Variational formula for the time-constant of first-passage percolation

by

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Dedication

To my mother.

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Abstract

We consider first-passage percolation with positive, stationary-ergodic weights on the square lattice $\mathbb{Z}^d$. Let $T(x)$ be the first-passage time from the origin to a point x in $\mathbb{Z}^d$. The convergence of the scaled first-passage time $T([nx])/n$ to the time-constant as n tends to infinity can be viewed as a problem of homogenization for a discrete Hamilton-Jacobi-Bellman (HJB) equation. By borrowing several tools from the continuum theory of stochastic homogenization for HJB equations, we derive an exact variational formula for the time-constant. We then construct an explicit iteration that produces the minimizer of the variational formula (under a symmetry assumption), thereby computing the time-constant. The variational formula may also be seen as a duality principle, and we discuss some aspects of this duality.

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Chapter 1

Introduction

1.1 Overview

First-passage percolation is a growth model in a random medium introduced by Hammersley and Welsh [19]. Consider the nearest-neighbor directed graph on the cubic lattice $\mathbb{Z}^d$. We will define the model when the random medium consists of positive edge-weights attached to the edges of this graph. For the purposes of this paper, first-passage percolation is better thought of as an optimal-control problem. Define the set of *control directions*

$$A := \{\pm e_1, \dots, \pm e_d\}, \tag{1.1}$$

where e_i are the canonical unit basis vectors for the lattice $\mathbb{Z}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space[1]. The weights will be given by a function $\tau \colon \mathbb{Z}^d \times A \times \Omega \to \mathbb{R}$, where $\tau(x, \alpha, \omega)$ refers to the weight on the edge from x to $x + \alpha$. We assume that

[1] We will frequently drop reference to the probability space when it plays no role in the argument.

the function $\tau(x, \alpha, \omega)$ is stationary-ergodic (see Section 1.3) under translation by $\mathbb{Z}^d$.

A path connecting x to y is a finite ordered set of nearest-neighbor vertices:

$$\gamma_{x,y} = \{x = v_1, \ldots, v_n = y\}. \tag{1.2}$$

The weight or total time of the path is

$$\mathcal{W}(\gamma_{x,y}) := \sum_{i=1}^{n-1} \tau(v_i, v_{i+1} - v_i, \omega).$$

The first-passage time from x to y is an infimum of the total time of the path taken over all paths from x to y:

$$\mathcal{T}(x, y) := \inf_{\gamma_{x,y}} \mathcal{W}(\gamma_{x,y}). \tag{1.3}$$

We will use $\mathcal{T}(x)$ to mean $\mathcal{T}(x, 0)$ unless otherwise specified. We're interested in the first-order asymptotics of $\mathcal{T}(x)$ as $|x| \to \infty$.

For any $x \in \mathbb{R}^d$, define the scaled first-passage time

$$\mathcal{T}_n(x) := \frac{\mathcal{T}([nx])}{n}, \tag{1.4}$$

where $[nx]$ represents the closest lattice point to nx (with some uniform way to break ties). The law of large numbers for $\mathcal{T}(x)$ has been the subject of a lot of research over the last 50 years and involves the existence of the so-called *time-constant* $m(x)$ given by

$$m(x) := \lim_{n \to \infty} \mathcal{T}_n(x). \tag{1.5}$$

The limit certainly exists in $d = 1$, since it is simply the usual law of large numbers. For general $d \geq 1$, Kingman's classical subadditive ergodic theorem [22] along with some simple estimates is enough to show the existence of $m(x)$ for all $x \in \mathbb{R}^d$. However, the theorem is merely an existence theorem; i.e., it does not give us any quantitative information about the limit, unlike the usual law of large numbers. Proving something substantial about the time-constant has been an open problem for the last several decades.

We prove that the time-constant satisfies a Hamilton-Jacobi-Bellmann (HJB) partial differential equation (PDE), and thus derive a new variational formula for the time-constant in part I. In part II, we first present a new explicit algorithm to produce a minimizer of the formula. Then, we discuss some aspects of the formula as a duality principle.

1.2 First-passage percolation as a homogenization problem

Since the first-passage time $\mathcal{T}(x)$ is an optimal-control problem (see Chapter 2), it has a dynamic programming principle (DPP) which says that

$$\mathcal{T}(x) = \inf_{\alpha \in A} \{\mathcal{T}(x + \alpha) + \tau(x, \alpha)\}.$$

We can rewrite the DPP as a difference equation in the so-called *metric* form of the HJB equation. Assuming $\tau(x, \alpha)$ are positive,

$$\sup_{\alpha \in A} \left\{ -\frac{(\mathcal{T}(x + \alpha) - \mathcal{T}(x))}{\tau(x, \alpha)} \right\} = 1. \tag{1.6}$$

Let's imagine that we were somehow able to extend $\mathcal{T}(x)$ as a smooth function on $\mathbb{R}^d$. Taylor expand $\mathcal{T}(x)$ at $[nx]$ to get

$$\sup_{\alpha\in A}\left\{-\frac{D\mathcal{T}([nx])\cdot\alpha + 1/2(\alpha, D^2\mathcal{T}(\xi)\alpha)}{\tau([nx],\alpha)}\right\} = 1, \tag{1.7}$$

where $(\cdot,\cdot)$ is the usual inner product on $\mathbb{R}^d$, and ξ is a point in $\mathbb{R}^d$. Introduce the scaled first-passage time $\mathcal{T}_n(x)$ into (1.7) to get

$$\sup_{\alpha\in A}\left\{-\frac{D\mathcal{T}_n(x)\cdot\alpha}{\tau([nx],\alpha)}\right\} + O(n^{-1}) = 1. \tag{1.8}$$

Equation (1.8) is reminiscent of a stochastic homogenization problem for a metric HJB equation in $\mathbb{R}^d$.

By considering the lattice to be embedded in $\mathbb{R}^d$, we can view the path $\gamma_{x,y}$ in (1.2) as a continuous curve moving along the edges of the lattice from x to y. Let $g_{x,y}(s)$ be a parametrization of this path satisfying

$$g'_{x,y}(s) = \frac{1}{\tau(z,\alpha)}\alpha,$$

when $g_{x,y}(s) = z + \lambda\alpha$ for $z \in \mathbb{Z}^d$, $0 < \lambda < 1$, and $\alpha \in A$. It's clear that if $g_{x,y}(0) = x$,

$$g_{x,y}\left(\sum_{i=1}^{n-1}\tau(v_i, v_{i+1})\right) = y.$$

Motivated by this interpretation, we can formulate a continuous version of first-passage percolation in $\mathbb{R}^d$. Let $t\colon \mathbb{R}^d \times A \to \mathbb{R}$ be a Lipschitz function in $\mathbb{R}^d$

(uniformly in A) satisfying for some constants $a, b > 0$

$$a \leq t(x, \alpha) \leq b.$$

Let the set of allowable paths be

$$\mathcal{A} := \left\{ g \in C^{0,1}([0,\infty), \mathbb{R}^d) : g'(s) = \frac{\alpha}{t(g(s), \alpha)} \ a.e.\ s \in [0,\infty),\ \alpha \in A \right\}.$$

Define the continuous version of the first-passage time as

$$T(x) := \inf_{g \in \mathcal{A}} \{ s : g(0) = x,\ g(s) = 0 \}. \tag{1.9}$$

Define the Hamiltonian for this continuous first-passage percolation to be

$$H(p, x) := \sup_{\alpha \in A} \frac{p \cdot \alpha}{t(x, \alpha)}. \tag{1.10}$$

It's a classical fact in optimal-control theory that $T(x)$ is the (unique) viscosity solution of the metric HJB equation [5]

$$\begin{aligned} H(DT(x), x) &= 1, \\ T(0) &= 0. \end{aligned} \tag{1.11}$$

Let $T_n(x) = T(nx)/n$ be the scaled continuous first-passage time. For each n, $T_n(x)$ solves

$$\begin{aligned} H(DT_n(x), nx) &= 1, \\ T_n(0) &= 0. \end{aligned} \tag{1.12}$$

The set of equations in (1.12) constitute a homogenization problem for the Hamiltonian in (1.10). The theory of stochastic homogenization states that $T_n(x) \to m(x)$ locally uniformly, and further, that there is a deterministic Hamiltonian $\overline{H}(p)$ such that $m(x)$ is the viscosity solution of

$$\begin{aligned} \overline{H}(Dm) &= 1, \\ m(0) &= 0. \end{aligned} \tag{1.13}$$

Importantly, one can characterize $\overline{H}(p)$ using a variational formula.

We will first prove that the time-constant of discrete first-passage percolation satisfies a HJB equation of the form (1.13). Proving that a continuous, but possibly non-smooth function like the time-constant is a solution of a HJB equation is most easily done using viscosity solution theory [11]. However, this is a continuum theory, and first-passage percolation is on the lattice. Constructing a continuous version of first-passage percolation allows us to embed the discrete problem in $\mathbb{R}^d$, and borrow the tools we need from the continuum theory.

1.3 Stochastic homogenization on $\mathbb{R}^d$

Fairly general stochastic homogenization theorems about HJB equations have been proved in recent years. We will state a special case of the theorem from Lions and Souganidis [27] that is relevant to our problem, although the later paper by Armstrong and Souganidis [3] would have been just as appropriate.

For a group G, let

$$\mathcal{G} := \{V_g : \Omega \to \Omega\}_{g \in G} \tag{1.14}$$

be a family of invertible measure-preserving maps satisfying

$$V_{gh} = V_g \circ V_h \quad \forall\ g, h \in G.$$

That is, $V_\cdot$ is a homomorphism from G to the group of all measure-preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$. In our case, G will either be $\mathbb{Z}^d$ or $\mathbb{R}^d$. Let $X = \mathbb{R}^d$ or $\mathbb{Z}^d$, and suppose $G \subset X$. A random function $f\colon X \times \Omega \to \mathbb{R}$ is said to be stationary with respect to G if it satisfies

$$f(x + g, \omega) = f(x, V_g \omega) \quad \forall\ x \in X,\ g \in G. \tag{1.15}$$

We say $B \in \mathcal{F}$ is an invariant set if it satisfies $V_g B = B$ for any $g \in G \backslash \{e\}$ where e is the identity element of G. The family of maps $\mathcal{G}$ is called (strongly) ergodic if invariant sets are either null or have full measure. A process $f(x, \omega)$ is called stationary-ergodic if it's stationary with respect to a group G, and $\mathcal{G}$ is ergodic.

Let $G = \mathbb{R}^d$, and suppose the Hamiltonian $H\colon \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is

1. stationary-ergodic,

2. convex in p for each x and ω,

3. coercive in p; i.e., uniformly in x and ω,

$$\lim_{|p| \to \infty} H(p, x, \omega) = +\infty,$$

4. and regular; i.e., for each ω,

$$H(\cdot, \cdot, \omega) \in C^{0,1}_{loc}(\mathbb{R}^d \times \mathbb{R}^d) \cap C^{0,1}(\overline{B(0, R)} \times \mathbb{R}^d).$$

Consider the homogenization problem in (1.12) for $T_n(x)$. The following is a special case of Theorem 3.1, Lions and Souganidis [27]:

Theorem 1.1. *There exists a deterministic, convex, Lipschitz $\overline{H}(p)$ with viscosity solution $m(x)$ of (1.13), such that $T_n(x) \to m(x)$ locally uniformly in $\mathbb{R}^d$.*

There is also a variational characterization of $\overline{H}(p)$. Define the set of functions with stationary and mean-zero gradients:

$$S := \left\{ f(\cdot,\omega) \in C^{0,1}(\mathbb{R}^d) \;\middle|\; \begin{array}{l} Df(x+z,\omega) = Df(x, V_z\omega),\ \forall x, z \in \mathbb{R}^d \\ E[Df(x,\omega)] = 0 \ \forall x \in \mathbb{R}^d \end{array} \right\}. \tag{1.16}$$

Proposition 3.2 from Lions and Souganidis [27] states that

Proposition 1.2. *For each $p \in \mathbb{R}^d$,*

$$\overline{H}(p) = \inf_{f \in S} \operatorname*{ess\,sup}_{\omega} \sup_{x \in \mathbb{R}^d} H(Df + p, x, \omega). \tag{1.17}$$

To apply Theorem 1.1 and Prop. 1.2 to first-passage percolation, we show that these results apply when $G = \mathbb{Z}^d$ by making the necessary minor modifications (see Section 3.1 and Section A.1).

1.4 Main results

Our first result is the homogenization theorem for the time-constant of discrete first-passage percolation. Let the edge-weights $\tau \colon \mathbb{Z}^d \times A \times \Omega \to \mathbb{R}$ be

1. (essentially) bounded above and below; i.e.,

$$
\begin{aligned}
0 < \quad a &= \operatorname*{ess\,inf}_{x,\alpha,\omega} \tau(x,\alpha,\omega), \\
b &= \operatorname*{ess\,sup}_{x,\alpha,\omega} \tau(x,\alpha,\omega) < \infty,
\end{aligned} \tag{1.18}
$$

   and

2. stationary-ergodic with $G = \mathbb{Z}^d$.

Theorem 1.3. *The time-constant $m(x)$ solves a Hamilton-Jacobi equation*

$$
\begin{aligned}
\overline{H}(Dm(x)) &= 1, \\
m(0) &= 0.
\end{aligned} \tag{1.19}
$$

The next result is a discrete variational formula for $\overline{H}(p)$.

Definition 1.4 (Discrete derivative)**.** For a function $\phi \colon \mathbb{Z}^d \to \mathbb{R}$, let

$$\mathcal{D}\phi(x,\alpha) = \phi(x+\alpha) - \phi(x)$$

be its discrete derivative at $x \in \mathbb{Z}^d$ in the direction $\alpha \in A$.

Let the discrete Hamiltonian for first-passage percolation be

$$\mathcal{H}(\phi, p, x, \omega) = \sup_{\alpha \in A} \left\{ \frac{-\mathcal{D}\phi(x,\alpha,\omega) - p \cdot \alpha}{\tau(x,\alpha,\omega)} \right\}, \tag{1.20}$$

and define the discrete counterpart of the set (1.16)

$$S := \left\{ \phi : \mathbb{Z}^d \times \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \mathcal{D}\phi(x+z,\omega) = \mathcal{D}\phi(x, V_z\omega),\ \forall x, z \in \mathbb{Z}^d \\ E[\mathcal{D}\phi(x,\alpha)] = 0\ \forall x \in \mathbb{Z}^d \text{ and } \alpha \in A, \end{array} \right\} \tag{1.21}$$

where A is defined in (1.1). Then,

Theorem 1.5. *the limiting Hamiltonian $\overline{H}(p)$ is given by*

$$\overline{H}(p) = \inf_{\phi \in S} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in \mathbb{Z}^d} \mathcal{H}(\phi, p, x, \omega). \tag{1.22}$$

The variational formula tells us that $\overline{H}(p)$ is positive 1-homogeneous, convex, and $\overline{H}(p) = 0$ iff $p = 0$. This means that it is a norm on $\mathbb{R}^d$, and indeed, the same is true of $m(x)$. By an elementary Hopf-Lax formula for the PDE in (1.19), we find that

Corollary 1.6. *$\overline{H}(p)$ is the dual norm of $m(x)$ on $\mathbb{R}^d$, defined as usual by*

$$\overline{H}(p) = \sup_{m(x)=1} p \cdot x.$$

In part II we present a new explicit algorithm that produces a minimizer of the variational formula under a symmetry assumption. It also contains a discussion of the formula as a duality principle.

1.5 Background on the time-constant

We give a brief overview of results about the time-constant in first-passage percolation. In this section, unless otherwise specified, we will assume that the nearest-neighbor graph on $\mathbb{Z}^d$ is undirected and that the edge-weights are i.i.d. Cox and Durrett [10] proved a celebrated result about the relationship between the time-constant and the so-called limit-shape of first-passage percolation. Let

$$R_t := \{x \in \mathbb{R}^2 : \mathcal{T}([x]) \leq t\} \tag{1.23}$$

be the *reachable set.* It is a fattened version of the sites reached by the percolation before time t. We're interested in the limiting behavior of the set $t^{-1}R_t$ as $t \to \infty$. Let $F(t)$ be the cumulative distribution of the edge-weights. Define the distribution G by $(1 - G(t)) = (1 - F(t))^4$. The following theorem holds iff the second moment of G is finite.

Theorem (Cox and Durrett [10])**.** *Fix any $\epsilon > 0$. If $m(x) > 0$ for all $x \in \mathbb{R}^2$,*

$$\{x : m(x) \leq 1 - \epsilon\} \subset \frac{R_t}{t} \subset \{x : m(x) \leq 1 + \epsilon\} \text{ as } t \to \infty \text{ a.s.} \tag{1.24}$$

Otherwise $m(x)$ is identically 0, and for every compact $K \subset \mathbb{R}^2$,

$$K \subset \frac{R_t}{t} \text{ as } t \to \infty \text{ a.s.}$$

Under the conditions of the above theorem, the sublevel sets of the time-constant

$$B_0 = \{x : m(x) \leq 1\}$$

can be thought of as the limit-shape. The extension of the Cox and Durrett [10] theorem to $\mathbb{Z}^d$ was shown by Kesten [21]. Boivin [7] proved the result for stationary-ergodic media. Despite these strong existence results on the time-constant and limit-shape, surprisingly little else is known in sufficient generality [39].

The following is a selection of facts that are known about the time-constant. It's known that $m(e_1) = 0$ iff $F(0) \geq p_T$, where p_T is closely related to the critical probability for bond percolation on $\mathbb{Z}^d$ [21][2]. Durrett and Liggett [13] described an interesting class of examples where B_0 has flat-spots. Marchand [29] and subsequently, Auffinger and Damron [4] have recently explored several aspects of this class of examples in great detail. It's also known that if F is an exponential distribution, B_0 is not a Euclidean ball in high-enough dimensions [21]. Exact results for the limit-shape are only available for "up-and-right" directed percolation with special edge weights [20, 37]. In fact, Johansson [20] not only obtains the limit-shape, but also shows

$$\mathcal{T}(x) \sim m(x) + |x|^{1/3}\xi,$$

where ξ is distributed according to the (GUE) Tracy-Widom distribution. Hence, first-passage percolation is thought to be in the KPZ universality class.

Several theorems can be proved assuming properties of the limit-shape. For example, results about the fluctuations of $\mathcal{T}(x)$ can be obtained if it's known that the limit-shape has a "curvature" that's uniformly bounded [4, 30]. Chatterjee and Dey [9] prove Gaussian fluctuations for first-passage percolation in thin-cylinders under the hypothesis that the limit-shape is strictly convex in the e_1 direction. Properties like strict convexity, regularity or the curvature of the limit-shape have

[2] It's the largest p such that the expected size of the cluster containing the origin is finite.

not been proved, and are of great interest.

We suggest the lecture notes of Kesten [21], and the review papers by Grimmett and Kesten [18], and Blair-Stahn [6] for a more exhaustive survey of the many aspects of first-passage percolation.

1.6 Background on stochastic homogenization

Stochastic homogenization has been an active field of research in recent years, and there have been several results and methods of proof. Periodic homogenization of HJB equations was studied first by Lions et al. [25]. The first results on stochastic HJB equations were obtained by Souganidis [38], and Rezakhanlou and Tarver [34]. These and other results [36] were about the "non-viscous" problem, and require super-linear growth of the Hamiltonian. That is, for positive constants $C_1, C_2, C_3, C_4 > 0$ and $\alpha_1, \alpha_2 > 1$,

$$C_1|p|^{\alpha_1} - C_2 \leq H(p, x, \omega) \leq C_3|p|^{\alpha_2} + C_4. \tag{1.25}$$

for all $p, x \in \mathbb{R}^d$ and a.s ω. These results do not apply directly to our situation since we have exactly linear growth in (1.10).

The "viscous" version of the problem includes a second-order term:

$$-\epsilon \ \mathrm{tr} A(\epsilon^{-1}x, \omega) D^2 u^\epsilon(x, t, \omega) + H(Du^\epsilon, \epsilon^{-1}x, \omega) = 1, \tag{1.26}$$

where $A(x, \omega)$ is a symmetric matrix. This problem is considered in Kosygina et al. [23], Caffarelli et al. [8], Lions and Souganidis [27], and Lions and Souganidis

[28]. Caffarelli et al. [8] and Kosygina et al. [23] require uniform ellipticity of the matrix A; i.e., they assume $\exists\ \lambda_1,\ \lambda_2 > 0$ such that for all x and ω,

$$\lambda_1|\xi|^2 \leq (A(x,\omega)\xi,\xi) \leq \lambda_2|\xi|^2. \tag{1.27}$$

Lions and Souganidis [27] allow for $A = 0$ (degenerate-ellipticity), and only linear growth of the Hamiltonian. Their method relies heavily on the optimal-control interpretation, and we were therefore able to borrow several ideas from them. The variational formula for the time-constant is a discrete version of theirs. We must also mention the work of Armstrong and Souganidis [3] that focuses specifically on metric Hamiltonians like the one for first-passage percolation. In fact, it was brought to our attention that Armstrong et al. [2] made the following observation: since $\mathcal{T}(x,y)$ induces a random metric on the lattice, it's reasonable to believe that there ought to be some relation to metric HJB equations. This is exactly what we prove.

1.7 Other variational formulas

Once we posted our preprint [24] on the arXiv, the concurrent but independent work of Georgiou et al. [16] appeared. They prove discrete variational formulas for the directed polymer model at zero and finite temperature, and for the closely related last-passage percolation model. Their ideas originate in the works of Rosenbluth [35], Rassoul-Agha and Seppäläinen [32] and Rassoul-Agha et al. [33] for quenched large-deviation principles for random-walk in random environment.

It is interesting to note that quite coincidentally, our results and those of Georgiou et al. [16] almost exactly parallel the development of stochastic homogenization results in the continuum. Lions and Souganidis [27] published their viscous homogenization results in 2005, using the classical cell-problem idea and the viscosity solution framework. Concurrently and independently in 2006, Kosygina et al. [23] published their viscous stochastic homogenization result. In contrast to Lions and Souganidis [27], their proof technique has the flavor of a duality principle and has a minimax theorem at its core. Both our results and those of Georgiou et al. [16] are discrete adaptations of Lions and Souganidis [27] and Kosygina et al. [23] respectively.

Chapter 2

Setup and notation

2.1 Continuum optimal-control problems

We introduce the classical optimal-control framework here, since it plays a major-role in our proofs. We follow Bardi and Capuzzo-Dolcetta [5] and Evans [14] for the setup. The evolution of the state of a control system $g(s)$ is governed by a system of ordinary differential equations

$$\begin{aligned} g'(s) &= f(g(s), a(s)), \\ g(0) &= x, \end{aligned} \tag{2.1}$$

where $a(s)$ is known as the control. A is typically a compact subset of a topological space like the one in (1.1), and the space of allowable controls consists of all measurable functions

$$\mathcal{A} := \{a \colon \mathbb{R}^+ \to A\}. \tag{2.2}$$

The function $f(y, \alpha)$ is assumed to be bounded and Lipschitz in y (uniformly in α). Hence for fixed $a \in \mathcal{A}$, (2.1) has a unique (global) solution $g_{a,x}(s)$. Define the total cost to be

$$I_{x,t}(a) := \int_0^t l(g_{a,x}(s), a(s))ds, \tag{2.3}$$

where $l(x, \alpha)$ is called the *running cost* and satisfies for some $C > 0$, and all $x, y \in \mathbb{R}^d$ and $\alpha \in A$:

$$\begin{aligned} & -C \leq l(x, \alpha) \leq C, \\ & |l(x, \alpha) - l(y, \alpha)| \leq C|x - y|. \end{aligned} \tag{2.4}$$

Let $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ be the *terminal cost*. The finite time-horizon problem is defined to be

$$u(x, t) = \inf_{a \in \mathcal{A}} \left\{ I_{x,t}(a) + u_0(g(t)) \right\}. \tag{2.5}$$

We will usually assume that u_0 is globally Lipschitz continuous.

There is a dynamic programming principle (DPP) for $u(x, t)$ and consequently, it is the viscosity solution of a HJB equation

$$\begin{aligned} & u_t(x, t) + H(Du, x) = 0, \\ & u(x, 0) = u_0(x), \end{aligned} \tag{2.6}$$

where

$$H(p, x) = \sup_{a \in A} (-p \cdot f(x, a) - l(x, a)). \tag{2.7}$$

We will have use for another type of optimal-control problem called the infinite-horizon or stationary problem. For $\epsilon > 0$, let

$$v(x) = \inf_{a \in \mathcal{A}} \int_0^\infty e^{-\epsilon s} l(g_{a,x}(s), a(s))\ ds. \tag{2.8}$$

$v(x)$ also has a DPP and is the unique viscosity solution of

$$\epsilon v(x) + H(Dv, x) = 0. \tag{2.9}$$

with the same Hamiltonian defined in (2.7). The functions $v(x)$ and $u(x,t)$ defined above are usually called *value functions*. Randomness is usually introduced into the problem by requiring $l(x, a, \omega)$ and $f(x, a, w)$ to be stationary-ergodic processes.

For the optimal-control problems that are of interest to us, l and f take the particular forms (for fixed $p \in \mathbb{R}^d$):

$$f(x, \alpha, \omega) = \frac{\alpha}{t(x, \alpha, \omega)}, \tag{2.10}$$

$$l(x, \alpha, \omega) = \frac{p \cdot \alpha}{t(x, \alpha, \omega)}, \tag{2.11}$$

where $t(x, \alpha, \omega)$ is the continuous edge-weight function discussed in Section 1.2.

2.2 Discrete optimal-control problems

Next, we define the discrete counterparts to continuum optimal-control problems we defined in the previous section. Let the state $\gamma_{\alpha,x} \colon \mathbb{Z}^+ \to \mathbb{Z}^d$ satisfy the difference equation

$$\begin{aligned}\gamma_{\alpha,x}(j+1) &= \gamma_\alpha(j) + \alpha(j) \qquad \forall\ j \geq 0,\\ \gamma_{\alpha,x}(0) &= x,\end{aligned} \tag{2.12}$$

The controls lie in the set

$$\mathcal{A} := \{\alpha \colon \mathbb{Z}^+ \to A\},$$

where A is defined in (1.1).

Suppose we have edge-weights $\tau(x,\alpha)$ as in first-passage percolation, and discrete running costs $\lambda(x,\alpha)$ satisfying

$$|\lambda(x,\alpha)| \leq C. \tag{2.13}$$

Assume that $\tau(x,\alpha)$ is positive and bounded as in (1.18). For a control $\alpha \in \mathcal{A}$, let

$$\mathcal{W}_{x,k}(\alpha) = \sum_{i=1}^{k} \tau(\gamma_{\alpha,x}(i), \alpha(i))$$

be the total time for k steps of the path γ. For any $\mu_0 \colon \mathbb{Z}^d \to \mathbb{R}$ the finite time-horizon problem is

$$\mu(x,t) = \inf_{\alpha\in\mathcal{A}} \inf_{k\in\mathbb{Z}^+} \left\{ \sum_{i=0}^{k} \lambda(\gamma_{\alpha,x}(i), \alpha(i)) + \mu_0(\gamma_{\alpha,x}(k)) : \mathcal{W}_{x,k}(\alpha) \leq t \right\}. \tag{2.14}$$

Again, we will assume that the discrete Lipschitz norm $\|\mu_0\|_{\text{Lip}}$ is finite (see (2.16)).

The stationary problem is defined to be

$$\nu(x) = \inf_{\alpha \in \mathcal{A}} \left(\sum_{i=0}^{\infty} e^{-\epsilon \mathcal{W}_{x,i}(\alpha)} \lambda(\gamma_{\alpha,x}(i), \alpha(i)) \right). \tag{2.15}$$

2.3 Generalization of our setup

We've formulated the problem so that it applies to first-passage percolation on the directed nearest-neighbor graph of $\mathbb{Z}^d$. It covers the following situations:

- Regular first-passage percolation on the undirected nearest-neighbor graph of $\mathbb{Z}^d$ if the edge-weights satisfy

$$\tau(x, \alpha) = \tau(x + \alpha, -\alpha) \quad \forall x \in \mathbb{Z}^d \text{ and } \alpha \in A$$

- Site first-passage percolation (weights are on the vertices of $\mathbb{Z}^d$) if

$$\tau(x, \alpha) = \tau(x) \quad \forall x \in \mathbb{Z}^d \text{ and } \alpha \in A.$$

These are by no means the most general problems that comes under the optimal-control framework. Specializing to nearest-neighbor first-passage percolation has mostly been a matter of convenience and taste.

For example, the e_i in the definition of A (1.1) could be any basis for $\mathbb{R}^d$ — i.e., any lattice— and our main theorems would hold with little modification. If $A = \{e_1, \ldots, e_d\}$ and we consider $\mathcal{T}(0, x)$, we get directed first-passage percolation; i.e., paths are only allowed to go up or right at any point. Versions of the theorems in Section 1.4 do indeed hold for such A, but the first-passage time $\mathcal{T}(x)$ is only defined for x in the convex cone of A. If A is enlarged to allow for long-range jumps —and very large jumps are appropriately penalized— we obtain long-range percolation. We avoid handling such subtleties here.

The $d+1$ dimensional directed random polymer assigns a random cost to randomly chosen paths in $\mathbb{Z}^d$. At zero-temperature, this too can be seen as an optimal-control problem. However, as mentioned earlier, variational formulas for directed last-passage percolation and zero-temperature polymer models have been proved in considerable generality by Georgiou et al. [16].

2.4 Notation

We will frequently need to compare discrete and continuous optimal-control problems. So we've tried to keep our notation as consistent as possible. Discrete objects —functions with at least one input taking values in $\mathbb{Z}^d$— will be either a Greek or a calligraphic version of a Latin letter. Objects that are not discrete will mostly use the Latin letters. For example, the function $t(x, \alpha)$ in (2.11) will be built out of the edge-weights $\tau(x, \alpha)$, and the running costs $l(x, \alpha)$ will be built out of $\lambda(x, \alpha)$.

Stated as a general rule of thumb: if it's a squiggly variable it's usually discrete and if it's Latin it's usually continuous. Discrete objects and their continuous counterparts are summarized below:

Description	**Discrete**	**Continuous**
Edge-weight function	$\tau(x,\alpha,\omega)$	$t(x,\alpha,\omega)$
Running costs	$\lambda(x,\alpha)$	$l(x,\alpha,\omega)$
Paths	$\gamma(i)$	$g(t)$
Weight of a path	$\mathcal{W}(\gamma)$	$W(g)$
First-passage time	$\mathcal{T}(x,y)$	$T(x,y)$
Total cost of a path	$\mathcal{I}$	I
Time-constant		$m(x)$
Finite time-horizon problem	$\mu(x,t)$	$u(x,t)$
Stationary problem	$\nu(x)$	$v(x)$
Hamiltonian	$\mathcal{H}(f,p,x,\omega)$	$H(Df,p,x,\omega)$
Homogenized Hamiltonian		$\overline{H}(p)$
Derivative	$\mathcal{D}$	D

Other notations and conventions are summarized below.

$\mathbb{R}^+$ and $\mathbb{Z}^+$ refer to the nonnegative real numbers and integers respectively. Leb$[a,b]$ represents Lebesgue measure on the interval $[a,b]$. $B_R(x)$ is the Euclidean ball on $\mathbb{R}^d$ that has radius R and is centered at x.

Integrals with respect to the probability measure will be written as $E[X]$, as $\int X d\mathbb{P}$, or as $\int X\mathbb{P}(d\omega)$.

$|\cdot|_p$ will refer to usual the l^p on $\mathbb{R}^d$. $|\cdot|$ without a subscript will either mean the l^2 norm or the absolute value of a number, depending on the context. $(\cdot,\cdot)$ is the usual dot product on $\mathbb{R}^d$. L^p refers to the space of functions over a measure space with the usual $\|\cdot\|_p$ norm.

The Lipschitz norm of a function $f \colon X \to \mathbb{R}$ on a metric space (X, ρ) is defined as

$$\|\cdot\|_{Lip}(f) := \inf\{C : |f(x) - f(y)| \le C\rho(x, y) \ \forall\ x, y \in X\}. \tag{2.16}$$

For us, (X, ρ) will be either $(\mathbb{R}^d, |\cdot|)$ or $(\mathbb{Z}^d, |\cdot|_1)$.

The symbol $\varnothing$ refers to the empty set.

The initialism DPP refers to the dynamic programming principle, and HJB stands for Hamilton-Jacobi-Bellmann.

Chapter 3

Outline of Proof

3.1 Continuum homogenization with $G = \mathbb{Z}^d$

As described in the introduction, we will construct a function $t(x, \alpha, \omega) : \mathbb{R}^d \times A \to \mathbb{R}$ using the edge-weights $\tau(x, \alpha, \omega)$. Hence, $t(x, \alpha, \omega)$ will only inherit the stationarity of the edge-weights on the lattice; i.e.,

$$t(x + z, \cdot, \omega) = t(x, \cdot, V_z \omega) \quad \forall\ z \in \mathbb{Z}^d.$$

Therefore, we first observe that

Proposition 3.1. *The homogenization theorem (Theorem* 3.1*) from Lions and Souganidis [27] holds with* $G = \mathbb{Z}^d$.

The proof of Prop. 3.1 can be summarized as follows: as the functions $T(x)$ are scaled by n, it's as if the lattice is scaled to have size $1/n$. The optimal-control interpretation gives us a uniform in n Lipschitz continuity estimate, and hence the scaled functions $T_n(x)$ do not fluctuate too much on the scaled lattice. Therefore, the discrete subadditive ergodic theorem is enough to prove the homogenization theorem. To flesh out some of the details, we'll identify where exactly the subadditive ergodic theorem is used in Lions and Souganidis [27].

The classical approach to proving homogenization is to find a corrector to the cell-problem; i.e., to find a function $v(y)$ satisfying

$$H(p + Dv(y), y) = \overline{H}(p),$$
$$\lim_{|y|\to\infty} \frac{v(y)}{|y|} = 0. \tag{3.1}$$

However, correctors with sublinear growth at infinity do not exist in general [26]. To get around this problem, Lions and Souganidis [27] consider the equation:

$$u_t(x,t) + H(p + Du, x) = 0, \quad u(x,0) = 0. \tag{3.2}$$

A (sub)solution of this equation can be written in terms of a subadditive quantity, and it follows from the subadditive ergodic theorem that

Proposition 3.2. *for any $R > 0$ and $\epsilon > 0$, there is a (random) t_0 large enough so that for all $t \geq t_0$*

$$\left| \frac{u(x,t)}{t} + \overline{H}(p) \right| \leq \epsilon \quad \forall x \in B_{Rt}(0) \ \text{a.s.}$$

The main ingredient needed to prove Theorem 1.1, are approximate corrector-like functions. One way to construct them is through the stationary equation for the cell-problem:

$$\epsilon v_\epsilon(x) + H(p + Dv_\epsilon(x)) = 0.$$

Lions and Souganidis [27] quote a Abelian-Tauberian theorem for the variational problems in (2.5) and (2.8), which says that

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = \lim_{\epsilon\to 0} \epsilon v_\epsilon(x) = -\overline{H}(p). \tag{3.3}$$

Hence, all we need to do is to prove Prop. 3.2 when $G = \mathbb{Z}^d$, and this will give the approximate corrector-like functions.

The variational formula for the limiting Hamiltonian (Prop. 1.2) also holds with $G = \mathbb{Z}^d$ in (1.16). This requires a little work, rather than merely being an observation like the homogenization theorem. We don't need the continuum variational formula in this paper. We present it here because we'll take an analogous route to prove the discrete variational formula. The proof closely follows Lions and Souganidis [27].

The sketch proof for Prop. 3.2 (and hence Prop. 3.1), and the proof of the continuum variational formula are in Appendix A.1.

3.2 Embedding the discrete problem in the continuum

Now that we have the appropriate version of Theorem 1.1 on $\mathbb{R}^d$ with $G = \mathbb{Z}^d$, we have to take discrete first-passage percolation into the continuum. In addition to defining a continuum version of first-passage percolation —as we've sketched in the introduction— we also need a discrete cell-problem. We will first "reverse engineer" the discrete cell-problem from the continuum Hamiltonian.

For fixed $p \in \mathbb{R}^d$, the shifted Hamiltonian for first-passage percolation in (1.10) can be written as

$$H(p+q,x) = \sup_{\alpha \in A} \frac{-p\cdot\alpha - q\cdot\alpha}{t(x,\alpha)}.$$

Comparing this formula with the definition of the continuum Hamiltonian in the optimal-control formulation (2.7), it follows that we must take the continuous running-cost to be

$$l(x,\alpha) = \frac{p\cdot\alpha}{t(x,\alpha)}. \tag{3.4}$$

Since we'll eventually require the continuous first-passage percolation to mimic discrete first-passage percolation, we'll define $t(x,\alpha) = \tau(x,\alpha)$ along the edges (see Section 4). For such a $t(x,\alpha)$ and $l(x,\alpha)$, let $u(x,t)$ be the finite time-horizon variational problem defined in (2.5). Suppose a path $g_{x,\alpha}$ traverses an edge $(x, x+\alpha)$. Along this edge, we accumulate cost

$$\int_0^{\tau(x,\alpha)} \frac{p\cdot\alpha}{\tau(x,\alpha)} = p\cdot\alpha.$$

This indicates that we must consider discrete running costs of the form

$$\lambda(x,\alpha) = p\cdot\alpha. \tag{3.5}$$

For the edge-weights $\tau(x,\alpha)$ and the cell-problem running-cost $\lambda(x,\alpha) = p\cdot\alpha$, we'll consider three optimal-control problems: first-passage percolation $\mathcal{T}(x)$, the finite time cell-problem $\mu(x,t)$, and the stationary cell-problem $\nu(x)$. The latter two are defined in (2.14) and (2.15).

Now that we've reverse engineered the cell-problem running costs (3.5), we turn to constructing continuous approximations of our three discrete optimal-control problems. Using the edge-weights $\tau(x,\alpha)$ and the running-costs $\lambda(x,\alpha)$, we will define (precisely in Section 4) families of functions $t^\delta : \mathbb{R}^d \times A \to \mathbb{R}$ and $l^\delta : \mathbb{R}^d \times A \to \mathbb{R}$ parametrized by δ. Let $T^\delta(x)$, $u^\delta(x,t)$ and $v^\delta(x)$ be defined by (1.9), (2.5) and (2.8) (with $\epsilon = 1$) respectively. As $\delta \to 0$, T^δ, u^δ and v^δ will approach $\mathcal{T}$, μ and ν. Hence, we will frequently refer to the continuum problems as "δ-approximations" to first-passage percolation.

Next, we define the scaling for the three functions $T^\delta(x), u^\delta(x,t)$ and $v^\delta(x)$ and their discrete counterparts $\mathcal{T}(x)$, $\mu(x)$ and $\nu(x)$. The function $\mathcal{T}_n(x)$ has already been defined in (1.4); $T_n^\delta(x)$ is similarly defined in terms of $T^\delta(x)$. For the finite-time horizon problems we similarly define

$$u_n^\delta(x,t) = \frac{u^\delta([nx], nt)}{n} \quad \text{and} \quad \mu_n(x,t) = \frac{\mu([nx], nt)}{n}. \tag{3.6}$$

For the stationary problem, the scaled versions $v_n^\delta(x)$ and $\nu_n(x)$ are obtained by setting $\epsilon = 1/n$ in the variational problems in (2.8) and (2.15) respectively.

The following interchange of limits (or commutation diagram) is the main ingredient in the proofs of the discrete homogenization theorem and variational formula (Theorem 1.3 and Theorem 1.5). The theorem compares each of the three sequences of continuum functions

$$b_n^\delta = T_n^\delta(x),\ u_n^\delta(x,t),\ \text{or}\ \nu_n^\delta(x) \tag{3.7}$$

with the corresponding discrete versions

$$\beta_n = \mathcal{T}_n(x), \mu_n(x,t), \text{ or } \nu_n(x). \tag{3.8}$$

Theorem 3.3. *For each fixed $x \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, every pair (b_n^δ, β_n) in* (3.7) *and* (3.8) *homogenizes. That is, $b_n^\delta \to \overline{b}^\delta$ and $\beta_n \to \overline{b}$. Further,*

$$\lim_{\delta\to 0}\lim_{n\to\infty} b_n^\delta = \lim_{n\to\infty}\lim_{\delta\to 0} b_n^\delta = \lim_{n\to\infty} \beta_n = \overline{b}.$$

Stated as a commutative diagram, this is

$$\begin{array}{ccc} b_n^\delta & \xrightarrow{\;n\;} & \overline{b}^\delta \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\ \beta_n & \xrightarrow[\;n\;]{} & \overline{b} \end{array}$$

We will prove theorem Theorem 3.3 when $(b,\beta) = (u,\mu)$, and when $(b,\beta) = (T,\mathcal{T})$. The proof for $(b,\beta) = (v,\nu)$ is nearly identical, and to avoid repetition of ideas, we will omit it.

Proposition 3.2 says that

$$\lim_{t\to\infty} u_n^\delta(x,t) = -\overline{H}^\delta(p).$$

Let m^δ and m be the time-constants of T and $\mathcal{T}$. The continuum homogenization theorem says that m^δ is the viscosity solution of

$$\overline{H}^\delta(Dm^\delta(x)) = 0,\ m^\delta(0) = 0.$$

Theorem 3.3 applied to (u, μ) says that there exists a function $\overline{H}(p)$ such that

$$\overline{H}^{\delta}(p) \to \overline{H}(p).$$

Theorem 3.3 applied to $(T, \mathcal{T})$ to says that for each x,

$$m^{\delta}(x) \to m(x).$$

We would like to show that $m(x)$ solves the PDE corresponding to $\overline{H}(p)$. For this, a uniform in δ Lipschitz estimate for both $\overline{H}^{\delta}(p)$ and $m^{\delta}(x)$ is sufficient. We state this as two separate propositions below.

Proposition 3.4. $\overline{H}^{\delta}(p)$ *is Lipschitz continuous in* p *with constant bounded above by* $1/a$*, where* a *is defined in* (1.18).

Proposition 3.5. $m^{\delta}(x)$ *is Lipschitz in* x *with constant bounded above by* b*, where* b *is defined in* (1.18).

Since $\overline{H}^{\delta}$ and m^{δ} converge locally uniformly to $\overline{H}$ and m respectively, the standard stability theorem for viscosity solutions [11] implies Theorem 1.3.

The proofs of the results in this section are in Chapter 4.

3.3 Discrete variational formula and solution of the limiting PDE

The commutation theorem also transfers the Abelian-Tauberian theorem over from the continuum relating the limits of $u_n^{\delta}(x,t)$ and $v_{\epsilon}^{\delta}(x)$ (3.3). That is, both $\mu_n(x,t)$

and $\nu_\epsilon(x)$ converge to $\overline{H}(p)$ almost surely as $n \to \infty$ and $\epsilon \to 0$. This is very useful in establishing the discrete version of the variational formula. We have

Proposition 3.6. *for each $R > 0$ and $\eta > 0$, there is a small enough (random) ϵ_0 such that for all $\epsilon \leq \epsilon_0$,*

$$\left|\epsilon\nu_\epsilon(x) + \overline{H}(p)\right| < \eta \quad \forall x \in B_{\epsilon^{-1}R}(0).$$

The stationary cell-problem has the discrete (DPP)

$$\nu_\epsilon(x) = \inf_{\alpha \in A} \left(\alpha \cdot p + e^{-\epsilon\tau(x,\alpha)}\nu_\epsilon(x+\alpha)\right). \tag{3.9}$$

With a little manipulation of the DPP, we can obtain a discrete version of the stationary PDE in (2.9).

Proposition 3.7. *For all $x \in \mathbb{Z}^d$, there is a constant $C > 0$ uniform in ϵ and ω such that*

$$-C\epsilon \leq \epsilon\nu_\epsilon(x) + \mathcal{H}(\nu_\epsilon, p, x) \leq C\epsilon,$$

where $\mathcal{H}$ is the discrete Hamiltonian (1.20).

The final ingredient for the variational formula is what is usually called a comparison principle for HJB equations. In the continuum, the comparison principle is stated for sub- and supersolutions of the PDE. The discrete comparison principle we prove is a less general version that suffices for our purposes. Consider the discrete problem in (2.14) for any $\phi \colon \mathbb{Z}^d \to \mathbb{R}$ such that $\|\phi\|_{Lip} < \infty$. Then,

Proposition 3.8.

$$\mu(x,t) \geq \phi(x) - t \sup_{x\in\mathbb{Z}^d} \mathcal{H}(\phi,p,x) \quad \forall x \in \mathbb{Z}^d \text{ and } t \in \mathbb{R}^+.$$

Using these facts, the discrete variational formula in Theorem 1.5 is easy to prove.

Now that we have a formula for the limiting Hamiltonian, we can solve the PDE in (1.13) to obtain the time-constant $m(x)$. It follows directly from the variational formula that

Proposition 3.9. *$\overline{H}(p)$ is a norm on $\mathbb{R}^d$.*

Let $L(x)$ be the dual norm of $\overline{H}(p)$ on $\mathbb{R}^d$. Consider the set of paths

$$\mathcal{A} := \left\{ g \in C^1([0,\infty), \mathbb{R}^d) : L(g'(s)) = 1 \ \forall \ s \in [0,t) \right\}.$$

Let $T(x)$ be the minimum time function defined by (1.9). From standard optimal-control theory [5], it follows that $T(x)$ is the unique viscosity solution of the metric HJB equation (1.19). A standard Hopf-Lax formula gives

Proposition 3.10. *$T(x) = L(x)$.*

The fact that $\overline{H}(p)$ is the dual norm of $m(x)$ follows immediately, and Corollary 1.6 is proved.

Remark 3.11. It is natural to question the necessity of taking the discrete problem into the continuum; the PDE is unnecessary once the Hamiltonian has been identified as the dual norm of the time-constant[1].

[1]observation due to S.R.S Varadhan.

There is a direct proof based on the continuum homogenization theorem of Kosygina et al. [23]. Such a route has been taken to prove variational formulas for the large deviations of random walks in random environments by Rosenbluth [35]. This work was considerably generalized by Rassoul-Agha et al. [33] and Rassoul-Agha and Seppäläinen [32]. Subsequently, Georgiou et al. [16] extended these ideas to prove variational formulas for the directed random polymer, and for last-passage percolation.

Remark 3.12. We chose the metric-form of the HJB equation so that the limiting Hamiltonian $\overline{H}(p)$ could be interpreted as a norm. This allowed us to solve the PDE for the time-constant. We can make the assumption on the edge-weights $\tau(x,\alpha,\omega)$ in (1.18) less restrictive if the Hamiltonian is written in the more standard form

$$\mathcal{H}(p,x) = \sup_{\alpha \in A} \left\{ -p \cdot \alpha - \tau(x,\alpha) \right\}.$$

Here, $\tau(x,\alpha,\omega)$ can take the value 0 without making the Hamiltonian blow-up. However, even though the homogenization theorem and variational formula are still valid, we do not know the Hopf-Lax formula for the limiting PDE.

The proofs of the results in this section can be found in Chapter 5.

Chapter 4

Proofs related to the discrete homogenization theorem

4.1 Fattening the unit-cell

In this section, all the estimates will hold almost surely, and hence we'll not explicitly refer to ω or Ω. It will be useful for the reader to visualize the lattice as being embedded in $\mathbb{R}^d$. Discrete paths on the lattice will now be allowed to wander away from the edges of the graph on $\mathbb{Z}^d$ and into $\mathbb{R}^d$. This will let us define continuum variational problems —what we've called δ-approximations in Section 3— that will approximate discrete first-passage percolation and its associated cell-problem.

Consider a unit-cell of the lattice embedded in $\mathbb{R}^d$. We will "fatten" the edges and vertices of the lattice into tubes and corners. The remaining space in the unit-cell contains its center point, and so we will call this region a center (see Fig. 4.1). Let $0 < \delta < 1/2$ be a parameter describing the size of the tubes and corners. Define

1. The *tube* at $x \in \mathbb{Z}^d$ in the $\alpha \in A$ direction as:

$$TU^\delta_{x,\alpha} := \big\{x + \lambda\alpha + y : \delta < \lambda < 1 - \delta,\ y \in \{\alpha\}^\perp,\ |y|_\infty \leq \delta\big\}.$$

   The tubes have width 2δ and length $1 - 2\delta$.

2. The *corner* around a vertex x as:

$$CO^\delta_x := \{y : |y - x|_\infty \leq \delta\}.$$

3. The *center* of the cell as:

$$CE^\delta_x := \left\{x + \sum_{i=1}^{d} \lambda_i e_i : \delta < \lambda_i < 1 - \delta,\ \forall\ i\right\}.$$

The three regions are disjoint. That is, for any $x, y, z \in \mathbb{R}^d$ and $\alpha \in A$,

$$TU^\delta_{x,\alpha} \cap CO^\delta_y \cap CE^\delta_z = \varnothing.$$

Next, we need to define the edge-weight function $t^\delta(x, \alpha)$ and the cell-problem running cost $l^\delta(x, \alpha)$ in the δ-approximation. We'd like the value functions in the δ-approximation to be close to the discrete problem, and so we would like paths to avoid the centers of cells and stick close to the edges. By penalizing paths that cross into centers of cells with additional cost, we'll ensure that they stay inside

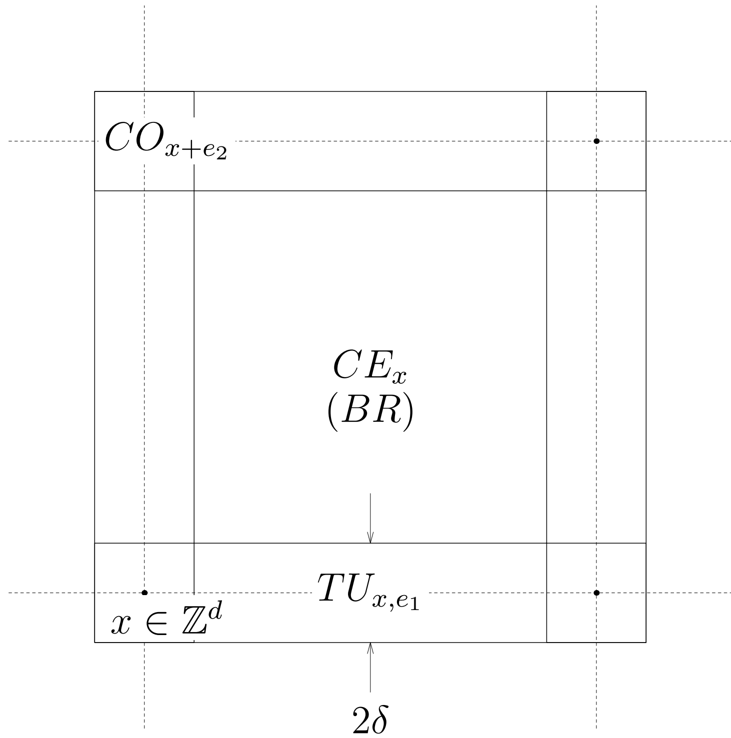

Figure 4.1: Sketch of fattened unit-cell. The dotted lines represent the edges of the lattice. The solid lines show the boundaries of the corners, tubes and centers. When δ is expanded to η, the center becomes a bad region.

the tubes and corners. For each $\alpha \in A$, let

$$t_c^\delta(x,\alpha) := \begin{cases} \tau(x,\alpha) & \text{if } x \in TU_{z,\alpha}^\delta \text{ for some } z \in \mathbb{Z}^d \\ a & \text{if } x \in CO_z \text{ for some } z \in \mathbb{Z}^d \\ \delta^{-1} & \text{otherwise} \end{cases} . \tag{4.1}$$

Fix $p \in \mathbb{R}^d$, and let $\lambda(x,\alpha) = p \cdot \alpha$ be the discrete cell-problem weight defined in (3.5). We define the running costs as

$$l_c^\delta(x,\alpha) = \begin{cases} \lambda(x,\alpha) & \text{if } x \in TU_{z,\alpha} \text{ for some } z \in \mathbb{Z}^d \\ -C & \text{if } x \in CO_z \text{ for some } z \in \mathbb{Z}^d \\ \delta^{-1} & \text{otherwise} \end{cases}, \tag{4.2}$$

where $C = -|p|_\infty$ is the lower bound on $\lambda(x,\alpha)$. The functions t_c^δ and l_c^δ represent piecewise extensions of the discrete edge-weights and running costs. Define the mollified functions

$$l^\delta(x,\alpha) = \eta_{\delta/2} * l_c^\delta(x,\alpha) \quad \text{and} \quad t^\delta(x,\alpha) = \eta_{\delta/2} * t_c^\delta(x,\alpha),$$

where η_δ is the standard mollifier with support δ. The Hamiltonian H^δ obtained from l^δ and t^δ using (2.7) satisfies the hypotheses of the Lions-Souganidis continuum homogenization theorem (Theorem 1.1).

For the finite time-horizon cell-problem, there is a final cost given by $\mu_0\colon \mathbb{Z}^d \to \mathbb{R}$. We can extend the final cost smoothly to $\mathbb{R}^d$ by defining

$$u_0^\delta(x) = \eta_{\delta/2} * \mu_0([x]).$$

The continuous variational problems in Section 2.1 can now be defined using the smooth functions l^δ, t^δ and u_0^δ.

The main focus of this section is Theorem 3.3. We will first prove it for when $\beta = \mu(x,s)$ and $b^\delta = u^\delta(x,s)$, the finite time-horizon cell-problems.

Remark 4.1. Although trivial, we remark that

$$u^\delta(x,s) \le \mu(x,s),$$

since we can always take paths going along edges in the δ-approximation.

Then, the proof of Theorem 3.3 is easy given that

Lemma 4.2. *for δ small enough, we have the estimate*

$$\frac{\mu([nx],ns)}{n} - \frac{u^\delta([nx],ns)}{n} \le C\left(\sqrt{\delta}s + \frac{1}{n}\right).$$

We will complete the proof of the main theorem before proving Lemma 4.2.

Proof of Theorem 3.3. While the homogenization theorem applies directly to $u^\delta(x,s)$, our scaling in (3.6) is slightly different. We scaled it differently so that it is enough to compare the discrete problem $\mu(z,s)$ to the continuum problem $u^\delta(z,s)$ on lattice points $z \in \mathbb{Z}^d$. So we account for this first. Since l^δ is bounded above by δ^{-1}, we have

$$\left|\frac{u^\delta([nx],ns)}{n} - \frac{u^\delta(nx,ns)}{n}\right| \le \frac{Cs}{\delta n}.$$

It follows from Prop. 3.2 (or the continuum homogenization theorem) that $u^\delta(nx,ns)/n$ has a limit, and hence

$$\lim_{n\to\infty} \frac{u^\delta([nx],ns)}{n} = \lim_{n\to\infty} \frac{u^\delta(nx,ns)}{n} =: \overline{u}^\delta(x,s).$$

From Lemma 4.2 and Remark 4.8, we have the following inequality for the scaled functions (3.6)

$$u_n^\delta(x,s) \le \mu_n(x,s) \le u_n^\delta(x,s) + C\left(\sqrt{\delta}s + \frac{1}{n}\right) \quad \forall x \in \mathbb{R}^d. \tag{4.3}$$

Taking a limit in n first, and then in δ (limsups and liminfs as appropriate), we get

$$\begin{aligned} \overline{u}^\delta(x,s) \le \varliminf_{n\to\infty} \mu_n(x,s), \\ \varlimsup_{n\to\infty} \mu_n(x,s) \le \overline{u}^\delta(x,s) + C\sqrt{\delta}s. \end{aligned}$$

Since δ is arbitrary, it follows that $\mu_n(x,s) \to \overline{u}(x,s)$ as $n \to \infty$. Taking limits in the reverse order and using the fact that $\mu_n(x,s)$ has a limit completes the proof. □

4.2 Integral formulation of variational problem

It's easier to work with an integral formulation of first-passage percolation and its cell-problem that will allow us to drop reference to the control $a(s)$ in (2.3). This is easily done by extending $l^\delta(x,\alpha)$ and $t^\delta(x,\alpha)$ one-homogeneously from $\mathbb{R}^d \times A$ to $\mathbb{R}^d \times \mathbb{R}^d$. For $x, r \in \mathbb{R}^d$ we redefine

$$\begin{aligned} l^\delta(x,r) &= l^\delta\left(x, \frac{r}{|r|}\right)|r|, \\ t^\delta(x,r) &= t^\delta\left(x, \frac{r}{|r|}\right)|r|. \end{aligned} \tag{4.4}$$

Hence, we may write the total cost of a path parametrized by $g\colon [0,s] \to \mathbb{R}^d$

(see (2.3)) as

$$I^{\delta}(g) = \int_0^s l^{\delta}(g(r), g'(r))dr. \tag{4.5}$$

We have to prove that dropping reference to the control in (4.5) will not affect our problem. That is we've to show that if we take a path g realized by a control and reparametrize it, its total cost will be unaffected. Let $r = h(q)$ be a smooth change of time parametrization, where h is an increasing function. Let $y(r) = g(h^{-1}(r))$ be the path. Then, a simple change of variable gives

Proposition 4.3 (Cost is independent of path parametrization)**.**

$$I^{\delta}(g) = \int_0^s l^{\delta}(g(q), g'(q))dq = \int_0^{h(s)} l^{\delta}(y(r), y'(r))dr.$$

Since we can dispense with the controls and talk directly about paths, define

$$U := \left\{ g(s) \in C^{0,1}([0,1], \mathbb{R}^d) : \frac{g'(s)}{|g'(s)|} \in A \text{ a.e. Leb}[0,1] \right\}.$$

This restricts us to paths that can be "realized" with controls. Let U_x be the subset of paths that start at x, and let $U_{x,y}$ be the subset of paths that go from x to y. Let $\mathcal{U}$, $\mathcal{U}_x$ and $\mathcal{U}_{x,y}$ be the corresponding subsets of U where paths are only allowed to go on the edges of the lattice $\mathbb{Z}^d$.

The total-time or weight of a path $g \in U$ is

$$W^{\delta}(g) = \int_0^1 t^{\delta} \left(g(r), \frac{g'(r)}{|g'(r)|} \right) |g'(r)| \, dr. \tag{4.6}$$

The cost of g is given by (4.5) with $s = 1$. Let the L^1 length of g be

$$d(g) = \int_0^1 |g'(r)| \, dr. \tag{4.7}$$

This allows us the following reformulation of the discrete and continuous cell-problems:

Proposition 4.4. *For $x \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$ and $s \in \mathbb{R}^+$, we have*

$$u^\delta(x, s) = \inf_{g \in U_x} \left\{ I^\delta(g) : W^\delta(g) \leq s \right\},$$
$$\mu(z, s) = \inf_{\gamma \in \mathcal{U}_z} \left\{ I^\delta(\gamma) : W^\delta(\gamma) \leq s \right\}.$$

Proof. The proof follows from Prop. 4.3 and the definitions of u and μ in (2.5) and (2.14). □

Remark 4.5. There are two things to notice about Prop. 4.4. First, notice that $\mu(x, s)$ is "embedded" in the continuum problem for every δ, and further, $u^\delta(x, s) \leq \mu(x, s) \; \forall \; x \in \mathbb{Z}^d$. Second, we no longer need $t^\delta(x, \alpha)$ and $l^\delta(x, \alpha)$ to be smooth in x for the variational problem to be well-defined. This allows us to work with the piecewise versions $l_c^\delta(x, \alpha)$ and $t_c^\delta(x, \alpha)$, since these are easier to compare to the discrete problems.

Remark 4.6. One could argue that we needn't have introduced the optimal-control framework since our costs depend only on the graph of the path and not its parametrization. However, DPPs and Hamiltonians are conventionally represented using the control interpretation. Several other models can be naturally formulated using the optimal-control language, and our approach ought to work for these (see Section 2.3).

4.3 Proof of Lemma 4.2

We state a simple comparison result first. Suppose $t_1(x,\alpha) \le t_2(x,\alpha)$ and $l_1(x,\alpha) \le l_2(x,\alpha)$. Let $u_1(x,s)$ and $u_2(x,s)$ be the corresponding variational problems defined by Prop. 4.4. Then, it follows quite easily that

Proposition 4.7 (Comparison of variational problems)**.** *for all $x \in \mathbb{R}^d$ and $s > 0$,*

$$u_1(x,s) \le u_2(x,s).$$

Remark 4.8. Let $w^\delta(x,s) \le u^\delta(x,s)$ for all $x \in \mathbb{R}^d$ and $s > 0$. To prove Lemma 4.2, it's enough to show that for any $s > 0$,

$$\mu(x,s) - w^\delta(x,s) \le C(\sqrt{\delta}s + 1) \quad \forall x \in \mathbb{Z}^d.$$

Proof of Lemma 4.2. We will use the observation made in Remark 4.8 repeatedly in the proof. We will also refer to the bounds on $\tau(x,\alpha)$ assumed in (1.18). The proof takes several steps.

1. Let us first reduce to the case where the cost and time functions are piecewise

constant. For this, we will show that

$$\begin{aligned} l_c^{2\delta}(x,\alpha) &\le l^{\delta}(x,\alpha), \\ t_c^{2\delta}(x,\alpha) &\le t^{\delta}(x,\alpha). \end{aligned} \tag{4.8}$$

Then, for a fixed path g, it would follow from Prop. 4.7 that

$$I_c^{2\delta}(g) \le I^{\delta}(g) \text{ and } W_c^{2\delta}(g) \le W^{\delta}(g),$$

where the subscript c in I and W represent the cost and time corresponding to the piecewise versions of l and τ. Then, Prop. 4.7 and Remark 4.8 imply that we can just work with $t_c^{2\delta}$ and $l_c^{2\delta}$.

To show the two inequalities in (4.8), it's enough to show that $t_c^{2\delta}(x,\alpha) \le t_c^{\delta}(y)$ and $l_c^{2\delta}(x) \le l_c^{\delta}(y)$ for all y in the ball $B_\delta(x)$. This is clear by drawing a picture. Then, multiply with the standard mollifier $\eta_{\delta/2}(y-x)$ and integrate over y.

2. Create boxes of side-length $1-2(\delta+\sqrt{\delta})$ called *bad regions* (BRs) contained inside centers (see Fig. 4.1). We show that if a path goes through a BR, it will do so badly that it will make more sense to stick to the tubes and corners. If g visits a BR, it will take at least time

$$\sqrt{\delta} t_c^{2\delta}(x,\alpha) = \frac{1}{\sqrt{\delta}}.$$

Hence it accumulates cost

$$\sqrt{\delta} t_c^{\delta}(x,\alpha) \cdot \frac{\delta^{-1}}{t_c^{\delta}(x,\alpha)} = \frac{1}{\sqrt{\delta}}.$$

A path g that visits a BR must leave a tube at some point A to enter the BR. Once it's done being bad, it must re-enter another tube or corner in the same cell at a point B. We will form a new path g^* that connects A and B by a path that only goes through tubes and corners. Making crude estimates, we see that the new path g^* has distance at most 3 to travel. Since the edge-weight function is bounded, it takes time at most $3b$, and costs at most $3C$. Since g^* does not take more time than g and costs less, we might as well assume that paths do not enter BRs. Henceforth, we will assume that U contains only such good paths.

3. Expand the tubes $TU^{2\delta}$ and corners $CO^{2\delta}$ to have thickness η, where

$$\eta = 2\delta + \sqrt{\delta}. \tag{4.9}$$

That is, we've expanded the tubes $TU^{2\delta}$ and corners $CO^{2\delta}$ so that the centers shrink to the BRs. Consider the problem t_c^η, l_c^η defined by (4.2) and (4.1). Then for δ small enough,

$$u^\eta(x,s) \le u^{2\delta}(x,s).$$

4. For a path $g \in U$, we need to construct an edge-path $\gamma \in \mathcal{U}$ that has a similar cost and time. We will consider each tube and corner that g passes through (since it avoids BRs), and construct γ in each region so that it follows g around. Fix such a tube TU or corner CO, and continue to write g for just the section of the path going through it. Then,

Claim 4.9. We can find $\gamma \in \mathcal{U}$ such that

$$I_c^\eta(\gamma) \le I_c^\eta(g) + C\eta,$$

$$W_c^\eta(\gamma) \leq W_c^\eta(g) + C\eta,$$

inside each tube or a corner that g goes through.

We will prove Claim 4.9 after completing the proof of this lemma.

5. A path $g \in U$ can travel a distance at most s/a in time s. So the number of tubes and corners the path can visit is at most $2s/a$. Using Claim 4.9, we will approximate g by an edge-path γ in each corner and tube that it goes through, except for possibly $C\eta s$ tubes and corners (since γ is typically slower through the corners). Hence, the most cost that γ could have missed out on is $C\eta s$, since $l_c^\eta(x,\alpha)$ is bounded below.

We must also take the final cost due to u_0 into account. Recall that we've assumed that $\|u_0\|_{Lip} < \infty$. The final locations of g and γ cannot differ by more than $C\eta s$ and hence, neither can the final cost. Finally, γ ought to end on a lattice point, whereas g need not. Accounting for all this, we get that

$$I_c^\eta(\gamma) \leq I_c^\eta(g) + C\eta s + C.$$

Since $I_c^\eta(\gamma)$ is exactly the cost accumulated by a path in the discrete problem, scaling by n completes the proof. □

Proof of claim 4.9. Corners are of size η, and the inequalities in Claim 4.9 are immediate. Hence, we only need to prove it for a tube, which wlog, we can assume to be TU_{0,e_1}^η. The end-caps of the tube of size η at the origin in the e_1 direction are $\{\eta e_1 + y : y \in \{e_1\}^\perp, |y|_\infty \leq \eta\}$, and $\{(1-\eta)e_1 + y : y \in \{e_i\}^\perp, |y|_\infty \leq \eta\}$. Let $A = (A_i)_{i=1}^d$ be the last point on the end-cap before g enters the tube, and let

$B = (B_i)_{i=1}^d$ be the first point on the end-cap when g exits the tube. Assume a parametrization such that $g(0) = A$ and $g(1) = B$. The total cost of g is

$$I_c^\eta(g) = \int_0^1 p \cdot g'(s)\ ds = p \cdot (B - A).$$

If A and B are on the same end-cap, the edge-path γ does not move at all, and if they're on different end-caps, it travels from end-cap to end-cap along the edge.

The tube is aligned with e_1 by construction, and hence $|A_1 - B_1| = 1 - 2\eta$ and $|A_j - B_j| \leq 2\eta\ \forall j \neq 1$. Hence,

$$I_c^\eta(g) - I_c^\eta(\gamma) \leq C\eta.$$

It's also clear that the L^1 length of the path is

$$1 - 2\eta = d(\gamma) \leq d(g).$$

It follows that

$$W_c^\eta(\gamma) \leq W_c^\eta(g). \tag{4.10}$$

□

To obtain a version of Lemma 4.2 for $T_n^\delta(x)$ and $\mathcal{T}_n(x)$, first replace U by $U_{0,x}$. The essential estimate is already present in Claim 4.9, and the rest of the argument is identical.

4.4 Lipschitz estimates on Hamiltonians and time-constants

We next prove the Lipschitz estimates on the time-constants m^δ and limiting Hamiltonians $\overline{H}^\delta$.

Proof of Prop. 3.4. Fix p_1 and $p_2 \in \mathbb{R}^d$. Let $l^\delta_{i,c}(x,\alpha)$ be the piecewise function defined in (4.2) for each p_i. Let l^δ_i be the mollified versions, and let u^δ_i be the corresponding finite time cell-problems with the same $t^\delta(x,\alpha)$ (for $i = 1, 2$). Then, Prop. 3.2 states that

$$\lim_{t\to\infty} \frac{u^\delta_i(x,s)}{t} = -\overline{H}^\delta(p_i) \quad \text{for } i = 1, 2.$$

Now, $l^\delta_{1,c}$ and $l^\delta_{2,c}$ differ only in the tubes, and hence satisfy

$$|l^\delta_{1,c}(x,r) - l^\delta_{2,c}(x,r)| \le |p_1 - p_2||r| \quad \forall x, r \in \mathbb{R}^d.$$

It follows that the same inequality applies for the mollified versions l^δ_i. So for any path $g(s)$,

$$\begin{aligned} I^\delta_{p_1}(g) - I^\delta_{p_2}(g) &= \int_0^s l^\delta_1(g(r), g'(r)) - l^\delta_2(g(r), g'(r)) dr \\ &\le |p_1 - p_2| d(g). \end{aligned}$$

Since the total length of the path is at most $d(g) \le s/a$, the Lipschitz estimate follows. □

Proof of Prop. 3.5. Let $T^\delta(x,y)$ be the first-passage time from x to y in the δ-

approximation. Since

$$|T^{\delta}(0,[x]) - T^{\delta}(0,x)| \leq \frac{1}{\delta},$$

we have that

$$\lim_{n\to\infty} \frac{T^{\delta}(0,[nx])}{n} = m^{\delta}(x).$$

For any $x, y \in \mathbb{R}^d$, using subadditivity, we get the estimate

$$T^{\delta}(0,[nx]) \leq T^{\delta}(0,[ny]) + T^{\delta}([ny],[nx]).$$

Then, using the fact that we can take an edge path from $[ny]$ to $[nx]$, and that the time to cross each edge is at most b, we get

$$T^{\delta}(0,[nx]) \leq T^{\delta}(0,[ny]) + b|[ny] - [nx]|.$$

Dividing by n and taking a limit as $n \to \infty$ gives us the Lipschitz estimate. $\square$

Chapter 5

Proofs related to the discrete variational formula

5.1 Proof of the discrete variational formula

In the following, constants will all be called C and will frequently change from line-to-line. We begin with the proof of Prop. 3.7, which says that the discrete stationary problem ν_ϵ approximately satisfies a HJB equation.

Proof of Prop. 3.7. We will first derive a bound and a Lipschitz estimate for $\nu_\epsilon(x)$. From the variational definition of ν_ϵ in (2.15), it's easy to see that

$$-\frac{|p|_\infty}{\epsilon a} \leq \nu_\epsilon(x) \leq -\frac{|p|_\infty}{\epsilon b} \quad \forall\ x \in \mathbb{Z}^d$$

where b and a are the upper and lower bounds on $\tau(x, \alpha)$ (see (1.18)). This simple upper bound can be used to derive a Lipschitz estimate

Claim 5.1. The functions $\nu_\epsilon(x)$ satisfy (uniformly in ϵ) for all $x \in \mathbb{Z}^d$,

$$\mathcal{H}(\nu_\epsilon, p, x) = \sup_{\alpha \in A} \left(\frac{-\mathcal{D}\nu_\epsilon(x,\alpha) - p \cdot \alpha}{\tau(x,\alpha)} \right) \leq \frac{1}{a}|p|_\infty, \tag{5.1}$$

$$0 \leq \sup_{\alpha \in A} \left(-\mathcal{D}\nu_\epsilon(x,\alpha) - p \cdot \alpha \right). \tag{5.2}$$

We will prove Claim 5.1 after completing the proof of the proposition. Applying (5.1) at x and $x + \alpha$ gives us the discrete Lipschitz estimate

$$\|\nu_\epsilon\|_{Lip} \leq \frac{a+b}{a}|p|_\infty. \tag{5.3}$$

Recall the DPP

$$\nu_\epsilon(x) = \inf_\alpha \left(\alpha \cdot p + e^{-\epsilon\tau(x,\alpha)}\nu_\epsilon(x+\alpha) \right).$$

Expand the exponential in the DPP in a Taylor series, and use the bound on ν_ϵ to get

$$-C\epsilon \leq \nu_\epsilon(x) + \sup_{\alpha \in A} \big(-\alpha \cdot p - (1 - \epsilon\tau(x,\alpha))\nu_\epsilon(x+\alpha) \big) \leq C\epsilon,$$

$$-C\epsilon \leq \sup_{\alpha \in A} \big(-\alpha \cdot p - \mathcal{D}\nu_\epsilon(x,\alpha) + \epsilon\tau(x,\alpha)\mathcal{D}\nu_\epsilon + \epsilon\tau(x,\alpha)\nu_\epsilon(x) \big) \leq C\epsilon.$$

Divide through by $\tau(x,\alpha)$, and then use the Lipschitz estimate on ν_ϵ and the bound on $\tau(x,\alpha)$ to get

$$-C\epsilon \leq \sup_{\alpha \in A} \tau(x,\alpha) \left(\frac{-\alpha \cdot p - \mathcal{D}\nu_\epsilon(x,\alpha)}{\tau(x,\alpha)} + \epsilon\nu_\epsilon(x) \right) \leq C\epsilon,$$

$$-C\epsilon \leq \epsilon\nu_\epsilon(x) + \sup_{\alpha \in A} \left(\frac{-\alpha \cdot p - \mathcal{D}\nu_\epsilon(x,\alpha)}{\tau(x,\alpha)} \right) \leq C\epsilon.$$

□

Proof of Claim 5.1. Using the DPP, we get for fixed $\alpha \in A$,

$$\begin{aligned} \nu_\epsilon(x) &\leq p \cdot \alpha + e^{-\epsilon\tau(x,\alpha)}\nu_\epsilon(x+\alpha), \\ \nu_\epsilon(x) &\leq p \cdot \alpha + (1 - \epsilon\tau(x,\alpha))\nu_\epsilon(x+\alpha), \\ -\mathcal{D}\nu_\epsilon(x,\alpha) - p \cdot \alpha &\leq \frac{\tau(x,\alpha)}{a}|p|_\infty. \end{aligned}$$

This proves the upper bound. The lower bound that will be useful in part II. Since $\nu_\epsilon(x)$ is negative, for each α,

$$e^{-\epsilon\tau(x,\alpha)}\nu_\epsilon(x+\alpha) \geq \nu_\epsilon(x+\alpha).$$

Hence,

$$\begin{aligned} &\nu_\epsilon(x) \geq \inf_{\alpha \in A} p \cdot \alpha + \nu_\epsilon(x+\alpha), \\ &\sup_{\alpha \in A}\left(-\mathcal{D}\nu_\epsilon(x,\alpha) - p \cdot \alpha\right) \geq 0. \end{aligned} \tag{5.4}$$

□

Before proving the comparison principle, we first finish the proof of the discrete variational formula in Theorem 1.5. In the following proof we'll need to use probability, so we'll reintroduce ω wherever necessary.

Proof of Theorem 1.5. Let's first prove the upper bound. Let $\phi \in S$, where S is defined in (1.21). Suppose ϕ is such that

$$\sup_x \mathcal{H}(\phi, p, x, \omega) < \infty \text{ a.s.}$$

The form of $\mathcal{H}(\phi, p, x, \omega)$ (see (1.20)) implies that it's coercive. Hence, we must have $\|\phi\|_{Lip} < \infty$. Then, the comparison principle for the finite time-horizon cell-problem in Prop. 3.8 gives

$$\phi(x) - t \sup_x \mathcal{H}(\phi, p, x, \omega) \leq \mu(x, t, \omega) \quad \forall x \in \mathbb{Z}^d.$$

Divide the inequality by t, take a limit as $t \to \infty$, and use Prop. 3.2 and Theorem 3.3. Then, rearrange the inequality and take a sup over x to get

$$\overline{H}(p) \leq \sup_{x \in \mathbb{Z}^d} \mathcal{H}(\phi, p, x, \omega).$$

Now, consider the discrete stationary problem given by (2.15) with edge-weights $\lambda(x, \alpha) = p \cdot \alpha$. Using the discrete HJB equation for ν_ϵ from Prop. 3.7, we get

$$\epsilon \nu_\epsilon(x, \omega) + \mathcal{H}(\nu_\epsilon, p, x, \omega) \leq C\epsilon \quad \forall\ x \in \mathbb{Z}^d.$$

As in the proof of the continuous variational formula in Section A.1, we can normalize this set of functions so that they're zero at the origin. Letting $\hat{\nu}_\epsilon(x, \omega) = \nu_\epsilon(x, \omega) - \nu_\epsilon(0, \omega)$, we get

$$\epsilon \hat{\nu}_\epsilon(x, \omega) + \mathcal{H}(\hat{\nu}_\epsilon, p, x, \omega) \leq C\epsilon - \epsilon \nu_\epsilon(0, \omega).$$

Using the definition of the discrete Hamiltonian, we get for each $\alpha \in A$,

$$\epsilon \hat{\nu}_\epsilon(x, \omega) + \frac{-p \cdot \alpha - \mathcal{D}\hat{\nu}_\epsilon(x, \alpha, \omega)}{\tau(x, \alpha, \omega)} \leq C\epsilon - \epsilon \nu_\epsilon(0, \omega). \tag{5.5}$$

$\hat{\nu}_\epsilon$ is normalized to zero at the origin, and inherits the discrete Lipschitz estimate on ν_e (5.3). Hence,

$$C = \sup_{\epsilon} \left\{ \left\| \hat{\nu}_\epsilon(y,\omega)(1+|y|)^{-1} \right\|_\infty + \|\mathcal{D}\hat{\nu}_\epsilon\|_\infty \right\} < \infty.$$

Let $\psi(\alpha,\omega)$ be an L^2 weak limit of $\mathcal{D}\hat{\nu}_\epsilon(0,\alpha,\omega)$ (as $\epsilon \to 0$) for each $\alpha \in \{e_1,\ldots,e_d\}$. With a slight abuse of notation, we use the translation group to define $\psi(x,\alpha,\omega) = \psi(\alpha, V_x\omega)$. Consider a control $\alpha \in \mathcal{A}$ such that for some $k > 0$, $\gamma_{\alpha,x}(k) = x$; i.e., it forms a loop. For fixed ϵ,

$$\sum_{i=0}^{k} \mathcal{D}\hat{\nu}_\epsilon(\gamma_{\alpha,x}(i), \alpha(i)) = 0 \quad \text{a.s } \omega.$$

Since the measure is translation invariant, each $\psi(\gamma_{\alpha,x}(i),\alpha(i),\omega)$ is an L^2 weak limit of $\mathcal{D}\hat{\nu}_\epsilon(\gamma_{\alpha,x}(i),\alpha(i),\omega)$ for $i \le k$. Then, for any $h \in L^2(\Omega)$,

$$\begin{aligned} \lim_{\epsilon\to 0} \int h(\omega) \sum_{i=0}^{k} \mathcal{D}\hat{\nu}_\epsilon(\gamma_{\alpha,x}(i),\alpha(i))\mathbb{P}(d\omega) &= \int h(\omega) \sum_{i=0}^{k} \psi(\gamma_{\alpha,x}(i),\alpha(i),\omega)\mathbb{P}(d\omega) \\ &= 0. \end{aligned}$$

Since there are only a countable number of loops and a countable number of points, ψ sums over all loops at every location to zero almost surely. Hence, there is a function $\phi(x,\omega)$ such that $\mathcal{D}\phi(x,\alpha,\omega) = \psi(x,\alpha,\omega)$. By L^2 weak-convergence and (5.5), we have for each fixed x,α and any nonnegative function $g(\omega) \in L^2(\Omega)$,

$$\int g(\omega) \left(\frac{p\cdot\alpha - \mathcal{D}\phi(x,\alpha,\omega)}{\tau(x,\alpha,\omega)} \right) \mathbb{P}(d\omega) \le \overline{H}(p) \int g(\omega)\mathbb{P}(d\omega).$$

using Prop. 3.6. We can take a supremum over $x \in \mathbb{Z}^d$ and $\alpha \in A$ to get

$$\sup_x \mathcal{H}(\phi, p, x, \omega) \le \overline{H}(p) \quad \text{a.s.}$$

This proves the other inequality and completes the proof. $\square$

5.2 Proof of the comparison principle

The results in this section make no use of probability and hence we'll ignore the ω dependence. Recall the discrete finite time-horizon variational problem from (2.14):

$$\mu(x,t) = \inf_{\alpha \in \mathcal{A}} \inf_{k \in \mathbb{Z}^+} \left\{ \sum_{i=0}^{k} \lambda(\gamma_{\alpha,x}(i), \alpha(i)) + \phi(\gamma_{\alpha,x}(k)) : \mathcal{T}(\gamma_{\alpha,x}, k) \le t \right\}.$$

Since the time-parameter $t \in \mathbb{R}^+$ is continuous, the DPP for $\mu(x,t)$ is slightly different.

Proposition 5.2. *The DPP for $\mu(x,t)$ takes the form*

$$\mu(x,t) = \begin{cases} \inf_{c \in A}\{\mu(x+c, t-\tau(x,c)) + \lambda(x,c)\} & t \ge \min_{c \in A} \tau(x,c) \\ \phi(x) & \textit{otherwise} \end{cases}.$$

Proof. If $t < \min_c \tau(x,c)$, then no neighbor of x can be reached, and $\mu(x,t) = \phi(x)$. So assume that at least one neighbor $x+c$ can be reached. For fixed $c \in A$, consider the set of controls whose first step is in the c direction:

$$\mathcal{B}_c := \{\alpha \in \mathcal{A} : \alpha(0) = c\}.$$

There is an obvious map from $\mathcal{B}_c$ onto $\mathcal{A}$, obtained by shifting the control $\alpha(\cdot) \to \alpha(\cdot + 1)$ and forgetting the first step. It follows immediately that

$$\mu(x,t) \le \inf_{c \in \mathcal{A}} \mu(x + c, t - \tau(x,c)) + \lambda(x,c).$$

For the opposite inequality, for any $\epsilon > 0$ pick α and k such that

$$\begin{aligned}\mu(x,t) &\ge \sum_{i=0}^{k} \lambda(\gamma_{\alpha,x}(i), \alpha(i)) + \phi(\gamma_{\alpha,x}(k)) - \epsilon, \\ &\ge \lambda(x, x + \alpha(0)) + \mu(x, x + \alpha(0), t - \tau(x, \alpha(0))) - \epsilon.\end{aligned}$$

□

We next prove the comparison principle in Prop. 3.8.

Definition 5.3 (Reachable set)**.** Following Bardi and Capuzzo-Dolcetta [5], define

$$R(x,t) := \{y \in \mathbb{Z}^d : \mathcal{T}(x,y) \le t\}$$

to be the set of sites that can be reached from x within time t.

Proof of Prop. 3.8. Let $\phi(x)$ have bounded discrete derivatives and define

$$\zeta(x,t) = \phi(x) - t \sup_x \mathcal{H}(\phi, p, x). \tag{5.6}$$

We need to show that $\zeta(x,t) \leq \mu(x,t)$. Let $N(x,t)$ be the cardinality of the reachable set $R(x,t)$. Suppose $N(x,t)$ jumps in value on a finite set of times contained in $(0,t]$. Then, $\mu(x,t)$ can only decrease at these times and remains constant otherwise. So it is enough to do an induction on this set of times to show that $\zeta(x,t) \leq \mu(x,t)$. However, it may well happen that $\mu(x,t)$ decreases on a possibly uncountable set of times, and the induction becomes harder to do. To handle this subtlety, we introduce a truncation of the problem.

For large $K > 0$, we'll define a truncated variational problem $\mu_K(x,t)$ as follows. Let $B_K(0)$ be the ball of radius K centered at the origin. Inside $Z_K := B_K(0) \cap \mathbb{Z}^d$, paths are allowed to wander freely, but once a path exits Z_K, it cannot move further. If a path starts in the set $\mathbb{Z}^d \setminus Z_K$, it cannot move at all. More succinctly, the set of control directions A_K is

$$A_K := \begin{cases} A & \text{inside } Z_K \\ \varnothing & \text{otherwise} \end{cases}.$$

Clearly $\mu_K(x) = \phi(x)\ \forall x \in \mathbb{Z}^d \setminus Z_K$ and for all $x \in Z^d$,

$$\mu(x,t) \leq \mu_K(x,t).$$

It is easy to verify that μ_K satisfies the same DPP as μ for all $x \in Z_K$. For any fixed $x \in \mathbb{Z}^d$ and $t \in \mathbb{R}^+$, we must have $R(x,t) \subset Z_K$ for large enough K. Therefore for K large enough, $\mu_K(x,t) = \mu(x,t)$. Hence, it's enough to show for any fixed $K > 0$ that

$$\zeta(x,t) \leq \mu_K(x,t)\ \forall\ x \in \mathbb{Z}^d.$$

We recursively define the sequence of times $\{t_{x,k}\}_{k\in\mathbb{Z}^+}$ at which $R(x,s)$ increases in size for $s \leq t$ as follows:

$$t_{x,k} = \inf\left\{s \leq t : N(x,s) > N(x,t_{x,k-1})\right\}, \quad t_{x,0} = 0.$$

Since $N(x,t) < \infty$, $t_{x,k}$ is finite only for a finite number of k; by convention, the infimum over an empty set is $+\infty$. Let $j(x) := \max\{k : t_{x,k} < \infty\}$ be the last jump of $N(x,\cdot)$ before time t.

Now, we look at the all the times at which the reachable set of any point $x \in Z_K$ expands. These are also all the possible times at which $\mu_K(x,s)$ can decrease for $s \leq t$. Order the finite set

$$\bigcup_{x\in Z_K}\bigcup_{k\leq j(x)} t_{x,k} =: s_1 \leq \ldots \leq s_N.$$

We will do induction on the ordered sequence $\{s_i\}$. Assume as the inductive hypothesis that $\zeta(x,r) \leq \mu_K(x,r)$ $\forall$ x and $r \leq s_{k-1}$. $\mu_K(x,r)$ does not decrease when $s_i < r < s_{i+1}$ because the reachable set $R_K(s)$ does not expand during this time. This implies that in fact,

$$\zeta(x,r) \leq \mu_K(x,r) \quad \forall\ x \text{ and } r < s_k.$$

Let $C = \sup_{x\in\mathbb{Z}^d} \mathcal{H}(\phi, p, x)$. Then,

$$\begin{aligned}
I &:= \sup_{c\in A}\left\{\frac{\zeta(x,s_k)-\zeta(x+c,s_k-\tau(x,c))-\lambda(x,c)}{\tau(x,c)}\right\}\\
&= \sup_{c\in A}\left\{\frac{(\phi(x)-Cs_k)-(\phi(x+\alpha)-C(s_k-\tau(x,c)))-\lambda(x,c)}{\tau(x,c)}\right\}\\
&= \sup_{c\in A}\left\{\frac{-(\phi(x+c)-\phi(x))-\lambda(x,c)}{\tau(x,c)}\right\}-C\\
&= \mathcal{H}(\phi,p,x)-C \le 0.
\end{aligned}$$

Since $\tau(x,y) > 0$, this means that for each c in the sup in I, we have

$$\zeta(x,s_k)-\zeta(x+c,t-\tau(x,c))-\lambda(x,c) \le 0.$$

Hence for all $x \in Z_K$,

$$\begin{aligned}
\zeta(x,s_k) &\le \inf_{c\in A}\left\{\zeta(x+c,s_k-\tau(x,c))+\lambda(x,c)\right\},\\
&\le \inf_{c\in A}\left\{\mu_K(x+c,s_k-\tau(x,c))+\lambda(x,c)\right\},\\
&= \mu_K(x,s_k),
\end{aligned}$$

where we've used the inductive hypothesis and the fact that μ_K also satisfies the DPP in Prop. 5.2. In case $s_k - \tau(x,c) < 0$ for all $c \in A$,

$$\mu_K(x,s_k) = \phi(x) \ge \zeta(x,s_k).$$

Letting $K \to \infty$ completes the proof. □

5.3 Solution of the HJB equation

Next, we prove that the limiting Hamiltonian is a norm on $\mathbb{R}^d$.

Proof of Prop. 3.9. Consider the variational formula again (dropping the sup over x doesn't make a difference, see (6.1)):

$$\overline{H}(p) = \inf_{\phi \in S} \operatorname{ess\,sup}_{w} \sup_{\alpha \in A} \frac{-\mathcal{D}\phi(0,\alpha,\omega) - p \cdot \alpha}{\tau(0,\alpha,\omega)},$$

Replacing $\phi \mapsto \lambda\phi$ leaves S invariant, and it follows that for $\lambda > 0$, $\overline{H}(\lambda p) = \lambda \overline{H}(p)$.

For any fixed ϕ, $E[p \cdot \alpha + \mathcal{D}\phi(0,\alpha)] = p \cdot \alpha$ and hence

$$\operatorname{ess\,sup}_{\omega \in \Omega} \sup_{\alpha} \left(-\mathcal{D}\phi(0,\alpha) - p \cdot \alpha\right) \geq \sup_{\alpha} E[-p \cdot \alpha - \mathcal{D}\phi(0,\alpha)] \geq |p|_\infty$$

Therefore,

$$\overline{H}(p) \geq \frac{|p|_\infty}{b}.$$

Finally, the triangle inequality for $\overline{H}$ follows from the fact that for each fixed p, $-\mu(x,t)/t$ converges to $\overline{H}(p)$ (see Chapter 3). For any $p, q \in \mathbb{R}^d$, we have

$$\begin{aligned}\sup_{\alpha \in \mathcal{A}} \sup_{k \in \mathbb{Z}^+} &\left\{ \sum_{i=0}^{k} -(p+q) \cdot \alpha(i) - \mu_0(\gamma_{\alpha,x}(k)) : \mathcal{W}_{x,k}(\alpha) \leq t \right\} \\ &\leq \sup_{\alpha \in \mathcal{A}} \sup_{k \in \mathbb{Z}^+} \left\{ \sum_{i=0}^{k} -p \cdot \alpha(i) - \mu_0(\gamma_{\alpha,x}(k)) : \mathcal{W}_{x,k}(\alpha) \right\} \\ &\quad + \sup_{\alpha \in \mathcal{A}} \sup_{k \in \mathbb{Z}^+} \left\{ \sum_{i=0}^{k} -q \cdot \alpha(i) - \mu_0(\gamma_{\alpha,x}(k)) : \mathcal{W}_{x,k}(\alpha) \right\}.\end{aligned}$$

Dividing by t and taking a limit $t \to \infty$ shows that $\overline{H}$ satisfies the triangle inequality. □

Proof of Prop. 3.10. We get $T(x) \leq L(x)$ by considering the straight line path from 0 to x. In fact, the straight line is the minimizing path, as can be seen by an application of the triangle inequality for $L(\cdot)$. □

Part II

Applications and Discussion

Chapter 6

Recap and some basic observations

In this section, we note some basic facts about the variational formula in Theorem 1.5, and some simple corollaries of its proof in Section 5.1. First, note that the sup over x can be dropped. That is, we can rewrite (1.22) as

$$\overline{H}(p) = \inf_{\phi \in S} \operatorname*{ess\,sup}_{\omega \in \Omega} \mathcal{H}(\phi, p, 0, \omega). \tag{6.1}$$

This is a simple consequence of the fact that $\sup_x \mathcal{H}(\phi, p, x, \omega)$ is translation invariant, and hence is a constant almost surely due to ergodicity.

We proved that the sequence of functions $\{\hat{\nu}_\epsilon\}_\epsilon$ defined in the proof of the variational formula in Section 5.1 is minimizing. $\hat{\nu}_\epsilon$ is a translate of ν_ϵ, the value function of the stationary cell-problem with DPP (3.9)

$$\nu_\epsilon(x) = \inf_{\alpha \in A} \left(\alpha \cdot p + e^{-\epsilon \tau(x,\alpha)} \nu_\epsilon(x + \alpha) \right).$$

The DPP gave the following estimate in Claim 5.1:

$$0 \leq \sup_{\alpha \in A} \left(-\mathcal{D}\nu_\epsilon(x,\alpha) - p\cdot\alpha\right) \leq \frac{b}{a}|p|_\infty.$$

$\hat{\nu}_\epsilon$ inherits this estimate, and this means that we may further restrict the set S of functions (1.21). We state this as a corollary of the variational formula.

Corollary 6.1 (of Theorem 1.5)**.** *The variational formula in* (1.22) *holds with*

$$S = \left\{ \phi\colon \mathbb{Z}^d \times \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \mathcal{D}\phi(x+z,\omega) = \mathcal{D}\phi(x, V_z\omega),\ \forall x,z \in \mathbb{Z}^d \\ E[\mathcal{D}\phi(x,\alpha)] = 0\ \forall x \in \mathbb{Z}^d \text{ and } \alpha \in A, \\ 0 \leq \sup_{\alpha \in A}\left(-\mathcal{D}\phi(x,\alpha) - p\cdot\alpha\right) \leq \dfrac{b}{a}|p|_\infty \end{array} \right\} \tag{6.2}$$

Now consider $\mu(x,t)$, the discrete finite time-horizon cell-problem. The discrete comparison principle in Prop. 5.2 says that for any $\phi\colon \mathbb{Z}^d \to \mathbb{R}$ such that $\|\phi\|_{Lip} < \infty$,

$$\mu(x,t) \geq \phi(x) - t \sup_{x \in \mathbb{Z}^d} \mathcal{H}(\phi,p,x) \quad \forall x,t.$$

An almost identical proof —which we will not repeat— but with a bunch of inequalities reversed, gives the following proposition:

Proposition 6.2. *Suppose $\phi \in S$, where S is defined in* (6.2)*. Then,*

$$\mu(x,t) \leq \phi(x) - t \inf_{x \in \mathbb{Z}^d} \mathcal{H}(\phi,p,x) \quad \forall x,t.$$

Then, following the same argument in the proof of the variational formula, we get

Corollary 6.3 (of Theorem 1.5)**.** *For each* $\phi \in S$ (6.2)*,*

$$\operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x} \mathcal{H}(\phi, p, x, \omega) \le \overline{H}(p) \le \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x} \mathcal{H}(\phi, p, x, \omega). \tag{6.3}$$

Proof. We've already proved the upper bound on $\overline{H}(p)$ in Section 5.1. Using the lower bound in Prop. 6.2, we have for any $x \in \mathbb{Z}^d$,

$$\mu(x,t) \le \phi(x) - t \inf_{x} \mathcal{H}(\phi, p, x, \omega)$$

Next, we divide the inequality by t, and take a limit as $t \to \infty$. Using $\mu(x,t)/t \to -\overline{H}(p)$ as $t \to \infty$ (Prop. 3.2 and Theorem 3.3) we get

$$\inf_{x\in\mathbb{Z}^d} \mathcal{H}(\phi, p, x, \omega) \le \overline{H}(p) \quad \text{a.s.}$$

□

Definition 6.4 (Discrete corrector)**.** For some constant C, if $\phi \in S$ satisfies

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}(\phi, p, x, \omega) = C \quad \text{a.s.},$$

ϕ is called a corrector for the variational formula.

This definition is consistent with the definition of corrector in continuum homogenization theory [25, 26]; i.e., it's a function that solves the discrete cell-problem. If ϕ is a corrector, then Corollary 6.3 tells us that it's a minimizer of the variational formula.

Chapter 7

Explicit algorithm to produce a minimizer

Let $\{V_{e_1}, \ldots, V_{e_d}\}$ be commuting, invertible, measure-preserving ergodic transformations on Ω. They generate the group of translation operators in (1.14) under composition. Suppose we have first-passage percolation on the undirected graph on $\mathbb{Z}^d$, i.e.,

$$\tau(x, \alpha, \omega) = \tau(x + \alpha, -\alpha, \omega). \tag{7.1}$$

Let $A_+ = \{e_1, \ldots, e_d\}$. Let $t \colon A_+ \times \Omega \to \mathbb{R}$ be a function representing the edge-weight at the origin. For example, it could consist of d i.i.d. edge-weights, one for each direction. Let $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. Then, the edge-weight function is given by

$$\tau(x, \alpha, \omega) = t\left(\alpha, V_{e_1}^{x_1} \cdots V_{e_d}^{x_d} \omega\right).$$

In this section, we will assume the following symmetry on the medium:

$$V_{e_1} = \cdots = V_{e_d} = V. \tag{7.2}$$

This means that for each ω the function $\tau(\cdot, \cdot, \omega)$ is constant along the hyperplanes $\{x \in \mathbb{Z}^d : \sum_{i=1}^d x_i = z\}$ for each $z \in \mathbb{Z}$. Despite this symmetry, the medium is still quite random, and it's not so obvious —although one ought to be able to calculate it— what the time-constant is. However, the set S in (1.21) is tremendously simplified.

Proposition 7.1. *If $\phi \in S$ and (7.2) holds, the derivative points in the $\sum_i e_i$ direction; i.e.,*

$$\mathcal{D}\phi(x, \alpha, \omega) = \mathcal{D}\phi(0, e_1, \omega) \quad \forall \alpha \in A \textit{ and } \forall x \in \mathbb{Z}^d \textit{ a.s.}$$

Proof. The derivative of ϕ sums to 0 over any discrete loop in $\mathbb{Z}^d$. In particular, for any $i \neq j \in \{1, \dots, d\}$

$$\mathcal{D}\phi(x, e_i, \omega) + \mathcal{D}\phi(x + e_i, e_j, \omega) + \\ + \mathcal{D}\phi(x + e_i + e_j, -e_i, \omega) + \mathcal{D}\phi(x + e_j, -e_j, \omega) = 0. \tag{7.3}$$

Since the derivative is stationary and $\mathcal{D}\phi(x, \alpha, \omega) = -\mathcal{D}\phi(x + \alpha, -\alpha)$, we have

$$\begin{aligned}\mathcal{D}\phi(x, e_i, \omega) - \mathcal{D}\phi(x, e_j, \omega) &= \mathcal{D}\phi(x, e_i, V_{e_j}\omega) - \mathcal{D}\phi(x, e_j, V_{e_i}\omega), \\ &= \mathcal{D}\phi(x, e_i, V\omega) - \mathcal{D}\phi(x, e_j, V\omega).\end{aligned}$$

Hence, $\mathcal{D}\phi(0, e_i, \omega) - \mathcal{D}\phi(0, e_j, \omega)$ is invariant under V. Since it also has zero mean, it follows from ergodicity that

$$\mathcal{D}\phi(x, \alpha, \omega) = \mathcal{D}\phi(x, e_1, \omega) \quad \forall \alpha \in A \text{ a.s.}$$

$\square$

Next, we simplify the variational formula under the symmetry assumption (7.2). Redefine the discrete Hamiltonian for $t \in \mathbb{R}$, $p \in \mathbb{R}^d$ to be

$$\mathcal{H}_{sym}(t, p, \omega) := \sup_{\alpha \in A^+} \frac{|t + p \cdot \alpha|}{\tau(0, \alpha, \omega)}. \tag{7.4}$$

Proposition 7.2. *If we assume* (7.2) *and* (7.1)*, the variational formula becomes*

$$\overline{H}(p) = \inf_{f \in F} \operatorname*{ess\,sup}_{w} \mathcal{H}_{sym}(f(\omega), p, \omega), \tag{7.5}$$

where

$$F := \left\{ f : \Omega \to \mathbb{R},\ E[f] = 0,\ \sup_{\alpha \in A^+} |f + p \cdot \alpha| \le (b/a)|p|_\infty \right\}. \tag{7.6}$$

Proof. For each $\phi \in S$, let $f(\omega) = D\phi(0, e_1, \omega)$. The proof is easy using the assumption (7.1) on the edge-weights and Prop. 7.1. To wit,

$$\begin{aligned}
\operatorname*{ess\,sup}_{\omega} \sup_{x} \mathcal{H}(\phi, p, x, \omega) &= \operatorname*{ess\,sup}_{\omega} \sup_{x} \sup_{\alpha \in A} \left\{ \frac{-\mathcal{D}\phi(x, \alpha, \omega) - p \cdot \alpha}{\tau(x, \alpha, \omega)} \right\}, \\
&= \operatorname*{ess\,sup}_{\omega} \sup_{\alpha \in A^+} \left\{ \frac{|\mathcal{D}\phi(0, e_1, \omega) + p \cdot \alpha|}{\tau(0, \alpha, \omega)} \right\}, \\
&= \operatorname*{ess\,sup}_{\omega} \mathcal{H}_{sym}(f(\omega), p, \omega).
\end{aligned}$$

□

In the following, we will write $\mathcal{H}_{sym}(f,\omega)$ and drop reference to p since it's irrelevant to our arguments. We present an algorithm that produces a minimizer for the variational problem under the symmetry assumption. The idea behind the algorithm is simple. At each iteration, we try to reduce the essential supremum over ω by modifying $f(\omega)$, while simultaneously keeping it inside the set F. If it fails to reduce the sup, we must be at a minimizer. We explain what we're trying to do in each step in the proof of convergence of the algorithm. So we suggest skimming the definition of the algorithm first, and returning to the definition of each step when reading the proof.

Start algorithm

1. Start with any $f_0 \in F$, for example, $f_0 = 0$. Let $\mu_0 = E[\mathcal{H}_{sym}(f_0,\omega)]$, and let

   $$d = \operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}_{sym}(f_0,\omega) - \mu_0.$$

   If $d = 0$, stop.

2. Define the sets

   $$MIN_0 := \{\omega : \mathcal{H}_{sym}(f_0,\omega) = \min_x \mathcal{H}_{sym}(x,\omega)\}, \tag{7.7}$$

   $$S := \{\omega : \mathcal{H}_{sym}(f_0,\omega) > \mu_0\}, \tag{7.8}$$

   $$I := \{\omega : \mathcal{H}_{sym}(f_0,\omega) < \mu_0\}. \tag{7.9}$$

If

$$\operatorname*{ess\,sup}_{\omega \in MIN_0} \mathcal{H}_{sym}(f_0, \omega) = \operatorname*{ess\,sup}_{\omega \in \Omega} \mathcal{H}_{sym}(f_0, \Omega),$$

stop.

3. Let $\Delta f^*(\omega)$ be such that

$$\mathcal{H}_{sym}(f_0 + \Delta f^*(\omega), \omega) = \min_x \mathcal{H}_{sym}(x, \omega).$$

Define the sets

$$S_+ := \{\omega \in S \setminus MIN_0 : D_+\mathcal{H}_{sym} \subset (-\infty, 0)\},$$
$$S_- := \{\omega \in S \setminus MIN_0 : D_-\mathcal{H}_{sym} \subset (0, \infty)\},$$

where D_+ and D_- are the left and right derivatives of the convex function $\mathcal{H}_{sym}(\cdot, \omega)$. Let

$$\Delta f(\omega) = \begin{cases} \max\left(-a(\mathcal{H}_{sym}(f_0, \omega) - \mu_0),\ \Delta f^*(\omega)\right) & \omega \in S_+ \\ \min\left(a(\mathcal{H}_{sym}(f_0, \omega) - \mu_0),\ \Delta f^*(\omega)\right) & \omega \in S_- \\ a\xi(\mu_0 - \mathcal{H}_{sym}(f_0, \omega)) & \omega \in I \\ 0 & \text{elsewhere} \end{cases},$$

where

$$\xi = -\frac{\int_{S_+ \cup S_-} \Delta f(\omega)\ \mathbb{P}(d\omega)}{\int_I a(\mu_0 - \mathcal{H}_{sym}(f_0, \omega))\ \mathbb{P}(d\omega)}.$$

Let $f_1 = f_0 + \Delta f(\omega)$. Return to step 1.

End algorithm

Theorem 7.3. *There are three possibilities for the algorithm:*

1. *If it terminates in a finite number of steps with $d = 0$, we have a minimizer that's a corrector.*

2. *If it terminates in a finite number of steps with $d > 0$, we have a minimizer that's not a corrector*

3. *If it does not terminate, we produce a corrector in the limit.*

We need the following lemma to prove Theorem 7.3.

Lemma 7.4. *The function $\mathcal{H}_{sym}(x, \omega)$ has the following properties:*

1. *For each ω, it is convex in x.*

2. *It has a unique measurable minimum $x^*(\omega)$.*

3. *Its left and right derivatives satisfy $D_-\mathcal{H}_{sym}(x, \omega) \in [b^{-1}, a^{-1}]$ or $D_+\mathcal{H}_{sym} \in [-a^{-1}, -b^{-1}]$ a.s. ω.*

We will prove Lemma 7.4 after proving Theorem 7.3.

Proof of Theorem 7.3.

1. In the first step, we compute d, the distance between the mean and supremum of $\mathcal{H}_{sym}(f, \omega)$. If $d = 0$, f must be a corrector and from Corollary 6.3, it must be a minimizer. Therefore, we stop the algorithm.

2. MIN_0 is the set on which $\mathcal{H}(f_0, \omega)$ cannot be lowered further. S and I are the sets on which $\mathcal{H}_{sym}(f_0, \omega)$ is bigger and lower than its mean μ_0. f will be modified on these two sets in step 3.

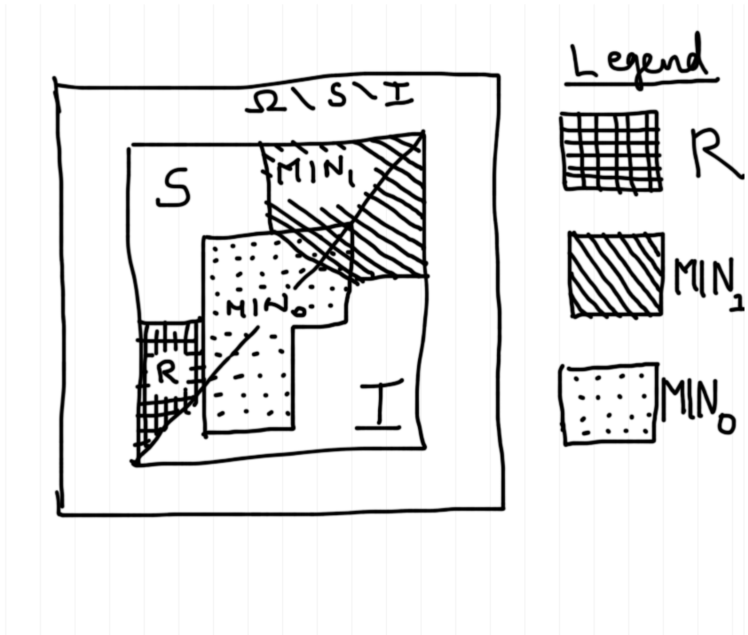

Figure 7.1: Sketch of sets in algorithm. The outermost square represents the probability space Ω. S is the upper triangle in the inner square, and I is the lower triangle in the inner square. $\Omega \backslash S \backslash I$ is the annular region between the two squares.

Lemma 4.2 says that $\mathcal{H}_{sym}(\cdot, \omega)$ is convex and has a minimum. So there is the possibility of the algorithm getting "stuck" at a minimum of $\mathcal{H}_{sym}$. That is, f_0 might be such that $\mathcal{H}_{sym}(f_0, \omega) = \mathcal{H}_{sym}(x^*(\omega), \omega)$ on a set of positive measure, and also

$$\operatorname*{ess\,sup}_{\omega \in MIN_0} \mathcal{H}_{sym}(f_0, \omega) = \operatorname*{ess\,sup}_{\omega \in \Omega} \mathcal{H}_{sym}(f_0, \Omega).$$

For any other $g \in F$, we clearly have $\mathcal{H}_{sym}(g(\omega), \omega) \geq \mathcal{H}_{sym}(f_0(\omega), \omega)$ on MIN_0. Hence f_0 must be a minimizer, and we stop the algorithm.

3. Δf is first defined on the sets S_+ and S_- so that the supremum falls. Then, Δf is defined on I so that it satisfies

$$E[\Delta f] = 0.$$

We need to make sure that ξ is not infinite. Notice that $E[(\mu_0 - \mathcal{H}_{sym}(f_0,\omega)), I] > 0$; for if not, $\operatorname{ess\,inf}_{\omega\in I}\mathcal{H}_{sym}(\omega) = \mu_0$. Hence Δf is well-defined on I. We derive a useful estimate on ξ next. Since

$$|E[(\mathcal{H}_{sym}(f_0,\omega) - \mu_0)\,, S \setminus MIN_0]| \leq E[(\mu_0 - \mathcal{H}_{sym}(f_0,\omega))\,, I],$$

we have

$$\begin{aligned} E[\Delta f, S_+ \cup S_-] &\leq aE[(\mathcal{H}_{sym}(f_0,\omega) - \mu_0), S_+ \cup S_-], \\ &\leq aE[\mu_0 - \mathcal{H}_{sym}(f_0,\omega), I]. \end{aligned}$$

Therefore,

$$-1 < \xi < 1. \tag{7.10}$$

Finally, we prove that if the algorithm does not terminate in either step 1 or 2, we produce a corrector in the limit. We claim that if at the end of step 3 of the algorithm, $\operatorname{ess\,sup}\mathcal{H}(f_1)$ does not fall enough, the algorithm will terminate at the next step.

Claim 7.5. If

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\mathcal{H}_{sym}(f_1,\omega) > \operatorname*{ess\,sup}_{\omega\in\Omega}\mathcal{H}_{sym}(f_0,\omega) - \frac{da}{b}, \tag{7.11}$$

the algorithm will terminate when it goes to step 2 in the following iteration. That is,

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\mathcal{H}_{sym}(f_1,\omega) = \operatorname*{ess\,sup}_{MIN_1}\mathcal{H}_{sym}(f_1,\omega),$$

where MIN_1 is defined in (7.9) with f_0 replaced by f_1.

Now suppose that the algorithm does not terminate, and let f_n be the n^{th} iterate. Claim 7.5 gives us the estimate

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}_{sym}(f_n,\omega) \le \operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}_{sym}(f_{n-1},\omega) - d_n \frac{a}{b}.$$

Since $\mathcal{H}_{sym} \ge 0$, we must have $d_n \to 0$. Since for all n, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}_{sym}(f_n,\omega) < \operatorname*{ess\,sup}_{\omega\in\Omega} \mathcal{H}_{sym}(f_0,\omega),$$

the coercivity of $\mathcal{H}_{sym}$ implies that f_n must be bounded uniformly in n. By our construction, $\{f_n\}$ is a bounded martingale with respect to the filtration $\mathcal{F}_n = \sigma(f_1,\ldots,f_n)$. Hence by the martingale convergence theorem, $f_\infty(\omega) = \lim_n f_n(\omega)$ exists a.s., and further the convergence is uniform in every L^p norm. Then, by the continuity of the $\mathcal{H}$, its nonnegativity, and its uniform boundedness on compact sets, we get for any $p \in \mathbb{R}$,

$$\begin{aligned}
0 &= \lim_n d_n \\
&= \lim_{n\to\infty} \operatorname*{ess\,sup} \mathcal{H}_{sym}(f_n(\omega),\omega) - \int \mathcal{H}_{sym}(f_n(\omega),\omega), \\
&\ge \|\mathcal{H}_{sym}(f_\infty,\omega)\|_p - \int \mathcal{H}_{sym}(f_\infty,\omega).
\end{aligned}$$

Taking $p \to \infty$ proves that f_∞ is a corrector. This completes the proof except for Claim 7.5. We prove this next. □

Proof of Claim 7.5. Let

$$R := \{\omega \in S \setminus MIN_0 : a\, |\mathcal{H}_{sym}(f_0,\omega) - \mu_0)| < \Delta f^*(\omega)\},$$

be the set on which we can modify f_0 without hitting the minimum of $\mathcal{H}(f_0,\omega)$; i.e., $\mathcal{H}_{sym}(f_1,\omega) > \mathcal{H}_{sym}(x^*(\omega),\omega)$. By the definition of Δf and the bound on the derivatives of $\mathcal{H}(\cdot,\omega)$ in Prop. 7.4, we have

$$\mathcal{H}(f_0,\omega) - \frac{1}{a}a(\mathcal{H}_{sym}(f_0,\omega)-\mu_0) \leq \mathcal{H}(f_1,\omega)$$
$$\leq \mathcal{H}(f_0,\omega) - \frac{1}{b}a(\mathcal{H}_{sym}(f_0,\omega)-\mu_0) \qquad \omega \in R \text{ a.s.}$$

Therefore,

$$\mu_0 \leq \operatorname*{ess\,sup}_{\omega\in R} \mathcal{H}(f_1,\omega) \leq \operatorname*{ess\,sup}_{\omega\in R} \mathcal{H}(f_0,\omega) - \frac{da}{b}. \tag{7.12}$$

Similarly for $\omega \in I$, we use the bound on ξ in (7.10) to get

$$\mathcal{H}_{sym}(f_1,\omega) \leq \mathcal{H}_{sym}(f_0,\omega) + \xi(\mu_0 - \mathcal{H}_{sym}(f_0,\omega)) \leq \mu_0 \quad \omega \in I \text{ a.s.} \tag{7.13}$$

From the definition of R, it follows that $S\backslash R \subset MIN_1$. Since $\Delta f = 0$ on $S \cap MIN_0$, we must have $S \cap MIN_0 \subset MIN_1$.

Consider the condition in (7.11) again. Equations (7.12) and (7.13) imply that we can ignore the sets R and I when taking a sup over Ω. It's clear that we can ignore $\Omega \setminus S \setminus I$ too, since $\mathcal{H}(f_0,\omega) = \mu_0$ on this set. Summarizing, we get

$$\operatorname*{ess\,sup}_{\Omega} \mathcal{H}_{sym}(f_1,\omega) = \operatorname*{ess\,sup}_{S\backslash R\ \cup\ MIN_0\cap S} \mathcal{H}_{sym}(f_1,\omega) = \operatorname*{ess\,sup}_{MIN_1} \mathcal{H}(f_1,\omega).$$

Hence, the algorithm terminates at step 2 in the next iteration. □

To finish, we complete the proof of Lemma 7.4.

Proof of Lemma 7.4. Since

$$\mathcal{H}_{sym}(t,p,\omega) = \sup_{\alpha \in A^+} \frac{|t + p \cdot \alpha|}{\tau(0,\alpha,\omega)},$$

Clearly $\mathcal{H}_{sym}$ is convex. Its minimum is unique since $\tau(0,\alpha,\omega) \geq a$, and therefore, it cannot have a "flat spot" parallel to the t-axis.

Now, $\mathcal{H}_{sym}$ can only take its minimum at a minimum of $|t + p \cdot \alpha|/\tau(0,\alpha,\omega)$ or when t is such that $|t+p\cdot\alpha|/\tau(0,\alpha_1,\omega) = |t+p\cdot\alpha|/\tau(0,\alpha_2,\omega)$ for any $\alpha_1, \alpha_2 \in A_+$. There are only a finite number of such possibilities, we can compute all of them, and hence its easy to see that $x^*(\omega)$ is measurable.

The fact that $D_-\mathcal{H}_{sym}(t,\omega) \in [b^{-1}, a^{-1}]$ or $D_+\mathcal{H}_{sym}(t,\omega) \in [-a^{-1}, -b^{-1}]$ for all t, follows easily from the form of $\mathcal{H}_{sym}$. $\square$

Suppose the vector $\vec{t}(\omega) = (t(e_1,\omega), \ldots, t(e_d,\omega))$ takes at most a finite number of different values $\{\vec{t}_0, \ldots, \vec{t}_{n-1}\} =: \Omega_0$. Let our probability space be $\Omega = (\mathbb{R}^d)^{\mathbb{Z}}$, let $\tau(z,\cdot,\omega) = \omega_{z_1}$ (z_1 is the first coordinate of z), and let the marginal of $\mathbb{P}$ on any coordinate of Ω be supported on Ω_0.

We show that even if $\mathbb{P}$ is a product measure, under the symmetry assumption, the structure of the problem is nearly equivalent to a periodic medium. Define the sets

$$A_i := \{\omega \in \Omega : \tau(0,\cdot,\omega) = \vec{t}_i\}, \quad i = 1, \ldots, n-1.$$

The set F of functions in (7.6) can be restricted to

$$F := \left\{ f(\omega) : f(\omega) = \sum_{i=0}^{n-1} f_i 1_{A_i}(\omega),\ f_i \in \mathbb{R},\ E[f] = 0 \right\}, \tag{7.14}$$

and the algorithm continues to produce a minimizer. Now suppose we have a periodic medium with equal periods in both directions; i.e., the translations satisfy

$$\tau(0,\cdot,V^n\omega) = \tau(0,\cdot,V\omega) \text{ a.s.}$$

in addition to (7.2). Periodicity only forces the additional constraint $\mathbb{P}(A_i) = 1/n$, and except for this, the problem is nearly unchanged. Periodic homogenization has been well-studied and there are many algorithms to produce the effective Hamiltonian; see for example, Gomes and Oberman [17] or Oberman et al. [31].

Our algorithm works even if $\tau(0,\cdot,\omega)$ takes an uncountable number of values; i.e., the period is infinite. Notice that what we have here is an n-dimensional deterministic convex minimization problem with linear constraints (see Prop. 7.2 and (7.14)). It's worth stating (without proof, of course) that our algorithm is computationally much faster than conjugate gradient and other standard constrained optimization methods.

Remark 7.6. The symmetry assumption is a massive simplification, and removing this is a real challenge. If the translations V_i are rationally related, we ought to be able to generalize the algorithm with a little work. However, taking this route —solving the loop/cocycle condition— in general is probably hopeless. It appears that working on an instance $\omega \in \Omega$ of the probability space would be the most convenient way to proceed, since we can work directly with a function $f\colon \mathbb{Z}^d \to \mathbb{R}$ (instead of its derivative) and forget the cocycle condition.

Chapter 8

Comparing two distributions

8.1 A simple coupling based argument

For $i = 1, 2$, let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ be probability spaces, and let $\tau_i \colon \mathbb{Z}^d \times A \times \Omega \to \mathbb{R}$ be two edge-weight functions. Assume that

$$0 < \; a_i = \operatorname*{ess\,inf}_{x,\alpha,\omega} \tau_i(x,\alpha,\omega),$$

$$b_i = \operatorname*{ess\,sup}_{x,\alpha,\omega} \tau_i(x,\alpha,\omega) < \infty.$$

We wish to compare $m_1(x)$ and $m_2(x)$, the corresponding time-constants. There is an elementary argument to obtain a very basic estimate between the two time-constants[1]. We will reproduce it using the variational formula to highlight the duality in the problem.

It will be easier to compare the two first-passage percolation problems if they're both on the space $(\mathbb{R}^d)^{\mathbb{Z}^d}$, and we first show that we can always assume this.

[1] told to me by M. Damron

Consider the map $M \colon \Omega \to (\mathbb{R}^d)^{\mathbb{Z}^d}$ defined as

$$M(\omega) = (\tau(V_z\omega))_{z\in\mathbb{Z}^d}. \tag{8.1}$$

Let

$$\mathcal{C} = \sigma(M) \subset \mathcal{F} \tag{8.2}$$

be the sigma-algebra generated by M. We next show that it's enough to consider functions $f \in S$ that are measurable with respect to $\mathcal{C}$.

Proposition 8.1. *Assume that every $f \in S$ (defined in Corollary 6.1) also satisfies the additional condition that $f(x, \omega)$ is $\mathcal{C}$ measurable for each $x \in \mathbb{Z}^d$. Then, the variational formula in* (6.1) *is unchanged.*

The proof is a simple consequence of convexity and can be found in Section A.3.

With Prop. 8.1, it's easy to show that pushing the problem forward to the space $(\mathbb{R}^{2d})^{\mathbb{Z}^d}$ does not change the first-passage percolation problem. Let $\text{Im}(M) \subset (\mathbb{R}^{2d})^{\mathbb{Z}^d}$ be the image of Ω under the map M. Let $\mathbb{P}_M$ be the push-forward measure of $\mathbb{P}$ under M. It is enough to show that for each $f \in S$, there is a $g \colon (\mathbb{R}^{2d})^{\mathbb{Z}^d} \to \mathbb{R}$ such that

$$g(M(\omega)) = f(\omega) \quad \text{a.s.}$$

By Prop. 8.1, we can assume that f is $\sigma(M)$ measurable and hence by an elementary measurability lemma (see, for example Williams [40]) there is a function g as required above.

Therefore, we will henceforth assume that $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^d}$, $\mathcal{F}$ is the infinite product σ-algebra, $\mathbb{P}_1$ and $\mathbb{P}_2$ are the probability measures, the group of translations are just shift maps, and

$$t_i(\omega) := (\tau_i(0, \alpha, \omega_i))_{\alpha \in A},$$

is the first coordinate of ω.

A coupling of two measures $\mathbb{P}_1$ and $\mathbb{P}_2$ is a probability measure on the product space $\Omega \times \Omega$ with product sigma-algebra $\mathcal{F} \times \mathcal{F}$ and $\mathbb{P}_1$ and $\mathbb{P}_2$ as marginals. Let $\Pi(\Omega \times \Omega)$ be the space of all couplings on $\Omega \times \Omega$.

Definition 8.2 (a type of W^∞ Wasserstein distance)**.**

$$d(\mathbb{P}_1, \mathbb{P}_2) = \inf_{\pi \in \Pi(\Omega \times \Omega)} \operatorname*{ess\,sup}_{(\omega_1, \omega_2) \in \Omega \times \Omega} \sup_{\alpha \in A} |t(\omega_1) - t(\omega_2)|.$$

The primal version of the comparison result is easily proved:

Proposition 8.3. *For all $x \in \mathbb{R}^d$,*

$$|m_1(x) - m_2(x)| \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) d(\mathbb{P}_1, \mathbb{P}_2)|x|_1.$$

Proof. Let γ be a path connecting the origin to $[nx]$, let π be a coupling, and let $d(\gamma)$ be the l^1 length of the path. Then,

$$\sum_{i=1}^{d(\gamma)} |\tau_1(\gamma_i, \gamma_{i+1} - \gamma_i, \omega_1) - \tau_2(\gamma_i, \gamma_{i+1} - \gamma_i, \omega_2)| \leq d(\gamma) \operatorname*{ess\,sup}_{\omega_1, \omega_2} |t(\omega_1) - t(\omega_2)|_\infty.$$

Since we can always take a shortest l^1 distance path between 0 and $[nx]$, its enough to consider paths from 0 to $[nx]$ that satisfy

$$d(\gamma) \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) ||[nx]||_1.$$

Take an inf over all such paths to get

$$|T_1([nx]) - T_2([nx])| \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) \operatorname*{ess\,sup}_{\omega_1,\omega_2} |t(\omega_1) - t(\omega_2)|_\infty |[nx]|_1$$

Divide by n and take a limit as $n \to \infty$. Taking an infimum over couplings π, we get the result. □

We can prove a similar version of Prop. 8.3 using just the variational formula.

Proposition 8.4. *For all $x \in \mathbb{R}^d$,*

$$|m_1(x) - m_2(x)| \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) \frac{b_1 b_2}{a_1 a_2} d(\mathbb{P}_1, \mathbb{P}_2)|x|_1$$

With the following lemma, the proof of Prop. 8.4 is easy.

Lemma 8.5. *Let H_1 and H_2 be the corresponding limiting Hamiltonians. Then,*

$$|\overline{H}_1(p) - \overline{H}_2(p)| \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) |p|_\infty \frac{1}{a_1 a_2} d(\mathbb{P}_1, \mathbb{P}_2)$$

Proof of Prop. 8.4. We established the elementary inequality

$$\overline{H}_i(p) \geq \frac{|p|_\infty}{b_i}$$

in the proof of Proposition 3.9 in part I. Hence for each $x, p \in \mathbb{R}^d$,

$$\begin{aligned}\left|\frac{p\cdot x}{\overline{H}_1(p)} - \frac{p\cdot x}{\overline{H}_2(p)}\right| &= \left|p\cdot x\left(\frac{\overline{H}_1(p)-\overline{H}_2(p)}{\overline{H}_1(p)H_2(p)}\right)\right| \\ &\leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right)\frac{1}{a_1a_2}|x|_1 d(\mathbb{P}_1, \mathbb{P}_2).\end{aligned}$$

We've used the Hölder inequality and Lemma 8.5 in the above computation. Since m_i are the dual norms of $\overline{H}_i$, the proof is complete. □

Proof of Lemma 8.5. First, fix $f \in S$, where S is defined in (6.2). For each measure $\mathbb{P}_1$ and $\mathbb{P}_2$, the constraint on S is different:

$$\sup_{\alpha\in A}\left(-\mathcal{D}f(x,\alpha) - p\cdot\alpha\right) \leq \frac{b_1}{a_1}|p|_\infty \qquad \forall x \in \mathbb{Z}^d.$$

Hence, we might as well assume that

$$\sup_{\alpha\in A}|\mathcal{D}f(x,\alpha)+p\cdot\alpha| \leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) \qquad \forall x \in \mathbb{Z}^d.$$

Then, for a fixed coupling $\pi \in \Pi(\Omega\times\Omega)$,

$$\begin{aligned}&\left|\frac{\mathcal{D}f(0,\alpha,\omega_1)+p\cdot\alpha}{\tau_1(0,\alpha,\omega_1)} - \frac{\mathcal{D}f(0,\alpha,\omega_2)+p\cdot\alpha}{\tau_2(0,\alpha,\omega_2)}\right| \\ &\qquad\leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right)|p|_\infty \frac{|\tau_1(0,\alpha,\omega_1)-\tau_2(0,\alpha,\omega_2)|}{\tau_1(0,\alpha,\omega_1)\tau_2(0,\alpha,\omega_2)} \\ &\qquad\leq \max\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right)|p|_\infty \frac{1}{a_1a_2}\operatorname*{ess\,sup}_{\omega_1,\omega_2}|t(\omega_1)-t(\omega_2)|_\infty,\end{aligned}$$

using Corollary 6.1. Since this is true for all functions in S, and all couplings in $\Pi(\Omega\times\Omega)$, we can take supremums and infimums as appropriate to get the result. □

Remark 8.6. The estimate through the variational formula in Prop. 8.4 is worse than the estimate in Prop. 8.3. However, the estimates used —the lower bound for $\overline{H}(p)$ and the bound for ν_ϵ from (5.1)— were quite crude, and these are easily improved.

Remark 8.7. The basic step in the primal argument was to take the worst case path in the x direction, and the corresponding step in the dual argument was to take the worst case function f in the p direction. This seems to indicate some (nonlinear) duality between paths on the lattice and functions in S. Is there a structural theory of this duality?

8.2 A more convenient coupling distance

The coupling distance in Definition 8.2 is not very useful in general. However, when the medium is i.i.d, it's easy to get an upper bound for it in terms of a more familiar distance on the marginal distribution of the edge-weight $\tau(0, \alpha, \omega)$. When $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$, where μ is a measure on $\mathbb{R}^d$, couplings on $\mathbb{R}^d \times \mathbb{R}^d$ can be turned into a coupling on $\Omega \times \Omega$ by taking a product. Suppose further that the marginal measure μ on $\mathbb{R}^d$ is also an i.i.d. product measure, and let F_i be the cumulative distribution function of $\tau_i(0, \alpha, \omega)$ for $i = 1,\ 2$. Then, for example, we can write $d(\mathbb{P}_1, \mathbb{P}_2)$ in terms of the Kolmogorov-Smirnov distance between F_1 and F_2, assuming F_1 and F_2 are nice enough.

Let F_i have density ρ_i, and assume

$$\min(\text{supp}(F_i)) = [a_i, b_i] \subset (0, \infty),$$

$$\rho^* = \min_i \min_{a_i \le x \le b_i} \rho_i(x) > 0,$$

where supp denotes the support of the distribution. Let

$$d_{Kol}(F_1, F_2) = \sup_x |F_1(x) - F_2(x)|$$

be the Kolmogorov-Smirnov distance between the two distributions.

We will use the standard Skorokhod representation to define random variables $Y_i(x)$ on $([0,1], \mathcal{F}, \text{Leb})$ with distributions F_i. Since $\rho^* > 0$, F_i is strictly monotone, and hence we define

Definition 8.8 (Skorokhod representation of edge-weights)**.**

$$Y_i(x) = F_i^{-1}(x).$$

It's clear that $\text{Leb}(Y_i(x) \leq c) = \text{Leb}(F_i^{-1}(x) \leq c) = F(c)$.

Proposition 8.9.

$$|Y_1(x) - Y_2(x)| \leq \frac{d_{Kol}(F_1, F_2)}{\rho^*}$$

Proof. Fix $s \in [0,1]$, let $x = F_1^{-1}(s)$, and use $d = d_{Kol}(F_1, F_2)$ as shorthand. From the definition of the Kolmogorov-Smirnov distance,

$$F_2(x) \geq F_1(x) - d = s - d.$$

If $r \geq d/\rho^* + x$, we have

$$F_2(r) \geq (s - d) + \rho^*(r - x) = s.$$

It follows that

$$F_2^{-1}(s) \leq \frac{d}{\rho^*} + F_1^{-1}(s).$$

Repeating the argument for $x = F_2^{-1}(s)$, we get the result. $\square$

Let π be the coupling on $\mathbb{R} \times \mathbb{R}$ defined as the pushforward measure of $\text{Leb}([0,1] \times [0,1]$ under the map $(x_1, x_2) \mapsto (Y_1(x_1), Y_2(x_2))$. We can take a product of π to get a coupling on $\Omega \times \Omega$. Thus, we have proved that

$$d(\mathbb{P}_1, \mathbb{P}_2) \leq d_{Kol}(F_1, F_2).$$

Chapter 9

Future work

There are several areas to explore, and we've listed a few below.

1. An interesting challenge is to remove the symmetry constraint in the algorithm. Since the idea behind the algorithm is so simple, it's reasonable to believe that it can be generalized.

2. Another related question is to find a rich enough subclass of problems (Hamiltonians) where correctors exist. Can the algorithm be tuned to produce correctors for these problems? Questions about regularity and strict convexity appear more accessible if the existence of correctors can be guaranteed.

3. What does the existence of correctors for the cell-problem tell us about the percolation problem?

4. As stated in Section 2.3, there are several possible generalizations of our work. It's probably quite easy to remove the bounds on $\tau(\cdot,\cdot,\cdot)$ in (1.18) and replace it with a moment condition. Other types of lattices and other

control problems can also be explored. For example, the directed versions of first-passage percolation results has a monotone Hamiltonian, and these appear to be easier to work with.

5. It will be interesting to explore the behavior of the so-called integrable models under the variational formula. This has already been begun in the context of last-passage percolation and polymer models by Georgiou et al. [16].

Appendix A

Miscellaneous Proofs

A.1 Proof of continuum homogenization with $G = \mathbb{Z}^d$

Lions and Souganidis [27] consider a much more general version of the homogenization theorem stated in Theorem 1.1: the problem includes a "viscous" second-order term, the Hamiltonian can depend on u, and it can have an unhomogenized variable. Their general version of Prop. 3.2 requires the analysis of an equation of the form

$$U_t^\epsilon - \epsilon \text{tr} A(\epsilon^{-1} y, \omega)\ D^2 U^\epsilon(y) + H(p + DU^\epsilon, \epsilon^{-1} y, \omega) = 0 \text{ in } \mathbb{R}^N \times (0, T] \quad \text{(A.1)}$$

$$U^\epsilon = u_0 \text{ in } \mathbb{R}^N \times \{0\} \quad \text{(A.2)}$$

where $A(y, \omega)$ is a symmetric matrix, and $T > 0$.

They first prove the theorem assuming that A and H are "nice", and then obtain the general version of the theorem through penalization arguments. The specifics can be found in Lions and Souganidis [27]. Their general result includes our case of interest: $A = 0$, and $H(p)$ given by (1.10).

When H and A are assumed to be nice, $H(p)$ satisfies the assumptions in Section 1.3, grows super-quadratically in $|p|$, and A is uniformly elliptic; i.e., for positive constants C_1 and C_2,

$$C_1|\xi|^2 \leq (A\xi, \xi) \leq C_2|\xi|^2.$$

Then, a (special) supersolution $\tilde{U}^\epsilon(y, t, \omega)$ of (A.2) has a representation in terms of a value function $L(y, y'; s, t)$ of a stochastic control problem [15]. Let

$$L^\epsilon(y, y'; s, t) = \epsilon L(\epsilon^{-1}y, \epsilon^{-1}y'; \epsilon^{-1}s, \epsilon^{-1}t).$$

Then, for $u_0 \in C^{1,1}(\mathbb{R}^d)$,

$$\tilde{U}^\epsilon(y, t, \omega) = \inf_{y'}\{u_0(y') + L^\epsilon(y, y'; s, t)\}. \tag{A.3}$$

$L(y, y'; s, t)$ has the following properties:

1. Stationarity: for all $y \in \mathbb{R}^d$ and $g \in \mathbb{R}^d$,

$$L(y + g, y' + g; s, t, \omega) = L(y, y'; s, t, V_g\omega). \tag{A.4}$$

2. Uniform Continuity: (Prop. 6.12 in Lions and Souganidis [27]) Fix any $R > 0$ and $h > 0$. Then, L^ϵ is uniformly continuous with respect to $(y, t), (y', s)$ where $h \leq s \leq t \leq T$ and $|y - y'| \leq R$, uniformly in ϵ and ω.

3. Boundedness: (follows from Prop. 6.9 from Lions and Souganidis [27] and an elementary estimate)

   For all $(y, y', s, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0,T] \times [0,T]$, there exist independent of ω and ϵ, constants $C_1, C_2, C_3 > 0$, and $k \in (1,2)$ such that

$$L^\epsilon(y, y'; s, t) \leq C_1[|y - y'|^k (t-s)^{1-k} + \epsilon^{k/2}(t-s)^{1-k/2} + (t-s)],$$

   and

$$L(y, y'; s, t) \geq C_2|y - y'| - C_3(t-s).$$

4. Subadditivity: for all $y, y', z \in \mathbb{R}^d$ and $0 < s < \tau < t$,

$$L(y, y'; s, t) \leq L(y, z; s, \tau) + L(z, y'; \tau, t).$$

Lions and Souganidis [27] use the subadditive ergodic theorem from Dal Maso and Modica [12] to prove

Proposition A.1.

$$\lim_{\epsilon \to 0} L^\epsilon(y, y'; 0, t; \omega) = t\overline{L}\left(\frac{y' - y}{t}\right).$$

When $G = \mathbb{Z}^d$, we can use the uniform continuity of L^ϵ, and the discrete subadditive ergodic theorem [1, 22] to prove Prop. A.1. We first fix $y, y' \in \mathbb{Q}^d$, and $s, t \in \mathbb{Q}$ such that $0 < s \leq t \leq T$. Then, we can apply the classical subadditive ergodic theorem [22] on the subsequence $\epsilon = n^{-1}$. The continuity estimates for

$L^\epsilon(y, y'; s, t)$ takes care of the rest. We will not repeat this standard argument here; a version of this argument, for example, appears in Seppäläinen [37].

Prop. A.1 allows Lions and Souganidis [27] to take a limit $\epsilon \to 0$ in (A.3). They show that the error between the supersolution $\tilde{U}^\epsilon$ and the actual solution U^ϵ remains small, and hence Prop. 3.2 follows. The rest of the proof of the homogenization theorem does not make use of specifics of the translation group.

Remark A.2. Lions and Souganidis [27] do not state Theorem 1.1 in the metric form; they state it for the stationary equation in (2.9). However, since both the metric problem in (1.11) and the finite-time horizon problem in (2.6) have comparison principles [5], their proof goes through without much alteration.

A.2 Variational formula on $\mathbb{R}^d$ with $G = \mathbb{Z}^d$

The argument we follow is again nearly identical to [27]. But it does involve a few subtle changes to make it work, and this is interesting to write down. In any case, no one reads appendices, so it doesn't hurt to repeat an argument.Following Lions and Souganidis [27], we begin with the approximate problem

$$\epsilon v_\epsilon + H(Dv_\epsilon, x) = 0 \quad \forall\, x \in \mathbb{R}^d. \tag{A.5}$$

From the variational interpretation of v_ϵ in (2.8) and its dynamic programming principle, it follows that v_ϵ is globally Lipschitz (uniformly in ϵ and ω). Define the normalized set of functions

$$\hat{v}_\epsilon(x) = v_\epsilon(x) - v_\epsilon(0).$$

Since $\hat{v}_\epsilon$ is also Lipschitz and normalized to 0 at the origin,

$$C := \sup_\epsilon \left\{ \left\| \hat{v}_\epsilon(y)(1+|y|)^{-1} \right\|_\infty + \| D\hat{v}_\epsilon \|_\infty \right\} < \infty. \tag{A.6}$$

From the PDE (A.5), it follows that functions $v_\epsilon(x, \omega)$ are stationary and hence have stationary, mean-zero increments. Hence, the normalized functions are in the set S defined in (1.16). We're now ready to prove the variational formula in Prop. 1.2 with $G = \mathbb{Z}^d$. In the following, all constants will be called C and might change value from line-to-line.

Proof of Prop. 1.2 with $G = \mathbb{Z}^d$. Denote the right side of (1.17) by RHS. Using the comparison principle for HJB equations, Lions and Souganidis [27] show that

$$\overline{H}(p) \leq RHS.$$

The same argument works for us.

Consider the normalized approximating functions $\hat{v}_\epsilon$ defined above. We will use these functions to construct functions in S that give the other inequality. Using the optimal-control characterization of H in (2.7) (or plain old convexity), we get for fixed $a \in A$

$$\epsilon \hat{v}_\epsilon(x, \omega) - f(x, a, \omega) \cdot (p + D\hat{v}_\epsilon) - l(x, a, \omega) \leq -\epsilon v_\epsilon(0, \omega) \quad x \in \mathbb{R}^d. \tag{A.7}$$

We require some extra smoothness on $\hat{v}_\epsilon$, and so we convolve it with the standard mollifier η_r, where r is the size of its support. Let $\bar{v}_\epsilon = \eta_r * \hat{v}_\epsilon$, and let $f(x, a, \omega)$ and $l(x, a, \omega)$ have Lipschitz constant C in x. For fixed y, multiply (A.7) by $\eta_r(x - y)$

and integrate over x to get

$$\epsilon\bar{v}_\epsilon(y,\omega) - f(y,a,\omega)\cdot(p + D\bar{v}_\epsilon) - l(y,a,\omega) - Cr \leq -\epsilon v_\epsilon(0,\omega). \tag{A.8}$$

The mollified functions also satisfy the bound in (A.6). Moreover,

$$\left|D^2\bar{v}_\epsilon\right| = |D\eta_r * D\hat{v}_\epsilon| \leq \int |D\eta_r(y-x)D\hat{v}_\epsilon(x)dx| \leq C(r).$$

We will take a weak limit (vague, to be precise) as $\epsilon \to 0$ on the patch $[0,1]^d \times \Omega$, and then translate it using the group of translation operators $\{V_z\}_{z\in\mathbb{Z}^d}$ to obtain a function on $\mathbb{R}^d \times \Omega$. Consider the complete separable metric space $W := C^1([0,1]^d)$ with metric corresponding to the norm $\|u\| = \|u\|_\infty + \||Du|\|_\infty$. The random functions $\bar{v}_\epsilon(x,\omega)$ are in the set

$$K_r := \left\{u(x) \in W : \|u\|_\infty + \|Du\|_\infty + \left\|D^2u\right\|_\infty \leq C + C(r)\right\}.$$

The set K_r is compact in the metric space by the Arzela-Ascoli theorem. Then, the family $\{\overline{v}_\epsilon\}_{\omega>0}$ is tight, and we can pass to a subsequence to obtain a weak limit $u_r(x,\omega)$. Since $f(x,r,w)$ and $l(x,r,\omega)$ are continuous, it follows that

$$f(x,a,\omega)\cdot(p + Dv_\epsilon) + l(x,a,\omega) \xrightarrow{w} f(x,a,\omega)\cdot(p + Du_r) + l(x,a,\omega)$$

as $\epsilon \to 0$ vaguely in $C([0,1]^d,\mathbb{R})$. Hence, it follows from (A.7) that for any fixed $\eta > 0$ and r small enough,

$$-f(x,a,\omega)\cdot(p + Du_r) - l(x,a,\omega) \leq \overline{H}(p) + \eta \quad \forall\ x \in [0,1]^d \quad \text{a.s.}$$

Now, extend u_r to all of $\mathbb{R}^d$ by defining $u_r(x+g,\omega) = u_r(x, V_g\omega)$. Take a sup over $r \in A$, followed by a sup over x to get for arbitrary $\eta > 0$

$$\sup_x H(p + Du_r, x, w) \leq \overline{H}(p) + \eta.$$

Letting $\eta \to 0$ gives us the other inequality and completes the proof. □

Remark A.3. When $G = \mathbb{R}^d$, there is a minimizer in S [27]. When $G = \mathbb{Z}^d$, we don't have the estimates to prove this.

A.3 Some proofs from Chapter 8

Proof of Prop. 8.1.

Claim A.4. Let $F\colon \mathbb{R}^n \times \Omega \to \mathbb{R}$ be convex in its first variable, and bounded in its first argument on any compact subset of $\mathbb{R}^n$ uniformly in ω. For each fixed $p \in \mathbb{R}^n$, let $F(p,\omega)$ be measurable with respect to a σ-algebra $\mathcal{C} \subset \mathcal{F}$. If $h\colon \Omega \to \mathbb{R}^n$ is any bounded $\mathcal{F}$ measurable function,

$$\operatorname*{ess\,sup}_{\omega\in\Omega} F(E[h|\mathcal{C}],\omega) \leq \operatorname*{ess\,sup}_{\omega\in\Omega} F(h,\omega).$$

We return to the proof of Claim A.4 after completing the proof of the proposition.

Let $\phi(x, \omega) \in S$, where S is defined in Corollary 6.1. We apply Claim A.4 with

$$h(\omega) = \mathcal{D}f = (\phi(x+\alpha) - \phi(x, \omega))_{\alpha \in A},$$
$$F(h, \omega) = \mathcal{H}(\mathcal{D}\phi, p, 0, \omega),$$
$$\mathcal{C} = \sigma(M)$$

where $\mathcal{H}$ is the discrete Hamiltonian in (1.20), and M is defined in (8.2). Claim A.4 implies that

$$\operatorname*{ess\,sup}_{\omega} \mathcal{H}(E[\phi|\mathcal{C}], \omega) \leq \operatorname*{ess\,sup}_{\omega} \mathcal{H}(\phi(\omega), \omega).$$

This means that we might as well take $\phi(x, \cdot)$ to be $\mathcal{C}$ measurable for every $x \in \mathbb{Z}^d$ in Theorem 1.5. Claim A.4 remains to be proved and this is done below. □

Proof of Claim A.4. We need a conditional version of Jensen's inequality which says that

$$E[F(h, \omega)|\mathcal{C}] \geq F(E[h|\mathcal{C}], \omega) \quad \text{a.s.} \tag{A.9}$$

For any constant c, suppose $A := \{\omega : F(E[h|\mathcal{C}], \omega) \geq c\}$ has positive measure. The set A is $\mathcal{C}$ measurable since both $F(p, \omega)$ and $E[h|\mathcal{C}]$ are. By (A.9), and the definition of conditional expectation

$$E[F(h, \omega), A] = E[F(E[h|\mathcal{C}], \omega), A] \geq c.$$

Hence, there is a subset of A of positive measure where $F(h, \omega) \geq c$. Letting c approach $\operatorname{ess\,sup} F(E[h|\mathcal{C}], \omega)]$ completes the proof.

It remains to prove (A.9). We mollify F with η_ϵ the standard mollifier on $\mathbb{R}^n$ with support in a ball of radius ϵ to obtain a smooth function F_ϵ. Then, for any measurable functions h and h_0 we have almost surely,

$$F_\epsilon(h,\omega) \geq DF_\epsilon(h_0,\omega)\cdot(h-h_0)+F_\epsilon(h_0,\omega),$$

Letting $h_0(\omega) = E[h|\mathcal{C}]$, taking conditional expectation and using the fact that $DF_\epsilon(h_0,\omega)$ is $\mathcal{C}$ measurable, we get

$$E[F_\epsilon(h,\omega)|\mathcal{C}] \geq F_\epsilon(E[h(\omega)|\mathcal{C}],\omega).$$

Finally letting $\epsilon \to 0$, and using the boundedness of h and the assumptions on F, we get (A.9). $\square$

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