

Harnack Inequalities and Heat-kernel Estimates for Degenerate Diffusion Operators Arising in Population Biology

Charles L. Epstein* and Rafe Mazzeo†

August 10, 2014

Abstract

This paper continues the analysis, started in [3, 4], of a class of degenerate elliptic operators defined on manifolds with corners, which arise in Population Biology. Using techniques pioneered by J. Moser, and extended and refined by L. Saloff-Coste, Grigor'yan, and Sturm, we show that weak solutions to the parabolic problem defined by a sub-class of these operators, which consists of those that can be defined by Dirichlet forms and have non-vanishing transverse vector field, satisfy a Harnack inequality. This allows us to conclude that the solutions to these equations belong, for positive times, to the natural anisotropic Hölder spaces, and also leads to upper and, in some cases, lower bounds for the heat kernels of these operators. These results imply that these operators have a compact resolvent when acting on C^0 or L^2 . The proof relies upon a scale invariant Poincaré inequality that we establish for a large class of weighted Dirichlet forms, as well as estimates to handle certain mildly singular perturbation terms. The weights that we consider are neither Ahlfors regular, nor do they generally belong to the Muckenhaupt class A_2 .

*Research partially supported by NSF grant DMS12-05851, and ARO grant W911NF-12-1-0552. Address: Department of Mathematics, University of Pennsylvania; e-mail: cle@math.upenn.edu

†Research partially supported by NSF grant DMS1105050. Address: Department of Mathematics, Stanford University; e-mail: mazzeo@math.stanford.edu

Keywords: degenerate diffusions, Kimura operator, Population Genetics, Harnack inequality, weighted Poincaré inequality, doubling measure, heat kernel bounds, eigenvalue asymptotics.

MSC-2010: 35K65, 35K08, 35B65, 35P20, 35Q92, 60J60, 92D10, 42B37

1 Introduction

In a series of paper and a book we have considered the analysis of a class of degenerate diffusion operators, which arise in Population Biology, see [9], which we call generalized Kimura diffusion operators. The typical examples that arise in population genetics act on functions defined on the n -simplex

$$\Sigma_n = \{(x_1, \dots, x_n) : 0 \leq x_j \text{ and } x_1 + \dots + x_n \leq 1\}, \quad (1)$$

and take the form

$$L = \sum_{i,j=1}^n (x_i \delta_{ij} - x_i x_j) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^n b_i(x) \partial_{x_i}. \quad (2)$$

The vector field is inward pointing, and the coefficient functions $\{b_i(x)\}$ are often either linear or quadratic polynomials. The class of operators we analyze includes these examples, but is considerably more general. They are defined on manifolds with corners by degenerate, elliptic, partial differential operators. In “adapted local coordinates” $(x; y) \in S_{n,m} = \mathbb{R}_+^n \times \mathbb{R}^m$, (where $\mathbb{R}_+ = [0, \infty)$), such an operator takes the form:

$$\begin{aligned} Lu = & \sum_{j=1}^n [x_j \partial_{x_j}^2 + b_j(x; y) \partial_{x_j}] u + \sum_{i,j=1}^n x_i x_j a_{ij}(x; y) \partial_{x_i} \partial_{x_j} u + \\ & \sum_{i=1}^n \sum_{l=1}^m x_i c_{il} \partial_{x_i} \partial_{y_l} u + \sum_{k,l=1}^m d_{kl}(x; y) \partial_{y_k} \partial_{y_l} u + \sum_{l=1}^m d_l(x; y) \partial_{y_l} u, \quad (3) \end{aligned}$$

in a neighborhood of $(0; 0)$.

In our work thus far we have assumed that the coefficients are smooth functions of the variables $(x; y)$, or of the “square root” variables,

$$(\sqrt{x}; y) \stackrel{d}{=} (\sqrt{x_1}, \dots, \sqrt{x_n}; y_1, \dots, y_m);$$

later in this paper we see that somewhat less regular coefficients arise naturally. The monograph [4] provides a starting point for the analysis of generalized Kimura diffusion operators by analyzing the so-called “backward Kolmogorov” operator acting on data belonging to a family of anisotropic Hölder spaces. Central to this study are the explicit heat kernels associated to the model operators

$$L_{b,m} = \sum_{j=1}^n [x_j \partial_{x_j}^2 + b_j \partial_{x_j}] + \sum_{l=1}^m \partial_{y_l}^2, \quad (4)$$

acting on functions defined on $S_{n,m}$. These kernels are used to construct parametrices for the heat and resolvent kernels for a generalized Kimura diffusion operator on a compact manifold with corners. This parametrix construction is far from sharp, but using it and various functional analytic arguments connected to anisotropic Hölder spaces, we establish existence, uniqueness and essentially optimal regularity results in this setting. This leads to a proof of existence of the Feller semigroup acting on C^0 , which is of importance in biological applications, but it is not informative as to the regularity properties of solutions to the parabolic problem with merely continuous initial data.

This parametrix approach does not give optimal regularity results for solutions with initial data in C^0 , or regularity results for local solutions, nor does it lead to pointwise estimates for the heat kernel. For many applications, such heat kernel estimates and local regularity results are quite important, which has motivated our further work on this problem. One step was taken in [5], where we treated the special case where P is a manifold with boundary. In that setting we were able to adapt the techniques of geometric microlocal analysis to give more precise information on the heat kernel, which then directly implies the various optimal regularity results for solutions of the heat equation, including the precise regularity for solutions with initial data in C^0 .

In the present paper we continue this program in a somewhat different setting using very different techniques. We use the formalism of Dirichlet forms, weak solutions, and Moser's approach to Harnack inequalities, as clarified and extended by Saloff-Coste, Grigor'yan, and Sturm, see [10, 11, 14] and [18, 19, 20], to prove that local solutions of the parabolic equations associated to certain generalized Kimura diffusion operators satisfy a Harnack inequality. We also adapt the results from the papers just cited to explain how this leads to upper and (sometimes) lower pointwise bounds for the heat kernel, and Hölder regularity at positive times for local, weak solutions of the Cauchy problem.

The analysis in this paper brings to the fore the mutation rates, which, in the mathematical formulation, appear as normalized coefficients of a vector field transverse to the boundary that we call *weights*. These are essentially the functions $\{b_i(x; y)\}$ appearing in (3), restricted to the respective subsets of ∂P , given by $\{x_i = 0\}$. In [4] no hypothesis is made on the weights, other than non-negativity, though it has been apparent for some time that the structure of the heat kernel is radically different along the part of the boundary where weights vanish. An early result along these lines is given in [16].

In the Dirichlet form approach the weights define a measure given locally by

$$d\mu_{\mathbf{b}}(x; y) = e^U(x; y) x_1^{b_1(x; y)-1} \cdots x_n^{b_n(x; y)-1} dx_1 \cdots dx_n \cdot dy_1 \cdots dy_m \quad (5)$$

on a neighborhood of $(0, 0)$ in $S_{n,m}$. Here U is a bounded function, which we take

to be zero for the remainder of the introduction. The second order part of the generalized Kimura diffusion operator, i.e. the principal symbol, defines a quadratic form on functions in $\mathcal{C}_c^1(S_{n,m})$ of the form

$$q(u, v)(x; y) = \left(\sum_{j=1}^n [x_j \partial_{x_i} u \partial_{x_i} v + \sum_{i,j=1}^n +x_i x_j a_{ij}(x; y) \partial_{x_i} u \partial_{x_j} v + \frac{1}{2} \sum_{i=1}^n \sum_{l=1}^m x_i c_{il} [\partial_{x_i} u \partial_{y_l} v + \partial_{x_i} v \partial_{y_l} u] + \sum_{k,l=1}^m d_{kl}(x; y) \partial_{y_k} u \partial_{y_l} v \right) (x; y). \quad (6)$$

In the body of the paper, this is abbreviated as

$$q(u, v)(x; y) = \langle A(x; y) \nabla u(x; y), \nabla v(x; y) \rangle. \quad (7)$$

The measure $d\mu_{\mathbf{b}}$ and the quadratic form q together define a Dirichlet form

$$Q(u, v) = \int \langle A(x; y) \nabla u(x; y), \nabla v(x; y) \rangle d\mu_{\mathbf{b}}(x; y). \quad (8)$$

Formally integrating by parts, we arrive at an operator, L_Q , with a densely defined domain in $L^2(S_{n,m}; d\mu_{\mathbf{b}})$, specified by a “natural” boundary condition. On sufficiently smooth initial data, it is easy to check that the solution of the parabolic problem defined by this operator agrees with the regular solution for “backward Kolmogorov” operator analyzed in [4]. Similar considerations apply to define self adjoint operators on $L^2(B; d\mu_{\mathbf{b}})$ for open sets $B \subset S_{n,m}$.

It is well known that Q can be modified by the addition of a non-symmetric term

$$\int \langle A(x; y) \nabla u(x; y), X(x; y) v(x; y) \rangle d\mu_{\mathbf{b}}(x; y), \quad (9)$$

where $X(x; y)$ is an \mathbb{R}^{n+m} -valued function. This has the effect of adding a tangential vector field, V_X , to L_Q . If a weight $b_j(x; y)$ is non-constant along a portion of the boundary where $x_j = 0$, then L_Q includes a vector field tangent to this boundary hypersurface, with mildly singular coefficients of the form

$$\sum_{i,j} \alpha_{ij}(x; y) \log x_j x_i \partial_{x_i} + \sum_{j,l} \beta_{j,l}(x; y) \log x_j \partial_{y_l}. \quad (10)$$

These terms do not appear if the weights $\{b_i(x; y)\}$ are constant along the appropriate boundary components. In other words, to obtain an arbitrary generalized Kimura diffusion operator, as in [4], using a Dirichlet form we must allow coefficient functions X with log singularities, i.e., which satisfy

$$\langle A(x; y) X(x; y), X(x; y) \rangle \leq M \left[\sum_{i=1}^n |\log x_i| + 1 \right]^2, \quad (11)$$

near the boundary of $S_{n,m}$.

If L is a generalized Kimura diffusion operator with smooth coefficients, acting as an unbounded operator on C^0 -functions, then its dual L^t acts naturally on a subspace of the regular Borel measures. Since C^0 is non-reflexive, the obvious dual semi-group is not strongly continuous as $t \rightarrow 0^+$. Following Phillips, one restricts to a subspace on which it is. Measures belonging to this subspace are absolutely continuous, away from the boundary, with respect to $d\mu_{\mathbf{b}}$. For this, and other reasons (see Theorem 1.2) it is natural to represent them in the form $w d\mu_{\mathbf{b}}$. If the weights are non-constant, then the differential operator representing the action of L^t on w has lower order terms with logarithmically singular coefficients. Such terms are therefore not simply an artifact of our method, but rather intrinsic to this class of operators. These types of singular terms can be controlled using several variants of the following lemma:

Lemma 1.1. [See Lemma B.4] Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(\mathbf{x}; \mathbf{y})$, with $0 < \beta_0 < b_j$, constant outside a compact set. Let q be a measurable function on $S_{n,m}$ that satisfies

$$|q(x; y)| \leq M \left[\chi_B(x; y) \sum_{j=1}^l |\log x_j|^k + 1 \right], \quad (12)$$

for some $k \in \mathbb{N}$, B a bounded set, and $M > 0$. Given $\eta > 0$ there is a $0 < \delta < \frac{1}{2}$, so that if $\text{supp } \chi \subset [0, \delta]^n \times (-1, 1)^m$, then there is a C_η so that

$$\int_{S_{n,m}} \chi^2(x; y) |q(x; y)| u^2(x; y) d\mu_{\mathbf{b}} \leq \eta \int_{S_{n,m}} \langle A \nabla u, \nabla u \rangle \chi^2 d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} [\langle A \nabla \chi, \nabla \chi \rangle + \chi^2] u^2 d\mu_{\mathbf{b}}, \quad (13)$$

for any positive differentiable function u .

Remark 1.1. This allows us to control the singular terms in a neighborhood of any boundary point. Since the weights \mathbf{b} are constant outside of a compact set, we can use this lemma along with a simple covering argument to show that these singular terms are bounded by a small multiple of $Q(u, u)$, plus a large multiple of the L^2 -norm of u .

Assuming that the weights are bounded below by a positive constant, and have a particular logarithmic modulus of continuity, we are able to show that $d\mu_{\mathbf{b}}$ is a doubling measure and the Dirichlet form satisfies a scale invariant L^2 -Poincaré

inequality. The Sobolev inequality and then the Harnack inequality follow from an argument of Saloff-Coste, which K. Sturm adapted to the metric-measure category. These are local estimates that can then be applied to solutions defined on a compact manifold with corners. As an important consequence we then show that the regular solution to the Cauchy problem with initial data in \mathcal{C}^0 is Hölder continuous for positive times.

A generalized Kimura diffusion operator, L on a compact manifold with corners, P , defines a measure $d\mu_L$, given locally by $e^{U(x;y)}d\mu_b$, as in (5). This is a finite measure if the weights are strictly positive. The following basic regularity result is a consequence of our local estimates:

Theorem 1.1. *[Theorem 5.1] Let P be a compact manifold with corners and L a generalized Kimura diffusion operator with smooth coefficients defined on P . Suppose that the weights defined by L are positive along every boundary component. If u is a weak solution to the initial value problem*

$$(\partial_t - L)u = 0 \text{ with } u(\xi, 0) = f(\xi) \in L^2(P; d\mu_L), \quad (14)$$

then $u \in \mathcal{C}^\infty(P \times (0, \infty))$.

Among other things we also show that $(\mu - L)^{-1}$ acting on $\mathcal{C}^0(P)$ is a compact operator. In addition we establish upper bounds for the “heat kernel,” i.e. the Schwartz kernel of e^{tL} . Our earlier work indicates that this heat kernel is smooth along the boundary in the outgoing variables, but somewhat singular along the boundary in the incoming variables.

Let $\rho_i(\xi, \eta)$ denote the distance between $\xi, \eta \in P$, with respect to the incomplete metric defined by dualizing the principal symbol of P .

Theorem 1.2. *[Theorem 5.2] Assume that P is a compact manifold with corners and L is a generalized Kimura diffusion defined on P with positive weights. If we represent the kernel of the operator e^{tL} as $p_t(\xi, \eta)d\mu_L(\eta)$, then there are positive constants C_0, C_1, C_2 so that, for all $t > 0$ and pairs $\xi, \eta \in P$ we have*

$$p_t(\xi, \eta) \leq \frac{C_0 \exp\left(-\frac{\rho_i^2(\xi, \eta)}{C_2 t}\right)}{\sqrt{\mu_L(B_{\sqrt{t}}^i(\xi))\mu_L(B_{\sqrt{t}}^i(\eta))}} \times \left(1 + \frac{\rho_i(\xi, \eta)}{\sqrt{t}}\right)^D \cdot \exp(C_1 t). \quad (15)$$

For each $\eta \in P$, the function $(\xi, t) \mapsto p_t(\xi, \eta)$ belongs to $\mathcal{C}^\infty(P \times (0, \infty))$.

In particular, $p_t(\xi, \eta)$ is bounded for positive times, which shows that the leading singularity of the heat kernel on the incoming face is captured by the measure $d\mu_L$. In [15] Shimakura gives a similar estimate for the heat kernel of the standard

Kimura diffusion operator on the simplex in \mathbb{R}^d , under the assumption that the weights are constant and at least $1/2$. In [2] Chen and Stroock prove an analogous result in the 1-dimensional case, with vanishing weights.

In a separate paper we treat a special subclass of “diagonal operators,” which act on functions defined on $S_{n,m}$ and take the special form

$$Lu = \sum_{j=1}^n [x_j \partial_{x_j}^2 + b_j(x; y) \partial_{x_j}] u + \sum_{l=1}^m [\partial_{y_l}^2 + d_l(x; y) \partial_{y_l}] u. \quad (16)$$

We analyze this special case using the kernel methods introduced in [3] and [4]. Assuming that the weights are bounded below, and that the coefficients, $\{b_i, d_l\}$, are constant outside a compact set, we establish the Hölder regularity of solutions to $(\partial_t - L)u = 0$ with initial data in $\mathcal{C}_c^0(S_{n,m})$.

Acknowledgements

We would like to thank Daniel Stroock for suggesting that we look at the works of Saloff-Coste, Grigor’yan and Sturm. We would like to thank Camelia Pop for many helpful discussions and her many independent contributions to this effort; we would also like to thank Phil Gressman for many useful conversations over the course of this research.

2 A Preliminary Result

The weights, which are the coefficients of the transverse components of the vector field along ∂P , play a central role in this paper. We first show that they are invariantly defined by the operator itself. Let P be a manifold with corners, and $\xi \in \partial P$ a boundary point of codimension n . If L is a generalized Kimura diffusion operator, then Proposition 2.2.3 in [4] shows that there are adapted local coordinates $(x_1, \dots, x_n; y_1, \dots, y_m)$ lying in $S_{n,m}$ with $p \leftrightarrow (0; 0)$, in which L takes the form

$$Lu = \sum_{j=1}^n [x_j \partial_{x_j}^2 + b_j(x; y) \partial_{x_j}] u + \sum_{i,j=1}^n +x_i x_j a_{ij}(x; y) \partial_{x_i} \partial_{x_j} u + \sum_{i=1}^n \sum_{l=1}^m x_i c_{il} \partial_{x_i} \partial_{y_l} u + \sum_{k,l=1}^m d_{kl}(x; y) \partial_{y_k} \partial_{y_l} u + \sum_{l=1}^m d_l(x; y) \partial_{y_l} u. \quad (17)$$

The operator is assumed to be elliptic where $\{x_i > 0 : i = 1, \dots, n\}$, and coefficients of the transverse vector field $\{b_i(x; y) : i = 1, \dots, n\}$ are non-negative along the boundary, i.e. $b_i(x; y) \geq 0$, where $x_i = 0$. A given point can belong to a variety of such coordinate charts, nonetheless, as shown below, these coefficients are invariantly defined.

Label the hypersurface boundary components of P by indices \mathcal{I} :

$$\partial P = \bigcup_{i \in \mathcal{I}} H_i. \quad (18)$$

To demonstrate this invariance, recall that the principal symbol of the operator L in the interior of P is a positive definite quadratic form on the fibers of T^*P ; by duality, it defines an incomplete metric on P . Let $r_i(\eta)$ denote the minimal distance from a point $\eta \in P$ to the boundary hypersurface with index i . Each r_i is smooth in a neighborhood of $\overline{H_i} \subset P$. Suppose that the point $\xi \in \partial P$ is of codimension n and $(x_1, \dots, x_n; y_1, \dots, y_m)$ are adapted local coordinates centered at ξ . There are distinct indices $\{i_1, \dots, i_n\}$ so that

$$\xi \in \bigcap_{j=1}^n H_{i_j}. \quad (19)$$

Moreover, upon relabeling we recall from the construction of adapted local coordinates that

$$2\sqrt{x_i} = r_{j_i}(x; y). \quad (20)$$

In these coordinates, the operator takes the form (17) from which it is clear that, for each i :

$$b_i(x; y) = Lx_i \upharpoonright_{x_i=0} = \frac{1}{4} Lr_{j_i}^2 \upharpoonright_{r_{j_i}=0}. \quad (21)$$

The last expression is globally defined along H_i , completing the proof of the following proposition:

Proposition 2.1. *Let P be a manifold with corners and L a generalized Kimura diffusion operator defined on P . The coefficients of the transverse vector field along the boundary of P , in any adapted coordinate system, are restrictions of functions defined globally on the hypersurfaces.*

These functions are of central importance in what follows, so we make the following definition:

Definition 2.1. The normalized coefficients of the transverse vector fields along ∂P defined by a generalized Kimura diffusion operator, $\{\frac{1}{4} Lr_i^2 \upharpoonright_{r_i=0} : i \in \mathcal{I}\}$ are called the *weights* of the Kimura operator.

Strictly speaking, the weights are invariantly defined only along ∂P , but we sometimes use the term to refer to the functions $\{b_i(x; y)\}$ defined in a neighborhood of a subset of ∂P , and which agree with the weights on ∂P . As described earlier,

these weights define a class of measures on P , the elements of which differ by a bounded, non-vanishing factor of the form $e^{U(\xi)}$.

Let dV_P be a smooth non-degenerate density on P . For each $i \in \mathcal{I}$, we let B_i be a smooth extension of the weight b_i from H_i to all of P . For simplicity, assume that B_i is independent of x_i in a small neighborhood of H_i and reduces to a positive constant outside of a slightly larger neighborhood; similarly, let R_i denote a smooth extension to P of r_i , the distance to H_i , which we again assume is a positive constant outside of a small neighborhood of H_i . Set

$$W(\xi) = \prod_{i \in \mathcal{I}} R_i(\xi)^{2B_i(\xi)-1}, \quad (22)$$

and define the measure $d\mu_L$ by

$$d\mu_L(\xi) = W(\xi)dV_P(\xi). \quad (23)$$

It follows from (20) that in any adapted coordinate system, $(\mathbf{x}; \mathbf{y})$, there is a bounded, continuous function $U(\mathbf{x}; \mathbf{y})$ so that $d\mu_L(\mathbf{x}; \mathbf{y}) = e^{U(\mathbf{x}; \mathbf{y})} d\mu_b(\mathbf{x}; \mathbf{y})$. The expansion of U along H_i typically takes the form $\gamma(\mathbf{x}; \mathbf{y})x_i \log x_i$, so it is not in general smooth. We speak of a “measure defined by the weights of L ” as any measure with this property.

3 Metric-measure estimates

We now turn to the analysis of the class of generalized Kimura diffusion operators that can be locally defined using a symmetric Dirichlet form, as in (8). Elements of this class of Kimura diffusion operators are both more general than the ones considered before, as certain coefficients of the first order terms are allowed to be singular, but also less general in that not every Kimura diffusion operator has such a description, even locally. This approach to proving estimates is an outgrowth of the pioneering work of John Nash and Jürgen Moser on estimates for elliptic and parabolic equations with bounded measurable coefficients. More recently these ideas have been recast by Fabes and Stroock on the one hand, and Davies, Saloff-Coste, and Grigor’yan on the other, as a way to obtain Harnack inequalities, Hölder estimates on solutions and kernel bounds for the Green and heat kernels defined by uniformly elliptic operators.

Briefly, this approach uses Moser’s iteration to obtain bounds on solutions to elliptic and parabolic equations via the Sobolev inequality and properties of doubling measures. For uniformly elliptic operators on manifolds, Saloff-Coste and Grigor’yan isolated the two essential ingredients: that the measure have the doubling property and that there is a scale-invariant L^2 -Poincaré inequality. This was

later generalized by Sturm, see [18, 19, 20], to the setting of metric measure spaces, with operators defined through strongly local Dirichlet forms. Sturm's work provided a strong impetus to adopt this general approach, but in the end, we often found it easier to adapt the proofs given in [14], rather than to use Sturm's results directly.

The underlying space is $S_{n,m}$, which is a manifold with corners, and should be understood as an adapted coordinate chart for P . We endow this chart with the measure

$$d\mu_{\mathbf{b}}(x; y) = \prod_{j=1}^n x_j^{b_j(x;y)-1} dx dy, \quad (24)$$

where

$$\mathbf{b} = (b_1(x; y), \dots, b_n(x; y))$$

is a vector of positive continuous functions, which are constant outside a bounded neighborhood of $(0; 0)$. Fix a constant $\beta_0 > 0$ such that

$$\beta_0 \leq b_i(x; y) \text{ for all } (x; y) \in S_{n,m}. \quad (25)$$

For many applications it is reasonable, even necessary, to assume that these functions are \mathcal{C}^1 in the variables $(\mathbf{x}; \mathbf{y})$, or else in the ‘‘square-root’’ variables $(\sqrt{\mathbf{x}}; \mathbf{y})$; however, many of the basic results below require far less regularity.

If $B \subset S_{n,m}$ is relatively open and $u, v \in \mathcal{C}_c^\infty(B)$, then we consider the Dirichlet form

$$Q_B(u, v) = \int_B \left[\sum_{j=1}^n x_j \partial_{x_j} u \partial_{x_j} v + \sum_{i,j} \sqrt{x_i x_j} a_{ij}(x; y) \partial_{x_i} u \partial_{x_j} v + \frac{1}{2} \sum_{j,l} \sqrt{x_j} c_{jl}(x; y) [\partial_{x_j} u \partial_{y_l} v + \partial_{x_j} v \partial_{y_l} u] + \sum_{l,m} d_{lm}(x; y) \partial_{y_l} u \partial_{y_m} v \right] d\mu_{\mathbf{b}}(x; y), \quad (26)$$

Note that u, v are *not* required to vanish along $\partial S_{n,m} \cap \overline{B}$.

We define the associated L^2 -inner product by setting:

$$(u, v)_{\mathbf{b}, B} = \int_B uv \, d\mu_{\mathbf{b}}(x; y). \quad (27)$$

The subscript B is omitted if the intended subset is clear from the context. Formally integrating by parts, assuming for example that v vanishes near $\partial S_{n,m}$, gives

$$Q_B(u, v) = -(L_Q u, v)_{\mathbf{b}, B}, \quad (28)$$

where

$$\begin{aligned}
L_Q u &= \sum_{j=1}^n [x_j \partial_{x_j}^2 + b_j(x; y) \partial_{x_j}] u + \sum_{i,j} \partial_{x_j} \sqrt{x_i x_j} a_{ij}(x; y) \partial_{x_i} u + \\
&\frac{1}{2} \sum_{j,l} [\partial_{y_l} \sqrt{x_j} c_{jl}(x; y) \partial_{x_j} u + \partial_{x_j} \sqrt{x_j} c_{jl}(x; y) \partial_{y_l} u] + \sum_{l,m} d_{lm}(x; y) \partial_{y_m} \partial_{y_l} u + V u.
\end{aligned} \tag{29}$$

The vector field V is tangent to $\partial S_{n,m}$, but note that if $\mathbf{b}(x; y)$ is non-constant near $\partial S_{n,m}$, then V may have singular coefficients and involve terms of the form

$$\log x_j (x_i \partial_{x_i}), \quad \log x_j \partial_{y_l}. \tag{30}$$

All of this works equally well on more general manifolds with corners. This setup is related to the ideas used by Shimakura in [15] to study certain simple Kimura-type operators defined on simplices. In Shimakura's work the weights are assumed to be constant.

If L is a generalized Kimura diffusion operator on P , then L determines a class of measures, as noted above, which can be taken to have the form $d\mu_{\mathbf{b}}$ in an adapted coordinate chart. This measure and the principal symbol of L then determine the symmetric quadratic form Q . Conversely, if L_Q is the second order operator determined from Q as above, then in a neighborhood of a point on ∂P , the difference $L - L_Q$ is a vector field tangent to ∂P , possibly with mildly singular coefficients, as in (30). The estimates produced by the Moser method are local, which allows us to establish Hölder regularity for weak solutions of many classes of generalized Kimura diffusion operators.

We assume that symmetric quadratic form is positive definite in the interior of $S_{n,m}$. Because of the form of the coefficients, there is a naturally induced quadratic form on any boundary stratum, and we assume that each of these is also positive definite on the interior of that stratum. We assume finally that the coefficients of the quadratic form,

$$\{a_{ij}(x; y), c_{jl}(x; y), d_{lm}(x; y)\},$$

are smooth functions of the variables $(\sqrt{\mathbf{x}}; \mathbf{y})$. Integrating by parts shows that the natural boundary conditions are the same as those defining the "regular solution" for a generalized Kimura operator introduced in [4], see Section 4.1.

Writing the integrand symbolically as $\langle A(x; y) \nabla u, \nabla v \rangle$, we also consider operators which include non-symmetric terms of the form

$$\int_B \langle A(x; y) \nabla u, X(x; y) \rangle d\mu_{\mathbf{b}}(x; y), \tag{31}$$

where $X(x; y) = (X_1(x; y), \dots, X_{n+m}(x; y))$ are continuous in the interior of $S_{n,m}$. That is, we allow the addition of an arbitrary continuous tangent vector field. We defer the development of this case to Section 4.3, and focus on the symmetric case.

Changing variables in the Dirichlet form by $w_i = \sqrt{x_i}$ gives

$$d\mu_{\mathbf{b}}(w; y) = 2^n \prod_{i=1}^n w_i^{2b_i(w; y)-1} dw dy, \quad (32)$$

and

$$\begin{aligned} Q_B(u, v) = \frac{1}{4} \int_B \left[\sum_{j=1}^n \partial_{w_j} u \partial_{w_j} v + \sum_{i,j} a_{ij} \partial_{w_i} u \partial_{w_j} v + \right. \\ \left. 2 \sum_{j,l} c_{jl} [\partial_{w_j} u \partial_{y_l} v + \partial_{w_j} v \partial_{y_l} u] + 4 \sum_{l,m} d_{lm} \partial_{y_l} u \partial_{y_m} v \right] d\mu_{\mathbf{b}}(w; y). \quad (33) \end{aligned}$$

The ellipticity hypothesis is that

$$q_{(w; y)}(\xi, \eta) = \frac{1}{4} \left[\sum_{j=1}^n \xi_j^2 + \sum_{i,j} a_{ij} \xi_i \xi_j + 4 \sum_{j,l} c_{jl} \xi_j \eta_l + 4 \sum_{l,m} d_{lm} \eta_l \eta_m \right] \quad (34)$$

(the coefficients of which are constant outside a compact set) is positive definite, i.e., there are positive constants λ, Λ so that

$$\lambda(|\xi|^2 + |\eta|^2) \leq q_{(w; y)}(\xi, \eta) \leq \Lambda(|\xi|^2 + |\eta|^2). \quad (35)$$

Since we are primarily interested in local estimates near the corner $(0; 0)$, we assume that

$$\text{The quadratic form in (34) is uniformly elliptic in } S_{n,m}. \quad (36)$$

Observe that this is invariant under the dilations $(w; y) = (\mu w'; \mu y')$, $\mu > 0$, which transform the measure $d\mu_{\mathbf{b}}(w; y)$ to

$$\mu^m 2^n \prod_{i=1}^n (w'_i)^{2\tilde{b}_i(w'; y')-1} e^{\phi(w', y') \log \mu} dw' dy', \quad (37)$$

where

$$\tilde{b}_i(w', y') = b_i(\mu w', \mu y'), \quad \text{and} \quad \phi(w', y') = 2 \sum_{i=1}^n b_i(\mu w', \mu y'). \quad (38)$$

Sturm introduces the notation that if u is in the domain of Q , then there is a measure $d\Gamma(u, u)$ so that

$$Q(u, u) = \int_{S_{n,m}} d\Gamma(u, u). \quad (39)$$

In our case

$$d\Gamma(u, u)(w; y) = \langle A(w; y)\nabla u, \nabla u \rangle d\mu_{\mathbf{b}}(w; y). \quad (40)$$

Next, in terms of the space of functions

$$\mathcal{U}_1 = \{u : \langle A(w; y)\nabla u, \nabla u \rangle \leq 1\}, \quad (41)$$

the *intrinsic metric* is defined by

$$\rho_i((w_1; y_1), (w_2; y_2)) = \sup\{u(w_1; y_1) - u(w_2; y_2) : u \in \mathcal{U}_1\}. \quad (42)$$

By (36), this intrinsic metric is uniformly equivalent to the Euclidean metric,

$$\rho_2^e((w_1; y_1), (w_2; y_2)) = (\|w_1 - w_2\|_2^2 + \|y_1 - y_2\|_2^2)^{\frac{1}{2}}, \quad (43)$$

or equivalently, in terms of the $(x; y)$ coordinates,

$$\rho_2^e((x_1; y_1), (x_2; y_2)) = \left(\sum_{j=1}^n |\sqrt{x_{1j}} - \sqrt{x_{2j}}|^2 + \|y_1 - y_2\|_2^2 \right)^{\frac{1}{2}}. \quad (44)$$

This determines the standard topology on $S_{n,m}$. It is equivalent to the metric used in [4] to define the anisotropic Hölder spaces $\mathcal{C}_{\text{WF}}^{k,\gamma}$ and $\mathcal{C}_{\text{WF}}^{k,2+\gamma}$, which play a key role in the analysis of generalized Kimura diffusion operators. The ball of radius r centered at $(w; y)$ with respect to ρ_2^e is denoted $B_r^e(w; y)$.

The main estimates on the heat kernel and solutions to the heat equations follow by a rather general argument once we prove that:

1. The measure, $d\mu_{\mathbf{b}}$ is a doubling measure, and
2. The Dirichlet forms, $Q_{B_r^i}$ satisfy scale-free L^2 Poincaré inequalities, for intrinsic-metric balls B_r^i .

The proofs of these facts both proceed by checking their validity when \mathbf{b} is constant and then using perturbative arguments to conclude their validity in general. The details of this analysis occupy the remainder of this section.

First observe that for the purposes of proving the Poincaré inequality, we may replace the quadratic form $q(\nabla_w u, \nabla_y u)$, defined in (34), with the standard Euclidean one, giving the equivalent Dirichlet form

$$Q_B^e(u, u) = \int_B \left[\sum_{j=1}^n |\partial_{w_j} u(\tilde{w}; \tilde{y})|^2 + \sum_{l=1}^m |\partial_{y_l} u(\tilde{w}; \tilde{y})|^2 \right] d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}), \quad (45)$$

while of course retaining the same measure $d\mu_{\mathbf{b}}$. We may also use the equivalent ℓ^∞ metric,

$$\rho_\infty^e((w_1; y_1), (w_2; y_2)) = \max\{\|w_1 - w_2\|_\infty, \|y_1 - y_2\|_\infty\}, \quad (46)$$

with respect to which closed balls now have the form

$$B_r(w; y) = \prod_{j=1}^n [\max\{w_j - r, 0\}, w_j + r] \times \prod_{l=1}^m [y_l - r, y_l + r]. \quad (47)$$

Indeed, the inclusions

$$B_r^e(w; y) \subset B_r(w; y) \subset B_{\frac{r}{\sqrt{n+m}}}^e(w; y)$$

show that $d\mu_{\mathbf{b}}$ is a doubling measure with respect to one set of balls if and only if it is a doubling measure with respect to the other. As for the Poincaré inequality, suppose that we prove that there exists a constant $C > 0$ such that

$$\int_{B_r(w; y)} |u(\tilde{w}; \tilde{y}) - u_{B_r(w; y)}|^2 d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}) \leq Cr^2 Q_{B_r(w; y)}(u, u) \quad (48)$$

for all $r > 0$ and $(w; y) \in S_{n,m}$, where (for any measurable set B), u_B is the average

$$u_B = \frac{1}{\mu_{\mathbf{b}}(B)} \int_B u(\tilde{w}; \tilde{y}) d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}), \quad \mu_{\mathbf{b}}(B) = \int_B d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}). \quad (49)$$

We then have that

$$\begin{aligned} \int_{B_r^e(w; y)} |u(\tilde{w}; \tilde{y}) - u_{B_r^e(w; y)}|^2 d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}) &\leq \int_{B_r^e(w; y)} |u(\tilde{w}; \tilde{y}) - u_{B_r(w; y)}|^2 d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}) \\ &\leq \frac{C}{n+m} [(n+m)r^2] Q_{B_{\frac{r}{\sqrt{n+m}}}^e(w; y)}(u, u). \end{aligned} \quad (50)$$

In other words, the strong Poincaré inequality for the family of $\{B_r(w; y)\}$ implies a weak Poincaré inequality for the balls $\{B_r^e(w; y)\}$. Theorem 2.4 from [20] then implies that the strong Poincaré also holds for the balls $B_r^e(w; y)$. We obtain the estimate (48) following a well-known argument of Jerison [8], who shows how to pass from a weak scale-invariant Poincaré inequality to a strong one.

There is one further preparatory remark. The volume doubling and Poincaré inequality, and hence the various conclusions that they imply, require very little regularity for the functions $\{b_i\}$. The minimal condition that naturally emerges here is that there is a constant C so that for each i ,

$$|b_i(w; y) - b_i(\tilde{w}; \tilde{y})| \leq \frac{C}{|\log \rho_\infty^e((w; y), (\tilde{w}; \tilde{y}))|}. \quad (51)$$

We first prove that $d\mu_b$ is a doubling measure:

Proposition 3.1. *Let $\{b_i(w; y)\}$ be positive functions which are constant outside a compact set and satisfy (51). Then there is a constant D so that for any $r > 0$ and $(w; y) \in S_{n,m}$,*

$$\mu_b(B_{2r}(w; y)) \leq 2^D \mu_b(B_r(w; y)). \quad (52)$$

Proof. We first verify this when the b_i are everywhere constant. Since each $B_r(w; y)$ is a product of intervals, we immediately reduce to the one-dimensional case, where $B_r(w_i) = (\max\{w_i - r, 0\}, w_i + r)$, and hence for the measure $\mu_b = w_i^{2b-1} dw_i$,

$$\mu_b(B_r(w_i)) = \begin{cases} \frac{(w_i+r)^{2b}}{2b} & \text{if } w_i \leq r \\ \frac{(w_i+r)^{2b} - (w_i-r)^{2b}}{2b} & \text{if } w_i > r. \end{cases} \quad (53)$$

It follows directly from this that for some constant $C_b > 0$,

$$\frac{1}{C_b} r^{2b} \leq \mu_b(B_r(w_i)) \leq C_b r^{2b}, \text{ if } w_i \leq 4r, \quad (54)$$

and

$$\frac{1}{C_b} w_i^{2b-1} r \leq \mu_b(B_r(w_i)) \leq C_b w_i^{2b-1} r \text{ if } w_i > r. \quad (55)$$

The doubling inequality (52) follows immediately from these estimates in this case.

For the general case, we need to show that the quotient

$$F(w; y, r) := \frac{\mu_b(B_{2r}(w; y))}{\mu_b(B_r(w; y))}$$

is uniformly bounded from above. Suppose that it is not, i.e., there exists a sequence of radii r_ℓ and centers of balls $(w^{(\ell)}; y^{(\ell)})$ such that $F(w^{(\ell)}; y^{(\ell)}, r_\ell)$ tends

to infinity. Since F is clearly continuous in its arguments $(w; y) \in S_{n,m}$ and $0 < r < \infty$, this unboundedness could only occur if either $(w^{(\ell)}; y^{(\ell)})$ diverges (to infinity or ∂P) or else r_ℓ tends to 0, or ∞ . We shall rule these possibilities out in turn.

The first case, where $r_\ell \nearrow \infty$, is easy. Suppose that the functions $\{b_i\}$ are constant outside the ball $B_R(0; 0)$. Consider the worst case, when $(w^{(\ell)}; y^{(\ell)}) = (0; 0)$. But then, for $\rho \gg R$,

$$\mu_{\mathbf{b}}(B_\rho(0; 0)) = \mu_{\mathbf{b}^0}(B_\rho(0; 0)) + A, \quad A = \mu_{\mathbf{b}}(B_R(0; 0)) - \mu_{\mathbf{b}^0}(B_R(0; 0)),$$

where \mathbf{b}^0 is the constant value of \mathbf{b} outside a compact set. The uniform upper bound for $F(0; 0; r_\ell)$ is then straightforward. A slightly more complicated estimate, which we leave to the reader, is required when the center of the ball does not lie at the origin, but the same conclusion still holds. The case where r_ℓ remains in a bounded interval $0 < \underline{r} \leq r_\ell \leq \bar{r} < \infty$, but $(w^{(\ell)}; y^{(\ell)}) \rightarrow \infty$, is covered by the computations when \mathbf{b} is constant.

Finally, suppose that the centers $(w^{(\ell)}; y^{(\ell)})$ remain in $B_{2R}(0; 0)$ and $r_\ell \searrow 0$. Change variables, setting $w_i = r_\ell \hat{w}_i$, $y_i = r_\ell \hat{y}_i$, where $(\hat{w}; \hat{y}) \in B_{2R/r_\ell}(0; 0)$. The centers $(w^{(\ell)}; y^{(\ell)})$ are transformed to new points $(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})$, and

$$F(w^{(\ell)}; y^{(\ell)}, r_\ell) = \frac{\int_{B_2(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})} \prod \hat{w}_i^{2\hat{b}_i(\hat{w}; \hat{y})-1} e^{(2\hat{b}_i(\hat{w}; \hat{y})-1) \log r_\ell} d\hat{w} d\hat{y}}{\int_{B_1(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})} \prod \hat{w}_i^{2\hat{b}_i(\hat{w}; \hat{y})-1} e^{(2\hat{b}_i(\hat{w}; \hat{y})-1) \log r_\ell} d\hat{w} d\hat{y}}.$$

Here $b_i(\hat{w}; \hat{y}) = b_i(r\hat{w}; r\hat{y}) = b_i(w; y)$. Note that a common factor of r_ℓ^{m+n} has been cancelled from both the numerator and denominator. The dependence on r_ℓ is now entirely contained in the functions $\hat{\mathbf{b}}(\hat{w}; \hat{y})$. We are aided by the fact that each \hat{b}_i takes values in some interval $0 < \underline{\beta} \leq \hat{b}_i \leq \bar{\beta} < \infty$. Now substitute

$$\hat{b}_i(\hat{w}; \hat{y}) = \hat{b}_i(\hat{w}^{(\ell)}; \hat{y}^{(\ell)}) + \beta_i(r_\ell(\hat{w}; \hat{y}), r_\ell(\hat{w}^{(\ell)}; \hat{y}^{(\ell)}))$$

into the final exponent in each integrand. The expression

$$\exp((2\hat{b}_i(\hat{w}^{(\ell)}; \hat{y}^{(\ell)}) - 1) \log r_\ell)$$

is constant and appears in both the numerator and denominator, hence may be cancelled. We are left with

$$\frac{\int_{B_2(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})} \prod \hat{w}_i^{2\hat{b}_i(\hat{w}; \hat{y})-1} e^{\beta_i \log r_\ell} d\hat{w} d\hat{y}}{\int_{B_1(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})} \prod \hat{w}_i^{2\hat{b}_i(\hat{w}; \hat{y})-1} e^{\beta_i \log r_\ell} d\hat{w} d\hat{y}},$$

where we omit the arguments of the β_i for simplicity. According to (51),

$$|\beta_i(r_\ell(\hat{w}; \hat{y}), r_\ell(\hat{w}^{(\ell)}; \hat{y}^{(\ell)}))| |\log r_\ell| \leq \frac{C |\log r_\ell|}{|\log r_\ell + \log \rho_\infty^e((\hat{w}; \hat{y}), (\hat{w}^{(\ell)}; \hat{y}^{(\ell)}))|} \leq C' \quad (56)$$

since $\rho_\infty^e((\hat{w}; \hat{y}), (\hat{w}^{(\ell)}; \hat{y}^{(\ell)})) \leq 2$. Hence these second factors are bounded above and below, and may be disregarded.

There are now two final cases to analyze. In the first, the centers $(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})$ remain bounded (relative to the $(\hat{w}; \hat{y})$ coordinate system), and in the second they do not. The first case is slightly easier, since we may assume that $\langle (\hat{w}^{(\ell)}; \hat{y}^{(\ell)}) \rangle$ converges, and then simply pass to the limit $r_\ell \rightarrow 0$. Both the numerator and denominator have finite, positive limits, and so we conclude that this sequence of quotients is bounded after all. In the second case, the numerator and denominator each tend to infinity with ℓ . The functions \hat{b} are constant outside the ball $B_{R/r_\ell}(0; 0)$. If the centers $(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})$ lie outside this ball, then the quotient is clearly bounded. So the only remaining case is when $(\hat{w}^{(\ell)}; \hat{y}^{(\ell)}) \in B_{R/r_\ell}(0; 0)$.

We may now perform the same substitution as above, writing each $\hat{w}_i^{2\hat{b}_i(\hat{w}; \hat{y})-1}$ as $\hat{w}_i^{2\hat{b}_i(\hat{w}^{(\ell)}; \hat{y}^{(\ell)})-1} e^{\beta_i \log \hat{w}_i}$. Since $\log \hat{w}_i \leq C(1 + \log r_\ell)$, we may apply exactly the same reasoning as above to neglect these error terms, and then cancel the remaining constant terms.

This contradiction demonstrates that the quotient is uniformly bounded as $r \searrow 0$, and therefore that μ_b is a doubling measure. \square

Remark 3.1. This Lemma is slightly more complicated than one might expect because the measures μ_b are not Ahlfors $(m+n)$ -regular. Indeed, $\mu_b(B_r)$ is bounded above and below by constant multiples of r^{m+n} provided the ball does not intersect the boundary, but these constants are not uniform. The decay rates of measures of small balls centered at any boundary point are given by different powers of r . Hence our problem provides an interesting example where the most common version of Moser's arguments to get heat kernel bounds does not apply, since these require Ahlfors regularity, so the variant of these arguments given in [14] is needed.

We now turn to the proof of the scale-invariant Poincaré inequality. As we have explained earlier, it suffices to prove the following result:

Theorem 3.1. *Suppose that the functions $\{b_i(w; y) : i = 1, \dots, n\}$, defined in $\mathbb{R}_+^n \times \mathbb{R}^m$, satisfy (51), are bounded below by a positive constant, and are constant outside a compact set. Then there is a constant C so that for any $0 < r$, and $(w; y) \in S_{n,m}$ and $u \in C^1(\overline{B_r^i(w; y)})$,*

$$\int_{B_r^i(w; y)} |u(\tilde{w}; \tilde{y}) - u_{B_r(w; y)}|^2 d\mu_b(\tilde{w}; \tilde{y}) \leq Cr^2 Q_{B_r^i(w; y)}^e(u, u). \quad (57)$$

The proof is somewhat more complicated than in the doubling measure result above; it uses a covering argument due to Jerison [8] which produces a cover of $B_r^i(w; y)$ by smaller balls where the approximation of the functions $\{b_i(\tilde{w}; \tilde{y})\}$ by constants is permissible. Thus the first step is to prove the result when the b_i are all constant for balls with respect to the ρ_∞^e -metric.

Proposition 3.2. *Let $\mathbf{b}_0 = (b_{01}, \dots, b_{0n})$ be a vector of positive constants. There is a constant $C_{\mathbf{b}_0}$ so that for all $0 < r, (w; y) \in S_{n,m}$ and $u \in C^1(\overline{B_r(w; y)})$ we have the estimate*

$$\int_{B_r(w; y)} |u - u_{\mathbf{b}_0}|^2 d\mu_{\mathbf{b}_0}(\tilde{w}, \tilde{y}) \leq C_{\mathbf{b}_0} r^2 \int_{B_r(w; y)} |\nabla_{\tilde{w}; \tilde{y}} u|^2 d\mu_{\mathbf{b}_0}(\tilde{w}, \tilde{y}), \quad (58)$$

where

$$u_{\mathbf{b}_0} = \frac{1}{\mu_{\mathbf{b}_0}(B_r(w; y))} \int_{B_r(w; y)} u d\mu_{\mathbf{b}_0}. \quad (59)$$

Moreover there is a constant $C_{\beta, B, m}$ so that if $0 < \beta < b_{0j} < B$ for $1 \leq j \leq n$, then

$$C_{\mathbf{b}_0} \leq C_{\beta, B, m}. \quad (60)$$

Proof of Proposition 3.2. The sharp constant $1/C_{\mathbf{b}_0}$ in (58) is the first non-zero eigenvalue of the operator $L_{\mathbf{b}_0}$ associated to this Dirichlet form, acting on functions on $B_r(w; y)$, which satisfy appropriate ‘‘Neumann’’ boundary conditions.

The formal operator is given by

$$L_{\mathbf{b}_0} u = - \left[\sum_{j=1}^n \left(\partial_{w_j}^2 + \frac{2b_{0j} - 1}{w_j} \partial_{w_j} \right) u + \Delta_y u \right]. \quad (61)$$

Recall that the ball is a product

$$B_r(w; y) = \prod_{j=1}^n [\max\{w_j - r, 0\}, w_j + r] \times \prod_{l=1}^m [y_l - r, y_l + r]. \quad (62)$$

Since the form domain is $C^\infty(\overline{B_r(w; y)})$, we see that on smooth elements in the domain of $L_{\mathbf{b}_0}$, the boundary condition is the standard Neumann one on the ‘‘tangential’’ boundary: $\tilde{y}_l = y_l \pm r$, i.e., $\partial_{\tilde{y}_l} u \upharpoonright_{\tilde{y}_l = y_l \pm r} = 0$. On the right ends of the intervals,

$$\lim_{\tilde{w}_j \rightarrow (w_j + r)^-} \partial_{w_j} u(\tilde{w}; \tilde{y}) = 0, \quad (63)$$

and similarly, on the left ends when $w_j - r > 0$,

$$\lim_{\tilde{w}_j \rightarrow (w_j - r)^+} \partial_{w_j} u(\tilde{w}; \tilde{y}) = 0. \quad (64)$$

However, when $w_j < r$, then the boundary condition at the left endpoint becomes

$$\lim_{\tilde{w}_j \rightarrow 0^+} \tilde{w}_j^{2b_0j-1} \partial_{w_j} u(\tilde{w}; \tilde{y}) = 0. \quad (65)$$

The domain of the Friedrichs extension of L_{b_0} is denoted $\mathcal{D}(L_{b_0})$. We are clearly in a setting where the spectral data for $(L_{b_0}, \mathcal{D}(L_{b_0}))$ can be determined by separation of variables. This is one reason why we replaced the Euclidean balls by sup-norm balls. The eigenfunctions take the form

$$f_1(\tilde{w}_1) \cdots f_n(\tilde{w}_n) g_1(\tilde{y}_1) \cdots g_m(\tilde{y}_m), \quad (66)$$

where each factor is an eigenfunction of the appropriate boundary value problem in 1-dimension. The first non-zero eigenvalue of $(L_{b_0}, \mathcal{D}(L_{b_0}))$ is then the minimum of the first non-trivial eigenvalues of these $(n+1)$ self adjoint operators. For the y -variable the first non-trivial eigenvalue for the Neumann operator on an interval of length $2r$ is $\frac{\pi^2}{4r^2}$. This leaves the 1-dimensional problems in the w -variables, which we treat in the following lemma.

Lemma 3.1. *If $0 < \beta < B$ then there is a positive constant $\lambda_{\beta, B}$ so that for $0 \leq x$, $\beta < b < B$ and $u \in \mathcal{C}^1([0 \vee (x-r), x+r])$ we have the estimate*

$$\lambda_{\beta, B} \leq \frac{r^2 \int_{\max\{x-r, 0\}}^{x+r} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{\max\{x-r, 0\}}^{x+r} |u(w) - \bar{u}_b|^2 w^{2b-1} dw}, \quad (67)$$

where

$$\bar{u}_b = \frac{\int_{\max\{x-r, 0\}}^{x+r} u(w) w^{2b-1} dw}{\int_{\max\{x-r, 0\}}^{x+r} w^{2b-1} dw}. \quad (68)$$

Proof. Fix $b > 0$. For each $0 \leq x$ and $0 < r$ we need to estimate the infimum, over functions with $w^{2b-1} dw$ -mean zero, of the quotient:

$$\frac{r^2 \int_{\max\{x-r, 0\}}^{x+r} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{\max\{x-r, 0\}}^{x+r} |u(w)|^2 w^{2b-1} dw} \quad (69)$$

Replacing x by x/r , we reduce to the case $r = 1$, but still with arbitrary center $x \in [0, \infty)$. Let \bar{u} denote the mean of u . As usual, there are three cases:

Case 1: If $x < 1$, then to estimate

$$\inf_{\{u:\bar{u}=0\}} \frac{\int_0^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_0^{x+1} |u(w)|^2 w^{2b-1} dw}. \quad (70)$$

we find eigenfunctions of the operator

$$\begin{aligned} L_b u &= -w^{1-2b} \partial_w w^{2b-1} \partial_w u \text{ with} \\ \lim_{w \rightarrow 0^+} w^{2b-1} \partial_w u(w) &= 0 \text{ and } \partial_w u(x+1) = 0. \end{aligned} \quad (71)$$

Solutions of the eigenvalue equation

$$\partial_w^2 u + \frac{2b-1}{w} \partial_w u + \lambda^2 u = 0 \quad (72)$$

are in terms of J -Bessel functions by

$$w^{1-b} [A J_{1-b}(\lambda w) + B J_{b-1}(\lambda w)], \quad (73)$$

at least for $b \notin \mathbb{N}$. The boundary condition at $w = 0$ implies that $A = 0$. Indeed, if $\nu \notin \mathbb{N}$, then $J_\nu(z) = a_\nu z^\nu (1 + O(z^2))$, so $w^{1-b} J_{1-b}(\lambda w) \sim c w^{2-2b}$, and the boundary condition at $w = 0$ eliminates this term. If $b = 1$, the singular solution has leading term $\log w$, which is again eliminated by the boundary condition. Finally, for $b \in \mathbb{N}$, $b > 1$, $w^{1-b} J_{b-1}(\lambda w)$ is the only regular solution. Thus, the solution is $(\lambda w)^{1-b} J_{b-1}(\lambda w)$ whenever $b > 0$. This is an entire function which oscillates infinitely many times as $w \rightarrow \infty$. Let $z_{1,b}$ be the smallest positive root of the equation:

$$\partial_z [z^{1-b} J_{b-1}(z)] = 0, \quad (74)$$

then the smallest non-trivial eigenvalue is:

$$\lambda_1^2 = \left(\frac{z_{1,b}}{1+x} \right)^2. \quad (75)$$

This gives the infimum of the functional in (70) for any $x \geq 0$, which proves useful in the analysis of the next case.

We now derive bounds for the constant $z_{1,b}$, depending on the upper and lower bounds of b . For any $b > 0$, define

$$\phi_b(\zeta) = \sum_{k=0}^{\infty} \frac{(-1)^k \zeta^k}{k! \Gamma(k+b)}. \quad (76)$$

This is an entire function satisfying the ODE

$$\zeta \partial_{\zeta}^2 \phi_b + b \partial_{\zeta} \phi_b + \phi_b = 0 \quad (77)$$

and the functional equation

$$\partial_{\zeta} \phi_b = -\phi_{b+1}. \quad (78)$$

A simple calculation shows that there is a constant C_b so that

$$z^{1-b} J_{b-1}(z) = C_b \phi_b \left(\frac{z^2}{4} \right) \quad (79)$$

Thus $\zeta_{1,b} = z_{1,b}^2/4$ is the smallest positive solution to $\phi_{b+1}(\zeta) = 0$. We can rewrite $\Gamma(b+1)\phi_{b+1}(\zeta)$ as

$$\begin{aligned} \Gamma(b+1)\phi_{b+1}(\zeta) &= 1 - \frac{\zeta}{b+1} + \\ &\sum_{k=1}^{\infty} \frac{\zeta^{2k}}{(2k)!(b+1) \cdots (b+2k)} \left[1 - \frac{\zeta}{(2k+1)(b+2k+1)} \right], \end{aligned} \quad (80)$$

from which we see that

$$(b+1) < \zeta_{1,b}. \quad (81)$$

Thus for any $0 < \beta < B$ there is a constant $1 \leq M_B$ so that

$$4(1+\beta) \leq z_{1,b}^2 \leq M_B \text{ if } \beta \leq b \leq B. \quad (82)$$

Case 2: If $1 < x < 2$, it is simpler to estimate

$$\inf_{\{u \in \mathcal{C}^1[x-1, x+1]\}} \frac{\int_{x-1}^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{x-1}^{x+1} |u(w) - \tilde{u}|^2 w^{2b-1} dw}, \quad (83)$$

from below; here

$$\tilde{u} = \frac{\int_{x-1}^{x+1} u(w)w^{2b-1}dw}{\int_{x-1}^{x+1} w^{2b-1}dw}. \quad (84)$$

The analysis in the previous case shows that if $\beta < b < B$, then for $x \geq 0$,

$$\frac{4(1+\beta)}{(1+x)^2} \int_0^{x+1} |u(w) - \bar{u}|^2 w^{2b-1} dw \leq \int_0^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw. \quad (85)$$

We define the extension of any $u \in \mathcal{C}^1([x-1, x+1])$ to

$$U(w) = \begin{cases} u(w) & \text{for } w \in [x-1, x+1] \\ u(x-1) & \text{for } w \in [0, x-1], \end{cases} \quad (86)$$

which is a function on $[0, x+1]$. This extension is admissible for the inequality in (85), so

$$\begin{aligned} \int_{x-1}^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw &= \int_0^{x+1} |\partial_w U(w)|^2 w^{2b-1} dw \\ &\geq \frac{4(1+\beta)}{(1+x)^2} \int_0^{x+1} |U(w) - \bar{U}|^2 w^{2b-1} dw \\ &\geq \frac{4(1+\beta)}{(1+x)^2} \int_{x-1}^{x+1} |u(w) - \bar{U}|^2 w^{2b-1} dw. \end{aligned} \quad (87)$$

It is a classical fact that the minimum of

$$\int_{x-1}^{x+1} |u(w) - a|^2 w^{2b-1} dw \quad (88)$$

is attained only when $a = \tilde{u}$, and therefore

$$\frac{4(1+\beta)}{(1+x)^2} \leq \frac{\int_{x-1}^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{x-1}^{x+1} |u(w) - \tilde{u}|^2 w^{2b-1} dw}, \quad (89)$$

completing the argument in this case as well.

Case 3: If $2 \leq x$, then observe that

$$\frac{\int_{x-1}^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{x-1}^{x+1} |u(w)|^2 w^{2b-1} dw} \geq \left(\frac{x-1}{x+1} \right)^{|2b-1|} \frac{\int_{x-1}^{x+1} |\partial_w u(w)|^2 dw}{\int_{x-1}^{x+1} |u(w)|^2 dw}, \quad (90)$$

to conclude that, via Fisher's min-max principle, that

$$\frac{\pi^2}{4 \cdot 3^{|2b-1|}} \leq \inf_{\{u: \bar{u}=0\}} \frac{\int_{x-1}^{x+1} |\partial_w u(w)|^2 w^{2b-1} dw}{\int_{x-1}^{x+1} |u(w)|^2 w^{2b-1} dw}. \quad (91)$$

□

Lemma 3.1 implies the result for the $n + m$ -dimensional case with constant weights \mathbf{b}_0 , which completes the proof of Proposition 3.2. □

We now give the proof of Theorem 3.1. Let $E = B_r^i(w_0; y_0)$ be the (intrinsic) ball with center $(w_0; y_0)$ and radius $r > 0$. As noted earlier, we use Jerison's covering argument, essentially as in [14, Theorem 5.3.4]. For the convenience of the reader we outline the argument, highlighting places where our argument differs from the standard one. If $B = B_r(w; y)$, then for any $k > 0$, write

$$kB = B_{kr}(w; y). \quad (92)$$

We let \mathcal{F} denote a collection of countably many ρ_∞^e -balls in E with the following properties:

1. The balls $B \in \mathcal{F}$ are disjoint.
2. The balls $\{2B : B \in \mathcal{F}\}$ are a cover of E .
3. If $B \in \mathcal{F}$, then its radius satisfies

$$r(B) = 10^{-3} d(B, \partial E). \quad (93)$$

4. There exists a constant K depending only on the doubling constant so that

$$\sup_{(w;y) \in E} \#\{B \in \mathcal{F} : (w; y) \in 10^2 B\} \leq K. \quad (94)$$

Here and throughout this argument $d(\cdot, \cdot)$ should be understood as the distance defined by the metric ρ_∞^e .

The existence of such a ‘Jerison covering’ \mathcal{F} satisfying these properties is standard. Several additional properties of \mathcal{F} are established in [8], and these are essential to the argument that follows. We are using the metric ρ_∞^e to define the balls in the covering. While the shortest paths for this metric are not unique. Euclidean geodesics (i.e., straight line segments) are length-minimizing paths for ρ_∞^e , so, by convention, we use these, thereby rendering the choice of shortest path unique.

Note that if $B_\sigma(w; y) \in \mathcal{F}$, then

$$\sigma \leq 10^{-2} d((w; y), E^c) \leq 10^{-2} d((w; y), \partial S_{n,m}). \quad (95)$$

Our use of Jerison’s argument rests on the following lemma:

Lemma 3.2. *There is a positive constant C_1 , so that if the ball $B = B_\sigma(w; y)$ has radius*

$$\sigma \leq 10^{-2} d((w; y), \partial S_{n,m}), \quad (96)$$

then for any $(\tilde{w}; \tilde{y}) \in 10B$, we have

$$\frac{1}{C_1} \leq \frac{\prod_{j=1}^n \tilde{w}_i^{b_i(\tilde{w}; \tilde{y})}}{\prod_{j=1}^n \tilde{w}_i^{b_i(w; y)}} \leq C_1. \quad (97)$$

Proof. Recall that there is a constant $0 < \beta_0$ and C so that for $1 \leq i \leq n$ and $(w; y) \in S_{n,m}$

$$\beta_0 \leq b_i(w; y) \text{ and } |b_i(w; y) - b_i(\tilde{w}; \tilde{y})| \leq \frac{C}{|\log \rho_\infty^e((w; y), (\tilde{w}; \tilde{y}))|}. \quad (98)$$

Moreover there is an R so that the functions $\{b_i(w; y) : i = 1, \dots, n\}$ are constant in $[B_R(0; 0)]^c$. From this it is clear that if we fix any positive number ρ , then there is a constant C_0 so that if $d((w; y), \partial S_{n,m}) > \rho$ and $\sigma \leq 10^{-2} d((w; y), \partial S_{n,m})$, then, for all $(\tilde{w}; \tilde{y}) \in 10B_\sigma(w; y)$ we have the estimate:

$$\frac{1}{C_0} \leq \frac{\prod_{j=1}^n \tilde{w}_i^{b_i(\tilde{w}; \tilde{y})}}{\prod_{j=1}^n \tilde{w}_i^{b_i(w; y)}} \leq C_0. \quad (99)$$

Thus we only need to consider balls with centers close to $\partial S_{n,m}$.

Let $\rho < 1/10$ and assume that $d((w; y), bS_{n,m}) < \rho$. Let $B_\sigma(w; y)$ be a ball with $\sigma \leq 10^{-2} d((w; y), \partial S_{n,m})$. And let $\bar{w} = \min\{w_1, \dots, w_n\}$, clearly $\bar{w} = d((w; y), bS_{n,m})$. If $(\tilde{w}; \tilde{y}) \in 10B_\sigma(w; y)$, then the w -coordinates satisfy

$$\frac{9\bar{w}}{10} < \tilde{w}_i < 1, \quad (100)$$

and therefore

$$|\log \tilde{w}_i| \leq \left| \log \frac{9\bar{w}}{10} \right|. \quad (101)$$

The ratio in (97) satisfies the estimate

$$\exp \left(-nC \frac{\log \left(\frac{9\bar{w}}{10} \right)}{\log \left(\frac{\bar{w}}{10} \right)} \right) \leq \frac{\prod_{j=1}^n \tilde{w}_i^{b_i(\tilde{w}; \tilde{y})}}{\prod_{j=1}^n \tilde{w}_i^{b_i(w; y)}} \leq \exp \left(nC \frac{\log \left(\frac{9\bar{w}}{10} \right)}{\log \left(\frac{\bar{w}}{10} \right)} \right). \quad (102)$$

The lemma follows easily from these bounds. \square

Combining this lemma with Proposition 3.2, and the Courant-Fisher min-max principle, we obtain the corollary:

Corollary 3.1. *Assume that the exponents $\{b_i(w; y)\}$ satisfy $0 < \beta_0 \leq b_i(w; y)$, the estimate in (51), and are bounded above by B . Let $1 \leq \kappa \leq 10$. There is a constant C depending on β_0, β_1, B and the dimension $(n + m)$ so that if $B_\sigma(w; y)$ is a ball with*

$$\sigma \leq 10^{-2} d((w; y), \partial S_{n,m}), \quad (103)$$

then for any $u \in \mathcal{C}^1(\overline{B_{\kappa\sigma}(w; y)})$ we have the estimate

$$\int_{B_{\kappa\sigma}(w; y)} |u(\tilde{w}; \tilde{y}) - \bar{u}|^2 d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}) \leq C(\kappa\sigma)^2 \int_{B_{\kappa\sigma}(w; y)} |\nabla u(\tilde{w}; \tilde{y})|^2 d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}), \quad (104)$$

where

$$\bar{u} = \frac{\int_{B_{\kappa\sigma}(w; y)} u(\tilde{w}; \tilde{y}) d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y})}{\mu_{\mathbf{b}}(B_{\kappa\sigma}(w; y))}. \quad (105)$$

In particular, this estimate holds for any ball B belonging to a covering \mathcal{F} as defined above.

The remaining lemmas needed to apply the argument from Section 5.3 of [14] to prove Theorem 3.1 are given in Appendix A, where we have sketched the details of the proof using this covering argument, in part, because it applies immediately to establish an important generalization of this inequality wherein we include cutoff functions in the integrals. The proof of the following result is very similar to the one given above and in Appendix A.

Proposition 3.3. *For $(x; y) \in S_{n,m}$ and $r > 0$, let*

$$\phi(\cdot) = \max \left\{ \left[1 - \frac{\rho_i(\cdot, (x; y))}{r} \right], 0 \right\}. \quad (106)$$

Under the hypotheses of the previous theorem, there is a constant C so that

$$\int_{B_r^i(x; y)} |u - u_\phi|^2 \phi^2 d\mu_{\mathbf{b}}(\tilde{x}, \tilde{y}) \leq Cr^2 \int_{B_r^i(x; y)} \phi^2(\tilde{x}, \tilde{y}) \langle A\nabla u, \nabla u \rangle d\mu_{\mathbf{b}}(\tilde{x}, \tilde{y}), \quad (107)$$

where

$$u_\phi = \frac{\int_{B_r^i(x; y)} \phi^2 u d\mu_{\mathbf{b}} \tilde{x}, \tilde{y}}{\int_{B_r^i(x; y)} \phi^2 d\mu_{\mathbf{b}} \tilde{x}, \tilde{y}}. \quad (108)$$

This is Corollary 2.5 in [20]. It is needed for Moser's proof of the parabolic Harnack inequality.

An important consequence of these results is the Sobolev inequality, Theorem 2.6 in [20]:

Theorem 3.2. *Let $D \geq 3$ be such that for all $0 < r$, and $(x; y) \in S_{n,m}$ we have the doubling property*

$$\mu_{\mathbf{b}}(B_{2r}^i(x; y)) \leq 2^D \mu_{\mathbf{b}}(B_r^i(x; y)). \quad (109)$$

For all functions in $\mathcal{D}(Q_{B_r^i(x; y)})$, we have the estimate

$$\left[\int_{B_r^i(x; y)} |u|^{\frac{2D}{D-2}} d\mu_{\mathbf{b}}(\tilde{x}, \tilde{y}) \right]^{\frac{D-2}{D}} \leq C_S \frac{r^2}{[\mu_{\mathbf{b}}(B_r^i(x; y))]^{\frac{2}{D}}} \left[Q_{B_r^i(x; y)}(u, u) + \frac{1}{r^2}(u, u)_{\mathbf{b}} \right]. \quad (110)$$

Remark 3.2. These proofs of the doubling property and scale-invariant L^2 Poincaré inequality readily adapt to allow the replacement of the class of measures $\{d\mu_{\mathbf{b}}\}$ by the slightly more general class of measures of the form $\{e^{U(w;y)}d\mu_{\mathbf{b}}(w;y)\}$, where $U(w;y)$ is a bounded C^0 -function that is constant outside a compact set. This vindicates our claim, made after (36), that the set of measures and quadratic forms to which our analysis applies is invariant under (bounded) dilations.

4 From Dirichlet Forms to Operators

The symmetric Dirichlet form $Q_B(u, v)$, with core $C^1(\overline{B})$ introduced above defines an unbounded self-adjoint operator, L_Q , acting on a dense domain $\mathcal{D}(L_Q) \subset L^2(B; d\mu_{\mathbf{b}})$. There are two features in the definition of $\mathcal{D}(L_Q)$: first, the formal symbol of the operator, and second, the natural boundary condition. The domain of the Dirichlet form $\mathcal{D}(Q)$ is the graph closure of $C^1(\overline{B})$ with respect to the norm

$$|u|_Q^2 = \|u\|_{L^2(B; d\mu_{\mathbf{b}})}^2 + Q_B(u, u). \quad (111)$$

The domain of the operator is defined by the condition: $u \in \mathcal{D}(L_Q)$, if there is a constant C so that

$$|Q_B(v, u)| \leq C \|v\|_{L^2(B; d\mu_{\mathbf{b}})}^2 \quad \text{for any } v \in \mathcal{D}(Q). \quad (112)$$

By the Riesz representation theorem there is a unique element $w \in L^2(B; d\mu_{\mathbf{b}})$ so that

$$Q_B(v, u) = -(v, w)_{\mathbf{b}, B}. \quad (113)$$

We define $L_Q u = w$.

4.1 The Second Order Operator

By considering smooth functions in the form domain we can use the condition in (112) to derive the formal symbol of the operator associated to the symmetric Dirichlet form Q , along with the boundary conditions that must be satisfied by smooth elements of $\mathcal{D}(L_Q)$. These conditions are then satisfied in a distributional sense by all elements of the operator domain. As it fits better with our earlier work, we derive these formulæ in the $(x; y)$ -variables.

After some calculation, the integration by parts gives that

$$L_Q u = \left[\sum_{j=1}^n [x_j \partial_{x_j}^2 u + b_j(x; y) \partial_{x_j} u] + \sum_{i,j=1}^n \partial_{x_j} \sqrt{x_i x_j} a_{ij} \partial_{x_j} u + \frac{1}{2} \sum_{i=1}^n \sum_{l=1}^m [\partial_{y_l} \sqrt{x_j} c_{jl} \partial_{x_j} u + \partial_{x_j} \sqrt{x_j} c_{jl} \partial_{y_l} u] + \sum_{k,l=1}^m \partial_{y_m} d_{lm} \partial_{y_l} u + V u \right], \quad (114)$$

where V is a vector field with possibly slightly singular coefficients. For each j the formal Neumann-type boundary condition along $\partial S_{n,m} \cap \bar{B}$ is given by

$$\lim_{x_j \rightarrow 0^+} \left[x_j^{b_j} \partial_{x_j} u + x_j^{b_j - \frac{1}{2}} \left(\sum_{i=1}^n a_{ij} \sqrt{x_i} \partial_{x_i} u + \sum_{l=1}^m c_{jl} \partial_{y_l} u \right) \right] = 0. \quad (115)$$

To make V more explicit, set

$$W_{\mathbf{b}} = \prod_{j=1}^n x_j^{b_j(x;y)-1}. \quad (116)$$

Then

$$V = \sum_{i,j=1}^n [\log x_i \partial_{x_j} b_i + \sqrt{\frac{x_i}{x_j}} a_{ji} (W_{\mathbf{b}}^{-1} \partial_{x_i} W_{\mathbf{b}})] x_j \partial_{x_j} + \sum_{k,l=1}^m d_{lm} (W_{\mathbf{b}}^{-1} \partial_{y_m} W_{\mathbf{b}}) \partial_{y_l} + \frac{1}{2} \sum_{j=1}^n \sum_{l=1}^m c_{jl} [(W_{\mathbf{b}}^{-1} \partial_{y_l} W_{\mathbf{b}}) \sqrt{x_j} \partial_{x_j} u + \sqrt{x_j} (W_{\mathbf{b}}^{-1} \partial_{x_j} W_{\mathbf{b}}) \partial_{y_l}], \quad (117)$$

where

$$W_{\mathbf{b}}^{-1} \partial_{x_j} W_{\mathbf{b}} = \frac{b_j - 1}{x_j} + \sum_{i=1}^n \log x_i \partial_{x_j} b_i \quad (118)$$

$$W_{\mathbf{b}}^{-1} \partial_{y_l} W_{\mathbf{b}} = \sum_{i=1}^n \log x_i \partial_{y_l} b_i.$$

A typical assumption in population genetics is that the coefficients $\{a_{ij}, c_{jl}\}$ in (114) can be written as $a_{ij} = \sqrt{x_i x_j} \alpha_{ij}(x; y)$, and $c_{jl} = \sqrt{x_j} \gamma_{jl}(x; y)$, where $\{\alpha_{ij}, \gamma_{jl}\}$ are smooth functions of $(x; y)$. Thus

$$V = \sum_{j=1}^n \left[\beta_j^0 + \sum_{i=1}^n \log x_i \beta_{ij}^1 \right] x_j \partial_{x_j} + \sum_{l=1}^m \left[\epsilon_j^0 + \sum_{i=1}^n \log x_i \epsilon_{il}^1 \right] \partial_{y_l}, \quad (119)$$

where $\{\beta_j^0, \beta_{ij}^1, \epsilon_l^0, \epsilon_{il}^1\}$ are smooth in $(x; y)$, so in this case V is tangent to $\partial S_{n,m}$, but has slightly singular coefficients. Under this hypothesis, the boundary condition along $\partial S_{n,m} \cap \bar{B}$ becomes

$$\lim_{x_j \rightarrow 0^+} x_j^{b_j} \left[\partial_{x_j} u + \left(\sum_{i=1}^n \alpha_{ij} x_i \partial_{x_i} u + \sum_{l=1}^m \gamma_{jl} \partial_{y_l} u \right) \right] = 0, \quad (120)$$

which is certainly satisfied if u is in $\mathcal{C}^1(S_{n,m})$. Indeed, it is a simple to show that a function $u \in \mathcal{C}^2(S_{n,m})$ whose derivatives decay rapidly enough belongs to $\text{Dom}(L_Q)$.

The log terms in these coefficients do not appear, at least to leading order, if the derivatives of the weights $\{b_i(x; y)\}$ vanish along the boundary, e.g. if these functions are constant and their gradients vanish on $\partial S_{n,m}$. If that is the case, then

$$V = \sum_{i,j=1}^n \alpha_{ij} (b_i - 1) x_j \partial_{x_j} + \sum_{l=1}^m \sum_{j=1}^n \gamma_{jl} (b_j - 1) \partial_{y_l}. \quad (121)$$

We now discuss two possible modifications to the form of this second order operator which may be directly handled by our methods. The first is to replace the measure $d\mu_b$ by a multiple

$$d\mu_{b,U}(x; y) = e^{U(x;y)} d\mu_b(x; y), \quad (122)$$

where, for example, U is \mathcal{C}^1 (as a function of $(x; y)$) and is constant outside of a compact set. The extra terms coming from this factor in the integration by parts leads to an additional ‘‘conservative’’ tangent vector field

$$\begin{aligned} V_U = & \sum_{j=1}^n (\partial_{x_j} U) x_j \partial_{x_j} + \sum_{i,j=1}^n \sqrt{x_i x_j} a_{ij} \partial_{x_j} U \partial_{x_i} + \\ & \frac{1}{2} \sum_{l=1}^m \sum_{j=1}^n \sqrt{x_j} c_{jl} [\partial_{y_l} U \partial_{x_j} + \partial_{x_j} U \partial_{y_l}] + \sum_{k,l=1}^m d_{lm} \partial_{y_m} U \partial_{y_l} \end{aligned} \quad (123)$$

The associated second order operator is denoted $L_{Q,U}$. It is quite straightforward to incorporate such a factor into all of the arguments above and below.

4.2 Non-self Adjoint Perturbations

A general Kimura operator L may deviate from the operator $L_{Q,U}$ defined by the symmetric Dirichlet form (8), with the modified measure $d\mu_{b,U}$, by a first order term. Indeed, it is typically impossible to write L as in (114), with V a sum of

two terms (117) and (123). To accommodate this, we use the formalism of non-symmetric Dirichlet forms. Many of the estimates proved in [14] and [20] extend to operators defined in this way, and indeed [19] proves some of these. In lieu of following Sturm's argument, we show that the proofs given in [14] can be adapted to the present circumstance.

To be more specific, consider a non self-adjoint operator

$$L_Q^{X,c} = L_{Q,U} - V_X - c, \quad (124)$$

where V_X is a tangent vector field, with possibly singular coefficients and c is a measurable function. The tangent part of the vector field in L_Q has coefficients with log-singularities, and hence for $L_Q^{X,c}$ to be an arbitrary generalized Kimura diffusion operator, as defined in [4], we must add a perturbation V_X that also has log-singularities. Using a simple integration by parts trick we are able to control such terms with mild singularities along the boundary. To that end we prove the following lemma in Appendix B:

Lemma 4.1. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(\mathbf{x}; \mathbf{y})$, with $0 < \beta_0 < b_j$, constant outside a compact set. Let q be a measurable function defined on $S_{n,m}$ that satisfies*

$$|q(x; y)| \leq M \left[\chi_B(x; y) \sum_{j=1}^n |\log x_j|^k + 1 \right]. \quad (125)$$

for some $k \in \mathbb{N}$, $0 < M$, and B a bounded neighborhood of $(0; 0)$, Given $\eta > 0$ there is a C_η so that for any $2 \leq p$, we have

$$\int_{S_{n,m}} |q(x; y)| u^p(x; y) d\mu_{\mathbf{b}} \leq \eta \int_{S_{n,m}} \langle A \nabla u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}} \rangle d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} u^p d\mu_{\mathbf{b}}, \quad (126)$$

for u a bounded, compactly supported, non-negative function in $\text{Dom}(Q)$.

We suppose that X is a continuous \mathbb{R}^{n+m} -valued function in $\text{int } S_{n,m}$, with $|X|_A^2 = \langle AX, X \rangle$ satisfying an estimate like in (125). We define the non-symmetric Dirichlet form

$$\begin{aligned} Q_B^X(u, v) = \int_B \left\{ \sum_{j=1}^n [x_j \partial_{x_j} u X_j + \sum_{i,j=1}^n +x_i x_j a_{ij}(x; y) \partial_{x_i} u X_j + \right. \\ \left. \frac{1}{2} \sum_{i=1}^n \sum_{l=1}^m x_i c_{il} [\partial_{x_i} u X_l + \partial_{y_l} u X_i] + \sum_{k,l=1}^m d_{kl}(x; y) \partial_{y_k} u X_l \right\} v \times \\ x_1^{b_1(x;y)-1} \dots x_n^{b_n(x;y)-1} dx dy; \quad (127) \end{aligned}$$

this represents the action of the vector field V_X , which is continuous and tangent to the boundary.

Representing the integrand in (127) as $\langle A\nabla u, Xv \rangle d\mu_{\mathbf{b}}$, and allowing also for a zeroth order term cu , where c is a measurable, real valued function, satisfying an estimate like that in (125), we define

$$\tilde{Q}_{U,B}^{X,c}(u, v) = \int_B [\langle A\nabla u, \nabla v \rangle + \langle A\nabla u, Xv \rangle + cuv] e^U d\mu_{\mathbf{b}}. \quad (128)$$

For simplicity of notation, and because it provides no additional generality, we shall omit the factor e^U in the measure.

A ‘‘sector condition’’ holds for $\tilde{Q}_B^{X,c}$: there is a constant $C > 0$ so that for any $u, v \in \mathcal{D}(Q)$,

$$|\tilde{Q}^{X,c}(u, v)| \leq C (Q(u, u) + (u, u)_{\mathbf{b}})^{\frac{1}{2}} (Q(v, v) + (v, v)_{\mathbf{b}})^{\frac{1}{2}}. \quad (129)$$

This is clear since the Cauchy-Schwarz inequality implies that

$$\begin{aligned} |\tilde{Q}^{X,c}(u, v)| &= \left| \int [\langle A\nabla u, \nabla v + vX \rangle + cuv] d\mu_{\mathbf{b}} \right| \\ &\leq \sqrt{2} (Q(u, u) + (u, u)_{\mathbf{b}})^{\frac{1}{2}} (Q(v, v) + (\sigma v, v)_{\mathbf{b}})^{\frac{1}{2}}, \end{aligned} \quad (130)$$

where $\sigma = |X|_A^2 + |c|^2$. By Lemma 4.1, there is a C' so that

$$(\sigma v, v)_{\mathbf{b}} \leq C' [Q(v, v) + (v, v)_{\mathbf{b}}], \quad (131)$$

which proves (129). From this it is immediate that the form domains of $\tilde{Q}^{X,c}$ and Q agree. A function $u \in \text{Dom}(\tilde{Q}^{X,c})$ is in the domain of the operator $L_Q^{X,c}$ if there is a constant C so that, for every $v \in \text{Dom}(\tilde{Q}^{X,c})$,

$$|\tilde{Q}^{X,c}(u, v)| \leq C \|v\|_{\mathbf{b}}. \quad (132)$$

This implies, as before, that there is a unique element, $w \in L^2$ so that

$$\tilde{Q}^{X,c}(u, v) = -(w, v)_{\mathbf{b}}; \quad (133)$$

we then define $L_Q^{X,c}u = w$.

For the associated operator to satisfy the Markov property, and hence define contractions on L^p -spaces, (see Lemmas 1.4 and 1.5 in [19]) we would need to assume that

$$c - \frac{1}{2} \text{Div}_{A,\mathbf{b}} X \geq 0, \quad (134)$$

where

$$\begin{aligned} \operatorname{Div}_{A,b} X &= \nabla_{x;y} \cdot (AX) + \\ &\mathbf{x}^{1-b} \left[\sum_{i,k} A_{ik} X_k \partial_{x_i} (\mathbf{x}^{b-1}) + \sum_{l,k} A_{(l+n)k} X_k \partial_{y_l} (\mathbf{x}^{b-1}) \right]. \end{aligned} \quad (135)$$

Writing out the second line in detail gives

$$\sum_{i,k} A_{ik} X_k \left[\frac{b_i - 1}{x_i} + \sum_{s=1}^n \log x_s \partial_{x_i} b_s \right] + \sum_{l,k} A_{(l+n)k} X_k \left[\sum_{s=1}^n \log x_s \partial_{y_l} b_s \right], \quad (136)$$

where $1 \leq i \leq n$, $1 \leq l \leq m$ and $1 \leq k \leq m + n$. For (134) to hold with a bounded function c , it is generally necessary that the following three conditions hold:

1. The weights must be constant along the appropriate boundary components.
2. The vector field AX must be Lipschitz.
3. The coefficients A_{ik} are (boundedly) divisible by x_i .

For our applications, these hypotheses are unnatural, but fortunately they are actually not necessary. Using the estimates that follow from Lemma 4.1, with $p = 2$, we easily establish that there is a constant m so that a weak local solution, in $[0, T] \times B$, to $u_t - L_Q^{X,c} u = 0$ satisfies,

$$\|u(t)\|_{L^2(B; d\mu_b)} \leq e^{mt} \|u(0)\|_{L^2(B; d\mu_b)}. \quad (137)$$

The operator adjoint to the one defined by $\tilde{Q}^{X,c}$, with respect to the $L^2(B; d\mu_b)$ -pairing, is

$$\widehat{L}_Q^{X,c} = L_Q + V_X - \widehat{c}; \quad (138)$$

this is defined by the Dirichlet form $\tilde{Q}^{-X, \widehat{c}}$, where

$$\widehat{c} = c - \operatorname{Div}_{A,b} X. \quad (139)$$

With this representation for the adjoint, $\widehat{L}_Q^{X,c}$ has an unbounded term of order zero, even if $c = 0$, unless the weights are constant. In the sequel we prove Harnack estimates for the operators $L_Q^{X,c}$ assuming that c is bounded and that X satisfies an estimate like that in (125). This enables us to prove the Harnack estimate and the Hölder continuity for solutions to a generalized Kimura diffusion on a compact manifold with corners, with initial data in L^2 . Since Lemma B.1 holds for potentials with log-singularities at the boundary, we can use the argument in [14] to prove upper bounds for the heat kernel in this more general case.

4.3 Consequences of the Doubling Property and Poincaré Inequality

The hypothesis that the functions $b_i(x; y)$ and $U(x; y)$ are constant outside a compact set implies that the doubling property and Poincaré inequality hold globally in $S_{n,m}$. Our main intention, however, is to apply these results to solutions of the parabolic equation

$$\partial_t u - Lu = 0 \text{ with } u|_{t=0} = f \quad (140)$$

on a compact manifold with corners P , where L is a generalized Kimura diffusion operator. We thus work in a boundary adapted coordinate system, and use the fact that these estimates hold for local solutions. Grigor'yan and Saloff-Coste, and in somewhat greater generality Sturm, show that the doubling property, (52), of the measure, and the scale-invariant L^2 Poincaré inequality, (57), imply a range of properties of solutions to both the parabolic and elliptic problems, including:

1. Harnack inequalities for non-negative solutions.
2. Hölder continuity for weak solutions with initial data in $L^2(B; d\mu_b)$.
3. Pointwise upper and lower bounds for the heat kernel itself.

It is shown in [19] that if a Dirichlet form satisfies the hypotheses of uniform parabolicity (UP) and strong uniform parabolicity (SUP), as well as the doubling property for $d\mu_b$ and the scale invariant L^2 -Poincaré inequality, established for Q in Theorem 3.1, then weak solutions satisfy the conclusions of Lemma 1 in [11]. Sturm did not derive all the conclusions that are available in the non-symmetric case, and the verification of the SUP condition requires the assumption that the weights $\{b_i(x; y)\}$ are constant and their gradients vanish at ∂P . We therefore show directly that analogues of Theorems 5.2.9, 5.2.16 and Lemma 5.4.1 in [14] hold for weak local solutions, i.e., $u \in \mathcal{D}(Q_{B_r^i(x;y)})$ for which

$$\tilde{Q}_{B_r^i(x;y)}^{X,c}(u, \varphi) = 0 \quad (141)$$

for all $\varphi \in \mathcal{D}(Q_{B_r^i(x;y)}) \cap L^\infty$, with support in $B_r^i(x; y)$. The proof of the Harnack inequality then follows, more or less functorially, from the argument in [14], which employs the lemma of Bombieri and Giusti (Lemma 2.2.6 in [14]). The proofs of the necessary lemmas are given in Appendix B. Here we simply state the consequences of these estimates. The first and most important is a Harnack inequality for local solutions.

Theorem 4.1. *Suppose that the functions $\{b_i(x; y) : i = 1, \dots, n\}$ defined in $\mathbb{R}_+^n \times \mathbb{R}^m$ are continuously differentiable functions of $(\mathbf{x}; \mathbf{y})$, bounded below by a positive constant and constant outside a compact set, and $X(w; y)$ is a continuous*

\mathbb{R}^{n+m} -valued function, satisfying (125) for some $k \in \mathbb{N}$, and B a bounded set, which vanishes outside of a compact set, and $c(w; y)$ is a bounded measurable function supported in a compact set. There is a constant C so that for any $0 < r$, and $(x; y) \in \mathbb{R}_+^n \times \mathbb{R}^m$, and u a non-negative, weak solution to

$$\partial_t u = (L_Q - V_X - c)u \quad (142)$$

in $W = (t - 4r^2, t) \times B_{2r}^i(x; y)$, we have the following estimate:

$$\sup_{W^-} u \leq C \inf_{W^+} u \quad (143)$$

where

$$\begin{aligned} W^+ &= (t - r^2, t) \times B_r^i(x; y) \\ W^- &= (t - 3r^2, t - 2r^2) \times B_r^i(x; y). \end{aligned} \quad (144)$$

Sketch of proof. Using Lemmas B.1, B.5 and B.6 we verify that the hypotheses of the lemma of Bombieri and Giusti (Lemma 2.2.6 in [14]) are satisfied with $\alpha_0 = \infty$. The proof of the inequality then follows exactly as in [11], which is essentially identical to the argument used in [14]. \square

As noted above this estimate has a wide range of consequences, among them the Hölder continuity of solutions to the initial value problem for the parabolic operator $\partial_t - (L_Q - V_X)$ and upper and lower bounds on the heat kernel. We first state the Hölder continuity result.

Corollary 4.1. *If $\{b_i(x; y)\}$ are positive C^1 -functions of $\{\mathbf{x}, \mathbf{y}\}$, which are constant outside of compact set, and $X(w; y)$ is a continuous \mathbb{R}^{n+m} -valued function, satisfying (125) for some $k \in \mathbb{N}$, which vanishes outside of a compact set, then there exists a $\gamma > 0$ and a constant C such that, for all balls $B_{2r}^i(x; y) \subset S_{n,m}$ and all $t \in \mathbb{R}$, if u is a weak solution to the equation*

$$\partial_t u - (L_Q - V_X)u = 0 \quad (145)$$

in the set $W = (t - 4r^2, t) \times B_{2r}^i(x; y)$, then for $(s_1, x_1; y_1), (s_2, x_2; y_2) \in (t - r^2, t) \times B_r^i(x; y)$,

$$|u(s_1, x_1; y_1) - u(s_2, x_2; y_2)| \leq C \sup_W |u| \left(\frac{|s_1 - s_2|^{\frac{1}{2}} + \rho_i((x_1; y_1), (x_2; y_2))}{r} \right)^\gamma. \quad (146)$$

Remark 4.1. Note that we need to take $c = 0$, as the proof requires that constant functions be solutions of the parabolic equation.

The proof exactly follows the proof of Theorem 5.4.7 in [14].

This corollary has a very useful corollary itself, which gives the rate at which the $\mathcal{C}_{\text{WF}}^{0,\gamma}$ -norm of a solution with initial data in \mathcal{C}^0 blows up.

Corollary 4.2. *Suppose that u is defined in $W = (0, t) \times B_2^i(0; 0)$, with $t < \frac{1}{2}$, satisfies the estimate in (146). There are constants C, C' independent of u and t so that if $\sup_{(x;y,t) \in W} |u(x; y, t)| < M$, then*

$$|u(t, x_1; y_1) - u(t, x_2; y_2)| \leq M \left[C \left(\frac{\rho_i((x_1; y_1), (x_2; y_2))}{\sqrt{t}} \right)^\gamma + C' \left(\frac{\rho_i((x_1; y_1), (x_2; y_2))}{\sqrt{t}} \right) \right], \quad (147)$$

for $(x_1; y_1), (x_2; y_2) \in B_1^i(0; 0)$.

Remark 4.2. As noted after hypothesis (36), the conditions under which this corollary holds are dilation invariant. This result also shows that

$$\int_0^{\frac{1}{2}} \|u(\cdot, t)\|_{\text{WF},0,\gamma} dt < C \|u\|_{L^\infty}, \quad (148)$$

which in turn implies that if $L_Q - V_X$ is a Kimura diffusion operator on the compact manifold with corners P , then its graph closure on $\mathcal{C}^0(P)$ has a compact resolvent.

Proof. From (146) it follows that for points $(x_1; y_1), (x_2; y_2) \in B_1^i(0; 0)$ with $\rho_i((x_1; y_1), (x_2; y_2)) < \sqrt{t/3}$, we have

$$|u(t, x_1; y_1) - u(t, x_2; y_2)| \leq MC \left(\frac{\rho_i((x_1; y_1), (x_2; y_2))}{\sqrt{t}} \right)^\gamma. \quad (149)$$

If $\rho_i((x_1; y_1), (x_2; y_2)) > \sqrt{t/3}$, then we can choose points $(x^{(j)}; y^{(j)})$, $j = 0, \dots, N$, on a length minimizing geodesic joining $(x^{(0)}; y^{(0)}) = (x_1; y_1)$, to the point $(x^{(N)}; y^{(N)}) = (x_2; y_2)$, with

$$\rho_i((x^{(j)}; y^{(j)}), (x^{(j+1)}; y^{(j+1)})) = \sqrt{\frac{t}{3}} \text{ for } j = 0, \dots, N-1, \quad (150)$$

and

$$\rho_i((x^{(N-1)}; y^{(N-1)}), (x^{(N)}; y^{(N)})) \leq \sqrt{\frac{t}{3}}. \quad (151)$$

Clearly

$$N \leq \frac{4\rho_i((x_1; y_1), (x_2; y_2))}{\sqrt{t}}. \quad (152)$$

Applying (149) for a sequence of points along the straight line geodesic from $(x_1; y_1)$ to $(x_2; y_2)$ gives

$$\begin{aligned} |u(t, x_1; y_1) - u(t, x_2; y_2)| &\leq MC \sum_{j=0}^{N-1} \left(\frac{\rho_i((x^{(j)}; y^{(j)}), (x^{(j+1)}; y^{(j+1)}))}{\sqrt{t}} \right)^\gamma \\ &\leq MCN^{1-\gamma} \left(\sum_{j=0}^{N-1} \frac{\rho_i((x^{(j)}; y^{(j)}), (x^{(j+1)}; y^{(j+1)}))}{\sqrt{t}} \right)^\gamma. \end{aligned} \quad (153)$$

Then (152) and the fact that the points $\{(x^{(j)}; y^{(j)})\}$ lie along a length minimizing geodesic, so the sum of the ρ_i distances between them telescope, shows that

$$|u(t, x_1; y_1) - u(t, x_2; y_2)| \leq MC' \left(\frac{\rho_i((x_1; y_1), (x_2; y_2))}{\sqrt{t}} \right); \quad (154)$$

this completes the proof of the corollary. \square

The final corollaries are upper and lower bounds for the heat kernel itself. The upper bound holds for the general class of operators $L_Q - V_X - c$ we have been considering, provided that the adjoint operator, $\widehat{L}_Q^{-X, \widehat{c}} = L_Q + V_X - \widehat{c}$, where \widehat{c} is given by (139), is an operator of the same type. The lower bounds only apply to the self adjoint case. The solution operator for the heat equation defines a semigroup $f \mapsto T_t f$, which is represented by a kernel function

$$T_t f(x; y) = \int_{S_{n,m}} p_t((x; y), (\tilde{x}; \tilde{y})) f(\tilde{x}; \tilde{y}) d\mu_{\mathbf{b}}(\tilde{x}; \tilde{y}). \quad (155)$$

As shown in Lemma 1.5 and Proposition 2.3 of [19], the operator

$$S_t f(\tilde{x}; \tilde{y}) = \int_{S_{n,m}} p_t((x; y), (\tilde{x}; \tilde{y})) f(x; y) d\mu_{\mathbf{b}}(x; y) \quad (156)$$

gives the semigroup for adjoint operator $\widehat{L}_Q^{-X, \widehat{c}}$.

Remark 4.3. Notice that the inclusion of the weight defining the measure gives a kernel of the form

$$p_t((x; y), (\tilde{x}; \tilde{y})) \tilde{x}_1^{b_1(\tilde{x}; \tilde{y})-1} \dots \tilde{x}_n^{b_n(\tilde{x}; \tilde{y})-1}, \quad (157)$$

which exactly mirrors the kernels that arise in the model case. The upper bound in (158) shows that the principal singularity of the heat kernel at the incoming face is no worse than that defined by the weight function, \mathbf{x}^{b-1} . For the self adjoint case, the lower bound (164) shows that this precisely captures the leading singularity.

The following estimates contain the doubling constant D from (52). The proofs of Theorem 5.2.10 and Corollary 5.2.11 in [14] give the upper bound:

Corollary 4.3. *Assume that the $\{b_i\}$ are positive C^1 functions of $\{x, y\}$, which are constant outside a bounded set, $X(x; y)$ is a C^1 -function, satisfying (125) for some $k \in \mathbb{N}$, vanishing outside a compact set; and c is a function also satisfying (125), for some $k' \in \mathbb{N}$, and vanishing outside a compact set. For any $0 < \eta$ there are constants C_0, C_1 so that, for all $t > 0$ and pairs $(x; y), (\tilde{x}; \tilde{y}) \in S_{n,m}$, we have*

$$p_t((x; y), (\tilde{x}; \tilde{y})) \leq \frac{C_0 \exp\left(-\frac{\rho_i^2((x; y), (\tilde{x}; \tilde{y}))}{4(1+\eta)t}\right)}{\sqrt{\mu_{\mathbf{b}}(B_{\sqrt{t}}^i(x; y))\mu_{\mathbf{b}}(B_{\sqrt{t}}^i(\tilde{x}; \tilde{y}))}} \times \left(1 + \frac{\rho_i((x; y), (\tilde{x}; \tilde{y}))}{\sqrt{t}}\right)^D \cdot \exp(C_1 t). \quad (158)$$

If $X = c = 0$, then we can take $C_1 = \eta = 0$ in this estimate.

Proof. The proof given in [14] for Corollary 5.2.11 applies with several modifications. If $X = 0$, then the kernel function is symmetric and, for $t > 0$ defines a weak solution to $(\partial_t - L_Q^c)u = 0$, in both the $(t, x; y)$ and $(t, \tilde{x}; \tilde{y})$ variables. We can therefore apply the estimates in (251) with $p = 1$ in both sets of variables.

If X is not zero, then the kernel weakly satisfies the equations

$$\begin{aligned} (\partial_t - L_{Q, (x; y)}^{X, c})p_t &= 0 \\ (\partial_t - \widehat{L}_{Q, (\tilde{x}; \tilde{y})}^{X, c})p_t &= 0. \end{aligned} \quad (159)$$

The vector field X and potential c are allowed to have log-singularities along the boundary. The function \widehat{c} defined in (139) also satisfies the estimate in (125), with some $k'' \in \mathbb{N}$. The adjoint operator is therefore defined by a Dirichlet form satisfying the hypotheses of Lemma B.1, and therefore weak solutions of $(\partial_t - \widehat{L}_Q^{X, c})u = 0$ also satisfy the estimates in (251) with $p = 1$.

Instead of the estimate [14, Lemma 4.2.1] for the L^2 operator norm for $T_t^{\alpha, \phi} = e^{-\alpha\phi}T_t e^{\alpha\phi}$, we have, for any $\epsilon > 0$, that

$$\|T_t^{\alpha, \phi}\|_{2 \rightarrow 2} \leq e^{((1+\epsilon)\alpha^2 + C_\epsilon)t}, \quad (160)$$

for a constant C_ϵ . To see this we observe that if $u(t) = T_t^{\alpha, \phi} f$, then

$$\begin{aligned} \partial_t \|u(t)\|_{\mathbf{b}}^2 &= \\ - \int_{S_{n,m}} [\langle A\nabla u, \nabla u \rangle + \langle A\nabla u, Xu \rangle + (\alpha \langle A\nabla \phi, X \rangle - \alpha^2 \langle A\nabla \phi, \nabla \phi \rangle + c)u^2] d\mu_{\mathbf{b}}. \end{aligned} \quad (161)$$

Using the Cauchy-Schwarz and arithmetic-geometric mean inequalities we see that

$$\begin{aligned} \partial_t \|u(t)\|_{\mathbf{b}}^2 \leq & -\frac{1}{2} \int_{S_{n,m}} \langle A \nabla u, \nabla u \rangle d\mu_{\mathbf{b}} + \alpha^2 \int_{S_{n,m}} \langle A \nabla \phi, \nabla \phi \rangle u^2 d\mu_{\mathbf{b}} + \\ & \int_{S_{n,m}} \left[|\alpha| |X|_A u^2 + \frac{1}{2} \langle AX, X \rangle + |c| u^2 \right] d\mu_{\mathbf{b}}. \end{aligned} \quad (162)$$

Applying Lemma 4.1 we easily show that for any $\epsilon > 0$ there is a constant C_ϵ so that

$$\partial_t \|u(t)\|_{\mathbf{b}}^2 \leq [(1 + \epsilon)\alpha^2 + C_\epsilon] \|u(t)\|_{\mathbf{b}}, \quad (163)$$

thus verifying (160). If $X = c = 0$, then clearly we can take $\epsilon = 0$, and $C_\epsilon = 0$. Apart from these modifications, the proof works exactly as in Saloff-Coste. \square

The lower bound, which follows from Corollary 4.10 in [20] is somewhat less general.

Corollary 4.4. *Suppose that $\{b_i\}$ are positive C^1 functions of $\{\sqrt{x}, \mathbf{y}\}$, which are constant outside a bounded set, and L_Q is a generalized Kimura diffusion operator defined by a symmetric Dirichlet form satisfying the hypotheses above. If p_t denotes the heat kernel for $\partial_t - L_Q$, then there is a constant C so that, for all $t > 0$ and $(x; y), (\tilde{x}; \tilde{y}) \in S_{n,m}$ we have*

$$p_t((x; y), (\tilde{x}; \tilde{y})) \geq \frac{\exp\left(-C \frac{\rho_i^2((x;y), (\tilde{x}; \tilde{y}))}{t}\right)}{C \mu_{\mathbf{b}}(B_{\sqrt{t}}^i(x; y))}. \quad (164)$$

Remark 4.4. The proof of the off-diagonal lower bound follows from the Harnack inequality and a lower bound for $p_t((x; y), (x; y))$. This diagonal estimate relies on the semi-group property *and* the self adjointness of the heat kernel with respect to the measure $d\mu_{\mathbf{b}}$. Generalizations of these lower bounds to non-self adjoint operators are given in [17]; we will return to this question in a later publication.

5 Applications to Population Genetics

The foregoing results have many applications to models in population genetics. Let P be a manifold with corners, and L a generalized Kimura diffusion operator defined on P . As shown in [4], we can introduce adapted coordinates near each boundary point so that the operator takes the form (17). Let $\{H_i : i = 1, \dots, I\}$ denote the hypersurface boundary components of P . As shown in Proposition 2.1,

the coefficient $b_i(x; y)$ of the vector field transverse vector to H_i , in adapted coordinates, has a natural meaning along each H_i .

The principal symbol of L , and hence the second order part of this operator, are globally defined throughout P . The weight functions are invariantly defined by L along the faces of P . In [4], we prove a tubular neighborhood theorem for each face of P , which implies that the weight functions have global extensions to non-negative functions on P , which can be taken to be positive if the weights themselves are. Throughout this section we assume that the weights are strictly positive. If a weight is constant, then it can be extended to be globally constant. As explained at the end of Section 2, these extended weights define a measure $d\mu_L$, which is locally of the form (5) and, in each coordinate chart, satisfies

$$C^{-1} \leq \frac{d\mu_L}{d\mu_b} \leq C. \quad (165)$$

The principal symbol of L , $q_L^{(2)}$, is a non-negative quadratic form on the fibers of T^*P . Its canonical dual defines an incomplete Riemannian metric on P , as discussed in Chapter 2 of [4]. We denote the distance between points $\xi, \eta \in P$ defined by this metric by $\rho_i(\xi, \eta)$, which is consistent with our usage of this notation in Sections 3–4. The compactness of P and Proposition 3.1 together imply that there is a constant D so that, for $\xi \in P$ and $0 < r$,

$$\mu_L(B_{2r}^i(\xi)) \leq 2^D \mu_L(B_r^i(\xi)), \quad (166)$$

i.e., μ_L is a doubling measure.

Using $d\mu_L$ and the principal symbol of L , we can define a Dirichlet form Q with core $\mathcal{C}^1(P)$:

$$Q(u, v) = \int_P q_L^{(2)}(du, dv) d\mu_L. \quad (167)$$

For an open set $B \subset P$, we use the notation Q_B for

$$Q_B(u, v) = \int_B q_L^{(2)}(du, dv) d\mu_L, \text{ for } u, v \in \mathcal{C}^1(\overline{B}). \quad (168)$$

If B is contained in an adapted local coordinate chart, then $Q_B(u, v)$ takes the form given in (8), with the measure $d\mu_b$ replaced by $e^U d\mu_b$, for a smooth function U . Using Jerison's covering argument and the scale invariant L^2 -Poincaré inequality, with a uniform constant, for sufficiently small balls, we can show that there is a constant C_P so that for any $\xi \in P$, $0 < r$, and $u \in \mathcal{C}^1(B_r^i(\xi))$ we have the estimate

$$\int_{B_r^i(\xi)} |u - \bar{u}|^2 d\mu_L \leq C_P r^2 Q_{B_r^i(\xi)}(u, u), \quad (169)$$

where

$$\bar{u} = \frac{\int_{B_r^i(\xi)} u d\mu_L}{\mu_L(B_r^i(\xi))}. \quad (170)$$

This Dirichlet form defines a second order, self-adjoint operator L_Q by

$$Q(u, v) = -(L_Q u, v) \text{ for all } u \in \text{Dom}(L_Q) \text{ and } v \in \text{Dom}(Q). \quad (171)$$

The difference $L_Q - L$ is a vector field V , tangent to the boundary of P . If the weights are constant, then V is a smooth tangent vector field, but otherwise it has logarithmically divergent coefficients as in (119). In any case there is a globally defined section Ξ of T^*P so that L is the second order operator defined by the non-symmetric Dirichlet form

$$\tilde{Q}^\Xi(u, v) = Q(u, v) + \int_P q_L^{(2)}(du, v\Xi) d\mu_L. \quad (172)$$

The ellipticity hypotheses imply that $q_L^{(2)}(\Xi, \Xi)^{\frac{1}{2}}$ diverges at worst logarithmically at ∂P . We let L_Q^Ξ denote the unbounded operator on $L^2(P; d\mu_L)$ defined by this Dirichlet form. The addition of such a vector field does not change the natural boundary condition that appears in the definition of domain of L_Q^Ξ , and it again follows that functions in $\mathcal{C}^2(P)$ automatically belongs to $\text{Dom}(L_Q^\Xi)$.

5.1 Regularity Results

We begin our analysis of regular solutions to the Cauchy problem for $\partial_t u - Lu = 0$ by considering the local regularity for solutions with initial data in $L^2(P)$, and then $\mathcal{C}^0(P)$. The following can be deduced from the results of Section 4.

Theorem 5.1. *Let P be a compact manifold with corners and L a generalized Kimura diffusion operator with smooth coefficients defined on P . Suppose that the weights defined by L are positive along every boundary component. If u is a weak solution to the initial value problem*

$$(\partial_t - L)u = 0 \text{ with } u(\xi, 0) = f(\xi) \in L^2(P, d\mu_L), \quad (173)$$

then $u \in \mathcal{C}^\infty(P \times (0, \infty))$.

Proof. Let $\{(\phi_j, U_j)\}$ be a cover of P by adapted coordinate charts, where $\phi_j : U_j \rightarrow W_j \subset S_{n_j, m_j}$. In each coordinate chart there is a measure and Dirichlet form Q_j defined in a neighborhood of $(0; 0)$ so that, in this chart, the operator L is of the form $L_{Q_j} - V_{X_j}$. As L is assumed to have smooth coefficients, the

coefficient functions X_j satisfy an estimate of the form given in (125), with $k = 1$. The measure is defined throughout S_{n_j, m_j} , and the Dirichlet form can be extended as well. The operator L_{Q_j} is a model operator outside of a compact neighborhood of W_j . Corollary 4.1 implies that for $t > 0$, the solution $\phi_j^* u(\cdot, t)$ belongs to the Hölder space $\mathcal{C}_{\text{WF}}^{0, \gamma}(W_j)$ for each j . This shows that $u(\cdot, t) \in \mathcal{C}_{\text{WF}}^{0, \gamma}(P)$ for $t > 0$. We can therefore apply Corollary 11.2.2 from [4] to conclude that $u \in \mathcal{C}^\infty(P \times (0, \infty))$. \square

We now use the estimates in Corollary 4.2 along with the maximum principle to conclude that there exist constants $0 < \gamma < 1$, and C , so that if $u(\xi, t)$ is the regular solution to $(\partial_t - L)u = 0$, in $P \times (0, \infty)$ with $u(\xi, 0) = f(\xi) \in \mathcal{C}^0(P)$, then

$$\|u(\cdot, t)\|_{\text{WF}, 0, \gamma} \leq C \|f\|_{\mathcal{C}^0} (t^{-\gamma/2} + t^{-1/2}). \quad (174)$$

Using this for small t and the estimates in [4] for $t \gg 0$, we see that when $\text{Re } \mu > 0$,

$$(\mu - L)^{-1} f = \int_0^\infty e^{-\mu t} u(\cdot, t) dt \in \mathcal{C}_{\text{WF}}^{0, \gamma}(P). \quad (175)$$

In fact there is a constant C_μ so that

$$\|(\mu - L)^{-1} f\|_{\text{WF}, 0, \gamma} \leq C_\mu \|f\|_{\mathcal{C}^0}, \quad (176)$$

which leads immediately to the following.

Corollary 5.1. *Let P be a compact manifold with corners and L is a generalized Kimura diffusion operator defined on P with positive weights. If \overline{L} is the \mathcal{C}^0 -graph closure of L acting on $\mathcal{C}^3(P)$, then for μ with $\text{Re } \mu > 0$, the resolvent operator $(\mu - \overline{L})^{-1}$ is bounded from $\mathcal{C}^0(P)$ to $\mathcal{C}_{\text{WF}}^{0, \gamma}(P)$, and is therefore a compact operator. For initial data in $\mathcal{C}^0(P)$, the regular solution to the initial value problem $\partial_t u - Lu = 0$ has an analytic extension to $\{t : \text{Re } t > 0\}$. The spectrum, $\sigma_{\mathcal{C}^0}(\overline{L})$, lies in a conic neighborhood of $(-\infty, 0]$.*

Proof. Since $u(\cdot, \epsilon) \in \mathcal{C}_{\text{WF}}^{0, \gamma}(P)$, for any $\epsilon > 0$, Theorem 11.2.1 in [4] shows that u extends to be analytic in sets of the form $\{t : \text{Re } t > \epsilon\}$. The analyticity assertion follows from this. Since \overline{L} has a compact resolvent, every point in $\sigma_{\mathcal{C}^0}(\overline{L})$ is an eigenvalue. The eigenvectors belong to $\mathcal{C}^\infty(P)$, and therefore the spectrum of \overline{L} is the same as its spectrum acting on $\mathcal{C}_{\text{WF}}^{0, \gamma}(P)$. In Theorem 11.1.1 of [4] this is shown to lie in a conic neighborhood of $(-\infty, 0]$. \square

5.2 Heat Kernel Estimates

As noted above, the kernel function for the semigroup $e^{tL_{\bar{Q}}}$ takes the form

$$p_t(\xi, \eta)d\mu_L(\eta).$$

Corollary 4.3 indicates that it is reasonable to expect that the kernel p_t is a bounded function for $t > 0$. The kernel function locally satisfies the equations

$$(\partial_t - L_{Q,\xi} + V_{X,\xi})p_t(\xi, \eta) = 0 \text{ and } (\partial_t - L_{Q,\eta} + \widehat{V}_{X,\eta})p_t(\xi, \eta) = 0. \quad (177)$$

The adjoint of the vector field, $\widehat{V}_{X,\eta}$, is computed with respect to the measure $d\mu_L$. It is of the form $-V_X + \widehat{c}$, where, in local coordinates, $\widehat{c} = \text{Div}_{A,b} X$. The operator $\widehat{L} = L_{Q,\eta} - \widehat{V}_{X,\eta}$ is the L^2 -adjoint of $L = L_{Q,\xi} - V_{X,\xi}$.

It is important to note that this representation for the adjoint operator is different from the one employed in [4]. In this paper the semigroup acts on $L^2(P; d\mu_L)$, and the adjoint \widehat{L} is defined with respect to this Hilbert space structure. The operator \overline{L} is defined as the C^0 -graph closure of L acting on $C^3(P)$. The adjoint, \overline{L}^t , acts canonically on the dual space, i.e., the space of regular Borel measures on P . If dV_P is a smooth non-degenerate measure on P , then $d\mu_L = WdV_P$. For v a smooth function, we then have the relation: $\widehat{L}v = W^{-1}\overline{L}^t(Wv)$.

It is clear from the discussion in 4.1 that functions in $C^3(P)$ belong to the domain of the operator $L_{\bar{Q}}$. For $f \in C^3(P)$, let $v(t) = e^{tL_{\bar{Q}}}f$, and let $u(t)$ be the regular solution to $(\partial_t u - Lu) = 0$, with $u(t) = f$, given by Theorem 11.2.1 in [4]. The regularity results in [4] show that $u(t) \in \text{Dom}(L_{\bar{Q}})$ for all $t \in [0, \infty)$. Thus, there is a constant m such that

$$\begin{aligned} \partial_t \|u(t) - v(t)\|_{L^2(P; d\mu_L)}^2 &= 2(L(u(t) - v(t)), (u(t) - v(t)))_{L^2(P; d\mu_L)} \\ &= -2\widetilde{Q}^\Xi(u(t) - v(t), u(t) - v(t)) \\ &\leq m \|u(t) - v(t)\|_{L^2(P; d\mu_L)}^2. \end{aligned} \quad (178)$$

The last line follows using the same argument used to prove (137). This then implies that $u(t) = v(t)$ for all $t \geq 0$. Hence, if $p_t(\xi, \eta)$ is kernel for $e^{tL_{\bar{Q}}}$, then the regular solution is given by

$$u(\xi, t) = \int_P f(\eta)p_t(\xi, \eta)d\mu_L(\eta). \quad (179)$$

That is, the heat kernel defined by the L^2 -semigroup is the same as that defined by the C^0 -theory.

In light of (166) and (169), the argument used to prove Corollary 4.3 can easily be adapted to prove the following upper bound on the heat kernel.

Theorem 5.2. *Assume that P is a compact manifold with corners and L is a generalized Kimura diffusion defined on P with positive weights. If we represent the kernel of the operator e^{tL} as $p_t(\xi, \eta)d\mu_L(\eta)$, then there are positive constants C_0, C_1, C_2 so that, for all $t > 0$ and pairs $\xi, \eta \in P$ we have*

$$p_t(\xi, \eta) \leq \frac{C_0 \exp\left(-\frac{\rho_i^2(\xi, \eta)}{C_2 t}\right)}{\sqrt{\mu_L(B_{\sqrt{t}}^i(\xi))\mu_L(B_{\sqrt{t}}^i(\eta))}} \times \left(1 + \frac{\rho_i(\xi, \eta)}{\sqrt{t}}\right)^D \cdot \exp(C_1 t). \quad (180)$$

For each $\eta \in P$, the function $(\xi, t) \mapsto p_t(\xi, \eta)$ belongs to $C^\infty(P \times (0, \infty))$.

Proof. All statements have been proved but the last. The first equation in (177) shows that $p_t(\cdot, \eta)$ is a weak solution to a parabolic equation to which Theorem 5.1 applies. This proves the last assertion. \square

In a neighborhood, $U \times U$, of $(0, 0)$ in the product coordinate chart $S_{n,m} \times S_{n,m} \simeq S_{2n,2m}$, the heat kernel satisfies the equation

$$(2\partial_t - L_{Q,\xi} - L_{Q,\eta} + V_{X,\xi} - V_{X,\eta})p_t(\xi, \eta) = -\widehat{c}(\eta)p_t(\xi, \eta) \quad (181)$$

If we extend our analysis slightly to include the inhomogeneous problem, and apply a bootstrap argument, then we can easily show that if the weights are constant (so that \widehat{c} and V_X are smooth), then

$$p_t(\xi, \eta) \in C^\infty(U \times U \times (0, \infty)). \quad (182)$$

Corollary 5.2. *Let P be a manifold with corners and L a generalized Kimura operator with smooth coefficients defined on P . Assume that L has constant weights along ∂P . Let \tilde{Q} be a globally defined (but possibly non-symmetric) Dirichlet form with measure $d\mu_L$ that defines L . The heat kernel for L has a representation as $p_t(\xi, \eta)d\mu_L(\eta)$, where*

$$p_t \in C^\infty(P \times P \times (0, \infty)). \quad (183)$$

As noted in the proof of Corollary 5.1, the spectrum of L acting on $C^0(P)$ agrees with its spectrum acting on $C_{\text{WF}}^{0,\gamma}(P)$ for any $0 < \gamma < 1$. Hence, Corollary 12.3.3 in [4] implies that there is an $\theta < 0$ so that

$$\text{spec}_{C^0(P)}(L) \setminus \{0\} \subset \{\mu : \text{Re } \mu < \theta\}. \quad (184)$$

We can also conclude that the constant functions span the null-space of L , and that there is a probability measure of the form $\nu = w(\eta)d\mu_L$ spanning the nullspace of

L^t . In fact the proof of Corollary 12.3.3 in [4] shows that if $(\partial_t - L)u = 0$ with $u(\xi, 0) = f(\xi)$, then

$$u(\xi, t) = \nu(f) + O(e^{\theta t}). \quad (185)$$

If the weights are not constant, then the heat kernel $p_t(\xi, \eta)$ is no longer smooth in the product space for $t > 0$. From Theorem 5.1 it follows that $p_t(\cdot, \eta) \in C^\infty(P)$ for fixed η and $t > 0$. In general it has a complicated singularity as η tends to ∂P . The stationary distribution ν is the push-forward of this kernel

$$\nu(\eta) = \left[\int_P p_t(\xi, \eta) d\sigma(\xi) \right] d\mu_L(\eta). \quad (186)$$

The singularities of $p_t(\xi, \eta)$, beyond those arising from the measure $d\mu_L(\eta)$, produce higher order terms in an asymptotic expansion of $\nu(\eta)$ as $q \rightarrow \partial P$.

To illustrate this we consider a simple 2d-case where the weights are non-constant; the Kimura operator is:

$$L = x\partial_x^2 + \partial_y^2 + b(y)\partial_x, \text{ with } b(y) > \beta > 0 \text{ and } b'(0) \neq 0, \quad (187)$$

which implies that $d\mu_L(\tilde{x}; \tilde{y}) = \tilde{x}^{b(\tilde{y})-1} d\tilde{x} d\tilde{y}$. Working formally one easily shows that the asymptotic expansion of ν takes the form:

$$\nu(\tilde{x}; \tilde{y}) \sim \left[1 + \sum_{j=1}^{\infty} \sum_{k=0}^{2j} \varphi_{jk}(\tilde{y}) \tilde{x}^j \log^k \tilde{x} \right] d\mu_L(\tilde{x}; \tilde{y}), \quad (188)$$

where the first few coefficients are given by

$$\begin{aligned} \varphi_{12}(\tilde{y}) &= -\frac{b'(\tilde{y})^2}{b(\tilde{y})} \\ \varphi_{11}(\tilde{y}) &= \frac{b''(\tilde{y})}{b(\tilde{y})} - 2 \left(\frac{b'(\tilde{y})}{b(\tilde{y})} \right)^2 \\ \varphi_{10}(\tilde{y}) &= (1 + b(\tilde{y})) \frac{b''(\tilde{y})}{b(\tilde{y})^2} - 2(2 + b(\tilde{y})) \frac{b'(\tilde{y})^2}{b(\tilde{y})^3}. \end{aligned} \quad (189)$$

This indicates the additional complexities one expects to see in this case.

5.3 Eigenvalue Asymptotics

If L is a generalized Kimura diffusion, with positive weights, defined by a possibly non-symmetric Dirichlet form on a compact manifold with corners, then the

operators e^{tL} are trace class for all $t > 0$. This does not require the weights to be constant. The estimate in (180) shows that, for all positive times, the heat kernels are square integrable with respect to the finite measure $d\mu_L$. Since $e^{tL} = e^{\frac{tL}{2}} e^{\frac{tL}{2}}$, it follows that e^{tL} is a product of Hilbert-Schmidt operators and therefore trace class. We do not pursue the non-symmetric case further here, as heat kernel asymptotics do not generally lead to eigenvalue asymptotics unless the spectrum is real.

Assuming now that $L = L_Q$ is defined globally by a symmetric Dirichlet form, it follows that the Friedrichs extension of L_Q is an unbounded self adjoint operator acting on $L^2(P; d\mu_b)$. As noted above, the trace of the heat semigroup, $\text{tr}(e^{tL})$ is finite for $\text{Re } t > 0$. From this it follows immediately that L has a compact resolvent acting on $L^2(P)$. Let $0 = \lambda_1 < \lambda_2 \leq \lambda_2 \leq \dots$ be the spectrum of $-L_Q$. Define the counting function

$$N(\lambda) = |\{i : \lambda_i \leq \lambda\}|. \quad (190)$$

Let $d = \dim P$. The heat kernel estimates in local coordinate charts (158) show that

$$\lim_{t \rightarrow 0^+} t^{\frac{d}{2}} \text{tr}(e^{tL}), \quad (191)$$

exists; the standard Tauberian argument then gives asymptotics for $N(\lambda)$ as $\lambda \rightarrow \infty$.

Theorem 5.3. *Let P be a compact manifold with corners of dimension d , and L a Kimura operator, self adjoint on $L^2(P; d\mu_P)$, defined by a globally defined symmetric Dirichlet form. Assume that the weights are strictly positive. The heat kernel e^{tL} is a trace class operator, and there is a dimensional constant K_d so that*

$$\lim_{t \rightarrow 0^+} t^{\frac{d}{2}} \text{tr } e^{tL} = K_d \mu_L(P). \quad (192)$$

The counting function $N(\lambda)$, for the eigenvalues of L , satisfies the asymptotic relation

$$N(\lambda) = \lambda^{\frac{d}{2}} \left(\frac{K_d \mu_L(P)}{\Gamma(1 + d/2)} + o(1) \right). \quad (193)$$

Remark 5.1. Note that this theorem does not require the assumption that the weights are constant along the boundary.

Proof. Let $p_t(\xi, \eta)$ be the heat kernel. For $\delta > 0$ define

$$P_\delta = \{\xi \in P : \rho_i(\xi, \partial P) \geq \delta\}. \quad (194)$$

Proposition 2.32 in [1] implies that the trace is given by the formula

$$\text{tr}(e^{tL}) = \int_L p_t(\xi, \xi) d\mu_L(\xi). \quad (195)$$

For $\delta > 0$, $L \upharpoonright_{P_\delta}$ is uniformly elliptic with smooth coefficients and therefore it follows from a classical argument that

$$p_t(\xi, \xi) = \frac{K_d}{t^{\frac{d}{2}}} + O\left(\frac{1}{t^{\frac{d}{2}-1}}\right), \quad (196)$$

with $t^{\frac{d}{2}}p_t(\xi, \xi)$ converging to K_d uniformly in P_δ , see [1]. Because $p_t(\xi, \xi)$ is non-negative we see that

$$\begin{aligned} \liminf_{t \rightarrow 0^+} t^{\frac{d}{2}} \operatorname{tr}(e^{tL}) &\geq \liminf_{t \rightarrow 0^+} \int_{P_\delta} t^{\frac{d}{2}} p_t(\xi, \xi) d\mu_L(\xi) \\ &= K_d \mu_L(P_\delta) = K_d \mu_L(P) - a_\delta, \end{aligned} \quad (197)$$

where $a_\delta \rightarrow 0$ as $\delta \rightarrow 0$.

On the other hand, the following lemma gives an estimate for

$$t^{\frac{d}{2}} \int_{P \setminus P_\delta} p_t(\xi, \xi) d\mu_L(\xi). \quad (198)$$

Lemma 5.1. *Suppose that $d\mu_L$ is a measure defined on P , a compact manifold with corners, with weights $\{b_j(\eta)\}$ bounded below by $\beta_0 > 0$, and \mathcal{C}^1 in the square-root variables. There is a constant C depending on the dimension, β_0 , and the upper bound β_1 on $\|\nabla b_j(w, y)\|$, so that*

$$t^{\frac{d}{2}} \int_{P \setminus P_\delta} p_t(\xi, \xi) d\mu_L(\xi) \leq C\delta \quad (199)$$

Remark 5.2. As before, we could replace the regularity assumption in this lemma with (51). For simplicity we use the bound on the gradient in the following argument.

Proof. To prove the lemma we cover $P \setminus P_\delta$ by a finite collection of coordinate charts, in which the Dirichlet form defining L takes the form given in the square root coordinates $(w_1, \dots, w_n; y_1, \dots, y_m)$ with $m + n = d$, in (33). The measure takes the form

$$d\mu_L(w; y) = \prod_{i=1}^n w_i^{2b_i(w; y)-1} e^{U(w; y)} dw dy, \quad (200)$$

where U is a bounded, continuous function and

$$0 < \beta_0 < b_i(w, y) < B, \text{ and } \sum_{j=1}^n |\partial_{w_j} b(w; y)| + \sum_{l=1}^m |\partial_{y_l} b_i(w; y)| \leq \beta_1. \quad (201)$$

Theorem 5.2 now gives

$$p_t((w; y), (w; y)) \leq \frac{C}{\mu_L(B_{\sqrt{t}}^i(w; y))}. \quad (202)$$

To prove the lemma we need to bound $\mu_L(B_{\sqrt{t}}^i(w; y))$ from below. To that end we observe that there is a positive constant η so that

$$\begin{aligned} B_{\eta\sqrt{t}}^{p,e}(w; y) &= \prod_{j=1}^n [(w_j - \eta\sqrt{t}) \vee 0, w_j + \eta\sqrt{t}] \times \prod_{l=1}^m [y_l - \eta\sqrt{t}, y_l + \eta\sqrt{t}] \\ &\subset B_{\sqrt{t}}^i(w; y) \end{aligned} \quad (203)$$

and therefore it suffices to bound $\mu_L(B_{\eta\sqrt{t}}^{p,e}(w; y))$ from below.

To bound the contribution to the trace from $P \setminus P_\delta$ we integrate in these coordinates over sets of the form

$$\left\{ [0, \delta] \times [0, \frac{1}{2}]^{n-1} \cup [0, \frac{1}{2}] \times [0, \delta] \times [0, \frac{1}{2}]^{n-2} \cup \dots \cup [0, \frac{1}{2}]^{n-1} \times [0, \delta] \right\} \times (-1, 1)^m. \quad (204)$$

Indeed, it obviously suffices to estimate the contribution from the first term: $[0, \delta] \times [0, \frac{1}{2}]^{n-1} \times (-1, 1)^m$. We first get a lower bound on

$$\begin{aligned} \mu_L(B_{\eta\sqrt{t}}^{p,e}(w; y)) &= \\ &\int_{y_1 - \eta\sqrt{t}}^{y_1 + \eta\sqrt{t}} \dots \int_{y_m - \eta\sqrt{t}}^{y_m + \eta\sqrt{t}} \int_{(w_1 - \eta\sqrt{t}) \vee 0}^{w_1 + \eta\sqrt{t}} \dots \int_{(w_n - \eta\sqrt{t}) \vee 0}^{w_n + \eta\sqrt{t}} \tilde{w}_1^{2b_1(\tilde{w}; \tilde{y})-1} \dots \tilde{w}_n^{2b_n(\tilde{w}; \tilde{y})-1} d\tilde{w} d\tilde{y} \end{aligned} \quad (205)$$

Writing

$$b_j(\tilde{w}; \tilde{y}) = b_j(\tilde{w}; \tilde{y}) - b_j(w; y) + b_j(w; y), \quad (206)$$

then within the domain of this integral,

$$b_j(\tilde{w}; \tilde{y}) \leq b_j(w; y) + \eta\sqrt{t}\beta_1. \quad (207)$$

Since the coordinates $\{\tilde{w}_j\}$ are less than 1 in the domain of the integral (at least for small t), we have the estimate

$$\begin{aligned} \mu_L(B_{\eta\sqrt{t}}^{p,e}(w; y)) &\geq \\ &(2\eta\sqrt{t})^m \int_{(w_1 - \eta\sqrt{t}) \vee 0}^{w_1 + \eta\sqrt{t}} \dots \int_{(w_n - \eta\sqrt{t}) \vee 0}^{w_n + \eta\sqrt{t}} \tilde{w}_1^{2b_1(w; y) - 1 + \alpha} \dots \tilde{w}_n^{2b_n(w; y) - 1 + \alpha} d\tilde{w} d\tilde{y}, \end{aligned} \quad (208)$$

where

$$\alpha = 2\eta\beta_1\sqrt{t}. \quad (209)$$

This shows that

$$\begin{aligned} \mu_L(B_{\eta\sqrt{t}}^{p,e}(w; y)) &\geq \\ (2\eta\sqrt{t})^m \frac{\prod_{j=1}^n [(w_j + \eta\sqrt{t})^{2b_j(w;y)+\alpha} - ((w_j - \eta\sqrt{t}) \vee 0)^{2b_j(w;y)+\alpha}]}{\prod_{j=1}^n (2b_j(w; y) + \alpha)}. \end{aligned} \quad (210)$$

Using this estimate in the integral we can show that there is a constant, C , depending only of the dimension, β_0 and β_1 so that

$$\int_{(-1,1)^m} \int_0^\delta \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \frac{d\mu_L(w; y)}{\mu_L(B_{\eta\sqrt{t}}^{p,e}(w; y))} \leq \frac{C\delta t^{-n\eta\beta_1\sqrt{t}}}{t^{\frac{d}{2}}}. \quad (211)$$

Since

$$\sup_{0 < t < 1} t^{-n\eta\beta_1\sqrt{t}} \leq e^{2n\eta\beta_1 e^{-2}}, \quad (212)$$

the lemma follows easily from this estimate and those above. \square

Using the lemma we see that

$$\limsup_{t \rightarrow 0^+} t^{\frac{d}{2}} \text{tr}(e^{tL}) \leq K_d \mu_L(P) + C\delta - a_\delta. \quad (213)$$

Letting $\delta \rightarrow 0$, we conclude that

$$\lim_{t \rightarrow 0^+} t^{\frac{d}{2}} \text{tr}(e^{tL}) = K_d \mu_L(P). \quad (214)$$

Since we can rewrite the trace as

$$\text{tr}(e^{tL}) = \int_0^\infty e^{-\lambda t} dN(\lambda), \quad (215)$$

the Tauberian theorem, see [7], implies that

$$N(\lambda) = \lambda^{\frac{d}{2}} \left(\frac{K_d \mu_L(P)}{\Gamma(1 + N/2)} + o(1) \right). \quad (216)$$

\square

In the 1-dimensional case the operator takes the form:

$$Lu = x(1-x)\partial_x^2 u + b(x)\partial_x u. \quad (217)$$

If $b \in \mathcal{C}^1([0, 1])$, then this operator is defined by the Dirichlet form

$$Q(u, v) = \int_0^1 x(1-x)\partial_x u \partial_x v x^{b_0-1}(1-x)^{b_1-1} e^{U(x)} dx, \quad (218)$$

where

$$b_0 = b(0), \quad b_1 = b(1) \quad \text{and} \quad \partial_x U(x) = \frac{b(x) - (b_0(1-x) - b_1x)}{x(1-x)}. \quad (219)$$

The operator can therefore be expressed as

$$Lu = x^{1-b_0}(1-x)^{1-b_1} e^{-U(x)} \partial_x \left(x^{b_0}(1-x)^{b_1} e^{U(x)} \partial_x \right) u. \quad (220)$$

From this formulation it is clear that the stationary distribution, defined as the unique probability measure ν satisfying $L^t \nu = 0$ is

$$\nu = c_0 x^{b_0-1} (1-x)^{b_1-1} e^{U(x)} dx, \quad (221)$$

where c_0 is chosen so that $\nu([0, 1]) = 1$.

This discussion applies equally well in higher dimensions. Suppose that P is a domain in \mathbb{R}^p , and that the operator L is globally defined by the Dirichlet form

$$Q(u, v) = \int_P \langle A \nabla u, \nabla v \rangle e^U W_{\mathbf{b}}(z) dz. \quad (222)$$

We assume that in local coordinates $W_{\mathbf{b}}$ takes the form

$$W_{\mathbf{b}}(x; y) = e^{w(x; y)} x_1^{b_1-1} \dots x_n^{b_n-1}, \quad (223)$$

with $w(x; y)$ a smooth function. Integrating by parts formally, we see that

$$Lu = W_{\mathbf{b}}^{-1} e^{-U} \nabla \cdot (e^U W_{\mathbf{b}} A \nabla u), \quad (224)$$

and therefore

$$L^t \nu = \nabla \cdot (e^U W_{\mathbf{b}} A \nabla W_{\mathbf{b}}^{-1} e^{-U} \nu). \quad (225)$$

From this it is evident that if $\nu = W_{\mathbf{b}} e^U dz$, then

$$L^t \nu = 0. \quad (226)$$

Under our usual assumption that each $b_i > \beta_0 > 0$, the measure ν is finite and therefore can be normalized to define the stationary distribution for L^t . This statement remains correct whether or not the $\{b_i\}$ are constants, though if they are not, then the resulting operator is not a standard generalized Kimura diffusion operator. Note that we have shown in [4] that, for generalized Kimura diffusion operators, there is a unique stationary distribution whenever $\mathbf{b} \geq \beta \mathbf{e} > 0$. See also [9, pg. 189].

In a forthcoming paper by the first author and Camelia Pop, [6], we give a probabilistic approach to handling the logarithmically divergent perturbations, and an independent proof of the Harnack inequality for generalized Kimura diffusion operators. This paper also establishes various properties of the Markov processes defined by these operators, and the solutions of the corresponding systems of SDEs. Pop has further analyzed the probabilistic aspects of Kimura diffusions in two additional papers, [13, 12], establishing among other things, that the Feynman-Kac and Girsanov formulæ can be used to represent solutions to these diffusion equations.

The results of this paper represent considerable progress in our understanding of the qualitative properties of solutions to Kimura diffusion equations, its kernel function, and the relationship between L and L^t , at least when the weights are positive. The main outstanding analytic questions seem to be:

1. What is the structure of the heat kernel, and what are the estimates for solutions of the parabolic problem when the weights vanish at some points of ∂P ? For biological applications it is reasonable to consider cases where the weights vanish on hypersurface boundary components, or on components of lower dimensional strata of the boundary. In these cases the measure $d\mu_L$ may not be finite.
2. For non-constant weights, what is the detailed behavior of the heat kernel near the incoming face?
3. What does the size of the gap in the spectrum around 0 depend upon?
4. Under what conditions is the span of the eigenfunctions of L dense in $C^0(P)$?

A Lemmas for the Proof of Theorem 3.1

The argument begins with a series of geometric lemmas, see [14] or [8], which we now recall. We let $E = B_r^i(w_0; y_0)$, and B_0 denote a ρ_∞^e -ball in the set \mathcal{F} , which we call a covering though really $\{2B : B \in \mathcal{F}\}$ is a covering, so that $(w_0; y_0) \in 2B_0$. Recall that $(w_0; y_0)$ is the center of the ball E . Let $B \in \mathcal{F}$ be another ball in the covering, with center $(w; y)$. We let γ_B denote the Euclidean

geodesic from $(w; y)$ to the center of B_0 . The following geometric lemma is proved in [14].

Lemma A.1. *For any $B \in \mathcal{F}$ we have that*

$$d(\gamma_B, \partial E) \geq \frac{1}{2}d(B, \partial E) = \frac{10^3}{2}r(B). \quad (227)$$

Moreover, any ball B' in \mathcal{F} such that $2B'$ intersects γ_B has radius bounded below by

$$r(B') \geq \frac{1}{4}r(B). \quad (228)$$

An important feature of this argument is the construction of a chain of balls in \mathcal{F} that join a given ball $B \in \mathcal{F}$ to the central ball, B_0 . For each $B \in \mathcal{F}$ we let $\mathcal{F}(B) = \{B_0, B_1, \dots, B_{\ell(B)-1}\}$, where $B_{\ell(B)-1} = B$, and

$$\overline{2B_i} \cap \overline{2B_{i+1}} \neq \emptyset. \quad (229)$$

This chain is constructed by following the intersections of the doubles of the balls along the geodesic γ_B . The details are in [14]. There are two further geometric properties of this cover that were already proved in [8].

Lemma A.2. *For any $B \in \mathcal{F}$ and two consecutive balls B_i, B_{i+1} in $\mathcal{F}(B)$ we have that $B_{i+1} \subset 4B_i$ and the estimate*

$$(1 + 10^{-2})^{-1}r(B_i) \leq r(B_{i+1}) \leq (1 + 10^{-2})r(B_i). \quad (230)$$

Moreover, there is a constant c independent of B , so that

$$\mu_{\mathbf{b}}(4B_i \cap 4B_{i+1}) \geq c \max\{\mu_{\mathbf{b}}(B_i), \mu_{\mathbf{b}}(B_{i+1})\}, \quad (231)$$

where

$$\mu_{\mathbf{b}}(B) = \int_B d\mu_{\mathbf{b}}(\tilde{w}; \tilde{y}). \quad (232)$$

Finally

Lemma A.3. *For any ball $B \in \mathcal{F}$ and any ball $A \in \mathcal{F}(B)$ we have that $B \subset 10^4 A$.*

The remainder of the proof of Theorem 3.1 proceeds very much as in [14], though our argument is a little simpler. For consecutive balls B_i, B_{i+1} in a chain, $\mathcal{F}(B)$, we need to compare the mean values $u_{4B_i}, u_{4B_{i+1}}$.

Lemma A.4. *Under the assumptions of Theorem 3.1, there exists a constant C independent of \mathcal{F} so that for any consecutive balls B_i, B_{i+1} in a chain, $\mathcal{F}(B)$, for $B \in \mathcal{F}$ we have the estimate*

$$|u_{4B_i} - u_{4B_{i+1}}| \leq C \frac{r(B_i)}{\sqrt{\mu_{\mathbf{b}}(B_i)}} \left(\int_{16B_i} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}}. \quad (233)$$

Proof. Following Saloff-Coste, we write

$$\begin{aligned} [\mu_{\mathbf{b}}(4B_i \cap 4B_{i+1})]^{\frac{1}{2}} |u_{4B_i} - u_{4B_{i+1}}| &= \left(\int_{4B_i \cap 4B_{i+1}} |u_{4B_i} - u_{4B_{i+1}}|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} \\ &\leq \left(\int_{4B_i \cap 4B_{i+1}} |u - u_{4B_i}|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} + \left(\int_{4B_i \cap 4B_{i+1}} |u - u_{4B_{i+1}}|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} \\ &\leq Cr(B_i) \left(\int_{4B_i} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} + Cr(B_{i+1}) \left(\int_{4B_{i+1}} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}}. \end{aligned} \quad (234)$$

We use Corollary 3.1 to pass from the second line to the third. The conclusion now follows from the Lemma A.2. \square

Recall that the maximal function is defined by

$$M_r f(x) = \sup_{\{B: x \in B, r(B) < r\}} \frac{1}{\mu_{\mathbf{b}}(B)} \int_B |f| d\mu_{\mathbf{b}} \quad (235)$$

Since $d\mu_{\mathbf{b}}$ is a doubling measure, it satisfies a maximal inequality. For $1 \leq p \leq \infty$, and $1 \leq K$ there is a constant $C(p, K)$ so that, for all $f \in C_0^\infty(S_{n,m})$

$$\|M_r f\|_{L^p(S_{n,m}; d\mu_{\mathbf{b}})} \leq C(p, K) \|f\|_{L^p(S_{n,m}; d\mu_{\mathbf{b}})}. \quad (236)$$

The maximal inequality has the following remarkable consequence:

Lemma A.5. *Fix $0 < R, 1 \leq K$ and $1 \leq p < \infty$, There is a constant $C(p, K)$ so that, for any sequence of balls $\{B_i\}$ of radius at most R , and any sequence of non-negative numbers $\{a_i\}$ we have the estimate*

$$\left\| \sum_i a_i \chi_{KB_i} \right\|_{L^p(S_{n,m}; d\mu_{\mathbf{b}})} \leq C(p, K) \left\| \sum_i a_i \chi_{B_i} \right\|_{L^p(S_{n,m}; d\mu_{\mathbf{b}})}. \quad (237)$$

Proof of Theorem 3.1. We can now give the final estimates needed to prove the theorem. Recall that $E = B_r(w_0; y_0)$ is a ball in $S_{n,m}$ and \mathcal{F} is a Jerison “covering” as described above. In fact the balls in \mathcal{F} are disjoint and $\{2B : B \in \mathcal{F}\}$ is a covering. We let B_0 be a ball in \mathcal{F} so that $(w_0; y_0) \in 2B_0$. For any ball $B \in \mathcal{F}$, we let $\mathcal{F}(B) = (B_0, B_1, \dots, B_{l(B)-1})$ denote a chain joining $B = B_{l(B)-1}$ to B_0 .

As $E = \cup_{B \in \mathcal{F}} 2B$, we see that

$$\begin{aligned} \int_E |u - u_{4B_0}|^2 d\mu_{\mathbf{b}} &\leq \sum_{B \in \mathcal{F}_{2B}} \int |u - u_{4B_0}|^2 d\mu_{\mathbf{b}} \\ &\leq 4 \sum_{B \in \mathcal{F}_{4B}} \int (|u - u_{4B}|^2 + |u_{4B} - u_{4B_0}|^2) d\mu_{\mathbf{b}} \\ &4 \sum_{B \in \mathcal{F}} \left[Cr(4B)^2 \int_{4B} |\nabla u|^2 d\mu_{\mathbf{b}} + |u_{4B} - u_{4B_0}|^2 d\mu_{\mathbf{b}}(4B) \right]. \end{aligned} \quad (238)$$

We use Corollary 3.1 to pass from the second to the third line. Since $4B \subset E$, we can use (94) to conclude that there is a constant C_0 so that

$$\sum_{B \in \mathcal{F}} r(4B)^2 \int_{4B} |\nabla u|^2 d\mu_{\mathbf{b}} \leq C_0 r^2 \int_E |\nabla u|^2 d\mu_{\mathbf{b}}. \quad (239)$$

The next step is to establish a similar estimate for

$$I = \sum_{B \in \mathcal{F}_{4B}} \int |u_{4B} - u_{4B_0}|^2 d\mu_{\mathbf{b}}. \quad (240)$$

Using Lemma A.4 we obtain the estimate

$$|u_{4B} - u_{4B_0}| \leq \sum_{i=0}^{l(B)-1} |u_{4B_i} - u_{4B_{i+1}}| \leq C \sum_{i=0}^{l(B)-1} \frac{r(B_i)}{\sqrt{\mu_{\mathbf{b}}(B_i)}} \left(\int_{16B_i} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}}. \quad (241)$$

According to Lemma A.3 the ball B is contained in $10^4 B_i$ for any i and therefore

$$|u_{4B} - u_{4B_0}| \chi_B \leq C \sum_{A \in \mathcal{F}} \frac{r(A)}{\sqrt{\mu_{\mathbf{b}}(A)}} \left(\int_{16A} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} \chi_{10^4 A \chi_B}. \quad (242)$$

As the balls in \mathcal{F} are disjoint, we see that

$$\sum_{B \in \mathcal{F}} |u_{4B} - u_{4B_0}|^2 \chi_B \leq C \left| \sum_{A \in \mathcal{F}} \frac{r(A)}{\sqrt{\mu_{\mathbf{b}}(A)}} \left(\int_{16A} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} \chi_{10^4 A} \right|^2. \quad (243)$$

Once again, since the balls in \mathcal{F} are disjoint, we can apply Lemma A.5 to conclude that there is a constant C_2 so that

$$\begin{aligned} \int_E \sum_{B \in \mathcal{F}} |u_{4B} - u_{4B_0}|^2 \chi_B d\mu_{\mathbf{b}} &\leq C_2 \int_E \left| \sum_{A \in \mathcal{F}} \frac{r(A)}{\sqrt{\mu_{\mathbf{b}}(A)}} \left(\int_{16A} |\nabla u|^2 d\mu_{\mathbf{b}} \right)^{\frac{1}{2}} \chi_A \right|^2 d\mu_{\mathbf{b}} \\ &\leq C_2 \int_E \sum_{A \in \mathcal{F}} \frac{r^2(A)}{\mu_{\mathbf{b}}(A)} \left(\int_{16A} |\nabla u|^2 d\mu_{\mathbf{b}} \right) \chi_A d\mu_{\mathbf{b}}. \end{aligned} \quad (244)$$

The key point here is that, in the first line, we have replaced the characteristic functions $\{\chi_{10^4 A}\}$ in the inner sum with $\{\chi_A\}$. We pass to the second line by using the fact that the balls in \mathcal{F} are disjoint. Using the doubling property one last time it follows that

$$I \leq C r^2 \int_E |\nabla u|^2 d\mu_{\mathbf{b}}. \quad (245)$$

Combining our results thus far, we have shown that that there is a constant C' so that

$$\int_E |u - u_{B_0}|^2 d\mu_{\mathbf{b}} \leq C' r^2 \int_E |\nabla u|^2 d\mu_{\mathbf{b}}. \quad (246)$$

The fact that the integral $\int_E |u - a|^2 d\mu_{\mathbf{b}}$, where $a \in \mathbb{R}$, is minimized when $a = u_E$, and the uniform equivalence of the metrics ρ_{∞}^e and ρ_i complete the proof of the theorem. \square

B Lemmas for Section 4.3

In this section we give analogues for Theorem 5.2.9, Theorem 5.2.16, Theorem 5.2.17 and Lemma 5.4.1 in [14]. These are the ingredients needed to apply Saloff-Coste's proof of the Harnack inequality for non-negative solutions to $\partial_t u - (L_Q - V_x - c)u = 0$, and certain estimates for the heat kernel. These results are largely

consequences of the following two estimates: 1. for balls $B_r^i(x; y)$ we have the doubling estimate:

$$\mu_{\mathbf{b}}(B_{2r}^i(x; y)) \leq 2^D \mu_{\mathbf{b}}(B_r^i(x; y)), \quad (247)$$

and 2. for all functions in $\mathcal{D}(Q_{B_r^i(x; y)})$ we have the Sobolev inequality:

$$\left[\int_{B_r^i(x; y)} |u|^{\frac{2D}{D-2}} d\mu_{\mathbf{b}}(\tilde{x}, \tilde{y}) \right]^{\frac{D-2}{D}} \leq C_S \frac{r^2}{[\mu_{\mathbf{b}}(B_r^i(x; y))]^{\frac{2}{D}}} \left[Q_{B_r^i(x; y)}(u, u) + \frac{1}{r^2}(u, u)_{\mathbf{b}} \right]. \quad (248)$$

We first begin with an argument due to Moser, and appearing, in the elliptic case, as part of the proof of Lemma 2.2.1 in [14], to show that weak nonnegative subsolutions to the equation $\partial_t u = (L_Q - V_X - c)u$ are locally bounded. We allow the vector field V_X and the scalar potential c to be somewhat singular. Assume that there is a $k \in \mathbb{N}$, and a constant M so that the coefficients satisfy the estimate in (125). This generality is essential for the applications to population genetics. We give a fairly detailed proof of this statement.

For $0 < r, \delta, s \in \mathbb{R}$, and $q \in S_{n,m}$ we let

$$W_r(s, q) = (s - r^2, s) \times B_r^i(q), \text{ and } W(\delta) = (s - \delta r^2, s) \times B_{\delta r}^i(q). \quad (249)$$

Lemma B.1. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(x; y)$. Suppose that $X(x; y) \in C^0(\text{int } S_{n,m}; \mathbb{R}^{n+m})$ is constant outside of a compact set, and $c(x; y)$ is a measurable function supported in a compact set, both of which satisfy (125). There is a constant C_1 that depends only on the doubling dimension, D , so that with $0 < \delta < 1$, and $r < R$, for u a non-negative weak subsolution of*

$$\partial_t u = (L_Q - V_X - c)u \quad (250)$$

in $W_r(s, q)$, we have the estimates, for $0 < p$:

$$\sup_{W(\delta)} u^p \leq \frac{C_1}{(1 - \delta)^{D+2} r^2 \mu_{\mathbf{b}}(B_r^i(q))} \iint_{W(1)} u^p d\mu_{\mathbf{b}} dt. \quad (251)$$

In particular, a weak subsolution is bounded for positive times.

Remark B.1. This is part of Theorem 2.1 in [19] in a more general context, but one that would require additional hypotheses in the present circumstance.

Proof. Recall that u is a weak subsolution in W if $u \in \text{Dom}(Q)$ and for any non-negative function in $\text{Dom}(Q)$ with compact support in W , we have, for a.e. t , that

$$\int_B [u_t \varphi + \langle A \nabla u, \nabla \varphi \rangle + \langle A \nabla u, X \varphi \rangle + c u \varphi] d\mu_{\mathbf{b}}(x; y) \leq 0. \quad (252)$$

Moser's trick to prove this lemma is to use test functions φ of the form $\varphi = \psi^2 G(u)$. Here $\psi \in \mathcal{C}_c^\infty(W)$, and $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a piecewise \mathcal{C}^1 function that satisfies the conditions:

1. $G(r) = ar$, a a positive constant, when r is sufficiently large.
2. $G'(r)$ is non-negative and non-decreasing, which implies
3. $G(r) \leq rG'(r)$.

We then set $H'(r) = \sqrt{G'(r)}$, with $H(0) = 0$. Once again, the mean value theorem implies that $H(r) \leq rH'(r)$. Finally we set

$$K(s) = \int_0^s G(\rho) d\rho, \quad (253)$$

and assume that there is a universal constant, C_0 , so that

$$H^2(r) \leq C_0 K(r), \quad (254)$$

for $r \in [0, \infty)$.

Since $G(r)$ grows linearly for large r , the function $\varphi = \psi^2 G(u)$ is an admissible test function, and therefore:

$$\int_B [\partial_t K(u) \psi^2 + \langle A \nabla u, \nabla \psi^2 G(u) \rangle + \langle A \nabla u, X \psi^2 G(u) \rangle + c u \psi^2 G(u)] d\mu_{\mathbf{b}}(x; y) \leq 0. \quad (255)$$

Using the argument on page 39 of [14] we can show that this implies that there is a constant C_1 so that

$$\begin{aligned} & \int_B [\partial_t K(u) \psi^2 + \frac{1}{4} \langle A \nabla \psi H(u), \nabla \psi H(u) \rangle] d\mu_{\mathbf{b}} \leq \\ & C_1 \int_B [(\langle A \nabla \psi, \nabla \psi \rangle + \psi^2) u^2 G'(u) + (\langle AX, X \rangle + |c|) \psi^2 u G(u)] d\mu_{\mathbf{b}}. \end{aligned} \quad (256)$$

From this point the argument would be standard, but for the fact that the $(\langle AX, X \rangle + |c|)$ -term is not required to be bounded near the boundary. Below we prove a lemma that allows us to handle the contribution from near the singular locus.

Following Moser, and Saloff-Coste, we take:

$$H_N(r) = \begin{cases} r^{\frac{p}{2}} & \text{for } 0 \leq r \leq N \\ N^{\frac{p}{2}-1}r & \text{for } r > N, \end{cases} \quad (257)$$

which implies that

$$G_N(r) = \begin{cases} \frac{p^2}{4(p-1)}r^{p-1} & \text{for } 0 \leq r \leq N \\ N^{p-2}(r-N) + \frac{p^2}{4(p-1)}N^{p-1} & \text{for } r > N, \end{cases} \quad (258)$$

and

$$K_N(r) = \begin{cases} \frac{p}{4(p-1)}r^p & \text{for } 0 \leq r \leq N \\ \frac{N^{p-2}}{2}(r-N)^2 + \frac{p^2}{4(p-1)}N^{p-1}(r-N) + \frac{p}{4(p-1)}N^p & \text{for } r > N. \end{cases} \quad (259)$$

We now show that there is constant C_0 independent of N and $2 \leq p$, so that

$$\frac{K_N(r)}{H_N^2(r)} \geq C_0. \quad (260)$$

For $0 \leq r \leq N$, we have that

$$\frac{K_N(r)}{H_N^2(r)} = \frac{p}{4(p-1)} \geq \frac{1}{4}. \quad (261)$$

For $r > N$, we let $r = N\rho$, and obtain that

$$\frac{K_N(N\rho)}{H_N^2(N\rho)} = \frac{\frac{(\rho-1)^2}{2} + \frac{p^2}{4(p-1)}(\rho-1) + \frac{p}{4(p-1)}}{\rho^2}, \quad (262)$$

from which is it clear that the minimum does not depend on N . A simple calculation shows that the minimum on $[1, \infty)$ is assumed at $\rho = 1$, and therefore (260) holds for $2 \leq p$, with $C_0 = \frac{1}{4}$.

The main new result is in the following lemma:

Lemma B.2. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(\mathbf{x}; \mathbf{y})$, with $0 < \beta_0 < b_j$, constant outside a compact set. Let q be a measurable function defined on $S_{n,m}$ that satisfies*

$$|q(x; y)| \leq M \left[\chi_B(x; y) \sum_{j=1}^n |\log x_j|^k + 1 \right]. \quad (263)$$

for some $k \in \mathbb{N}$, $0 < M$, and B a bounded neighborhood of $(0; 0)$, Given $\eta > 0$ there is a C_η so that we have

$$\int_{S_{n,m}} |q(x; y)| u G(u) d\mu_{\mathbf{b}} \leq \eta \int_{S_{n,m}} \langle A \nabla u, \nabla u \rangle G'(u) d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} u^2 G'(u) d\mu_{\mathbf{b}}, \quad (264)$$

for u a bounded, non-negative, compactly supported function in $\text{Dom}(Q)$.

We give the proof of the lemma below. With this bound we can estimate the contribution of the last term on the right hand side of (256) near the boundary and thereby show that there is a constant, independent of $2 \leq p$, and N so that

$$\int_W [\partial_t K_N(u) \psi^2 + \langle A \nabla \psi H_N(u), \nabla \psi H_N(u) \rangle] d\mu_{\mathbf{b}} \leq C \int_W [\langle A \nabla \psi, \nabla \psi \rangle + \psi^2] u^2 G'_N(u) d\mu_{\mathbf{b}}. \quad (265)$$

We first argue as on page 40, and then as on page 121-2, using the fact that $H_N(u) \leq 4K_N(u)$. Letting $N \rightarrow \infty$ we conclude that, with $\theta = 1 + 2/D$, we have:

$$\iint_{W(\delta)} u^{p\theta} d\mu_{\mathbf{b}} dt \leq \frac{C}{(\delta' - \delta)^{2+D} r^2 \mu_{\mathbf{b}}(B_r)} \left(\frac{A p^2}{r^2 (\delta' - \delta)^2} \iint_{W(\delta')} u^p d\mu_{\mathbf{b}} dt \right)^\theta \quad (266)$$

Starting with $p = 2$ we can iterate this inequality to conclude that u^p is integrable for any $2 \leq p$, and then apply the argument on page 122 of [14] to complete the proof of Lemma B.1 for $p = 2$.

In [14] the fact that u^p is a subsolution if $p > 1$ is employed to use the argument above to complete the proof of the lemma. Since we are allowing lower order terms, we cannot use this argument and use instead an argument given in [11]. We do not give the complete proof, but demonstrate that an exact analogue of the last formula on page 737 of [11] holds in the present context. From that point onward, the conclusion then follows, as in Moser, by employing the Sobolev inequality (110), which holds for Q_B . From the $p = 2$ case we can assume that u is bounded, and therefore $u^p \in \mathcal{D}(Q)$ if $p > 1$.

We let $\varphi = u^{p-1}\psi^2$ in (255). Here ψ is a smooth function that is 1 in $W(\delta)$ and zero outside of $W(1)$. A little algebra shows that the condition in (255) can be re-expressed as:

$$\begin{aligned} \int \left[\frac{1}{4} \partial_t (v^2) \psi^2 + \left(1 - \frac{1}{p} \right) \langle A \nabla v, \nabla v \rangle \psi^2 \right] d\mu_{\mathbf{b}} = \\ - \int \left[\langle A \nabla v, \nabla \psi \rangle v \psi + \frac{1}{2} \langle A \nabla v, X \rangle v \psi^2 + c v^2 \psi^2 \right] d\mu_{\mathbf{b}}, \end{aligned} \quad (267)$$

where $v = u^{\frac{p}{2}}$.

We let

$$\epsilon = \min \left\{ \frac{1}{4} \left| 1 - \frac{1}{p} \right|, \frac{1}{4} \right\}. \quad (268)$$

The Cauchy-Schwarz and arithmetic-geometric inequalities show that

$$|\langle A \nabla v, \nabla \psi \rangle v \psi| \leq \frac{1}{4\epsilon} \langle A \nabla \psi, \nabla \psi \rangle v^2 + \epsilon \langle A \nabla v, \nabla v \rangle \psi^2 \quad (269)$$

and

$$|\langle A \nabla v, X \rangle v \psi^2| \leq \frac{1}{4\epsilon} \langle AX, X \rangle \psi^2 v^2 + \epsilon \langle A \nabla v, \nabla v \rangle \psi^2. \quad (270)$$

This demonstrates that, for $2 < p$,

$$\begin{aligned} \frac{1}{4} \int \partial_t (\psi^2 v^2) d\mu_{\mathbf{b}} + 2\epsilon \int \langle A \nabla v, \nabla v \rangle \psi^2 d\mu_{\mathbf{b}} \\ \leq \frac{1}{4\epsilon} \int [\langle A \nabla \psi, \nabla \psi \rangle + \langle AX, X \rangle \psi^2 + 4\epsilon |c| \psi^2] v^2 d\mu_{\mathbf{b}} + \\ \frac{1}{2} \int |\psi \psi_t| v^2 d\mu_{\mathbf{b}}. \end{aligned} \quad (271)$$

From this point the argument goes very much as in the $p = 2$ case. In particular, we use Lemma B.4 to control the $q\psi^2 v^2$ term, where $q = [\langle AX, X \rangle + 4\epsilon |c|]$, obtaining the estimate

$$\begin{aligned} \frac{1}{4} \int \partial_t (\psi^2 v^2) d\mu_{\mathbf{b}} + \epsilon \int \langle A \nabla v, \nabla v \rangle \psi^2 d\mu_{\mathbf{b}} \\ \leq \frac{C}{4\epsilon} \int [\langle A \nabla \psi, \nabla \psi \rangle + \psi^2] v^2 d\mu_{\mathbf{b}} + \frac{1}{2} \int |\psi \psi_t| v^2 d\mu_{\mathbf{b}}. \end{aligned} \quad (272)$$

After integrating in t this is essentially the same as the estimate at the bottom of page 737 in [11], completing the proof of the lemma for $2 \leq p$.

To obtain the estimate for $0 < p < 2$ we employ the argument used to prove Theorem 2.2.3 in [14]. \square

We still need to prove Lemma B.2:

Proof of Lemma B.2. If q were bounded, i.e. $B = \emptyset$, then the estimate in (264), with $\eta = 0$, would follow from the fact that $G(u) \leq uG'(u)$. To treat the case where $B \neq \emptyset$, we begin with a local version of the lemma:

Lemma B.3. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(x; y)$, with $0 < \beta_0 < b_i$, constant outside a compact set. Let q be a measurable function defined on $S_{n,m}$ so that for some M, B and $0 < k$ it satisfies the estimate (263). Let $(x_0; y_0) \in \partial S_{n,m}$. Given $\eta > 0$ there is a open neighborhood*

$$U_\delta(x_0; y_0) = \{(x; y) \in S_{n,m} : |x_i - x_{0i}| < \delta, y_j \in (-1, 1)\},$$

so that if $\text{supp } \chi \subset U_\delta(x_0; y_0)$, then there is a C_η , independent of u, G , and χ so that

$$\int_{S_{n,m}} \chi^2(x; y) |q(x; y)| u G(u) d\mu_{\mathbf{b}} \leq \eta \int_{S_{n,m}} \langle A \nabla u, \nabla u \rangle G'(u) \chi^2 d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} [\langle A \nabla \chi, \nabla \chi \rangle + \chi^2] u^2 G'(u) d\mu_{\mathbf{b}}, \quad (273)$$

for u a non-negative function in $\text{Dom}(Q)$.

Proof of Lemma B.3. By relabeling, we can assume that

$$x_{01} = \dots = x_{0l} = 0 \text{ and } 0 < x_{0i} \text{ for } i = l+1, \dots, n. \quad (274)$$

The key observation is that for any $\eta' > 0$, and $a > 0$, there is a $\delta_0 > 0$ so that if $|x_i - x_{0i}| < \delta_0$, with $0 < x_i$, for $i = 1, \dots, n$, then

$$M \left[\sum_{i=1}^n |\log x_i|^k + 1 \right] \leq \eta' \sum_{i=1}^n x_i^{-a}. \quad (275)$$

For each $i = 1, \dots, n$, let $\underline{b}_i(x; y) = \min\{b_i(x; y) : x_i \in [0, x_{0i} + \delta_0]\}$; these are Lipschitz functions. Fix a positive number $0 < a < \min\{\beta_0/2, 1/4\}$, and let $0 < \delta_0 < 1/2$ be fixed so that (275) holds, and

$$a + b_i(x; y) - \underline{b}_i(x; y) < \frac{1}{2} \text{ for } (x; y) \in U_{\delta_0}(x_0; y_0). \quad (276)$$

We may need to reduce δ several times, but we first assume that χ is supported in the set $U_{\delta_0}(x_0; y_0)$. Under this assumption we see that

$$\int \chi^2 q u G(u) d\mu_{\mathbf{b}} \leq \eta' \int \chi^2 \left[\sum_{i=1}^n x_i^{-a} \right] u G(u) d\mu_{\mathbf{b}} \leq \eta' \int \chi^2 \left[\sum_{i=1}^n x_i^{-a} \right] u G(u) d\mu_{\underline{\mathbf{b}}}, \quad (277)$$

where $d\mu_{\underline{\mathbf{b}}} = x_1^{b_1} \cdots x_n^{b_n} dx dy$.

Recalling the form of the measure, and the fact that each b_i is Lipschitz and independent of x_i for $0 < x_i < \delta_0$, we can integrate by parts to obtain that

$$\begin{aligned} \int \chi^2 \left[\sum_{i=1}^n x_i^{-a} \right] u G(u) d\mu_{\underline{\mathbf{b}}} &= \sum_{i=1}^n \left[\int \frac{x_i^{b_i-a}}{b_i-a} \partial_{x_i} \left(\chi^2 u G(u) \prod_{k \neq i} x_k^{b_k} \right) dx dy \right] \\ &= \sum_{i=1}^n \left[\int \frac{x_i^{1-(a+b_i-b_i)}}{b_i-a} \left(\partial_{x_i} [\chi^2 u G(u)] + \chi^2 u G(u) \left(\sum_{k \neq i} \partial_{x_i} b_k \log x_k \right) \right) d\mu_{\mathbf{b}} \right] \end{aligned} \quad (278)$$

Choosing a $0 < \delta_1 \leq \delta_0$, we can arrange to have

$$\frac{2}{\beta_0} \sum_{i=1}^n \sum_{k \neq i} |\partial_{x_i} b_k \log x_k| \leq \frac{1}{2} \sum_{i=1}^n x_i^{-a} \text{ if } (x; y) \in U_{\delta_1}(x_0; y_0). \quad (279)$$

Now assuming that $\text{supp } \chi \subset U_{\delta_1}(x_0; y_0)$ these inequalities and (276) imply that

$$\int \chi^2 \left[\sum_{i=1}^n x_i^{-a} \right] u G(u) d\mu_{\mathbf{b}} \leq \frac{4}{\beta_0} \int \left[\sum_{i=1}^n x_i^{\frac{1}{2}} |\partial_{x_i} [\chi^2 u G(u)]| \right] d\mu_{\mathbf{b}}. \quad (280)$$

Using the Cauchy-Schwarz inequality and the properties of G we see that

$$\begin{aligned} \sum_{i=1}^n x_i^{\frac{1}{2}} |\partial_{x_i} [\chi^2 u G(u)]| &\leq \\ [n \chi^2 u^2 G'(u)]^{\frac{1}{2}} &\left[\left(\sum_{i=1}^n x_i (\partial_{x_i} \chi)^2 u^2 G'(u) \right)^{\frac{1}{2}} + \left(\sum_{i=1}^n x_i (\partial_{x_i} u)^2 \chi^2 G'(u) \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (281)$$

From ellipticity hypotheses on $q(\nabla u)$, see (35), it is clear that there is a constant M_1 so that for data, f , supported in a fixed small neighborhood of $\partial S_{n,m}$ we have the estimate

$$\sum_{i=1}^n x_i (\partial_{x_i} f)^2 \leq M_1 \langle A \nabla f, \nabla f \rangle. \quad (282)$$

and therefore

$$\begin{aligned} & \sum_{i=1}^n x_i^{\frac{1}{2}} |\partial_{x_i} [\chi^2 u G(u)]| \leq \\ & [nM_1 \chi^2 u^2 G'(u)]^{\frac{1}{2}} \left[(\langle A \nabla \chi, \nabla \chi \rangle u^2 G'(u))^{\frac{1}{2}} + (\langle A \nabla u, \nabla u \rangle \chi^2 G'(u))^{\frac{1}{2}} \right] \end{aligned} \quad (283)$$

Choosing $0 < \eta'$ sufficiently small, the assertion of the lemma follows easily from this estimate, the arithmetic-geometric mean inequality and the initial estimate (277). \square

The proof of Lemma B.2 follows from the local result and a simple covering argument. The set B , appearing in the estimate (263) is compact. For each $(u; v) \in \partial S_{n,m} \cap B$, the local result provides an open set $U_\delta(u; v)$ in which the estimate (273) holds. By compactness a finite collection $\{U_{\delta_i}(u_i, v_i) : i = 1 \dots, I\}$ covers $\partial S_{n,m} \cap B$. Since $\min\{\delta_1, \dots, \delta_I\} > 0$, we can choose a collection of smooth functions $\{\chi_i\}$ with $\text{supp } \chi_i \subset U_{\delta_i}(u_i, v_i)$, and

$$\tilde{\chi}^2(x; y) = \sum_{i=1}^I \chi_i^2(x; y) = 1 \quad (284)$$

in a neighborhood of $B \cap \partial S_{n,m}$. We therefore obtain that

$$\begin{aligned} & \sum_{i=1}^I \int_{S_{n,m}} q(x; y) \chi_i^2 u G(u) d\mu_{\mathbf{b}} < \\ & \sum_{i=1}^I \left[\eta \int_{S_{n,m}} \langle A \nabla u, \nabla u \rangle G'(u) \chi_i^2 d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} [\langle A \nabla \chi_i, \nabla \chi_i \rangle + \chi_i^2] u^2 G'(u) d\mu_{\mathbf{b}} \right]. \end{aligned} \quad (285)$$

Note that C_η depends only on q . With these choices, q is bounded in the $\text{supp}(1 - \tilde{\chi}^2)$, which completes the proof of the lemma. \square

The proof of this lemma is easily adapted to prove the following result:

Lemma B.4. *Assume that $\mathbf{b} = (b_1, \dots, b_l)$ are positive differentiable functions of $(x; y)$, with $0 < \beta_0 < b_j$, constant outside a compact set. Let q be a measurable*

function defined on $S_{n,m}$ that satisfies (263), for some $k \in \mathbb{N}$, B and M . Given $\eta > 0$ there is a C_η so that for any $2 \leq p$, we have

$$\int_{S_{n,m}} |q(x; y)| u^p(x; y) d\mu_{\mathbf{b}} \leq \eta \int_{S_{n,m}} \langle A \nabla u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}} \rangle d\mu_{\mathbf{b}} + C_\eta \int_{S_{n,m}} u^p d\mu_{\mathbf{b}}, \quad (286)$$

for u a bounded, non-negative, compactly supported function in $\text{Dom}(Q)$.

Remark B.2. It is evident that we can actually prove these results for somewhat more singular potentials, i.e. those satisfying an estimate of the form

$$|q(x; y)| \leq M \left[\sum_{i=1}^n x_i^{-a} + 1 \right], \quad (287)$$

for a constant $a < \beta_0$.

We have an estimate for supersolutions, which is the analogue of Saloff-Coste's Theorem 5.2.16 (or (2.11.c) in [19]):

Lemma B.5. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(x; y)$, satisfying (51), which are constant outside a compact set, $X(x; y)$ is a continuous \mathbb{R}^{n+m} -valued function, satisfying (263), which is constant outside of a compact set, and $c(x; y)$ is a bounded measurable function supported in a compact set. There is a constant $C(p, D)$ that depends on $0 < p$, the doubling dimension, D , and Sobolev constant so that with $0 < \delta < 1$, and $0 < r$, and u a bounded, positive, weak supersolution of*

$$\partial_t u = (L_Q - V_X - c)u \quad (288)$$

in $W_r(s, q)$, satisfies the estimate:

$$\sup_{W(\delta)} u^{-p} \leq \frac{C(p, D)}{(1 - \delta)^{D+2} r^2 \mu_{\mathbf{b}}(B_r^i(q))} \iint_{W(1)} u^{-p} dt d\mu_{\mathbf{b}}, \text{ for } p > 0. \quad (289)$$

Remark B.3. For this result we need to assume that c is bounded, for otherwise we could not begin the argument below by assuming that the supersolution is strictly positive.

Proof. Provided that c is non-negative, we can assume, by replacing u by $u + \epsilon$, that u is strictly positive. If c assumes negative values, then we first replace u by $e^{\mu t} u$, where $\mu > \|c\|_{L^\infty}$, which reduces us to the previous case. For any non-negative function φ with compact support in $B_r^i(q)$, a weak supersolution satisfies

$$\int [u_t \varphi + \langle A \nabla u, \nabla \varphi + \varphi X \rangle + c u \varphi] d\mu_{\mathbf{b}} \geq 0. \quad (290)$$

If we let $\varphi = p\psi^2 u^{-p-1}$, and set $v = u^{-\frac{p}{2}}$, then this is equivalent to

$$- \int \left[\psi^2 \partial_t v^2 + \frac{4(p+1)}{p} \psi^2 \langle A \nabla v, \nabla v \rangle + 4 \langle A \nabla v, \psi \nabla \psi \rangle v + 2\psi^2 \langle A \nabla v, Xv \rangle - pc\psi^2 v^2 \right] d\mu_{\mathbf{b}} \geq 0. \quad (291)$$

Once again, using Lemma B.4 and the Cauchy Schwarz and arithmetic-geometric mean inequalities, we show that there is a constant M for which

$$\int \left[\psi^2 \partial_t v^2 + \left(2 + \frac{4}{p}\right) \psi^2 \langle A \nabla v, \nabla v \rangle \right] d\mu_{\mathbf{b}} \leq M \int \left[(p+1)\psi^2 + \langle A \nabla \psi, \nabla \psi \rangle \right] v^2 d\mu_{\mathbf{b}}. \quad (292)$$

Arguing as above, we see that there is another constant M' so that

$$\int \left[\psi^2 \partial_t v^2 + \langle A \nabla \psi v, \nabla \psi v \rangle \right] d\mu_{\mathbf{b}} \leq M'(p+1) \|\langle A \nabla \psi, \nabla \psi \rangle\|_{L^\infty} \int_{\text{supp } \psi} v^2 d\mu_{\mathbf{b}}. \quad (293)$$

The statement of the lemma now follows from the iteration argument using the Sobolev inequality given on page 129 of [14]. \square

To complete the argument we need to show that the appropriate analogue of Saloff-Coste's Lemma 5.4.1 (which is Moser's Lemma 2) holds. In the present case this reads:

Lemma B.6. *Assume that $\mathbf{b} = (b_1, \dots, b_n)$ are positive differentiable functions of $(x; y)$, satisfying (51), which are constant outside a compact set, $X(x; y)$ is a continuous \mathbb{R}^{n+m} -valued function, satisfying (263), which is constant outside of a compact set, and $c(x; y)$ is a bounded measurable function supported in a compact set. For any weak positive supersolution u of (250) in $W_r(s, q)$, $0 < r < R$, $0 < \eta < 1$, and $0 < \delta < 1$, there is a constant $a(\eta, u)$ so that for all $0 < \lambda$ we have the estimates*

$$\begin{aligned} \mu_{\mathbf{b}} \times dt \{ (x; y, t) \in W_+ : \log u < -\lambda - a \} &\leq C \frac{r^2 \mu_{\mathbf{b}}(B_r^i)}{\lambda} \\ \mu_{\mathbf{b}} \times dt \{ (x; y, t) \in W_- : \log u > \lambda - a \} &\leq C \frac{r^2 \mu_{\mathbf{b}}(B_r^i)}{\lambda}, \end{aligned} \quad (294)$$

where $W_+ = (s - \eta r^2, s) \times B_{\delta r}^i$ and $W_- = (s - r^2, s - \eta r^2) \times B_{\delta r}^i$. Here C is independent of $\lambda > 0$, s , and r .

Proof. As with the proof of the previous lemma, we can assume that u is strictly positive and show that the fundamental inequality used in Saloff-Coste's proof holds in this case as well. The proof in [14] is contained on pages 143-145. We start, as before, with (290) and use the test function $\varphi = \psi^2(x; y)u^{-1}$. Letting $v = -\log u$, this equation takes the form

$$\int [v_t \psi^2 + \langle A \nabla v, \nabla v \rangle + \langle A \nabla v, \nabla \psi^2 \rangle + \langle A \nabla v, \psi^2 X \rangle - c \psi^2] d\mu_{\mathbf{b}} \leq 0. \quad (295)$$

Using the Cauchy-Schwarz and arithmetic-geometric mean inequalities we obtain that

$$\begin{aligned} \partial_t \int \psi^2 v d\mu_{\mathbf{b}} + \frac{1}{2} \int \langle A \nabla v, \nabla v \rangle d\mu_{\mathbf{b}} \\ \leq 8 \int [\langle A \nabla \psi, \nabla \psi \rangle + \psi^2 (\langle AX, X \rangle + |c|)] d\mu_{\mathbf{b}}. \end{aligned} \quad (296)$$

We once again use Lemma B.4 to control the $\langle AX, X \rangle$ -term and show that there is a constant C' independent of ψ, v, W, δ , so that

$$\begin{aligned} \partial_t \int \psi^2 v d\mu_{\mathbf{b}} + \frac{1}{2} \int \langle A \nabla v, \nabla v \rangle d\mu_{\mathbf{b}} \\ \leq C' \int [\langle A \nabla \psi, \nabla \psi \rangle + \psi^2] d\mu_{\mathbf{b}}. \end{aligned} \quad (297)$$

For ψ we use the function $\psi(\tilde{x}; \tilde{y}) = (1 - \rho_i((\tilde{x}; \tilde{y}), (x; y))/r)_+$. As $r < R$, it is clear that there is a constant C so that the right hand side in (296) is bounded by

$$C \|\langle A \nabla \psi, \nabla \psi \rangle\|_{L^\infty \mu_{\mathbf{b}}(\text{supp } \psi)}. \quad (298)$$

These estimates therefore imply that

$$\partial_t \int \psi^2 v d\mu_{\mathbf{b}} + \frac{1}{2} \int \langle A \nabla v, \nabla v \rangle d\mu_{\mathbf{b}} \leq C \|\langle A \nabla \psi, \nabla \psi \rangle\|_{L^\infty \mu_{\mathbf{b}}(\text{supp } \psi)}, \quad (299)$$

which is equivalent to the estimate (5.4.1) in [14]. Replacing Theorem 5.3.4 (The Weighted Poincaré Inequality) with our Proposition 3.3 (Sturm's Corollary 2.5), we complete the proof of this lemma exactly as in [14]. The only other ingredient used in the proof is the doubling property of the measure. \square

References

- [1] N. BERLINE, E. GETZLER, AND M. VERGNE, *Heat Kernels and Dirac Operators*, vol. 298 of Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin Heidelberg New York, 1992.

- [2] L. CHEN AND D. STROOCK, *The fundamental solution to the Wright-Fisher equation*, SIAM J. Math. Anal., 42 (2010), pp. 539–567.
- [3] C. L. EPSTEIN AND R. MAZZEO, *Wright-Fisher diffusion in one dimension*, SIAM J. Math. Anal., 42 (2010), pp. 568–608.
- [4] ———, *Degenerate Diffusion Operators Arising in Population Biology*, vol. 185 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2013.
- [5] ———, *The geometric microlocal analysis of generalized Kimura and Heston diffusions*, in Analysis and Topology in Nonlinear Differential Equations, D. G. de Figueiredo, J. M. do Ó, and C. Tomei, eds., vol. 85 of Progress in Nonlinear Differential Equations and Their Applications, Springer International Publishing AG, New York, NY, 2014, pp. 241–266.
- [6] C. L. EPSTEIN AND C. POP, *Harnack inequalities for degenerate diffusions*, (2014), p. 57pp. arXiv:1406.4759 [math.PR].
- [7] W. FELLER, *An Introduction to Probability Theory and Its Applications. Vol. II*, John Wiley & Sons, Inc., New York-London-Sydney, 1966.
- [8] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander’s condition*, Duke Math. J., 53 (1986), pp. 503–523.
- [9] M. KIMURA, *Diffusion models in population genetics*, Journal of Applied Probability, 1 (1964), pp. 177–232.
- [10] J. MOSER, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.
- [11] J. MOSER, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math., 24 (1971), pp. 727–740.
- [12] C. POP, *C^0 -estimates of solutions to the parabolic equation associated to Kimura diffusions*, preprint, (2014), p. 28pp. arXiv:1406.0742 [math.PR].
- [13] ———, *Existence, uniqueness and the strong Markov property of solutions to Kimura stochastic differential equations with singular drift*, (2014), p. 25pp. arXiv:1406.0745 [math.PR].
- [14] L. SALOFF-COSTE, *Aspects of Sobolev-type inequalities*, vol. 289 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2002.
- [15] N. SHIMAKURA, *Équations différentielles provenant de la génétique des populations*, Tôhoku Math. J., 29 (1977), pp. 287–318.
- [16] ———, *Formulas for diffusion approximations of some gene frequency models*, J. Math. Kyoto Univ., 21 (1981), pp. 19–45.
- [17] D. W. STROOCK, *Partial Differential Equations for Probabilists*, vol. 112 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2008.

- [18] K.-T. STURM, *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p -Liouville properties*, J. Reine Angew. Math., 456 (1994), pp. 173–196.
- [19] ———, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math., 32 (1995), pp. 275–312.
- [20] ———, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9), 75 (1996), pp. 273–297.