

COHEN-MACAULAY AND GORENSTEIN PROPERTIES UNDER THE AMALGAMATED CONSTRUCTION

P. SAHANDI, N. SHIRMOHAMMADI AND S. SOHRABI

ABSTRACT. Let A and B be commutative rings with unity, $f : A \rightarrow B$ a ring homomorphism and J an ideal of B . Then the subring $A \bowtie^f J := \{(a, f(a) + j) | a \in A \text{ and } j \in J\}$ of $A \times B$ is called the amalgamation of A with B along with J with respect to f . In this paper, among other things, we investigate the Cohen-Macaulay and (quasi-)Gorenstein properties on the ring $A \bowtie^f J$.

1. INTRODUCTION

In [7] and [8], D'Anna, Finocchiaro, and Fontana have introduced the following new ring construction. Let A and B be commutative rings with unity, let J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. They introduced the following subring

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A \text{ and } j \in J\}$$

of $A \times B$, called the *amalgamation of A with B along J with respect to f* . This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [6], [10]). Moreover, several classical constructions such as the Nagata's idealization (cf. [20, page 2], [19, Chapter VI, Section 25]), the $A + XB[X]$ and the $A + XB[[X]]$ constructions can be studied as particular cases of this new construction (see [7, Examples 2.5 and 2.6]).

Let M be an A -module. In 1955, Nagata introduced a ring extension of A called the *trivial extension* of A by M (or the *idealization* of M in A), denoted here by $A \times M$. Now, assume that A is Noetherian local and that M is finitely generated. It is well known that the trivial extension $A \times M$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and the module M is maximal Cohen-Macaulay. Furthermore, it is proved by Reiten [23] and Foxby [14] that $A \times M$ is Gorenstein if and only if A is Cohen-Macaulay and M is a canonical module of A . Next, in [2], Aoyama obtained a generalization of this result to quasi-Gorenstein rings. Indeed, he showed that $A \times M$ is a quasi-Gorenstein ring if and only if the completion \hat{A} satisfies Serre's condition (S_2) and M is a canonical module of A .

Let A be a Noetherian local ring and I be an ideal of A . Consider the amalgamated duplication $A \bowtie I := \{(a, a+i) | a \in A \text{ and } i \in I\}$ as in [6], [10]. D'Anna in [6] (see also [25]) proved that if A is a Cohen-Macaulay local ring and $\text{Ann}_A(I) = 0$, then the amalgamated duplication $A \bowtie I$ is Gorenstein if and only if A has a canonical ideal I . Next, in [3], the authors generalized this result as follows: if

2010 *Mathematics Subject Classification*. Primary 13A15, 13H10, 13C15.

Key words and phrases. Amalgamated algebra, Cohen-Macaulay ring, Gorenstein ring, quasi-Gorenstein ring, Serre condition, universally catenary ring.

$\text{Ann}_A(I) = 0$, then $A \bowtie I$ is a quasi-Gorenstein ring if and only if the \widehat{A} satisfies Serre's condition (S_2) and I is a canonical ideal of A . Finally, in [24], the authors determined when the amalgamated duplication $A \bowtie I$ is normal and satisfies Serre's conditions (S_n) and (R_n) .

In [9, Remark 5.1], assuming A is a Cohen-Macaulay local ring, J is finitely generated as an A -module, and J is contained in the Jacobson radical of B , it is observed that $A \bowtie^f J$ is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay A -module if and only if J is a maximal Cohen-Macaulay module. Moreover, it is shown in [9, Remark 5.4] that if A is a Cohen-Macaulay local ring, having a canonical module isomorphic (as an A -module) to J , then $A \bowtie^f J$ is Gorenstein. Also, it is observed, under the assumption $\text{Ann}_{f(A)+J}(J) = 0$, that if A is a Cohen-Macaulay local ring and $A \bowtie^f J$ is Gorenstein, then A has a canonical module isomorphic to $f^{-1}(J)$ [9, Proposition 5.5].

The above results lead us to investigate further when the amalgamated algebra $A \bowtie^f J$ is Cohen-Macaulay or (quasi-)Gorenstein.

More precisely, in Section 2, among other things, we prove that, for a Noetherian local ring (A, \mathfrak{m}) , let $f : A \rightarrow B$ be a ring homomorphism, and J be an ideal of B , contained in the Jacobson radical B , such that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$ and that $\text{depth } J < \infty$. Then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a big Cohen-Macaulay module. In particular if J is finitely generated A -module (with the structure naturally induced by f), then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is maximal Cohen-Macaulay. This improves [9, Remark 5.1], and [6, Discussion 10].

In Section 3, among other things, we show that, for a local ring (A, \mathfrak{m}) , $f : A \rightarrow B$ be a ring homomorphism, and J be an ideal of B contained in the Jacobson radical B , such that J is a finitely generated A -module, if \widehat{A} satisfies Serre's condition (S_2) and J is a canonical module of A , then $A \bowtie^f J$ is quasi-Gorenstein. Further, assume that $J^2 = 0$ and that $\text{Ann}_A(J) = 0$. If $A \bowtie^f J$ is quasi-Gorenstein, then A satisfies (S_2) and J is a canonical module of A . This generalizes [9, Remark 5.4 and Proposition 5.5], [6, Theorem 11] and [3, Theorem 3.3].

In Section 4, we describe the amalgamated algebra as a quotient of a polynomial ring. As a consequence, we derive a characterization of universally catenary property of the amalgamated algebra.

Now, in the following proposition, we collect some of the main properties of the amalgamated algebra $A \bowtie^f J$ needed in the present paper. We use it in this paper without comments.

Proposition 1.1. *Let A, B be commutative rings with unity, $f : A \rightarrow B$ be a ring homomorphism, and J be an ideal of B .*

- (1) ([7, Proposition 5.1(1)]) *Let $\iota_A : A \rightarrow A \bowtie^f J$ be the natural ring homomorphism defined by $\iota_A(a) = (a, f(a))$, for all $a \in A$. Then ι_A is an embedding, making $A \bowtie^f J$ a ring extension of A .*
- (2) ([7, Proposition 5.7(a)]) *If J is a finitely generated A -module (with the structure naturally induced by f), then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian.*
- (3) ([7, Lemma 2.3(4)]) *Let $\iota_J : J \rightarrow A \bowtie^f J$, defined by $j \mapsto (0, j)$ and $P_A : A \bowtie^f J \rightarrow A$, be the natural projection of $A \bowtie^f J \subseteq A \times B$ into A .*

Then the following is a split exact sequence of A -modules:

$$0 \longrightarrow J \xrightarrow{\iota_J} A \bowtie^f J \xrightarrow{P_A} A \longrightarrow 0.$$

(4) ([9, Corollaries 2.5 and 2.7]) For $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$, set

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie^f J := \{(p, f(p) + j) \mid p \in \mathfrak{p}, j \in J\},$$

$$\bar{\mathfrak{q}}^f := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in \mathfrak{q}\}.$$

Then, the following statements hold.

- (i) The prime ideals of $A \bowtie^f J$ are of the type $\bar{\mathfrak{q}}^f$ and \mathfrak{p}'^f , for \mathfrak{q} varying in $\text{Spec}(B) \setminus V(J)$ and \mathfrak{p} in $\text{Spec}(A)$.
 - (ii) $\text{Max}(A \bowtie^f J) = \{\mathfrak{p}'^f \mid \mathfrak{p} \in \text{Max}(A)\} \cup \{\bar{\mathfrak{q}}^f \mid \mathfrak{q} \in \text{Max}(B) \setminus V(J)\}$.
 - (iii) $A \bowtie^f J$ is a local ring if and only if A is local and $J \subseteq \text{Jac}(B)$.
- (5) ([9, Proposition 2.9]) The following statements hold.
- (i) For any prime ideal $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$, the localization $(A \bowtie^f J)_{\bar{\mathfrak{q}}^f}$ is canonically isomorphic to $B_{\mathfrak{q}}$.
 - (ii) For any prime ideal $\mathfrak{p} \in \text{Spec}(A) \setminus V(f^{-1}(J))$, the localization $(A \bowtie^f J)_{\mathfrak{p}'^f}$ is canonically isomorphic to $A_{\mathfrak{p}}$.
 - (iii) Let \mathfrak{p} be a prime ideal of A containing $f^{-1}(J)$. Consider the multiplicative subset $S_{\mathfrak{p}} := f(A \setminus \mathfrak{p}) + J$ of B and set $B_{S_{\mathfrak{p}}} := S_{\mathfrak{p}}^{-1}B$ and $J_{S_{\mathfrak{p}}} := S_{\mathfrak{p}}^{-1}J$. If $f_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{S_{\mathfrak{p}}}$ is the ring homomorphism induced by f , then the ring $(A \bowtie^f J)_{\mathfrak{p}'^f}$ is canonically isomorphic to $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}}$.

2. COHEN-MACAULAY PROPERTY UNDER AMALGAMATED CONSTRUCTION

Let us fix some notation which we shall use frequently throughout this section: A, B are two commutative rings with unity, A is Noetherian, $f : A \rightarrow B$ is a ring homomorphism, and J denotes an ideal of B , such that $A \bowtie^f J$ is Noetherian.

In [9, Remark 5.1], assuming A is a Cohen-Macaulay local ring, J is finitely generated as an A -module, and J is contained in the Jacobson radical of B , it is observed that $A \bowtie^f J$ is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay A -module if and only if J is a maximal Cohen-Macaulay module. Also, in [9, Remark 5.2], the authors mentioned that, if J is not finitely generated as A -module, it is more problematic to find conditions implying $A \bowtie^f J$ is Cohen-Macaulay. Our main result in this section improves their observation as well as, in case that J is not finitely generated, it provides conditions implying $A \bowtie^f J$ is Cohen-Macaulay.

To state the main result we need to introduce some terminology. Let (A, \mathfrak{m}) be a Noetherian local ring, and N an A -module (not necessarily finitely generated). The *depth* of N over A is defined by the non-vanishing of the local cohomology modules $H_{\mathfrak{m}}^i(N)$, with respect to \mathfrak{m} , in the way that

$$\text{depth } N := \text{depth}_A N := \inf\{i \mid H_{\mathfrak{m}}^i(N) \neq 0\},$$

see [5, Section 9.1] and [15]. When N is finitely generated, $\text{depth } N$ coincides with the usual depth defined by the common length of the maximal N -regular sequences in \mathfrak{m} by [5, Proposition 3.5.4(b)]. By [5, Exercise 9.1.12], if $\mathfrak{m}N \neq N$, then $\text{depth } N$ is finite, and when $\text{depth } N$ is finite, then $\text{depth } N \leq \dim A$.

It is well known that, over a local ring (A, \mathfrak{m}) , a finitely generated A -module M is Cohen-Macaulay if and only if $H_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M$ (see [4, Corollary 6.2.8]).

Recall that a finitely generated module M over a Noetherian local ring (A, \mathfrak{m}) is called a *maximal Cohen-Macaulay A -module* if $\text{depth } M = \dim A$. An A -module N is said to be *big Cohen-Macaulay* if $\text{depth } N = \dim A$. Such modules were constructed by Hochster [18], when A contains a field. The reader is referred to [5, Chapter 8] for details on the existence and properties of big Cohen-Macaulay modules.

The following auxiliary lemma which is interesting in itself is of fundamental importance in our study of Cohen-Macaulayness.

In the sequel, $\text{Jac}(B)$ will denote the Jacobson radical of B .

Lemma 2.1. *Let (A, \mathfrak{m}) be a local ring, and $J \subseteq \text{Jac}(B)$ be an ideal of B such that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$. Then $\dim A \bowtie^f J = \dim A$, and $\text{depth } A \bowtie^f J = \min\{\text{depth } A, \text{depth } J\}$.*

Proof. By [9, Corollary 3.2], we have $\mathfrak{m}'^f = \sqrt{\mathfrak{m}(A \bowtie^f J)}$. Using this, the Independence Theorem of local cohomology [4, Theorem 4.2.1] yields the isomorphism $H_{\mathfrak{m}'^f}^i(A \bowtie^f J) \cong H_{\mathfrak{m}}^i(A \bowtie^f J)$ for each i . On the other hand, applying the functor of local cohomology to the split exact sequence of A -modules appeared in Proposition 1.1(3), one obtains the isomorphism $H_{\mathfrak{m}}^i(A \oplus J) \cong H_{\mathfrak{m}}^i(A) \oplus H_{\mathfrak{m}}^i(J)$ for each i . Thus, for each i , we have shown the following isomorphism

$$H_{\mathfrak{m}'^f}^i(A \bowtie^f J) \cong H_{\mathfrak{m}}^i(A) \oplus H_{\mathfrak{m}}^i(J)$$

of A -modules. Using above isomorphism together with Grothendieck's Vanishing Theorem [4, Theorem 6.1.2], one concludes the first equality.

In order to prove the second equality, again, we use above isomorphism. First, note that $H_{\mathfrak{m}}^i(A) = H_{\mathfrak{m}}^i(J) = 0$ for all $i < \text{depth } A \bowtie^f J$. Hence $\text{depth } A \bowtie^f J \leq \text{depth } A$ and $\text{depth } A \bowtie^f J \leq \text{depth } J$. Now set $n := \text{depth } A \bowtie^f J$ and consider

$$H_{\mathfrak{m}}^n(A) \oplus H_{\mathfrak{m}}^n(J) \cong H_{\mathfrak{m}'^f}^n(A \bowtie^f J) \neq 0$$

to obtain $H_{\mathfrak{m}}^n(A) \neq 0$ or $H_{\mathfrak{m}}^n(J) \neq 0$. In the first case we deduce that $\text{depth } A = \text{depth } A \bowtie^f J \leq \text{depth } J$. Similarly, in the second case we have $\text{depth } J = \text{depth } A \bowtie^f J \leq \text{depth } A$. Therefore the second equality is obtained. \square

Remark 2.2. *Let A be a (not necessarily Noetherian) ring, and J be an ideal of B . It is observed, in [7, Propositions 4.1 and 4.2], that if f is surjective, or f is integral, then $\dim A \bowtie^f J = \dim A$.*

The next theorem is the main result of this section. With it, we not only offer an application of the above lemma, but we also provide more information about the Cohen-Macaulayness under the amalgamation than was given in [9, Remark 5.1].

Theorem 2.3. *With the assumptions of Lemma 2.1, the following statements hold.*

- (1) *If $A \bowtie^f J$ is Cohen-Macaulay, then so does A .*
- (2) *Further assume that $\text{depth } J < \infty$. Then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a big Cohen-Macaulay module.*

Proof. (1) Using Lemma 2.1, we have $\dim A = \dim A \bowtie^f J = \text{depth } A \bowtie^f J \leq \text{depth } A \leq \dim A$. Hence A is Cohen-Macaulay.

(2) Assume that $A \bowtie^f J$ is Cohen-Macaulay. Then, by (1), A is Cohen-Macaulay. To prove the big Cohen-Macaulayness of J , note that we have $\dim A = \dim A \bowtie^f J = \text{depth } A \bowtie^f J \leq \text{depth } J \leq \dim A$, where the last inequality comes from [5, Exercise 9.1.12]. Thus $\text{depth } J = \dim A$, that is, J is big Cohen-Macaulay.

Conversely, assume that A is Cohen-Macaulay and that J is a big Cohen-Macaulay module. It follows that $H_{\mathfrak{m}}^i(A) = H_{\mathfrak{m}}^i(J) = 0$ for all $i < \dim A$. Then, by the above isomorphism, one has $H_{\mathfrak{m}'_f}^i(A \bowtie^f J) = 0$ for all $i < \dim A = \dim A \bowtie^f J$, and hence $A \bowtie^f J$ is Cohen-Macaulay. \square

The next example shows that, if, in the above theorem, the hypothesis $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$, is dropped, then the corresponding statement is no longer always true.

Example 2.4. *Let k be a field and X, Y are algebraically independent indeterminates over k . Set $A := k[[X]]$, $B := k[[X, Y]]$ and let $J := (X, Y)$. Let $f : A \rightarrow B$ be the inclusion. Note that A is Cohen-Macaulay, and it is not hard to see that J is a big Cohen-Macaulay A -module. However, $A \bowtie^f J$ which is isomorphic to $k[[X, Y, Z]]/(Y, Z) \cap (X - Y)$ is not Cohen-Macaulay. It should be noted that, for $\mathfrak{q} := (X) \in \text{Spec}(B) \setminus V(J)$, one has $f^{-1}(\mathfrak{q}) = (X)$ which is the maximal ideal of A .*

It is clear that a finitely generated big Cohen-Macaulay module is maximal Cohen-Macaulay. Then we have the following corollary.

Corollary 2.5. *Let (A, \mathfrak{m}) be a local ring, and $J \subseteq \text{Jac}(B)$ be an ideal of B such that J is a finitely generated A -module. Then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a maximal Cohen-Macaulay A -module.*

Proof. By [9, Remark 3.3], we see that if J is finitely generated A -module, then $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$. So the assertion holds by Theorem 2.3. \square

Let M be a finitely generated A -module. In 1955, Nagata introduced a ring extension of A called the *trivial extension* of A by M (or the *idealization* of M in A), denoted here by $A \times M$ ([20, page 2], [19, Chapter VI, Section 25]). It should be noted that the module M becomes an ideal in $A \times M$ and $M^2 = 0$. As in [7, Example 2.8], if $B := A \times M$, $J := 0 \times M$, and $f : A \rightarrow B$ be the natural embedding, then $A \bowtie^f J \cong A \times M$. Therefore the following result follows from Corollary 2.5.

Corollary 2.6. *Let (A, \mathfrak{m}) be a local ring, and M be a finitely generated A -module. Then the trivial extension $A \times M$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and M is a maximal Cohen-Macaulay A -module.*

Corollary 2.7. *Assume that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$ and each $\mathfrak{m} \in \text{Max}(A)$. If $A \bowtie^f J$ is Cohen-Macaulay, then so does A .*

Proof. Let $\mathfrak{p} \in \text{Max}(A)$. If $\mathfrak{p} \notin V(f^{-1}(J))$, then $A_{\mathfrak{p}} \cong (A \bowtie^f J)_{\mathfrak{p}'_f}$ is Cohen-Macaulay. Now suppose $\mathfrak{p} \in V(f^{-1}(J))$. Then $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}} \cong (A \bowtie^f J)_{\mathfrak{p}'_f}$ is Cohen-Macaulay, where $S_{\mathfrak{p}} := f(A \setminus \mathfrak{p}) + J$. Note that, by [12, Remark 2.4], one has $f_{\mathfrak{p}}^{-1}(\mathfrak{q}B_{S_{\mathfrak{p}}}) = f^{-1}(\mathfrak{q})A_{\mathfrak{p}} \neq \mathfrak{p}A_{\mathfrak{p}}$, for each $\mathfrak{q}B_{S_{\mathfrak{p}}} \in \text{Spec}(B_{S_{\mathfrak{p}}}) \setminus V(J_{S_{\mathfrak{p}}})$. Hence, the Cohen-Macaulayness of $A_{\mathfrak{p}}$ results from Theorem 2.3. Therefore A is Cohen-Macaulay. \square

A finitely generated module M over a Noetherian ring A satisfies Serre's condition (S_n) if $\text{depth } M_{\mathfrak{p}} \geq \min\{n, \dim M_{\mathfrak{p}}\}$, for all $\mathfrak{p} \in \text{Spec}(A)$. Note that if M is Cohen-Macaulay, then it satisfies Serre's condition (S_n) for any integer n . Also, when $\dim M = d$ and M satisfies Serre's condition (S_d) , then M is Cohen-Macaulay. In the following results we investigate the property (S_n) for the amalgamated algebra.

Lemma 2.8. *Let $\mathfrak{p} \in \text{Spec}(A)$, and set $S_{\mathfrak{p}} := f(A \setminus \mathfrak{p}) + J$. If $J^2 = 0$, then $J_{\mathfrak{p}} \cong J_{S_{\mathfrak{p}}}$ as $A_{\mathfrak{p}}$ -modules.*

Proof. It is straightforward that the mapping $J_{\mathfrak{p}} \rightarrow J_{S_{\mathfrak{p}}}$ defined by $x/t \mapsto x/f(t)$, for all $x \in J$ and $t \in A \setminus \mathfrak{p}$, is an $A_{\mathfrak{p}}$ -isomorphism. \square

Recall that an ideal is called a *nil ideal* if each of its elements is nilpotent.

Corollary 2.9. *The following statements hold.*

- (1) *Assume that for each $\mathfrak{p} \in V(f^{-1}(J))$ and each $\mathfrak{q} \in \text{Spec}(B) \setminus V(J)$, $f^{-1}(\mathfrak{q}) \neq \mathfrak{p}$ (e.g. if f is surjective or J is a nil ideal of B). If $A \bowtie^f J$ satisfies (S_n) , then so does A .*
- (2) *Assume that $J^2 = 0$ and that J is a finitely generated A -module. If $A \bowtie^f J$ satisfies (S_n) , then so does J .*

Proof. (1) Let $\mathfrak{p} \in \text{Spec}(A)$. If $\mathfrak{p} \notin V(f^{-1}(J))$, then, by assumption, the inequality $\text{depth } A_{\mathfrak{p}} \geq \min\{n, \dim A_{\mathfrak{p}}\}$ holds for $A_{\mathfrak{p}} \cong (A \bowtie^f J)_{\mathfrak{p}'_f}$. Now suppose $\mathfrak{p} \in V(f^{-1}(J))$. This implies the isomorphism $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}} \cong (A \bowtie^f J)_{\mathfrak{p}'_f}$, where $S_{\mathfrak{p}} = f(A \setminus \mathfrak{p}) + J$. Meanwhile, by [12, Remark 2.4], we have the equality $f_{\mathfrak{p}}^{-1}(\mathfrak{q}B_{S_{\mathfrak{p}}}) = f^{-1}(\mathfrak{q})A_{\mathfrak{p}}$, for each $\mathfrak{q}B_{S_{\mathfrak{p}}} \in \text{Spec}(B_{S_{\mathfrak{p}}}) \setminus V(J_{S_{\mathfrak{p}}})$. Thus, the assumptions of Lemma 2.1 hold. Therefore we obtain

$$\begin{aligned} \text{depth } A_{\mathfrak{p}} &\geq \text{depth } A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}} \\ &= \text{depth}(A \bowtie^f J)_{\mathfrak{p}'_f} \\ &\geq \min\{n, \dim(A \bowtie^f J)_{\mathfrak{p}'_f}\} \\ &= \min\{n, \dim A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}}\} \\ &= \min\{n, \dim A_{\mathfrak{p}}\}. \end{aligned}$$

(2) Let $\mathfrak{p} \in \text{Spec}(A)$. Then, by Lemma 2.8, we have the isomorphism $J_{\mathfrak{p}} \cong J_{S_{\mathfrak{p}}}$ of $A_{\mathfrak{p}}$ -modules. If $\mathfrak{p} \notin V(f^{-1}(J))$, there is nothing to prove since $J_{\mathfrak{p}} \cong J_{S_{\mathfrak{p}}} = 0$ by [12, Remark 2.4]. So we can assume that $\mathfrak{p} \in V(f^{-1}(J))$. Thus, using Lemma 2.1, one gets

$$\begin{aligned} \text{depth } J_{\mathfrak{p}} &\geq \text{depth } A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{\mathfrak{p}} \\ &= \text{depth}(A \bowtie^f J)_{\mathfrak{p}'_f} \\ &\geq \min\{n, \dim(A \bowtie^f J)_{\mathfrak{p}'_f}\} \\ &= \min\{n, \dim A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{\mathfrak{p}}\} \\ &= \min\{n, \dim A_{\mathfrak{p}}\} \\ &\geq \min\{n, \dim J_{\mathfrak{p}}\}. \end{aligned}$$

Therefore J satisfies (S_n) . \square

One can employ Corollary 2.9 to deduce that the property (S_n) is retained under the trivial extension construction.

Corollary 2.10. *Let (A, \mathfrak{m}) be a local ring, and M be a finitely generated A -module. Then the trivial extension $A \times M$ satisfies (S_n) if and only if A and M satisfy (S_n) .*

Proposition 2.11. *If A, B satisfy (S_n) , and $J_{S_{\mathfrak{p}}}$ is a maximal Cohen-Macaulay $A_{\mathfrak{p}}$ -module for each prime ideal \mathfrak{p} of A , where $S_{\mathfrak{p}} = f(A \setminus \mathfrak{p}) + J$, then $A \bowtie^f J$ satisfies (S_n) .*

Proof. Each prime ideal of $A \bowtie^f J$ are of the type $\bar{\mathfrak{q}}^f$ or \mathfrak{p}'^f , for \mathfrak{q} varying in $\text{Spec}(B) \setminus V(J)$ and \mathfrak{p} in $\text{Spec}(A)$. The localization $(A \bowtie^f J)_{\bar{\mathfrak{q}}^f}$ is canonically isomorphic to $B_{\mathfrak{q}}$. Thus one gets

$$\text{depth}(A \bowtie^f J)_{\bar{\mathfrak{q}}^f} = \text{depth } B_{\mathfrak{q}} \geq \min\{n, \dim B_{\mathfrak{q}}\} = \min\{n, \dim(A \bowtie^f J)_{\bar{\mathfrak{q}}^f}\}.$$

Next, for $\mathfrak{p} \in \text{Spec}(A) \setminus V(f^{-1}(J))$, we have $(A \bowtie^f J)_{\mathfrak{p}'^f} \cong A_{\mathfrak{p}}$, hence, similarly, the inequality $\text{depth}(A \bowtie^f J)_{\mathfrak{p}'^f} \geq \min\{n, \dim(A \bowtie^f J)_{\mathfrak{p}'^f}\}$ holds. Now suppose $\mathfrak{p} \in V(f^{-1}(J))$. This implies the isomorphism $(A \bowtie^f J)_{\mathfrak{p}'^f} \cong A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}}$, where $S_{\mathfrak{p}} = f(A \setminus \mathfrak{p}) + J$. Thus, by assumption, one deduces

$$\begin{aligned} \text{depth}(A \bowtie^f J)_{\mathfrak{p}'^f} &= \min\{\text{depth } A_{\mathfrak{p}}, \text{depth } J_{S_{\mathfrak{p}}}\} \\ &\geq \min\{n, \dim A_{\mathfrak{p}}, \text{depth } J_{S_{\mathfrak{p}}}\} \\ &= \min\{n, \dim A_{\mathfrak{p}}\} \\ &= \min\{n, \dim(A \bowtie^f J)_{\mathfrak{p}'^f}\}. \end{aligned}$$

Therefore $A \bowtie^f J$ satisfies (S_n) . \square

In concluding this section, we want to generalize the above theorem to generalized Cohen-Macaulay rings. To this end, we need an auxiliary lemma.

Let us recall that a finitely generated module M over a Noetherian local ring (A, \mathfrak{m}) is said to be a *generalized Cohen-Macaulay* A -module if $H_{\mathfrak{m}}^i(M)$ is of finite length for all $i < \dim M$. A local ring is called generalized Cohen-Macaulay if it is a generalized Cohen-Macaulay module over itself. It is clear that every Cohen-Macaulay module is a generalized Cohen-Macaulay module.

In the course of next lemma and its proof, for a finite length module M over a ring R , we use $\ell_R(M)$ to denote the length of M over R .

Lemma 2.12. *Let $\varphi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a local homomorphism of local rings, such that the natural induced homomorphism $\bar{\varphi} : R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism. Let M be a S -module. If $\ell_S(M) < \infty$, then $\ell_R(M) < \infty$ and $\ell_S(M) = \ell_R(M)$. Here M is considered as an R -module via φ .*

Proof. Set $n := \ell_S(M)$ and suppose that $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a composition series of the S -module M . Then $M_i/M_{i-1} \cong S/\mathfrak{n}$ for all $i = 1, \dots, n$. On the other hand, it is clear that $\bar{\varphi}$ is R -module homomorphism. This means $S/\mathfrak{n} \cong R/\mathfrak{m}$ as R -modules. Thus, $M_i/M_{i-1} \cong R/\mathfrak{m}$ for all $i = 1, \dots, n$. Therefore $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a composition series of M as R -module. This yields $\ell_R(M) < \infty$ and $\ell_R(M) = n$. \square

Theorem 2.13. *Let (A, \mathfrak{m}) be a local ring, and $J \subseteq \text{Jac}(B)$ be an ideal of B such that J is a finitely generated A -module. Then $A \bowtie^f J$ is a generalized Cohen-Macaulay ring if and only if A and J are generalized Cohen-Macaulay and $\dim J \in \{0, \dim A\}$.*

Proof. Suppose that $A \bowtie^f J$ is generalized Cohen-Macaulay. Then $H_{\mathfrak{m}'^f}^i(A \bowtie^f J)$ is of finite length over $A \bowtie^f J$ for all $i < \dim(A \bowtie^f J)$. In conjunction with the previous lemma, this shows that $H_{\mathfrak{m}'^f}^i(A \bowtie^f J)$ has finite length over A for $i < \dim A$. Notice that, for all i , one has the isomorphism $H_{\mathfrak{m}'^f}^i(A \bowtie^f J) \cong H_{\mathfrak{m}}^i(A) \oplus H_{\mathfrak{m}}^i(J)$ of A -modules. Hence $H_{\mathfrak{m}}^i(A)$ and $H_{\mathfrak{m}}^i(J)$ have finite length over A for all $i < \dim A$. Therefore A and J are generalized Cohen-Macaulay and $\dim J = \dim A$ or 0 by [4, Corollary 7.3.3]. Conversely, suppose that A and J are generalized Cohen-Macaulay and $\dim J \in \{0, \dim A\}$; so that there exists a positive integer t such that $\mathfrak{m}^t H_{\mathfrak{m}}^i(A) = 0 = \mathfrak{m}^t H_{\mathfrak{m}}^i(J)$ for all $i < \dim A$. Hence $\mathfrak{m}^t H_{\mathfrak{m}'^f}^i(A \bowtie^f J) = 0$ for all $i < \dim(A \bowtie^f J)$. On the other hand, by [9, Corollary 3.2 and Remark 3.3], we know $\mathfrak{m}'^f = \sqrt{\mathfrak{m}(A \bowtie^f J)}$; so that there exists a positive integer s such that $(\mathfrak{m}'^f)^s \subseteq \mathfrak{m}(A \bowtie^f J)$. Consequently $(\mathfrak{m}'^f)^{st} H_{\mathfrak{m}'^f}^i(A \bowtie^f J) = 0$ for all $i < \dim(A \bowtie^f J)$. Therefore $A \bowtie^f J$ is generalized Cohen-Macaulay. \square

Corollary 2.14. *Let (A, \mathfrak{m}) be a local ring, and M be a finitely generated A -module. Then the trivial extension $A \times M$ is generalized Cohen-Macaulay if and only if A and M are generalized Cohen-Macaulay and $\dim M \in \{0, \dim A\}$.*

Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring. Then, using Corollaries 2.14 and 2.6, the trivial extension of A by A/\mathfrak{m} is a generalized Cohen-Macaulay ring which is not Cohen-Macaulay.

3. GORENSTEIN PROPERTY UNDER AMALGAMATED CONSTRUCTION

Let A, B be commutative rings with unity, A be Noetherian, $f : A \rightarrow B$ be a ring homomorphism, and $J \subseteq \text{Jac}(B)$ denotes an ideal of B , which is finitely generated A -module. It is shown in [9, Remark 5.4] that if A is a local Cohen-Macaulay ring, having a canonical module isomorphic (as an A -module) to J , then $A \bowtie^f J$ is Gorenstein. Also it is observed, under the assumption $\text{Ann}_{f(A)+J}(J) = 0$, that if A is a local Cohen-Macaulay ring and $A \bowtie^f J$ is Gorenstein, then A has a canonical module isomorphic to $f^{-1}(J)$ [9, Proposition 5.5].

In present section, under some assumptions, specified in Theorem 3.2, we give a sufficient condition and a necessary condition for the ring $A \bowtie^f J$ to be quasi-Gorenstein (definition recalled below). Consequently we improve the above mentioned results.

We now outline to recall some terminology. Let (A, \mathfrak{m}) be local. An A -module K is called a canonical module of A if

$$K \otimes_A \widehat{A} \cong \text{Hom}_A(H_{\mathfrak{m}}^{\dim A}(A), E_A(A/\mathfrak{m})),$$

where \widehat{A} is the \mathfrak{m} -adic completion of A , and $E_A(A/\mathfrak{m})$ is the injective hull of A/\mathfrak{m} over A . We say that A is a *quasi-Gorenstein ring* if a canonical module of A exists and it is a free A -module (of rank one). This is equivalent to saying that $H_{\mathfrak{m}}^{\dim A}(A) \cong E_A(A/\mathfrak{m})$. It is known that A is quasi-Gorenstein if and only if \widehat{A} is quasi-Gorenstein [2, Page 88].

The following lemma is of crucial importance in this section.

Lemma 3.1. *With the notation of Proposition 1.1, the following statements hold.*

(1) *If $\text{Ann}_{f(A)+J}(J) = 0$, then there is an $A \bowtie^f J$ -isomorphism*

$$\text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \cong f^{-1}(J) \times 0.$$

(2) If $J^2 = 0$, then there is an A -isomorphism

$$\mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \cong \mathrm{Ann}_A(J) \oplus J.$$

Proof. It is clear that $A \bowtie^f J/0 \times J \cong A$ as $A \bowtie^f J$ -modules [7, Proposition 5.1(2)]. Then we have

$$\begin{aligned} \mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) &\cong \mathrm{Hom}_{A \bowtie^f J}(A \bowtie^f J/0 \times J, A \bowtie^f J) \\ &\cong \mathrm{Ann}_{A \bowtie^f J}(0 \times J) \\ &= \{(a, f(a) + j) \in A \bowtie^f J \mid (f(a) + j)J = 0\} \\ &= \{(a, f(a) + j) \in A \bowtie^f J \mid f(a) + j \in \mathrm{Ann}_{f(A)+J}(J)\} =: X. \end{aligned}$$

Now assume that $\mathrm{Ann}_{f(A)+J}(J) = 0$. Then $X = f^{-1}(J) \times 0$. This completes the proof of (1). To prove (2), assume that $J^2 = 0$. It follows that in this case $X = \{(a, f(a) + j) \in A \bowtie^f J \mid f(a)J = 0\}$. Hence, one can define the A -homomorphism $\psi : X \rightarrow \mathrm{Ann}_A(J) \oplus J$ by $\psi((a, f(a) + j)) = (a, j)$. It is now easy to check that ψ is an isomorphism. Therefore (2) is obtained. \square

We are now ready for the proof of the main result of this section.

Theorem 3.2. *With the notation of Proposition 1.1, assume that A is Noetherian local, and $J \subseteq \mathrm{Jac}(B)$, which is finitely generated A -module. The following statements hold.*

- (1) *If \widehat{A} satisfies (S_2) and J is a canonical module of A , then $A \bowtie^f J$ is quasi-Gorenstein.*
- (2) *Assume that $J^2 = 0$ and that $\mathrm{Ann}_A(J) = 0$. If $A \bowtie^f J$ is quasi-Gorenstein, then A satisfies (S_2) and J is a canonical module of A .*
- (3) *Assume that $\mathrm{Ann}_{f(A)+J}(J) = 0$ and that $A \bowtie^f J$ is quasi-Gorenstein. Then $f^{-1}(J)$ is a canonical module of A . Furthermore, assume that f is surjective, then \widehat{A} satisfies (S_2) .*
- (4) *Assume that J is a flat A -module. If $A \bowtie^f J$ is quasi-Gorenstein, then A is quasi-Gorenstein.*

Proof. (1) Assume that \widehat{A} satisfies (S_2) and J is a canonical module of A . Using [17, Satz 5.12], this shows that $A \bowtie^f J$ has a canonical module $K_{A \bowtie^f J} \cong \mathrm{Hom}_A(A \bowtie^f J, J)$. On the other hand, applying the functor $\mathrm{Hom}_A(-, J)$ to the split exact sequence of A -modules considered in Proposition 1.1(3), one obtains the isomorphism $K_{A \bowtie^f J} \cong \mathrm{Hom}_A(J, J) \oplus J$. Since \widehat{A} satisfies (S_2) , using [1, Proposition 2], we have $\mathrm{Hom}_A(J, J) \cong A$. Thus $K_{A \bowtie^f J} \cong A \oplus J \cong A \bowtie^f J$ as A -modules. Note that $K_{A \bowtie^f J} \cong \mathrm{Hom}_A(A \bowtie^f J, J)$ is an $A \bowtie^f J$ -module by the usual way. Hence $K_{A \bowtie^f J} \cong A \bowtie^f J$ as $A \bowtie^f J$ -modules. This means that $A \bowtie^f J$ is quasi-Gorenstein.

(2) Assume that $A \bowtie^f J$ is quasi-Gorenstein. Then by [16, Lemma 2.1], we have $A \bowtie^f J$ satisfies (S_2) . Therefore by Corollary 2.9, A satisfies (S_2) . Since $A \bowtie^f J$ is quasi-Gorenstein, thus a canonical module of $A \bowtie^f J$ exists and it is isomorphic to $A \bowtie^f J$. By [17, Satz 5.12], A has a canonical module $K_A \cong \mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$. On the other hand, by Lemma 3.1(2), we have $\mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \cong J$. Therefore J is a canonical module of A .

(3) First we show that $f^{-1}(J)$ is a canonical module of A . We may use the same argument as employed in the proof of (2). One might take into consideration that

in the proof we appeal to Lemma 3.1(1) instead of Lemma 3.1(2) in the context of the proof of (2). Next, assuming the surjectivity of f , we show that \widehat{A} satisfies (S_2) . Since $A \bowtie^f J$ is quasi-Gorenstein, hence $\widehat{A \bowtie^f J}$ is also quasi-Gorenstein. Since f is surjective, B is a finitely generated A -module, and therefore there is an \widehat{A} -isomorphism $\widehat{A \bowtie^f J} \cong \widehat{A} \bowtie^{\widehat{f}} \widehat{J}$, where $\widehat{f} : \widehat{A} \rightarrow \widehat{B}$ be the induced natural surjective ring homomorphism. So that $\widehat{A} \bowtie^{\widehat{f}} \widehat{J}$ is quasi-Gorenstein. Then by [16, Lemma 2.1], we have $\widehat{A} \bowtie^{\widehat{f}} \widehat{J}$ satisfies (S_2) . Therefore by Corollary 2.9, \widehat{A} satisfies (S_2) .

(4) Assume that $A \bowtie^f J$ is quasi-Gorenstein. Consider the embedding $\iota_A : A \rightarrow A \bowtie^f J$. Since J is a flat A -module, ι_A is a flat local homomorphism. Using [9, Remark 3.3], the extension ideal $\mathfrak{m}(A \bowtie^f J)$ is $\mathfrak{m}^f = \mathfrak{m} \bowtie^f J$ -primary ideal. This in conjunction with [2, Theorem 2.3] implies that A is quasi-Gorenstein. \square

It is well known that A is Gorenstein if and only if it is Cohen-Macaulay and quasi-Gorenstein [2, Page 88]. Also, it is clear that A is Cohen-Macaulay if and only if \widehat{A} is Cohen-Macaulay. Therefore, one can use Theorem 2.3 together with Theorem 3.2 to derive the following corollaries.

Corollary 3.3. *Keep the assumptions of Theorem 3.2. Assume that A is Cohen-Macaulay and J is a canonical module of A . Then $A \bowtie^f J$ is Gorenstein.*

Corollary 3.4. *Keep the assumptions of Theorem 3.2. Assume that at least one of the following conditions holds*

- (1) f is an isomorphism and $\text{Ann}_B(J) = 0$; or
- (2) $J^2 = 0$ and $\text{Ann}_A(J) = 0$.

Then $A \bowtie^f J$ is Gorenstein if and only if A is Cohen-Macaulay and J is a canonical module of A .

Corollary 3.5. *Keep the assumptions of Theorem 3.2 and assume that $\text{Ann}_{f(A)+J}(J) = 0$. If $A \bowtie^f J$ is Gorenstein, then A is Cohen-Macaulay and $f^{-1}(J)$ is a canonical ideal of A .*

We recall that, if $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β . We refer the readers attention to [13], [11] and in the Noetherian setting, to [21], [22] for the properties and importance of pullback constructions. Let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . If $\pi : B \rightarrow B/J$ is the canonical projection and $\check{f} := \pi \circ f$, then $A \bowtie^f J \cong \check{f} \times_{B/J} \pi$, see [7, Proposition 4.2].

As an application of [22, Theorem 4], with some assumptions on the ideal $f^{-1}(J)$ and on the ring $f(A) + J$, the authors in [9] obtained the following characterization of when $A \bowtie^f J$ is Gorenstein.

Proposition 3.6. ([9, Proposition 5.7]) *With the notation of Proposition 1.1, assume that A is Noetherian local, and $J \subseteq \text{Jac}(B)$, which is finitely generated A -module and, moreover, assume that A is a Cohen-Macaulay ring, $f(A) + J$ is (S_1) and equidimensional, $J \neq 0$ and that $f^{-1}(J)$ is a regular ideal of A . Then, the following conditions are equivalent.*

- (1) $A \bowtie^f J$ is Gorenstein.

- (2) $f(A) + J$ is a Cohen-Macaulay ring, J is a canonical module of $f(A) + J$ and $f^{-1}(J)$ is a canonical module of A .

In concluding this section, as an application, we provide a method to construct a new quasi-Gorenstein ring from a given one. Let A be a local ring and X be an indeterminate over A . Set $B := A[X]/(X^2)$, $J := XB$, and $f : A \rightarrow B$ be the natural embedding. It is easy to see that J is isomorphic to A as an A -module. Then, using Theorem 3.2 (1) and (4), one sees that A is quasi-Gorenstein if and only if $A \bowtie^f J$ is quasi-Gorenstein. Note that we have $A \bowtie^f J \cong A[X]/(X^2)$.

4. UNIVERSALLY CATENARY PROPERTY UNDER AMALGAMATED CONSTRUCTION

Let us fix some notation which we shall use frequently throughout this section: A, B are two commutative rings with unity, $f : A \rightarrow B$ is a ring homomorphism, and J denotes an ideal of B .

In [10, Example 3.11] and [9, Example 2.6], the authors in some special examples illustrate the amalgamated algebras as a quotient of some polynomial rings. In the following proposition, we generally describe the amalgamated algebra $A \bowtie^f J$ as a quotient of a polynomial ring.

Proposition 4.1. *Let the notation just introduced and assume that J as an A -module is generated by $\{j_\lambda | \lambda \in \Lambda\}$. Then $A \bowtie^f J$ is a homomorphic image of $A[X_\lambda | \lambda \in \Lambda]$ the ring of polynomials over A in indeterminates X_λ , for each $\lambda \in \Lambda$.*

Proof. Consider the ring homomorphism $\varphi : A[X_\lambda | \lambda \in \Lambda] \rightarrow A \bowtie^f J$, defined by $\varphi(a) := (a, f(a))$, for $a \in A$, and $\varphi(X_\lambda) := (0, j_\lambda)$, for $\lambda \in \Lambda$. It is not difficult to see that φ is surjective. \square

Corollary 4.2. *Assume that I as an ideal of A is generated by $\{i_\lambda | \lambda \in \Lambda\}$. Then the amalgamated duplication $A \bowtie I := \{(a, a + i) | a \in A, i \in I\}$ ([6, 10]) is a homomorphic image of $A[X_\lambda | \lambda \in \Lambda]$ the ring of polynomials over A in indeterminates X_λ , for each $\lambda \in \Lambda$.*

As a consequence of the previous proposition, we provide a partial characterization of the Noetherianity of $A \bowtie^f J$. It should be noted that the authors in [7, Proposition 5.7] have already given a characterization of the Noetherianity of $A \bowtie^f J$. But, this is an obvious consequence of the above proposition provided that J is finitely generated as an A -module.

Corollary 4.3. *Assume that J is finitely generated as an A -module. Then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian.*

The next corollary investigates the behaviour of universally catenary property under amalgamated formation provided that J is finitely generated as an A -module. One says a locally finite dimensional ring A is *catenary* if every saturated chain joining prime ideals \mathfrak{p} and \mathfrak{q} , $\mathfrak{p} \subseteq \mathfrak{q}$, has (maximal) length $\text{height } \mathfrak{q}/\mathfrak{p}$; A is *universally catenary* if all the polynomial rings $A[X_1, \dots, X_n]$ are catenary.

Corollary 4.4. *Assume that J is finitely generated as an A -module. Then $A \bowtie^f J$ is universally catenary if and only if A is universally catenary.*

ACKNOWLEDGMENT

The authors are deeply grateful to the referee for his/her careful reading of the paper and valuable suggestions.

REFERENCES

1. Y. Aoyama, *On the depth and the projective dimension of the canonical module*, Japan. J. Math. **6**, (1980), 61–66.
2. Y. Aoyama, *Some basic results on canonical modules*, J. Math. Kyoto Univ. **23**, (1983), 85–94.
3. A. Bagheri, M. Salimi, E. Tavasoli and S. Yassemi, *A construction of quasi-Gorenstein rings*, J. Algebra Appl. **11**, No. 1, (2012), 1250013, (9 pages).
4. M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, **60**, Cambridge University Press, Cambridge, 1998.
5. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics. **39**, Cambridge University Press, Cambridge, 1998.
6. M. D’Anna, *A construction of Gorenstein rings*, J. Algebra, **306**, (2006), 507–519.
7. M. D’Anna, C. A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in: Commutative Algebra and Applications, Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, 2008, W. de Gruyter Publisher, Berlin, 2009, pp. 155–172.
8. M. D’Anna, C. A. Finocchiaro, and M. Fontana, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra, **214**, (2010), 1633–1641.
9. M. D’Anna, C. A. Finocchiaro, and M. Fontana, *New algebraic properties of an amalgamated algebra along an ideal*, to appear in Comm. Algebra.
10. M. D’Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6**, No.3, (2007), 443–459.
11. A. Facchini, *Fiber products and Morita duality for commutative rings*, Rend. Sem. Mat. Univ. Padova, **67**, (1982), 143–159.
12. C. A. Finocchiaro, *Prüfer-like conditions on an amalgamated algebra along an ideal*, Houston J. Math. **40**, No. 1, (2014), 63–79.
13. M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123**, (1980), 331–355.
14. H. B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31**, (1972), 267–284.
15. H. B. Foxby and S. Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001), 119–137, Contemp. Math., **331**, Amer. Math. Soc., Providence, RI, (2003).
16. S. H. Hassanzadeh, N. Shirmohammadi, and H. Zakeri, *A note on quasi-Gorenstein rings*, Arch. Math. (Basel), **91**, (2008), 318–322.
17. J. Herzog and E. Kunz (eds.), *Der kanonische Modul eines CohenMacaulay Rings*, Lecture Notes in Mathematics, **238**, Springer-Verlag, 1971.
18. M. Hochster, *Topics in the homological theory of modules over commutative rings*, Amer. Math. Soc. CBMS Regional conference series **24**, 1975.
19. J. Huckaba, *Commutative Rings with zero divisors*, M. Dekker, New York, 1988.
20. M. Nagata, *Local Rings*, Interscience, New York, 1962.
21. T. Ogoma, *Fibre products of Noetherian rings and their applications*, Math. Proc. Cambridge Philos. Soc. **97**, (1985), 231–241.
22. T. Ogoma, *Fiber products of Noetherian rings*, Adv. Stud. Pure Math. **11**, Commutative Algebra and Combinatorics (Edited by M. Nagata and H. Matsumura) (1987), 173–182.
23. I. Reiten, *The converse of a theorem of Sharp on Gorenstein modules*, Proc. Amer. Math. Soc. **32**, (1972), 417–420.
24. P. Sahandi and N. Shirmohammadi, *Notes on amalgamated duplication of a ring along an ideal*, to appear in Bull. Iranian Math. Soc.
25. J. Shapiro, *On a construction of Gorenstein rings proposed by M. D’Anna*, J. Algebra, **323**, (2010), 1155–1158.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
E-mail address: sahandi@tabrizu.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
E-mail address: shirmohammadi@tabrizu.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
E-mail address: sohrabi16@yahoo.com