 39–61.
- [2] Blanchet, Adrien; Calvez, Vincent; Carrillo, José A. Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model. *SIAM J. Numer. Anal.* 46 (2008), no. 2, 691–721.
- [3] Biler, Piotr; Karch, Grzegorz; Laurençot, Philippe; Nadzieja, Tadeusz. The 8π -problem for radially symmetric solutions of a chemotaxis model in a disc. *Topol. Methods Nonlinear Anal.* 27 (2006), no. 1, 133–147.
- [4] P. Biler, G. Karch, P. Laurençot, and T. Nadzieja, The 8π -problem for radially symmetric solutions of a chemotaxis model in the plane, *Math. Methods Appl. Sci.*, 29 (2006), pp. 1563-1583.

- [5] A. Blanchet, J.A. Carrillo, and N. Masmoudi, Infinite time aggregation for the critical two-dimensional Patlak-Keller-Segel model, *Comm. Pure Appl. Math.*, 61 (2008), pp. 1449-1481.
- [6] A. Blanchet, J. Dolbeault, and B. Perthame, Two dimensional Keller-Segel model: Optimal critical mass and qualitative properties of solutions, *Electron. J. Differential Equations*, 44 (2006), pp. 1-32.
- [7] Brenner, Michael P.; Constantin, Peter; Kadanoff, Leo P.; Schenkel, Alain; Venkataramani, Shankar C. Diffusion, attraction and collapse. *Nonlinearity* 12 (1999), no. 4, 1071–1098.
- [8] Calvez, Vincent; Carrillo, José Antonio. Refined asymptotics for the subcritical Keller-Segel system and related functional inequalities. *Proc. Amer. Math. Soc.* 140 (2012), no. 10, 3515–3530.
- [9] J. Dolbeault and B. Perthame, Optimal critical mass in the two-dimensional Keller-Segel model in \mathbb{R}^2 , *C. R. Math. Acad. Sci. Paris*, 339 (2004), pp. 611-616.
- [10] Cazenave, Thierry; Haraux, Alain. An introduction to semilinear evolution equations. Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998. xiv+186 pp.
- [11] Evans, Lawrence C. Partial differential equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.
- [12] Friedman, Avner. Partial differential equations of parabolic type. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964 xiv+347 pp.
- [13] Guerra, Ignacio A.; Peletier, Mark A. Self-similar blow-up for a diffusion-attraction problem. *Nonlinearity* 17 (2004), no. 6, 2137–2162.
- [14] M.A. Herrero, The mathematics of chemotaxis, *Handbook of differential equations: evolutionary equations*. Vol. III, 137-193, *Handb. Differ. Equ.*, Elsevier/North-Holland, Amsterdam, 2007.
- [15] M.A. Herrero and J.L. Velazquez, Singularity patterns in a chemotaxis model, *Math. Ann.* 306, pp. 583-623 (1996)
- [16] T. Hillen, K. J. Painter A user’s guide to PDE models for chemotaxis. *J. Math. Biol.* 58 (2009), no. 1-2, 183–217.
- [17] D. Horstmann, From 1970 until present : the Keller-Segel model in chemotaxis and its consequences I, *Jahresber. Deutsch. Math.-Verein.*, 105 (2003), pp. 103-165
- [18] D. Horstmann, From 1970 until present : the Keller-Segel model in chemotaxis and its consequences II, *Jahresber. Deutsch. Math.-Verein.*, 106 (2004), pp. 51-69
- [19] Horstmann, Dirk; Winkler, Michael. Boundedness vs. blow-up in a chemotaxis system. *J. Differential Equations* 215 (2005), no. 1, 52–107.
- [20] Kavallaris, Nikos I.; Souplet, Philippe. Grow-up rate and refined asymptotics for a two-dimensional Patlak-Keller-Segel model in a disk. *SIAM J. Math. Anal.* 40 (2008/09), no. 5, 1852–1881.
- [21] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.*, 26 (1970), pp. 399-415
- [22] Lunardi, Alessandra. Analytic semigroups and optimal regularity in parabolic problems. [2013 reprint of the 1995 original] [MR1329547 (96e:47039)]. *Modern Birkhäuser Classics*. Birkhäuser/Springer Basel AG, Basel, 1995. xviii+424 pp.

- [23] Montaru, Alexandre. Wellposedness and regularity for a degenerate parabolic equation arising in a model of chemotaxis with nonlinear sensitivity. *Discrete and Continuous Dynamical Systems - Series B* Volume 19, Issue 1, Pages : 231 - 256, 2013
- [24] Montaru, Alexandre. A semilinear parabolic-elliptic chemotaxis system with critical mass in any space dimension. *Nonlinearity* 26 (2013), no. 9, 2669–2701.
- [25] Otto, Felix. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* 26 (2001), no. 1-2, 101–174.
- [26] C.S. Patlak, Random walk with persistence and external bias, *Bull. Math. Biol. Biophys.*,15 (1953), pp. 311-338
- [27] Protter, Murray H.; Weinberger, Hans F. *Maximum principles in differential equations.* Corrected reprint of the 1967 original. Springer-Verlag, New York, 1984. x+261 pp.
- [28] Quittner, Pavol; Souplet, Philippe. *Superlinear parabolic problems. Blow-up, global existence and steady states.* Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel, 2007.
- [29] T. Suzuki, *Free energy and self-interacting particles.* Progress in Nonlinear Differential Equations and their Applications, 62. Birkhäuser Boston, Inc., Boston, MA, 2005.