arXiv:1406.2186v1 [math.PR] 9 Jun 2014

Normal approximation for the net flux through a random conductor

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June 6, 2104

Abstract

We consider solutions of an elliptic partial differential equation in $\mathbb{R}^d$ with a stationary, random conductivity coefficient. The boundary condition on a square domain of width L is chosen so that the solution has a macroscopic unit gradient. We then consider the average flux through the domain. It is known that in the limit $L \to \infty$, this quantity converges to a deterministic constant, almost surely. Our main result is about normal approximation for this flux when L is large: we give an estimate of the Kantorovich-Wasserstein distance between the law of this random variable and that of a normal random variable. This extends a previous result of the author [29] to a much larger class of random conductivity coefficients. The analysis relies on elliptic regularity, on bounds for the Green's function, and on a normal approximation method developed by S. Chatterjee [8] based on Stein's method.

1 Introduction

This paper pertains to solutions of the random partial differential equation

$$-\nabla \cdot (a(x)(\nabla \phi(x) + e_1)) + \beta \phi(x) = 0, \quad x \in D_L \subset \mathbb{R}^d, \tag{1.1}$$

where the coefficient $a(x) = (a_{ij}(x)) \in (L^\infty(\mathbb{R}^d))^{d\times d}$ is a stationary random matrix satisfying a uniform ellipticity condition. The parameter $\beta \geq 0$ is deterministic. The set $D_L = [0, L)^d$ is the domain, and we require that ϕ satisfies periodic boundary conditions on the boundary of D_L. Our main result is about the statistical behavior of the quantity

$$\Gamma_{L,\beta} = \frac{1}{|D_L|} \int_{D_L} (\nabla \phi + e_1) \cdot a(x)(\nabla \phi + e_1) + \beta \phi^2 \, dx \tag{1.2}$$

for large L. Using (1.1) and the periodicity of ϕ we see that $\Gamma_{L,\beta}$ may also be written as

$$\Gamma_{L,\beta} = \frac{1}{|D_L|} \int_{D_L} e_1 \cdot a(x)(\nabla \phi(x) + e_1) \, dx. \tag{1.3}$$

This is a random variable, as the coefficient $a(x)$ and the solution ϕ are random.

Partial differential equations like (1.1) arise in physical applications where the coefficient $a(x)$ may be modeled best as a random field, due to inherent uncertainty and complexity of the physical medium [36]. If we interpret (1.1) in terms of electrical conductivity, then ϕ is a potential, $a(x)$ is the conductivity, and the vector field $-a(x)(\nabla \phi + e_1)$ is a current density. The unit vector e_1 is deterministic, the gradient of the linear potential $x \cdot e_1$. Considering (1.3), we interpret $\Gamma_{L,\beta}$ as an

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average flux in the direction e_1 that results from a macroscopic potential gradient imposed in the direction of e_1.

The equation (1.1) plays an important role in the homogenization theory for the random elliptic operator $u \mapsto -\nabla \cdot (a(x/\epsilon)\nabla u)$ in the limit $\epsilon \to 0$ [32, 22]. It is well-known that the homogenized conductivity tensor $\bar{a}$ for that operator can be expressed in terms of functions ϕ, called "correctors", which solve (1.1) with e_1 being one of the d standard basis vectors and which have stationary gradient. On the other hand, in a numerical computation of $\bar{a}$ one must approximate the true correctors by solving (1.1) in a bounded domain D_L with suitable boundary condition. The parameter $\beta \geq 0$ is a kind of regularizing parameter that sometimes is used in approximation theory. The periodic boundary condition that we impose here is one choice that allows accurate approximation of the effective coefficient $\bar{a}$ in the limit $L \to \infty$ [7, 31, 15].

The results of [7, 31] imply that for $\beta \geq 0$ fixed, $\Gamma_{L,\beta}$ converges almost surely, as $L \to \infty$, to a deterministic constant $\bar{\Gamma}_\beta > 0$. For $\beta = 0$, the limit $\bar{\Gamma}_0$ is one of the diagonal entries of the homogenized tensor $\bar{a}$ described above. For finite L, it is interesting to understand how $\Gamma_{L,\beta}$ and ϕ fluctuate around their means. Our main result is an estimate showing that for $L \gg 1$, the distribution of $\Gamma_{L,\beta}$ is very close to that of a normal random variable. In [29], we proved a similar result under strong assumptions about the random coefficient $a(x)$. In the present paper, however, we develop a more general approach which yields normal approximation for $\Gamma_{L,\beta}$ under much weaker assumptions about the law of $a(x)$.

Before we present the main result and explain its relation to other works, let us define the problem precisely and establish notation.

The random coefficient $a(x)$

For $L \in \mathbb{Z}^+$, let $D_L = [0, L)^d \subset \mathbb{R}^d$ and let $L^\infty_{per}(D_L)$ denote the set of functions in $L^\infty(\mathbb{R}^d)$ which are periodic with period L in each direction. That is, for all $f \in L^\infty_{per}(D_L)$, $f(x + Lk) = f(x)$ holds for all $k \in \mathbb{Z}^d$ and almost every $x \in \mathbb{R}^d$. The coefficient $a(x)$ in (1.1) will be a random symmetric matrix with entries $a_{ij} \in L^\infty_{per}(D_L)$. Since we will be working with functions that are periodic over D_L, we use $dist(x, y)$ to refer to the periodized distance function:

$$dist(x, y) = \min_{k \in \mathbb{Z}^d} d_2(x, y + kL), \qquad x, y \in \mathbb{R}^d, \tag{1.4}$$

where $d_2(x, y)$ is the standard Euclidean metric in $\mathbb{R}^d$. Also, when working in the torus D_L, we use the notation $B_r(x)$ to refer to the ball of radius r in this metric on the torus:

$$B_r(x) = \{y \in D_L \mid dist(x, y) < r\}. \tag{1.5}$$

We suppose that the random nature of $a(x)$ comes from its dependence on a collection of L^d independent random variables $Z = \{Z_k\}_{k \in \mathbb{Z}^d \cap D_L}$ taking values in a set $\mathcal{Z}$, and defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, $Z : \Omega \to \mathcal{Z}^{L^d}$. We often will write $a(x)$ for $a(x, Z)$, the dependence on Z being understood. Let $\mathbb{E}[f(Z)]$ denote expectation with respect to the probability measure $\mathbb{P}$ defining the law of Z. We will make three additional structural assumptions about the random matrix $a(x)$. First, we require that $a(x)$ is statistically stationary with respect to integer shifts in x: for every $k \in \mathbb{Z}^d$ and $a(\cdot + k)$ is equal in law to $a(\cdot)$. Second, we suppose boundedness and uniform ellipticity: there are positive constants $a^*, a_* > 0$ such that for any nonzero $\xi \in \mathbb{R}^d$

$$a_*|\xi|^2 \leq \xi \cdot a(x)\xi \leq a^*|\xi|^2, \quad x \in D_L \tag{1.6}$$

holds $\mathbb{P}$-almost surely. Third, we suppose that there is a constant $\tau > \sqrt{d} > 0$ such that for all $k \in \mathbb{Z}^d$

$$a(x, Z) - a(x, Z') = 0 \quad \text{if } \; dist(x, k) \geq \tau \tag{1.7}$$

holds whenever $Z_j = Z'_j$ for all $j \neq k$. One consequence of this last assumption is that $x \mapsto a(x, Z)$ does not depend of Z_k if $dist(x, k) \geq \tau$. Moreover, $a(x, Z)$ and $a(y, Z)$ are statistically independent if $dist(x, y) \geq 2\tau$. In other words, the dependence of $a(x, Z)$ on Z is local: $a(x, Z)$ depends only on Z_j for indices $j \in \mathbb{Z}^d$ that are sufficiently near $x \in \mathbb{R}^d$.

For clarity, let us highlight some simple examples for which these assumptions hold. First, suppose that $a(x)$ is scalar and has the form of a random checkerboard

$$a(x) = \sum_{k \in \mathbb{Z}^d \cap D_L} Z_k \mathbb{I}_{Q_k}(x \bmod L) \tag{1.8}$$

where $\{Z_k\}_{k \in \mathbb{Z}^d \cap D_L}$ is a family of independent and identically distributed real-valued random variables satisfying $a_* \leq Z_k \leq a^*$ almost surely. The set $Q_k = k + [0, 1)^d$ is the unit cube with a corner at $k \in \mathbb{Z}^d$, and $(x \bmod L)$ denotes the point $(x_1 \bmod L, \ldots, x_d \bmod L) \in D_L$. So, $a(x)$ is a piecewise constant function, taking random values on the cubes Q_k. It is also periodic over D_L. In this example, $\mathcal{Z} = [a_*, a^*]$, but we need not make any further assumptions about regularity of the law of Z_k, as was required in [29].

In the next example, $a(x, \omega)$ is scalar and represents pores of conductivity a^* distributed randomly within a material having background conductivity $a_* > 0$. The pores are spheres having random radii, whose centers are determined by a Poisson point process with intensity $\mu > 0$. To construct such a conductivity function, let $\{X_j^k \mid k \in \mathbb{Z}^d,\ j \in \mathbb{N}\}$ be a collection of independent random variables that are each uniformly distributed on the cube $Q_0 = [0, 1)^d$. Let $\{N_k\}_{k \in \mathbb{Z}^d}$ be an independent set of Poisson random variables with mean $\mu > 0$, defined on the same probability space. That is, $\mathbb{P}(N_k = n) = (n!)^{-1} e^{-\mu} \mu^n$ for $n = 0, 1, 2, \ldots$. The random integer N_k will be the number of pores with centers in the cube $Q_k = k + [0, 1)^d$. The random measure

$$\rho_k(A) = \sum_{j=1}^{N_k} \mathbb{I}_A(k + X_j^k)$$

on Borel sets $A \subset Q_k$ is a homogeneous Poisson point process on Q_k with intensity μ. Let $\{R_j^k \mid k \in \mathbb{Z}^d,\ j \in \mathbb{N}\}$ be an independent collection of identically distributed, real-valued random variables such that $\mathbb{P}(0 < R_j^k \leq R_{max}) = 1$ for some constant R_{max}; these are the radii of the pores. Finally, we define

$$a(x) = a_* + (a^* - a_*) \min\left(1\ ,\ \sum_{k \in \mathbb{Z}^d \cap D_L} \sum_{j=1}^{N_k} \mathbb{I}_{B_{R_j^k}(X_j^k)}(x - k)\right). \tag{1.9}$$

Thus, $a(x) = a^*$ if and only if

$$dist(x, (k + X_j^k)) < R_j^k, \quad \text{for some } j \leq N_k \text{ and } k \in \mathbb{Z}^d \cap D_L.$$

Otherwise, $a(x) = a_*$. In this case, the random variables $\{Z_k\}_{k \in \mathbb{Z}^d \cap D_L}$ are the collections of (shifted) pore centers and radii: $Z_k = \{(X_j^k, R_j^k) \mid 0 \leq j \leq N_k\}$, and we may take the set $\mathcal{Z}$ to be the set of all finite sequences $((x_1, r_1), \ldots, (x_n, r_n))$ where $x_i \in Q_0$ and $r_i \in (0, R_{max})$. Recalling (1.5), we see that $a \in L^\infty_{per}(D_L)$ almost surely, and the stationarity property holds. The condition $\mathbb{P}(R_j \leq R_{max}) = 1$ guarantees that (1.7) holds with $\tau = R_{max} + \sqrt{d}$. There are many variations of this construction which fit into the framework described above, such as random rods having random orientation and length, as in the experiments described in [2].

The energy functional

Let $H^1_{per}(D_L)$ denote the set of L-periodic functions in $H^1_{loc}(\mathbb{R}^d)$. That is, $\phi \in H^1_{per}(D_L)$ if $\phi \in H^1_{loc}(\mathbb{R}^d)$ and $\phi(x + Lk) = \phi(x)$ a.e. $\mathbb{R}^d$ for every $k \in \mathbb{Z}^d$. If $a_{ij}(x) \in (L^\infty(D_L))^{d\times d}$ and satisfies (1.6), then there exists a weak solution $\phi \in H^1_{per}(D_L)$ to (1.1):

$$\int_{D_L} \nabla v \cdot a(x)(\nabla\phi + e_1) + \beta\phi v \, dx = 0, \quad \forall\, v \in H^1_{per}(D_L). \tag{1.10}$$

For $\beta > 0$, the solution is unique. For $\beta = 0$, the solution is not unique, but any two solutions in $H^1_{per}(D_L)$ must differ by a constant. So, under the normalization condition

$$\int_{D_L} \phi(x)\, dx = 0, \tag{1.11}$$

and for fixed L, the solution is unique in $H^1_{per}(D_L)$ for all $\beta \geq 0$. With $a(x) = a(x, Z)$ satisfying the conditions above, this unique solution $\phi(x) = \phi(x, a, L, \beta)$ depends on the parameters L and β, on $x \in D_L$, and on the random variables $Z = \{Z_j\}_{j\in D_L\cap\mathbb{Z}^d}$ which determine a. The uniqueness of the solution and the stationarity of a implies that $\phi(x)$ is statistically stationary with respect to integer shifts: the law of $\phi(x)$ is the same as that of $\phi(x + k)$ for any $k \in \mathbb{Z}^d$.

Having defined both $a(x)$ and $\phi(x)$, we now define the random variable $\Gamma_{L,\beta}$ by (1.2), which is equivalent to (1.3). This also is a function of the L^d random variables $\{Z_j\}_{j\in D_L\cap\mathbb{Z}^d}$. We will use Φ_j and Φ'_j to refer to the integrals

$$\Phi_j = \left(\int_{Q_j} |\nabla\phi(x) + e_1|^2\, dx\right)^{1/2}, \qquad \hat{\Phi}_j = \left(\int_{B_\tau(j)} |\nabla\phi(x) + e_1|^2\, dx\right)^{1/2} \tag{1.12}$$

which appear frequently in the analysis. Recall that $B_\tau(j) \supset Q_j$, so $\hat{\Phi}_j \geq \Phi_j$.

Main result

Our main result is the following theorem. Suppose W and Y are two real-valued random variables and that μ_W and μ_Y denote the laws on $\mathbb{R}$ of W and Y, respectively. The Kantorovich-Wasserstein distance between μ_W and μ_Y is

$$\begin{aligned} d_{\mathcal{W}}(W, Y) &= \sup\{\, |\mathbb{E}h(W) - \mathbb{E}h(Y)| \;\mid\; \|h\|_{Lip} \leq 1\} \\ &= \sup\left\{ \left|\int_{\mathbb{R}} h(w)\, d\mu_W(w) - \int_{\mathbb{R}} h(y)\, d\mu_Y(z)\right| \;\mid\; \|h\|_{Lip} \leq 1 \right\}. \end{aligned}$$

Theorem 1.1 *Let $d \geq 2$. Let $m_{L,\beta} = \mathbb{E}[\Gamma_{L,\beta}]$ and $\sigma^2_{L,\beta} = Var(\Gamma_{L,\beta})$. Let Y denote a standard normal random variable, $N(0, 1)$. There is a constant $C > 0$ (depending only on d, a_*, and a^*) and a constant $q > 2$ such that*

$$d_{\mathcal{W}}\left(\frac{\Gamma_{L,\beta} - m_{L,\beta}}{\sigma_{L,\beta}}, Y\right) \leq C\frac{L^{-2d}}{\sigma^3_{L,\beta}}\mathbb{E}[\Phi_0^6] + C\frac{L^{-3d/2}\log(L)}{\sigma^2_{L,\beta}}\mathbb{E}[\Phi_0^{8q}]^{\frac{1}{2q}} \tag{1.13}$$

holds for all $L > 2$ and $\beta \geq 0$.

In [29], we obtained a similar result under more restrictive structural assumptions about the law of the coefficient a. Specifically, the approach in [29] required that the law of $a(x)$ be obtained by a sufficiently smooth mapping of normally distributed random variables. Those assumptions excluded cases like (1.9) where the law of $a(x)$ may have no absolutely continuous part (with respect to Lebesgue measure on $[a_*, a^*]$); the assumptions on $a(x, Z)$ in the present setting are significantly less restrictive. Regularity of the law of $a(x)$ in [29] made it possible to differentiate $\Gamma_{L,\beta}$ with respect to the Z_k and to apply a "second order Poincaré inequality" developed by Chatterjee in [9]. In the present setting, the more general assumptions on the law of $a(x)$ do not allow us to apply the same approach. Consequently, the proof of Theorem 1.1 is based on a more general normal approximation technique from [8], which is suitable for fully discrete distributions.

The variance $\sigma^2_{L,\beta}$ and the moments of the random variable Φ_0 which appear in (1.13) depend on both L and β. If the moments of Φ are bounded by a constant, independent of L and β, and if the variance is bounded from below by $\sigma^2_{L,\beta} \geq CL^{-d}$, then the bound (1.13) becomes

$$d_{\mathcal{W}}\left(\frac{\Gamma_{L,\beta} - m_{L,\beta}}{\sigma_{L,\beta}}, Y\right) \leq CL^{-d/2}\log L.$$

For all dimensions $d \geq 1$, if $\beta \geq \beta_0 > 0$ is bounded away from zero independently of L, then all moments $\mathbb{E}[\Phi_0^q]$ are bounded independently of $L > 1$ (for example, see [29]). If $\beta = 0$ or if $\beta > 0$ is allowed to vanish as $L \to \infty$, estimating the moments $\mathbb{E}[\Phi_0^q]$ is a delicate issue. Elliptic regularity helps a bit. Meyers' estimate [26] implies that $\nabla\phi \in L^{p^*}$ for some $p^* > 2$. If the ratio $\frac{a^*}{a_*}$ is sufficiently close to 1, this p^* may be arbitrarily large. As a result, a uniform (in L and $\beta \geq 0$) bound on $\mathbb{E}[\Phi_0^q]$ follows from this regularity estimate if $\frac{a^*}{a_*} \approx 1$ (see Lemma 4.3 of [29], for example). This observation goes back to the work of Naddaf and Spencer [28]. On the other hand, without the assumption $\frac{a^*}{a_*} \approx 1$, the regularity only goes so far. To estimate $\mathbb{E}[\Phi_0^q]$ in this situation one can use the arguments developed recently by Gloria and Otto in [17]. In that work, the authors derive variance bounds for a discrete functional similar to $\Gamma_{L,\beta}$, involving an infinite network of random resistors on the bonds of the integer lattice $\mathbb{Z}^d$. The PDE (1.1) is replaced by a discrete difference equation on all of $\mathbb{Z}^d$, without the periodicity assumption. The stationary potential field $\phi(x)$ is defined at points $x \in \mathbb{Z}^d$; the gradient and divergence have interpretations as difference operators. A key point in their analysis is the following bound on moments of the discrete corrector ϕ:

$$\mathbb{E}[\,|\phi(0)|^q\,] \leq \begin{cases} C_q, & \text{if } d \geq 3 \\ C_q|\log(\beta)|^{\gamma_q}, & \text{if } d = 2. \end{cases} \tag{1.14}$$

The constants $C_q, \gamma_q > 0$ are independent of $L > 1$ and $\beta > 0$. The analysis of [17] can be extended to the present setting (spatial continuum, with periodicity on D_L) to estimate moments of both $\int_{Q_0} \phi(x)\,dx$ and Φ_0 (see [29] for some discussion of this). The argument shows that moments of $\int_{Q_0} \phi(x)\,dx$ satisfy the same bound as (1.14), which diverges as $\beta \to 0$ if $d = 2$. On the other hand, Φ_0 involves the gradient $\nabla\phi$, and it can be shown that for all $d \geq 2$, all moments $\mathbb{E}[\Phi_0^q]$ are bounded independently of $L > 1$ and $\beta \geq 0$ [19]. In the discrete setting, the uniform control (in L and β) of $\nabla\phi$ for all $d \geq 2$ was observed already by Gloria, Otto, Neukamm [15] (see Proposition 1 therein).

In view of $\sigma^2_{L,\beta}$ appearing in (1.13), let us note that in many cases it is expected that the variance of $\Gamma_{L,\beta}$ is bounded below by $\sigma^2_{L,\beta} \geq CL^{-d}$. Indeed, in [29] we proved that this is the case for the random checkerboard model (1.8). This bound is closely related to earlier work of Wehr [37] in the discrete setting. In a forthcoming work [30], we will give a more general sufficient condition under which $\sigma^2_{L,\beta} \geq CL^{-d}$ holds for the continuum setting; in particular, this lower bound holds for the coefficient (1.9) constructed from Poisson scatter. It is not known what is the most general class of stationary random fields $a(x, \omega)$ for which the lower bound $\mathrm{Var}(\Gamma_{L,\beta}) \geq CL^{-d}$ holds. It is conceivable that there are random fields $a(x, Z)$ satisfying both (1.6) and (1.7) such that $\mathrm{Var}(\Gamma_{L,\beta}) = o(L^{-d})$ as $L \to \infty$, due to some short-range correlation in the variables Z_k; the same phenomenon is possible even for simple averages of L^d identically distributed random variables when the variables may be dependent on one-another. For example, suppose $d = 1$, and consider the simple case of a sequence of L resistors wired together, in series, having conductivity $a_1, a_2, \ldots, a_L$. The effective conductivity of the series is just the harmonic mean $\Gamma_L = (L^{-1}\sum_{k=1}^{L}\frac{1}{a_k})^{-1}$. Suppose that $\{Z_k\}$ is a sequence of independent, Bernoulli-p random variables. Thus, $\mathbb{P}(Z_k = 1) = p$, $\mathbb{P}(Z_k = 0) = 1 - p$. Suppose that $a_k = (2 + Z_{k+1} - Z_k)^{-1}$. These a_k are dependent, but a_k and a_j are independent if $|k - j| > 1$. However, by definition of a_k, $(\Gamma_L)^{-1}$ is a telescoping sum, and the variance of the effective conductivity satisfies $\text{Var}\,(\Gamma_L) = O(L^{-2}) = o(L^{-d})$. Moreover, the distribution of Γ_L (after normalization) in this simple example is not asymptotically Gaussian.

In addition to the works we have mentioned already, the two works most closely related to Theorem 1.1 are those of Biskup, Salvi, and Wolff [4] and Rossignol [33] regarding discrete resistor network models. By making use of the martingale central limit theorem, Biskup, Salvi, and Wolff [4] have proved a central limit theorem for a discrete quantity similar to $\Gamma_{L,\beta}$ when ϕ satisfies linear Dirichlet boundary conditions on a square box, in the regime of small ellipticity contrast (i.e. $|\frac{a^*}{a_*} - 1|$ is sufficiently small). Using different techniques, including generalized Walsh decomposition and concentration bounds, Rossignol [33] has proved a variance bound and a central limit theorem for effective resistance of a resistor network on the discrete torus. We refer to the recent review paper [3] for many other references on the random conductance model. Also in the discrete setting, Mourrat and Otto [27] have studied the correlation structure of the corrector itself. Delmotte and Deuschel [11] and, more recently, Marahrens and Otto [25] derived some annealed estimates of the mixed second derivatives $\nabla_x \nabla_y G(x, y)$ of the Green function for the discrete random elliptic operator; as we mention just after Lemma 4.7, there is a step in our proof which involves bounding a similar quantity.

Other works related to Theorem 1.1 include those of Naddaf and Spencer [28], Conlon and Naddaf [10], and Boivin [5] in the discrete case and Yurinskii [38] in the continuum setting; they also derive upper bounds on the variance of quantities similar to $\tilde{\Gamma}_{L,\beta}$ and $\Gamma_{L,\beta}$. Komorowski and Ryzhik [23] have proved some related moment bounds on ϕ in the discrete case when $d = 1$. If $\beta = 0$ and the dimension is $d = 1$, then equation (1.1) can be integrated, with the solution ϕ written in terms of integrals of $1/a(x)$. In that case it is known that the solution itself may satisfy a central limit theorem after suitable renormalization; see Borgeat and Piatnitski [6] Bal, Garnier, Motsch, Perrier [1] for precise statement of these results. In the multidimensional setting, however, those techniques do not apply.

The basis for our proof of Theorem 1.1 is a normal-approximation technique of Chatterjee [8] (see Theorem 2.2 therein), based on Stein's method of normal approximation. This tool and related notation is explained in Section 2. In Section 3 we give some deterministic PDE estimates (Caccioppoli's inequality and Meyers' estimate) which are used later in the analysis. Section 4 contains the main argument in the proof of Theorem 1.1. Finally, in Section 5 we prove some facts about the periodic Green's function which are used in Section 4.

A few more comments about notation: throughout the article we will use the convention that summation over indices $j \in D_L$ means a summation over $j \in \mathbb{Z}^d \cap D_L$, with $j \in \mathbb{Z}^d$ being understood. For convenience we will also use brackets $\langle f \rangle = \mathbb{E}[f]$ to denote expectation. We also use C to denote deterministic constants that may change from line to line, but do not depend on L or β.

Acknowledgment. I am grateful to Felix Otto for very helpful discussions. This work was partially funded by grant DMS-1007572 from the US National Science Foundation.

2 Normal approximation

In this section we summarize a general approach to normal approximation based on Stein's method, and we establish some notation that will be used throughout the paper. Suppose W is a random variable with $\mathbb{E}[W] = 0$ and $\mathbb{E}[W^2] = 1$, and we wish to estimate

$$\mathbb{E}h(W) - \mathbb{E}h(Y) \tag{2.15}$$

where $Y \sim N(0, 1)$ is a standard normal random variable, h is a Lipschitz continuous function on $\mathbb{R}$ and satisfying $\|h'\|_\infty \leq 1$. Stein's method of normal approximation [35] is based on the following:

Lemma 2.1 (See [8], Lemma 4.2) *Suppose $h : \mathbb{R} \to \mathbb{R}$ is absolutely continuous with bounded derivative, and $Y \sim N(0, 1)$. There exists a solution to*

$$\psi'(x) - x\psi(x) = h(x) - \mathbb{E}[h(Y)], \quad x \in \mathbb{R} \tag{2.16}$$

which satisfies $\|\psi'\|_\infty \leq \sqrt{\frac{2}{\pi}}\, \|h'\|_\infty$ and $\|\psi''\|_\infty \leq 2\|h'\|_\infty$.

Therefore, to estimate (2.15) it suffices to estimate

$$\langle h(W) - h(Y)\rangle = \langle \psi'(W) - W\psi(W)\rangle = \mathrm{Cov}(\langle \psi'(W)\rangle W - \psi(W), W) \tag{2.17}$$

where ψ solves (2.16). In particular, a bound on $\mathrm{Cov}(\langle \psi'(W)\rangle W - \psi(W), W)$ which is independent of h satisfying $\|h'\|_\infty \leq 1$ will imply a bound on $d_{\mathcal{W}}(W, Y)$. Lemma 2.2 below gives us a way of estimating $\mathrm{Cov}(\langle \psi'(W)\rangle W - \psi(W), W)$ when $W = f(Z_1, \ldots, Z_n)$ is a function of a collection of independent random variables. Suppose $Z = (Z_1, \ldots, Z_n) \in \mathcal{Z}^n$ is a random n-tuple in $\mathcal{Z}^n$ having components that are independent, $\mathcal{Z}$ being a given set. Suppose $Z' = (Z_1', \ldots, Z_n') \in \mathcal{Z}^n$ is an independent copy of Z. Define

$$Z^j = (Z_1, \ldots, Z_{j-1}, Z_j', Z_{j+1}, \ldots, Z_n). \tag{2.18}$$

Similarly, for a set $A \subset \{1, \ldots, n\}$, the random n-tuple Z^A is defined by replacing Z_ℓ by Z_ℓ', for all indices $\ell \in A$. For any function $f : \mathcal{Z}^n \to \mathbb{R}$, define

$$\Delta_j f(Z) = f(Z^j) - f(Z).$$

This is a function of both Z and Z' and we sometimes write $\Delta_j f(Z, Z')$ to emphasize this point. If $j \notin A$, then define

$$\Delta_j f(Z^A) = f(Z^{A\cup\{j\}}) - f(Z^A).$$

Let $[n] = \{1, 2, \ldots, n\}$. The following identity is due to Chatterjee [8]:

Lemma 2.2 (See [8] Lemma 2.3) *Suppose $g, f : \mathcal{Z}^n \to \mathbb{R}$ and $\langle g(Z)^2\rangle < \infty$, $\langle f(Z)^2\rangle < \infty$. Then*

$$Cov(g(Z), f(Z)) = \frac{1}{2}\sum_{j=1}^{n} \sum_{\substack{A\subset[n] \\ j\notin A}} K_{n,A}\langle \Delta_j g(Z)\Delta_j f(Z^A)\rangle. \tag{2.19}$$

where $K_{n,A} = |A|!(n - |A| - 1)!/(n!)$.

By applying Lemma 2.2 with $g = f$, one can derive the well-known Efron-Stein inequality [12, 34]:

Lemma 2.3 *Suppose $f : \mathcal{Z}^n \to \mathbb{R}$ and $\langle f(Z)^2 \rangle < \infty$. Then*

$$Var(f(Z)) \leq \frac{1}{2} \sum_{j=1}^{n} \langle |\Delta_j f(Z)|^2 \rangle. \tag{2.20}$$

By applying Lemma 2.2 to $\mathrm{Cov}(\langle \psi'(W) \rangle W - \psi(W), W)$ in (2.17), one obtains the following normal approximation bound, due to Chatterjee [8]:

Theorem 2.4 (See [8] Theorem 2.2) *Let $W = f(Z)$ satisfy $\langle W \rangle = \mu$ and $\langle W^2 \rangle = \sigma^2$. Then*

$$d_{\mathcal{W}}\left(\frac{W-\mu}{\sigma}, Y\right) \leq \frac{1}{2\sigma^3} \sum_{j=1}^{n} \left\langle |\Delta_j f(Z)|^3 \right\rangle + \frac{2}{\sigma^2} Var(\mathbb{E}[T(Z,Z')|Z])^{1/2} \tag{2.21}$$

where $Y \sim N(0,1)$ and

$$T(Z,Z') = \frac{1}{2} \sum_{j=1}^{n} \sum_{\substack{A \subset [n] \\ j \notin A}} K_{n,A} \Delta_j f(Z) \Delta_j f(Z^A).$$

Our goal will be to prove Theorem 1.1 by applying Theorem 2.4 to the random variable $f(Z) = \Gamma_{L,\beta}$. The term $\mathrm{Var}(\mathbb{E}[T(Z,Z')|Z])$ in Theorem 2.4 can be estimated by the Efron-Stein inequality (2.20). To this end, we introduce a third n-tuple $Z'' = (Z_1'', Z_2'', \ldots, Z_n'')$ which is an independent copy of Z, independent of Z'. Let us define

$$Z^k = (Z_1, \ldots, Z_{k-1}, Z_k'', Z_{k+1}, \ldots). \tag{2.22}$$

For any function $g(Z,Z') : \mathcal{Z}^n \times \mathcal{Z}^n \to \mathbb{R}$ we define

$$\Delta_k g(Z,Z') = g(Z^k, Z') - g(Z,Z'). \tag{2.23}$$

In particular, $\Delta_k g(Z,Z') = 0$ if $g(Z,Z')$ does not depend on Z_k. We use the notation g^k to denote the action of replacing Z_k by Z_k'' in the argument of g:

$$g(Z,Z')^k = g(Z^k, Z').$$

Thus, $\Delta_k g(Z,Z') = (g(Z,Z'))^k - g(Z,Z')$. Let us emphasize that Z^k will always refer to (2.22) while Z^j refers to (2.18). The n-tuples Z^j and Z^k have the same law, but the n-tuple denoted by Z^j is not equivalent to Z^k even when the values of the indices k and j are the same.

Now, Lemma 2.3 implies

$$\mathrm{Var}(\mathbb{E}[T(Z,Z')|Z]) \leq \frac{1}{2} \sum_k \mathbb{E}[|\Delta_k h(Z)|^2]$$

where $h(Z) = \mathbb{E}[T(Z,Z')|Z]$ and

$$\Delta_k h(Z) = \mathbb{E}[T(Z^k, Z') \mid Z^k] - \mathbb{E}[T(Z,Z') \mid Z] = \mathbb{E}_{Z'}[T(Z^k, Z') - T(Z,Z') \mid Z, Z^k].$$

Hence,

$$\begin{aligned} \mathrm{Var}(\mathbb{E}[T(Z,Z')|Z]) &\leq \frac{1}{2}\mathbb{E}\left[\sum_k \mathbb{E}_{Z'}[T(Z^k,Z') - T(Z,Z') \mid Z, Z^k]^2\right] \\ &\leq \frac{1}{2} \sum_k \mathbb{E}\left[|T(Z^k,Z') - T(Z,Z')|^2\right]. \end{aligned} \tag{2.24}$$

Recalling that

$$T(Z,Z') = \frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{A\subset[n]\\ j\notin A}} K_{n,A}\Delta_j f(Z)\Delta_j f(Z^A),$$

we conclude that

$$\begin{aligned}
\mathrm{Var}(\mathbb{E}[T(Z,Z')|Z]) &\leq \frac{1}{8}\sum_k \mathbb{E}\left[\left|\sum_{j=1}^{n}\sum_{\substack{A\subset[n]\\ j\notin A}} K_{n,A}\left(\Delta_j f(Z)\Delta_j f(Z^A) - (\Delta_j f(Z))^k(\Delta_j f(Z^A))^k\right)\right|^2\right] \\
&\leq \sum_k \mathbb{E}\left[\left|\sum_{j=1}^{n}\sum_{\substack{A\subset[n]\\ j\notin A}} K_{n,A}(\Delta_j f(Z) - (\Delta_j f(Z))^k)\Delta_j f(Z^A)\right|^2\right] \\
&\quad + \sum_k \mathbb{E}\left[\left|\sum_{j=1}^{n}\sum_{\substack{A\subset[n]\\ j\notin A}} K_{n,A}(\Delta_j f(Z))^k(\Delta_j f(Z^A) - (\Delta_j f(Z^A))^k)\right|^2\right]. \qquad (2.25)
\end{aligned}$$

Let us clarify the notation here. In the case $k \neq j$, we have

$$(\Delta_j f(Z))^k = f(Z_1,\ldots,Z_j',\ldots,Z_k'',\ldots,Z_n) - f(Z_1,\ldots,Z_j,\ldots,Z_k'',\ldots,Z_n) = \Delta_j f(Z^k,Z').$$

If $k = j$, then we have $(\Delta_j f(Z))^k = \Delta_j f(Z^k,Z') = f(Z^j) - f(Z^k)$. Nevertheless, for all j and k we have

$$\Delta_j f(Z) - (\Delta_j f(Z))^k = -\Delta_k\Delta_j f(Z) = -\Delta_k(\Delta_j f(Z,Z')).$$

So, the first sum on the right side of (2.25) is

$$\sum_k \mathbb{E}\left[\left|\sum_{j=1}^{n}(\Delta_k(\Delta_j f(Z)))\overline{\Delta_j f(Z^A)}\right|^2\right], \qquad (2.26)$$

and the second sum is

$$\sum_k \mathbb{E}\left[\left|\sum_{j=1}^{n}(\Delta_j f(Z))^k\Delta_k\overline{\Delta_j f(Z^A)}\right|^2\right], \qquad (2.27)$$

where we have used the notation $\overline{\Delta_j f(Z^A)}$ to indicate averaging with respect to the set A. Specifically, if $S_{n,j}$ denotes the collection of all subsets $A \subset \{1,\ldots,n\}$ which do not contain the index j, and $H_A : S_{n,j} \to \mathbb{R}$, then

$$\overline{H_A} = \sum_{\substack{A\subset[n]\\ j\notin A}} K_{n,A}H_A = \sum_{A\in S_{n,j}} K_{n,A}H_A. \qquad (2.28)$$

The weights $K_{n,A} \geq 0$ define a probability measure on $S_{n,j}$: $\sum_{A\in S_{n,j}} K_{n,A} = 1$.

3 Deterministic estimates for solutions of the elliptic equation

In proving Theorem 1.1 we will make use of some regularity estimates – Caccioppoli's inequality and Meyers' estimate – that apply to solutions of elliptic PDEs. These estimates rely only on the uniform ellipticity assumption, not on the statistical structure of the coefficient $a(x)$ or on the periodicity.

Cacciopoli's inequality

if $\bar{u}_D$ is the average of a function u over a bounded domain D, then the Poincaré inequality is $\|u - \bar{u}_D\|_{L^2(D)} \le C_D \|\nabla u\|_{L^2(D)}$. For solutions of elliptic equations, Cacciopoli's inequality gives the reverse inequality, enabling control of ∇u by u itself. The basic estimate is:

Lemma 3.1 *Let $d \ge 1$. There is a constant K such that if $R > 0$ and $u \in H^1(B_R(x_0))$ is a weak solution to $-\nabla \cdot (a\nabla u) + \beta u = \nabla \cdot \xi$ for $x \in B_R(x_0)$, with $\xi \in (L^2(B_R))^d$, then*

$$\int_{B_{\frac{R}{2}}(x_0)} |\nabla u|^2 \, dx \le K \left(\int_{B_R(x_0)} |\xi|^2 \, dx + \frac{1}{R^2} \int_{B_R(x_0)} (u(x) - b)^2 \, dx + \beta b^2 R^d \right) \tag{3.29}$$

holds for any constant $b \in \mathbb{R}$.

Lemma 3.1 and variants are a consequence of the following:

Lemma 3.2 *Let $K_1 = 2/a_*$, $K_2 = (2/a_*) + 8(a^*/a_*)^2$, and $K_3 = (2/a_*) + 2/(a_*)^2$. Let Q be a bounded open subset of $\mathbb{R}^d$ with smooth boundary. If $\beta \ge 0$ and $u \in H^1(Q)$ is a weak solution to $-\nabla \cdot (a\nabla u) + \beta u = f + \nabla \cdot \xi$ for $x \in Q$, with $f \in L^2(Q)$ and $\xi \in (L^2(Q))^d$, then*

$$\begin{aligned} \int_Q \varphi^2 |\nabla u|^2 \, dx &\le K_1 \int_Q f(u-b)\varphi^2 \, dx - K_1 \beta \int_Q u(u-b)\varphi^2 \, dx \\ &\quad + K_2 \int_Q |\nabla \varphi|^2 (u-b)^2 \, dx + K_3 \int_Q |\xi|^2 \varphi^2 \, dx \end{aligned} \tag{3.30}$$

holds for any smooth function $\varphi \ge 0$ which vanishes on the boundary of Q, and any constant $b \in \mathbb{R}$.

For proofs of Lemma 3.1 and Lemma 3.2, see [29] (also [14], for example). The factor R^{-2} in (3.29) comes from choosing a test function φ in (3.30) with $|\nabla \varphi| \le R^{-1}$. There is nothing special about the balls B_R and B_{2R} in Lemma 3.1; for other nested domains whose boundaries are separated by distance R, a similar bound follows directly from Lemma 3.2.

Meyers' Estimate

We also will make use of a well-known regularity estimate of Meyers [26] which shows that if $u \in H^1_{loc}$ satisfies $-\nabla \cdot (a\nabla u) + \beta u = 0$, then $\nabla u \in L^p_{loc}$ for some $p > 2$. Moreover, ∇u may be bounded as follows:

Lemma 3.3 *There is a constant $p^* > 2$, depending on d and a^*/a_*, such that the following holds for all $p \in [2, p^*]$: there is C such that if $R > 0$ and $u \in H^1(B_{4R}(y))$ satisfies $-\nabla \cdot (a\nabla u) + \beta u = 0$ in $B_{4R}(y)$, then*

$$\left(⨍_{B_R} |\nabla u|^p \, dx \right)^{1/p} \le C R^{-1} \left(⨍_{B_{4R}} u^2 \, dx \right)^{1/2}.$$

Proof of Lemma 3.3: This is a consequence of Theorem 2 of [26] and Lemma 3.2. Since u satisfies $-\nabla \cdot a\nabla u = h$ with $h = -\beta u$, we may apply Theorem 2 of Meyers' [26] to u (with $p_1 = 2$, $r = 2$), to conclude that for $p > 2$ sufficiently small,

$$\begin{aligned} \left(⨍_{B_R} |\nabla u|^p \right)^{1/p} &\le C R^{-1} \left(⨍_{B_{2R}} |u|^2 \right)^{1/2} + C R \left(⨍_{B_{2R}} |h|^2 \, dx \right)^{1/2} \\ &= C R^{-1} \left(⨍_{B_{2R}} |u|^2 \right)^{1/2} + C R^{-1} \left(R^4 \beta^2 ⨍_{B_{2R}} |u|^2 \, dx \right)^{1/2}. \end{aligned} \tag{3.31}$$

Now we estimate the last term in (3.31). Let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth function supported in $B_{3R}(y)$ and satisfying $\varphi(x) = 1$ for all $x \in B_{2R}(y)$ and satisfying $|\nabla\varphi| \le C/R$. Applying Lemma 3.2 with this function φ, with $b = 0$ and with $Q = B_{3R}(y)$, we conclude that

$$\beta \int_{B_{2R}} u^2\,dx \le \beta \int_{B_{3R}} u^2\varphi^2\,dx \le C\int_{B_{3R}} u^2|\nabla\varphi|^2\,dx \le CR^{-2}\int_{B_{3R}} u^2\,dx. \tag{3.32}$$

Now we apply Lemma 3.2 once more, this time in $Q = B_{4R}$, using a function $\varphi : \mathbb{R}^d \to [0,1]$ supported in $B_{4R}(y)$ and satisfying $\varphi = 1$ in $B_{3R}(y)$ and $|\nabla\varphi| \le C/R$. We conclude

$$\beta \int_{B_{3R}} u^2\,dx \le \beta \int_{B_{4R}} u^2\varphi^2\,dx \le C\int_{B_{4R}} u^2|\nabla\varphi|^2\,dx \le CR^{-2}\int_{B_{4R}} u^2\,dx. \tag{3.33}$$

Combining (3.32) and (3.33) we obtain

$$R^4\beta^2 \fint_{B_{2R}} u^2\,dx \le C\fint_{B_{4R}} u^2\,dx.$$

This combined with (3.31) implies the result. □

4 Application to the elliptic problem

In this section we prove Theorem 1.1 by applying Theorem 2.4 to the random variable $f(Z) = \Gamma_{L,\beta}(Z)$ defined by (1.2). In this case, the indices j in Theorem 2.4 now run over the set $\mathbb{Z}^d \cap D_L$. The first step is to compute and estimate the terms $\Delta_j\Gamma$ and $\Delta_k\Delta_j\Gamma$ which appear in the sums (2.26) and (2.27).

Estimating $\Delta_j\Gamma$ and $\Delta_k\Delta_j\Gamma$

We will make use of the following chain rule and product rule for discrete differences:

$$\Delta_j(f(Z)^2) = (\Delta_j f(Z))(f(Z^j) + f(Z)) \tag{4.34}$$

and

$$\Delta_j(f(Z)g(Z)) = \frac{1}{2}(\Delta_j f(Z))(g(Z^j) + g(Z)) + \frac{1}{2}(f(Z^j) + f(Z))(\Delta_j g(Z)). \tag{4.35}$$

Let us introduce the notation $a^j = a(x, Z^j)$, $a^k = a(x, Z^k)$, $\phi^j = \phi(x, Z^j)$, $\phi^k = \phi(x, Z^k)$, according to (2.18) and (2.22). By the structural condition (1.7) observe that the functions

$$\Delta_j a = a(x, Z^j) - a(x, Z), \qquad \text{and} \qquad \Delta_k a = a(x, Z^k) - a(x, Z)$$

are supported on the sets $B_\tau(j)$ and $B_\tau(k)$ respectively. Furthermore, (1.7) implies that

$$\Delta_k\Delta_j a = \Delta_k(a(x, Z^j) - a(x, Z)) \equiv 0, \quad \text{if } \; dist(k, j) \ge 2\tau. \tag{4.36}$$

Lemma 4.1 *There is a constant C such that*

$$L^d|\Delta_j\Gamma(Z)| \le C(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j))$$

holds for all $L > 1$, $\beta \ge 0$, $j \in \mathbb{Z}^d$, where

$$\hat{\Phi}_j(Z) = \left(\int_{B_\tau(j)} |\nabla\phi(x, Z) + e_1|^2\,dx\right)^{1/2}.$$

Moreover, for any $q > 1$, there is C_q such that

$$L^{qd}\mathbb{E}[|\Delta_j \Gamma(Z)|^q] \le C_q \mathbb{E}[|\hat{\Phi}_0(Z)|^{2q}]. \tag{4.37}$$

and

$$L^{qd}\mathbb{E}[|\overline{\Delta_j \Gamma(Z^A)}|^q] \le C_q \mathbb{E}[|\hat{\Phi}_0(Z)|^{2q}]. \tag{4.38}$$

hold for all $L > 1$, $\beta \ge 0$, $j \in \mathbb{Z}^d$.

Lemma 4.2 *There is a constant C, independent of $L > 1$ and $\beta \ge 0$ such that*

$$\begin{aligned} L^d|\Delta_k \Delta_j \Gamma(Z)| &\le C\left(\hat{\Phi}_k^2(Z) + \hat{\Phi}_k^2(Z^j) + \hat{\Phi}_k^2(Z^k) + \hat{\Phi}_k^2(Z^{jk})\right) \\ &\quad + C\left(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j) + \hat{\Phi}_j^2(Z^k) + \hat{\Phi}_j^2(Z^{jk})\right) \end{aligned} \tag{4.39}$$

hold for all $k, j \in \mathbb{Z}^d$. Moreover,

$$\begin{aligned} L^{2d}|\Delta_k \Delta_j \Gamma(Z)|^2 &\le C\left(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j) + \hat{\Phi}_j^2(Z^k) + \hat{\Phi}_j^2(Z^{jk})\right) \\ &\quad \times \int_{B_\tau(j)} |\nabla \Delta_k \phi|^2 + |\nabla \Delta_k \phi^j|^2 \, dx \end{aligned} \tag{4.40}$$

holds for all j, k with $dist(k, j) \ge 2\tau$.

Proof of Lemma 4.1: Using (4.34) and (4.35) and the symmetry of a we compute:

$$\begin{aligned} L^d \Delta_j \Gamma(Z) &= \frac{1}{2}\int_{D_L} (\nabla\phi^j + e_1)\cdot(\Delta_j a)(\nabla\phi^j + e_1) + (\nabla\phi + e_1)\cdot(\Delta_j a)(\nabla\phi + e_1)\, dx \\ &\quad + \int_{D_L} \frac{1}{2}(\nabla\Delta_j\phi)\cdot(a^j + a)(\nabla\phi^j + \nabla\phi + 2e_1)\, dx + \beta\int_{D_L}(\Delta_j\phi)(\phi^j + \phi)\, dx. \end{aligned} \tag{4.41}$$

Due to (1.10), we have

$$\begin{aligned} &\int_{D_L} (\nabla\Delta_j\phi)\cdot a^j(\nabla\phi^j + e_1) + \beta(\Delta_j\phi)\phi^j\, dx = 0, \\ &\int_{D_L} (\nabla\Delta_j\phi)\cdot a(\nabla\phi + e_1) + \beta(\Delta_j\phi)\phi\, dx = 0. \end{aligned}$$

Using that observation we simplify (4.41) to

$$\begin{aligned} L^d \Delta_j \Gamma(Z) &= \frac{1}{2}\int_{D_L} (\nabla\phi^j + e_1)\cdot(\Delta_j a)(\nabla\phi^j + e_1) + (\nabla\phi + e_1)\cdot(\Delta_j a)(\nabla\phi + e_1)\, dx \\ &\quad + \frac{1}{2}\int_{D_L} \nabla\Delta_j\phi\cdot(\Delta_j a)(\nabla\phi + e_1)\, dx - \frac{1}{2}\int_{D_L} \nabla\Delta_j\phi\cdot(\Delta_j a)(\nabla\phi^j + e_1)\, dx \\ &= \frac{1}{2}\int_{D_L} (\nabla\phi^j + e_1)\cdot(\Delta_j a)(\nabla\phi^j + e_1) + (\nabla\phi + e_1)\cdot(\Delta_j a)(\nabla\phi + e_1)\, dx \\ &\quad - \frac{1}{2}\int_{D_L} \nabla\Delta_j\phi\cdot(\Delta_j a)\nabla\Delta_j\phi\, dx \\ &= \int_{D_L} (\nabla\phi^j + e_1)\cdot(\Delta_j a)(\nabla\phi + e_1)\, dx. \end{aligned} \tag{4.42}$$

Because $\Delta_j a = a(x, Z^j) - a(x, Z)$ vanishes outside $B_\tau(j)$ (by (1.7)), we then infer that

$$L^d|\Delta_j \Gamma(Z)| \leq C \int_{B_\tau(j)} |\nabla \phi^j + e_1|^2 \, dx + C \int_{B_\tau(j)} |\nabla \phi + e_1|^2 \, dx = C(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j)).$$

Since ϕ is stationary with respect to integer shifts and because Z and Z^j have the same law, the random variables $\hat{\Phi}_j(Z)$, $\hat{\Phi}_j(Z^j)$, and $\hat{\Phi}_0(Z)$ are identically distributed. Therefore, for any $q > 1$ there is a constant C_q such that

$$L^{qd}\mathbb{E}[|\Delta_j \Gamma(Z)|^q] \leq C_q \mathbb{E}[|\hat{\Phi}_j(Z)|^{2q}] + C_q \mathbb{E}[|\hat{\Phi}_j(Z^j)|^{2q}] = 2C_q \mathbb{E}[|\hat{\Phi}_0(Z)|^{2q}], \tag{4.43}$$

which is (4.37).

Now we prove (4.38). Jensen's inequality implies

$$|\overline{\Delta_j \Gamma(Z^A)}|^q = |\sum_{\substack{A \subset [n] \\ j \notin A}} K_{n,A} \Delta_j \Gamma(Z^A)|^q \leq \sum_{\substack{A \subset [n] \\ j \notin A}} K_{n,A} |\Delta_j \Gamma(Z^A)|^q.$$

Therefore from (4.37) we obtain

$$L^{qd}\mathbb{E}[|\overline{\Delta_j \Gamma(Z^A)}|^q] \leq L^{qd} \sum_{\substack{A \subset [n] \\ j \notin A}} K_{n,A} \mathbb{E}[|\Delta_j \Gamma(Z^A)|^q] \leq C \mathbb{E}[|\hat{\Phi}_0(Z)|^{2q}].$$

□

Proof of Lemma 4.2: Starting from (4.42) and using (4.34) and (4.35) we compute

$$\begin{aligned} L^d \Delta_k \Delta_j \Gamma(Z) &= \frac{1}{2} \int_{D_L} (\nabla \phi^{jk} + e_1) \cdot (\Delta_k \Delta_j a)(\nabla \phi^k + e_1) + (\nabla \phi^j + e_1) \cdot (\Delta_k \Delta_j a)(\nabla \phi + e_1) \, dx \\ &+ \frac{1}{4} \int_{D_L} \Delta_k \nabla \phi^j \cdot ((\Delta_j a)^k + (\Delta_j a))(\nabla \phi^k + e_1 + \nabla \phi + e_1) \\ &+ \frac{1}{4} \int_{D_L} \Delta_k \nabla \phi \cdot ((\Delta_j a)^k + (\Delta_j a))(\nabla \phi^{jk} + e_1 + \nabla \phi^j + e_1). \end{aligned}$$

The matrices $\Delta_j a$ and $\Delta_k a$ are zero outside $B_\tau(j)$ and $B_\tau(k)$, respectively. Also, $\Delta_k \nabla \phi = \Delta_k(\nabla \phi + e_1) = (\nabla \phi^k + e_1) - (\nabla \phi + e_1)$. Therefore, by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} L^d|\Delta_k \Delta_j \Gamma(Z)| &\leq C \left(\hat{\Phi}_k^2(Z) + \hat{\Phi}_k^2(Z^j) + \hat{\Phi}_k^2(Z^k) + \hat{\Phi}_k^2(Z^{jk}) \right) \\ &\quad + C \left(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j) + \hat{\Phi}_j^2(Z^k) + \hat{\Phi}_j^2(Z^{jk}) \right). \end{aligned} \tag{4.44}$$

for all $j, k \in \mathbb{Z}^d$. This is (4.39).

If $dist(k, j) \geq 2\tau$ then $(\Delta_k \Delta_j a) \equiv 0$ and $(\Delta_j a)^k = \Delta_j a$, by (4.36). So, in this case we have

$$\begin{aligned} L^d \Delta_k \Delta_j \Gamma(Z) &= \frac{1}{2} \int_{D_L} (\Delta_k \nabla \phi^j) \cdot (\Delta_j a)(\nabla \phi^k + e_1 + \nabla \phi + e_1) \\ &+ \frac{1}{2} \int_{D_L} (\Delta_k \nabla \phi) \cdot (\Delta_j a)(\nabla \phi^{jk} + e_1 + \nabla \phi^j + e_1). \end{aligned}$$

Applying Cauchy-Schwarz to this, using the fact that $\Delta_j a$ is supported on $B_\tau(j)$, we obtain

$$\begin{aligned} L^{2d}|\Delta_k\Delta_j\Gamma(Z)|^2 &\leq C\int_{B_\tau(j)}|\nabla(\Delta_k\phi^j)|^2\,dx\left(\int_{B_\tau(j)}|\nabla\phi^k+e_1|^2\,dx+\int_{B_\tau(j)}|\nabla\phi+e_1|^2\,dx\right)\\ &\quad +C\int_{B_\tau(j)}|\nabla(\Delta_k\phi)|^2\,dx\left(\int_{B_\tau(j)}|\nabla\phi^{jk}+e_1|^2\,dx+\int_{B_\tau(j)}|\nabla\phi^j+e_1|^2\,dx\right) \end{aligned} \tag{4.45}$$

if $dist(k,j)\geq 2\tau$. The implies (4.40). □

Relation to the periodic Green's function

The function $w_k=\Delta_k\phi=\phi^k-\phi\in H^1_{per}(D_L)$ which appears in Lemma 4.2 satisfies the equation

$$-\nabla\cdot(a\nabla w_k)+\beta w_k=\nabla\cdot(\Delta_k a)(\nabla\phi^k+e_1), \tag{4.46}$$

and the distribution on the right side of (4.46) is supported on $\overline{B_\tau(k)}$. Choosing w_k itself as a test function for (4.46), we obtain the bound

$$\int_{D_L}|\nabla w_k|^2\,dx\leq\left(\frac{a^*}{a_*}\right)^2\int_{B_\tau(k)}|\nabla\phi^k+e_1|^2\,dx=\left(\frac{a^*}{a_*}\right)^2\hat{\Phi}_k^2(Z^k). \tag{4.47}$$

Later it will be convenient to normalize the function $w_k=\Delta_k\phi$ by defining

$$\tilde{w}_k=\hat{\Phi}_k(Z^k)^{-1}w_k=\left(\int_{B_\tau(k)}|\nabla\phi^k+e_1|^2\,dx\right)^{-1/2}w_k. \tag{4.48}$$

The following estimate relates w_k to the periodic Green's function, and it will enable us to control the decay of $|\nabla w_k|^2$ away from $B_\tau(k)$ (using Caccioppoli's inequality). This connection between the Green's function and quantities analogous to $\Delta_k\phi$ has been used in other works, as well (e.g. [28, 17, 16]).

Lemma 4.3 *Let $d\geq 1$, and let $G=G(x,y,Z)$ be the periodic Green's function associated with the coefficient $a(x,Z)$:*

$$-\nabla_x\cdot(a(x,Z)\nabla_x G)+\beta G=\delta_y(x)-|D_L|^{-1},$$

normalized by $\int_{D_L}G(x,y)\,dx=0$ in the case $\beta=0$. There is a constant C (depending only on d,a_*,a^*) such that for any $L>2$, any $k\in D_L\cap\mathbb{Z}^d$, and any open set $A\subset D_L$ with $dist(A,B_\tau(k))>0$, we have

$$\int_A(\Delta_k\phi)^2\,dy\leq C\hat{\Phi}_k^2(Z^k)\int_{y\in A}\int_{x\in B_\tau(k)}|\nabla_x G(x,y)|^2\,dx\,dy. \tag{4.49}$$

with probability one.

Proof of Lemma 4.3: Let us define $\xi_k=(\Delta_k a)(\nabla\phi^k+e_1)$ which is supported in $B_\tau(k)$. Let $v\in H^1_{per}(D_L)$ satisfy

$$-\nabla\cdot(a\nabla v)+\beta v=\Delta_k\phi\mathbb{I}_A(x)-\frac{1}{D_L}\int_A\Delta_k\phi(x)\,dx.$$

By using (4.46) and the fact that $\int_{D_L} \Delta_k \phi(x)\, dx = 0$, we have

$$
\begin{aligned}
\int_A (\Delta_k \phi(x))^2\, dx &= \int_{D_L} (\mathbb{I}_A(x) \Delta_k \phi(x)) \Delta_k \phi(x)\, dx \\
&= \int_{D_L} \nabla v \cdot a(x) \nabla (\Delta_k \phi) + \beta v (\Delta_k \phi)\, dx \\
&= -\int_{D_L} \xi_k(x) \cdot \nabla v(x)\, dx \le \left(\int_{B_\tau(k)} |\xi_k|^2 \right)^{1/2} \left(\int_{B_\tau(k)} |\nabla v|^2 \right)^{1/2}. \quad (4.50)
\end{aligned}
$$

On the other hand,

$$
v(x) = \int_A G(x,y) \Delta_k \phi(y)\, dy, \qquad \nabla v(x) = \int_A \nabla_x G(x,y) \Delta_k \phi(y)\, dy
$$

hold for almost every x outside A. Therefore, by Cauchy-Schwarz we have

$$
|\nabla v(x)|^2 \le \int_A |\nabla_x G(x,y)|^2\, dy \int_A (\Delta_k \phi(y))^2\, dy.
$$

for almost every x in $B_\tau(j)$. Also, $\int_{B_\tau(k)} |\xi_k|^2\, dx \le C_2^2 \hat{\Phi}_k$, by (1.7). Combining this with (4.50) we obtain (4.49). □

In view of Lemma 4.2 and Lemma 4.3, we see that estimates of the Green's function will play an important role in estimating $\Delta_k \Delta_j \Gamma$. We will make use of the following bounds, proved later in Section 5. The first is a bound on the decay of $G(x,y)$ which is uniform with respect to the probability measure $\mathbb{P}$. The second, is a version of Lemma 2.9 in [17], and it is also uniform with respect to the probability measure $\mathbb{P}$. Recall the definitions (1.4) and (1.5) of $dist(x,y)$ and $B_r(x)$.

Lemma 4.4 *Let $d \ge 3$. There is a constant $C > 0$, depending only on d, a^*, and a_*, such that*

$$
|G(x,y)| \le C\, dist(x,y)^{2-d}
$$

holds for all $x, y \in D_L$ with $x \ne y$, all $L \ge 1$ and $\beta \ge 0$.

Lemma 4.5 *Let $d = 2$. There is a constant $C > 0$, depending only on d, a^*, and a_*, such that for all $R > 0$, $\beta \ge 0$, $L \ge 1$,*

$$
\int_{B_R(x_0)} |\nabla_x G(x,y)|^2\, dx \le C
$$

holds for all $x_0 \in D_L$ and $y \in D_L \setminus \overline{B_{2R}(x_0)}$.

Proof of Theorem 1.1.

Because of the stationarity assumption, moments of $\hat{\Phi}_0$ are controlled by the same moments of Φ_0: for any $q \ge 1$ there is a constant C_q such that

$$
\mathbb{E}[|\hat{\Phi}_0|^q] \le C_q \mathbb{E}[|\Phi_0|^q] \quad (4.51)
$$

for all $L \geq 1$ and $\beta \geq 0$. This is proved in Lemma 4.2 of [29], for example. Therefore, according to Lemma 4.1, we can bound the first term on the right side of (2.21) as

$$\frac{1}{2\sigma^3} \sum_{j \in D_L} \langle |\Delta_j \Gamma(Z)|^3 \rangle \leq C \frac{L^d}{\sigma^3} L^{-3d} \mathbb{E}[|\hat{\Phi}_0(Z)|^6] \leq C \frac{L^d}{\sigma^3} L^{-3d} \mathbb{E}[|\Phi_0(Z)|^6]. \tag{4.52}$$

As shown already, the term $\mathrm{Var}(\mathbb{E}[T(Z,Z')|Z])$ in (2.21) is controlled by the sum of (2.26) and (2.27). We now focus on estimating (2.26). By Minkowsi's inequality we have

$$\left\langle \left| \sum_{j \in D_L} (\Delta_k \Delta_j \Gamma(Z)) \overline{\Delta_j \Gamma(Z^A)} \right|^2 \right\rangle \leq \left(\sum_{j \in D_L} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle^{1/2} \right)^2 . \tag{4.53}$$

It will be convenient to split up this sum over domains resembling dyadic annuli centered around the cube $Q_k = k + [0,1)^d$. Let N denote the smallest integer such that $2^N \tau \geq L/4$. Hence, $N = O(\log L)$ and $L/4 \leq 2^N \tau \leq L/2$. Then, let A_0^k denote the union of cubes that are close to Q_k:

$$A_0^k = \{x \in D_L \mid \; x \in Q_j, \;\; 0 \leq dist(Q_j, Q_k) \leq 2\tau\},$$

and for $\ell = 1, 2, \ldots, N-1$ let A_ℓ^k denote the set

$$A_\ell^k = \{x \in D_L \mid \; x \in Q_j, \;\; 2^\ell \tau < dist(Q_j, Q_k) \leq 2^{\ell+1}\tau\}.$$

Again, we use $dist(Q_j, Q_k)$ to refer to distance on the torus D_L (modulo $L\mathbb{Z}^d$) between sets Q_j and Q_k. Finally, define A_N^k by

$$A_N^k = \{x \in D_L \mid \; x \in Q_j, \;\; 2^N \tau < dist(Q_j, Q_k)\}.$$

Each set A_ℓ^k is a union of cubes, and has Lebesgue measure $|A_\ell^k| = O(2^{\ell d})$. Let $A_+^k = D_L \setminus A_0^k$. Observe that $j \in A_\ell^k$ if and only if $Q_j \subset A_\ell^k$. Similarly, $j \in A_+^k$ if and only if $Q_j \in A_\ell^k$ for some $\ell \geq 1$. Thus,

$$D_L = A_0^k \cup A_+^k = \bigcup_{\ell=0}^N A_\ell^k.$$

In this way, we write the sum appearing in (4.53) as:

$$\begin{aligned} \sum_{j \in D_L} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle^{1/2} \;\; &= \;\; \sum_{j \in A_0^k \cup A_1^k} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle^{1/2} \\ &\quad + \sum_{\ell=2}^N \sum_{j \in A_\ell^k} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle^{1/2}. \end{aligned} \tag{4.54}$$

We will bound the terms in (4.54) using the following Lemma. The first estimate will bound the terms with indices $j \in A_0^k \cup A_1^k$. The second estimate will be used for the other indices.

Lemma 4.6 *For $p > 1$ there is a constant C_p such that if $k, j \in D_L$ with $dist(k,j) \geq 2\tau$ then*

$$L^{4d} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle \leq C_p \langle \hat{\Phi}_0^{4q} \rangle^{\frac{4}{2q}} \left\langle \left(\int_{B_\tau(j)} |\nabla \tilde{w}_k|^2 \, dx \right)^p \right\rangle^{1/p} \tag{4.55}$$

where $q = 2p/(p-1)$. Also, there is a constant C such that

$$L^{4d} \langle |\Delta_k \Delta_j \Gamma(Z) \overline{\Delta_j \Gamma(Z^A)}|^2 \rangle \leq C \langle \hat{\Phi}_0^8 \rangle \tag{4.56}$$

holds for all $j, k \in D_L$, $L > 1$, $\beta \geq 0$.

Proof: First we prove (4.56). By Lemma 4.2, we always have

$$\begin{aligned} L^{2d}|\Delta_k\Delta_j\Gamma(Z)|^2 &\leq C\left(\hat\Phi_j^2(Z)+\hat\Phi_j^2(Z^j)+\hat\Phi_j^2(Z^k)+\hat\Phi_j^2(Z^{jk})\right) \\ &\qquad \times\left(\hat\Phi_k^2(Z)+\hat\Phi_k^2(Z^j)+\hat\Phi_k^2(Z^k)+\hat\Phi_k^2(Z^{jk})\right). \end{aligned} \tag{4.57}$$

Moreover, the terms $\hat\Phi_j^2(Z)$, $\hat\Phi_j^2(Z^j)$, $\hat\Phi_j^2(Z^k)$, $\hat\Phi_j^2(Z^{jk})$, $\hat\Phi_k^2(Z)$, $\hat\Phi_k^2(Z^j)$, $\hat\Phi_k^2(Z^k)$, $\hat\Phi_k^2(Z^{jk})$ are identically distributed, all having the same distribution as $\hat\Phi_0(Z)$. By Lemma 4.1, we know that

$$L^{2d}\langle|\overline{\Delta_j\Gamma(Z^A)}|^{2q}\rangle^{1/q}\leq C\langle\hat\Phi_0^{4q}\rangle^{\frac{1}{q}}. \tag{4.58}$$

Therefore, by applying the generalized Hölder inequality with $\frac14+\frac14+\frac12=1$ we obtain

$$\begin{aligned} L^{2d}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle &\leq C\Big\langle\left(\hat\Phi_j^2(Z)+\hat\Phi_j^2(Z^j)+\hat\Phi_j^2(Z^k)+\hat\Phi_j^2(Z^{jk})\right) \\ &\qquad\times\left(\hat\Phi_k^2(Z)+\hat\Phi_k^2(Z^j)+\hat\Phi_k^2(Z^k)+\hat\Phi_k^2(Z^{jk})\right)|\overline{\Delta_j\Gamma(Z^A)}|^2\Big\rangle \\ &\leq C\langle\hat\Phi_0^8\rangle^{\frac14}\langle\hat\Phi_0^8\rangle^{\frac14}\langle|\overline{\Delta_j\Gamma(Z^A)}|^4\rangle^{1/2} \\ &\leq C\langle\hat\Phi_0^8\rangle^{\frac14}\langle\hat\Phi_0^8\rangle^{\frac14}L^{-2d}\langle\hat\Phi_0^8\rangle^{\frac12}. \end{aligned} \tag{4.59}$$

This proves (4.56).

If $dist(k,j)\geq 2\tau$, Lemma 4.2 tells us that

$$\begin{aligned} L^{2d}|\Delta_k\Delta_j\Gamma(Z)|^2 &\leq C\left(\hat\Phi_j^2(Z)+\hat\Phi_j^2(Z^j)+\hat\Phi_j^2(Z^k)+\hat\Phi_j^2(Z^{jk})\right) \\ &\qquad\times\left(\hat\Phi_k^2(Z^k)\int_{B_\tau(j)}|\nabla\tilde w_k|^2\,dx+\hat\Phi_k^2(Z^{jk})\int_{B_\tau(j)}|\nabla\tilde w_k^j|^2\,dx\right), \end{aligned} \tag{4.60}$$

where

$$\tilde w_k=\hat\Phi_k(Z^k)^{-1}\Delta_k\phi,\qquad \tilde w_k^j=\hat\Phi_k(Z^{jk})^{-1}\Delta_k\phi^j.$$

Let $p>1$, let $q=2p/(p-1)$ so that $\frac1p+\frac1{2q}+\frac1{2q}+\frac1q=1$. Then by (4.60) and the generalized Hölder inequality,

$$\begin{aligned} &L^{2d}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle \\ &\quad\leq C\langle\hat\Phi_0^{4q}\rangle^{\frac{1}{2q}}\langle\hat\Phi_0^{4q}\rangle^{\frac{1}{2q}}\langle(\int_{B_\tau(j)}|\nabla\tilde w_k|^2\,dx)^p\rangle^{1/p}\langle|\overline{\Delta_j\Gamma(Z^A)}|^{2q}\rangle^{1/q} \\ &\qquad+C\langle\hat\Phi_0^{4q}\rangle^{\frac{1}{2q}}\langle\hat\Phi_0^{4q}\rangle^{\frac{1}{2q}}\langle(\int_{B_\tau(j)}|\nabla\tilde w_k^j|^2\,dx)^p\rangle^{1/p}\langle|\overline{\Delta_j\Gamma(Z^A)}|^{2q}\rangle^{1/q}. \end{aligned} \tag{4.61}$$

If $j\neq k$, then $\Delta_k\phi$ and $\Delta_k\phi^j$ have the same distribution (since (Z,Z^k) and (Z^j,Z^{jk}) have the same joint distribution). Similarly, $\tilde w_k$ and $\tilde w_k^j$ must have the same distribution. Therefore,

$$\Big\langle(\int_{B_\tau(j)}|\nabla\tilde w_k^j|^2\,dx)^p\Big\rangle^{1/p}=\Big\langle(\int_{B_\tau(j)}|\nabla\tilde w_k|^2\,dx)^p\Big\rangle^{1/p}.$$

holds for all $j\neq k$. Combining this observation with (4.61) and (4.58) we obtain

$$L^{4d}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle\leq C\langle\hat\Phi_0^{4q}\rangle^{\frac{4}{2q}}\Big\langle(\int_{B_\tau(j)}|\nabla\tilde w_k|^2\,dx)^p\Big\rangle^{1/p}. \tag{4.62}$$

This completes the proof of Lemma 4.6. □

Now we return to (4.54). For the first sum on the right side of (4.54), over indices j near k, we apply Lemma 4.6 to obtain

$$L^{2d} \sum_{j\in A_0^k\cup A_1^k} \langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2} \le C(|A_0^k|+|A_1^k|)\langle\hat{\Phi}_0^8\rangle^{1/2}. \tag{4.63}$$

For the second sum in (4.54), we apply Lemma 4.6 again to obtain

$$L^{2d}\sum_{\ell=2}^{N}\sum_{j\in A_\ell^k}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2} \quad\le\quad C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}\sum_{\ell=2}^{N}\sum_{j\in A_\ell^k}\Big\langle\int_{B_\tau(j)}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}. \tag{4.64}$$

From our definition of the annuli A_ℓ^k and $\tau>\sqrt{d}$, we see that

$$\bigcup_{\substack{j\in A_\ell^k\\ \ell\ge 2}} B_\tau(j)\subset\bigcup_{\substack{j\in A_\ell^k\\ \ell\ge 1}} Q_j.$$

Furthermore, each ball $B_\tau(j)$ intersects only finitely many cubes ($O(\tau^d)$ of them). So, the last integral in (4.64) can be replaced by an integral over Q_j, at the expense of a constant factor of order $O(\tau^d)$. Indeed, by Minkowski's inequality,

$$\Big\langle\int_{B_\tau(j)}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}\le\sum_{\substack{n\in D_L\\ |B_\tau(j)\cap Q_n|>0}}\Big\langle\int_{Q_n}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}. \tag{4.65}$$

Therefore, (4.64) yields

$$\begin{aligned}
L^{2d}\sum_{\ell=2}^{N}\sum_{j\in A_\ell^k}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2} \quad&\le\quad C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}\sum_{\ell=2}^{N}\sum_{j\in A_\ell^k}\sum_{\substack{n\in D_L\\ |B_\tau(j)\cap Q_n|>0}}\Big\langle\int_{Q_n}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}\\
&\le\quad C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}\sum_{\ell=2}^{N}\sum_{j\in A_\ell^k}\sum_{\substack{n\in D_L\\ dist(Q_j,Q_n)<\tau}}\Big\langle\int_{Q_n}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}\\
&\le\quad C\tau^d\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}\sum_{\ell=1}^{N}\sum_{j\in A_\ell^k}\Big\langle\int_{Q_j}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}\\
&=\quad C\tau^d\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}\sum_{j\in A_+^k}\Big\langle\int_{Q_j}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}.
\end{aligned} \tag{4.66}$$

We will now show that the last sum in (4.66) is $O(\log L)$.

Lemma 4.7 *There is are constants $C>0$ and $p>1$ such that*

$$\sum_{j\in A_+^k}\Big\langle\int_{Q_j}|\nabla\tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}\le C\log L \tag{4.67}$$

and

$$\sum_{k\in A_+^0}\Big\langle \int_{Q_0} |\nabla \tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}} \le C\log L \tag{4.68}$$

for all $\beta \ge 0$, $L \ge 2$, $k \in D_L \cap \mathbb{Z}^d$, where $\tilde{w}_k$ is defined by (4.48).

Lemma 4.3 gives control of $\tilde{w}_k(y)$ in terms of $\nabla_x G(x = k, y)$. So, thinking heuristically, we expect that for $dist(y,k) \gg 1$, $\nabla_y \tilde{w}_k(y)$ should decay like the mixed second derivative $\nabla_y\nabla_x G(k,y)$ of the Green function. So, if the constant-coefficient case is any guide, we should hope that $\nabla_y \tilde{w}_k(y)$ decays like $O(|y-k|^{-d})$. Although we do not have uniform pointwise bounds on $\nabla_y \tilde{w}_k(y)$ of this sort, we still obtain (4.67), which is what we would obtain if we did have the uniform bound $|\nabla_y \tilde{w}_k(y)| \le C(1+|y-k|)^{-d}$. In the proof below, the strategy is to use Caccioploi's inequality to control $\nabla \tilde{w}_k$ by $\tilde{w}_k$, then Lemma 4.3 to control $\tilde{w}_k$ by ∇G. Then we use stationarity and Caccioploi's inequality again to control ∇G by G, for which we have uniform bounds in Lemma 4.4 ($d \ge 3$). Caccioploi's inequality is applied over a large domain (the dyadic annuli) to take advantage of the R^{-2} factor in Lemma 3.1. In the context of the discrete version of this elliptic problem, a similar strategy is employed by Gloria and Otto [17] to control the decay of $\nabla_x G(x,y)$ in terms of the uniform decay of $G(x,y)$ and by Marahrens and Otto [25] to estimate moments $\langle|\nabla_x\nabla_y G(x,y)|^{2p}\rangle^{1/(2p)} \le O((1+|x-y|)^{-d})$ of the discrete second derivative of G.

Proof of Lemma 4.7. By stationarity, we have

$$\Big\langle \int_{Q_j} |\nabla \tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}} = \Big\langle \int_{Q_0} |\nabla \tilde{w}_{k-j}|^{2p}\,dx\Big\rangle^{\frac{1}{2p}},$$

so the bound (4.68) is equivalent to (4.67). Therefore, we focus on proving (4.67).

The constant $p > 1$ may be chosen so that $2p \in (0, p^*)$, where $p^* > 2$ is as in Lemma 3.3. We split the (4.67) over the diadic annuli, and apply Hölder's inequality with $2p$ and $\frac{2p}{2p-1}$:

$$\begin{aligned}\sum_{\ell=1}^N \sum_{j\in A_\ell^k}\Big\langle \int_{Q_j} |\nabla \tilde{w}_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}} &\le \sum_{\ell=1}^N \left(\sum_{j\in A_\ell^k} 1^{2p/(2p-1)}\right)^{(2p-1)/(2p)} \left(\sum_{j\in A_\ell^k}\Big\langle \int_{Q_j} |\nabla \tilde{w}_k|^{2p}\,dx\Big\rangle\right)^{1/2p} \\ &= \sum_{\ell=1}^N |A_\ell^k|^{(2p-1)/(2p)} \left(\Big\langle \int_{A_\ell^k} |\nabla \tilde{w}_k|^{2p}\,dx\Big\rangle\right)^{1/(2p)}. \end{aligned} \tag{4.69}$$

For $\ell \ge 1$, let us use the notation $2A_\ell^k$ to refer to the fattened annuli:

$$2A_\ell^k = \{x \in D_L \mid x \in Q_j,\ \ 2^{\ell-1}\tau < dist(Q_j,Q_k) \le 3\cdot 2^\ell \tau\}, \quad \ell = 1,\dots,N-1,$$

and

$$2A_N^k = \{x \in D_L \mid x \in Q_j,\ \ 2^{N-1}\tau < dist(Q_j,Q_k)\}.$$

Observe that $A_\ell^k \subset 2A_\ell^k$ and $dist(A_\ell^k, \partial(2A_\ell^k)) \ge C2^\ell$. Also, $dist(2A_\ell^k, B_\tau(k)) > 0$. By Lemma 3.3 applied to $\tilde{w}_k$ and by Lemma 4.3, we know that

$$\begin{aligned}\int_{A_\ell^k} |\nabla \tilde{w}_k|^{2p}\,dx &\le C(2^\ell)^{d-p(2+d)}\left(\int_{2A_\ell^k} (\tilde{w}_k)^2\,dy\right)^p \\ &\le C(2^\ell)^{d-p(2+d)}\left(\int_{y\in 2A_\ell^k}\int_{x\in Q_k} |\nabla_x G(x,y)|^2\,dx\,dy\right)^p.\end{aligned}$$

Hence,

$$\sum_{j\in A_+^k}\Big\langle \int_{Q_j}|\nabla\tilde w_k|^{2p}\,dx\Big\rangle^{\frac{1}{2p}}$$
$$\le C\sum_{\ell=1}^N |A_\ell^k|^{(2p-1)/(2p)}\left(\Big\langle \int_{A_\ell^k}|\nabla\tilde w_k|^{2p}\,dx\Big\rangle\right)^{1/(2p)}$$
$$\le C\sum_{\ell=1}^N (2^\ell)^{d(2p-1)/(2p)}(2^\ell)^{(d-p(2+d))/(2p)}\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle^{1/2}$$
$$= C\sum_{\ell=1}^N (2^\ell)^{d/2-1}\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle^{1/2}. \tag{4.70}$$

By stationarity we have

$$\Big\langle \int_{y\in Q_j}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle = \Big\langle \int_{y\in Q_0}\int_{x\in Q_{k-j}}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle.$$

Therefore,

$$\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle \;=\; \Big\langle \int_{y\in Q_0}\int_{x\in 2A_\ell^0}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle. \tag{4.71}$$

The point here is that the integral in x is now over the annulus A_ℓ^0 of diameter $O(2^\ell)$, rather than over the unit cube.

For $d\ge 3$, we combine (4.71) with Cacciopoli's inequality to $x\mapsto G(x,y)$. The result is:

$$\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle \;\le\; C(2^\ell)^{-2}\Big\langle \int_{y\in Q_0}\int_{x\in 2A_\ell^0}|G(x,y)|^2\,dx\,dy\Big\rangle$$
$$+C|D_L|^{-1}\Big\langle \int_{y\in Q_0}\int_{x\in 2A_\ell^0}|G(x,y)|\,dx\,dy\Big\rangle.$$

By Lemma 4.4, we have a uniform decay estimates for $|G(x,y)|\le C dist(x,y)^{2-d}$ for $d\ge 3$. Therefore,

$$\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle \le C(2^\ell)^{-2}(2^\ell)^{d+2(2-d)} + CL^{-d}(2^\ell)^d(2^\ell)^{2-d} \le (2^\ell)^{2-d}.$$

So, returning to (4.70), we obtain

$$\sum_{j\in A_+^k}\Big\langle \int_{Q_j}|\nabla\tilde w_k|^{2p}\,dx\Big\rangle^{1/2p} \le C\sum_{\ell=1}^N (2^\ell)^{d/2-1}(2^\ell)^{1-d/2} = O(\log L). \tag{4.72}$$

In the case $d=2$, we apply Lemma 4.5 directly to (4.71) and conclude

$$\Big\langle \int_{y\in 2A_\ell^k}\int_{x\in Q_k}|\nabla_x G(x,y)|^2\,dx\,dy\Big\rangle \le C.$$

So, returning to (4.70), we still obtain

$$\sum_{j\in A_+^k}\Big\langle\int_{Q_j}|\nabla\tilde w_k|^2\,dx\Big\rangle^{1/2}\le C\sum_{\ell=1}^{O(\log L)}(2^\ell)^{d/2-1}=O(\log L). \tag{4.73}$$

This completes the proof of Lemma 4.7. □

Now we combine (4.53), (4.54), (4.63), (4.66), (4.51) and Lemma 4.7 to conclude that

$$\begin{aligned}
&\sum_k\mathbb{E}\left[\left|\sum_{j\in D_L}(\Delta_k\Delta_j\Gamma(Z))\overline{\Delta_j\Gamma(Z^A)}\right|^2\right]\\
&\quad\le\sum_k\left(\sum_{j\in A_0^k\cup A_1^k}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2}+\sum_{\ell=2}^N\sum_{j\in A_\ell^k}\langle|\Delta_k\Delta_j\Gamma(Z)\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2}\right)^2\\
&\quad\le\sum_k\Big(CL^{-2d}\langle\Phi_0^8\rangle^{1/2}+C\langle\Phi_0^{8q}\rangle^{1/2q}L^{-2d}\log L\Big)^2\\
&\quad\le CL^{-3d}\langle\Phi_0^{8q}\rangle^{1/q}(\log L)^2
\end{aligned}\tag{4.74}$$

holds for $d\ge 2$, for all $L\ge 2$, $\beta\ge 0$.

Finally, we estimate (2.27). By Minkowsi's inequality we have

$$\mathbb{E}\left[\left|\sum_{j\in D_L}(\Delta_j\Gamma(Z))^k\Delta_k\overline{\Delta_j\Gamma(Z^A)}\right|^2\right]\le\left(\sum_{j\in D_L}\langle|(\Delta_j\Gamma(Z))^k\Delta_k\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2}\right)^2. \tag{4.75}$$

Recall the notation (2.28) for the average with respect to sets A not containing index j. In particular, the weights $K_{n,A}$ define a probability distribution over the index sets A not containing j. By applying Jensen' inequality to (4.75) we obtain

$$\begin{aligned}
\left(\sum_{j\in D_L}\langle|(\Delta_j\Gamma(Z))^k\Delta_k\overline{\Delta_j\Gamma(Z^A)}|^2\rangle^{1/2}\right)^2&=\left(\sum_{j\in D_L}\langle|\overline{(\Delta_j\Gamma(Z))^k(\Delta_k(\Delta_j\Gamma))(Z^A,Z',Z'')}|^2\rangle^{1/2}\right)^2\\
&\le\left(\sum_{j\in D_L}\langle\overline{|(\Delta_j\Gamma(Z))^k(\Delta_k(\Delta_j\Gamma))(Z^A,Z',Z'')|^2}\rangle^{1/2}\right)^2\\
&=\left(\sum_{j\in D_L}\langle\!\langle|(\Delta_j\Gamma)^k(Z,Z',Z'')(\Delta_k(\Delta_j\Gamma))(Z^A,Z',Z'')|^2\rangle\!\rangle^{1/2}\right)^2,
\end{aligned}$$

where we have introduced the notation

$$\begin{aligned}
\langle\!\langle H_A(Z,Z',Z'')\rangle\!\rangle&=\mathbb{E}[\overline{H_A(Z,Z',Z'')}]\\
&=\mathbb{E}[\sum_{\substack{A\subset[n]\\ j\notin A}}K_{n,A}H_A(Z,Z',Z'')]=\sum_{\substack{A\subset[n]\\ j\notin A}}K_{n,A}\mathbb{E}[H_A(Z,Z',Z'')].
\end{aligned}\tag{4.76}$$

The rest proceeds exactly as in the proof of (4.74)), the only difference being the following modification of Lemma 4.6:

Lemma 4.8 *For $p > 1$ there is a constant C_p such that if $k, j \in D_L$ with $|k - j| > 2\tau$ then*

$$L^{4d}\langle\langle|(\Delta_k\Delta_j\Gamma)(Z^A, Z', Z'')(\Delta_j\Gamma)^k(Z, Z', Z'')|^2\rangle\rangle \leq C_p\langle\Phi_0^{4q}\rangle^{\frac{4}{2q}} \left\langle \left(\int_{B_\tau(j)} |\nabla\tilde{w}_k|^2\, dx\right)^p \right\rangle^{1/p} \quad (4.77)$$

where $q = 2p/(p-1)$. Also, there is a constant C such that

$$L^{4d}\langle\langle|(\Delta_k\Delta_j\Gamma)(Z^A, Z', Z'')(\Delta_j\Gamma)^k(Z, Z', Z'')|^2\rangle\rangle \leq C\langle\Phi_0^8\rangle \quad (4.78)$$

holds for all $j, k \in D_L$, $L > 1$, $\beta \geq 0$.

Proof: The proof is almost identical to that of Lemma 4.6. We only need to observe that, for any pair of indices $j, k \in D_L$ and any set $A \subset D_L \cap \mathbb{Z}^d$, if $g(Z, Z', Z'')$ denotes any of the random variables $\hat{\Phi}_j^2(Z)$, $\hat{\Phi}_j^2(Z^j)$, $\hat{\Phi}_j^2(Z^k)$, $\hat{\Phi}_j^2(Z^{jk})$, $\hat{\Phi}_k^2(Z)$, $\hat{\Phi}_k^2(Z^j)$, $\hat{\Phi}_k^2(Z^k)$, or $\hat{\Phi}_k^2(Z^{jk})$, then $g(Z, Z', Z'')$ and $g(Z^A, Z', Z'')$ have the same distribution. In particular, for any power p,

$$\langle\langle g(Z^A, Z', Z'')^p\rangle\rangle = \langle\langle g(Z, Z', Z'')^p\rangle\rangle = \langle g(Z, Z', Z'')^p\rangle = \langle\hat{\Phi}_0^{2p}\rangle. \quad (4.79)$$

Similarly, the random variables $(\Delta_j\Gamma)^k(Z, Z', Z'')$ and $\Delta_j\Gamma(Z, Z')$ have the same distribution. As before, by Lemma 4.2 and Lemma 4.1, we have

$$\begin{aligned} L^{2d}|(\Delta_k\Delta_j\Gamma)(Z, Z', Z'')|^2 \quad &\leq \quad C\left(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j) + \hat{\Phi}_j^2(Z^k) + \hat{\Phi}_j^2(Z^{jk})\right) \\ &\qquad \times\left(\hat{\Phi}_k^2(Z) + \hat{\Phi}_k^2(Z^j) + \hat{\Phi}_k^2(Z^k) + \hat{\Phi}_k^2(Z^{jk})\right) \end{aligned}$$

and

$$L^{2d}\langle|(\Delta_j\Gamma)^k(Z, Z', Z'')|^{2q}\rangle^{1/q} = L^{2d}\langle|\Delta_j\Gamma(Z, Z')|^{2q}\rangle^{1/q} \leq C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{q}}.$$

Therefore, as in the proof of Lemma 4.6, by applying the generalized Hölder inequality and (4.79) we obtain

$$\begin{aligned} L^{2d}\langle\langle|(\Delta_k\Delta_j\Gamma)(Z^A, Z', Z'')(\Delta_j\Gamma)^k(Z, Z', Z'')|^2\rangle\rangle \quad &\leq \quad C\langle\hat{\Phi}_0^8\rangle^{\frac{1}{4}}\langle\hat{\Phi}_0^8\rangle^{\frac{1}{4}}\langle|(\Delta_j\Gamma)^k|^4\rangle^{1/2} \\ &\leq \quad C\langle\hat{\Phi}_0^8\rangle^{\frac{1}{4}}\langle\hat{\Phi}_0^8\rangle^{\frac{1}{4}}L^{-2d}\langle\hat{\Phi}_0^8\rangle^{\frac{1}{2}}. \qquad (4.80) \end{aligned}$$

This and (4.51) imply (4.78).

If $dist(k, j) > 2\tau$, Lemma 4.2 tells us that

$$\begin{aligned} L^{2d}|\Delta_k\Delta_j\Gamma(Z)|^2 \quad &\leq \quad C\left(\hat{\Phi}_j^2(Z) + \hat{\Phi}_j^2(Z^j) + \hat{\Phi}_j^2(Z^k) + \hat{\Phi}_j^2(Z^{jk})\right) \\ &\qquad \times\left(\hat{\Phi}_k^2(Z^k)\int_{B_\tau(j)}|\nabla\tilde{w}_k|^2\, dx + \Phi_k^2(Z^{jk})\int_{B_\tau(j)}|\nabla\tilde{w}_k^j|^2\, dx\right), \qquad (4.81) \end{aligned}$$

where

$$\tilde{w}_k = \hat{\Phi}_k(Z^k)^{-1}\Delta_k\phi, \qquad \tilde{w}_k^j = \hat{\Phi}_k(Z^{jk})^{-1}\Delta_k\phi^j.$$

Let $p > 1$, let $q = 2p/(p-1)$ so that $\frac{1}{p} + \frac{1}{2q} + \frac{1}{2q} + \frac{1}{q} = 1$. Then by (4.81) and the generalized Hölder inquality,

$$\begin{aligned} &L^{2d}\langle\langle|(\Delta_k\Delta_j\Gamma)(Z^A, Z', Z'')(\Delta_j\Gamma)^k(Z, Z', Z'')|^2\rangle\rangle \\ &\quad\leq C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{2q}}\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{2q}}\langle\langle g_{jk}(Z^A, Z_k'')^p\rangle\rangle^{1/p}\langle|(\Delta_j\Gamma)^k(Z, Z', Z'')|^{2q}\rangle^{1/q} \\ &\qquad + C\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{2q}}\langle\hat{\Phi}_0^{4q}\rangle^{\frac{1}{2q}}\langle\langle g_{jk}(Z^{j\cup A}, Z_k'')^p\rangle\rangle^{1/p}\langle|(\Delta_j\Gamma)^k(Z, Z', Z'')|^{2q}\rangle^{1/q}, \qquad (4.82) \end{aligned}$$

where

$$g_{jk}(Z, Z_k'') = \int_{B_\tau(j)} |\nabla_x \tilde{w}_k(x, Z, Z_k'')|^2 \, dx.$$

On the other hand, $g_{jk}(Z, Z_k'')$ and $g_{jk}(Z^A, Z_k'')$ and $g_{jk}(Z^{j\cup A}, Z_k'')$ all have the same distribution. Hence

$$\langle\!\langle g_{jk}(Z^{j\cup A}, Z_k'')^p \rangle\!\rangle^{1/p} = \langle\!\langle g_{jk}(Z^A, Z_k'')^p \rangle\!\rangle^{1/p} = \Big\langle \Big(\int_{B_\tau(j)} |\nabla \tilde{w}_k|^2 \, dx \Big)^p \Big\rangle^{1/p}.$$

We conclude that

$$\begin{aligned} L^{2d} \langle\!\langle |(\Delta_k \Delta_j \Gamma)(Z^A, Z', Z'')(\Delta_j \Gamma)^k(Z, Z', Z'')|^2 \rangle\!\rangle & \\ \leq C \langle \hat{\Phi}_0^{4q} \rangle^{\frac{1}{2q}} \langle \hat{\Phi}_0^{4q} \rangle^{\frac{1}{2q}} \Big\langle \Big(\int_{B_\tau(j)} |\nabla \tilde{w}_k|^2 \, dx \Big)^p \Big\rangle^{1/p} \langle |\Delta_j \Gamma(Z)|^{2q} \rangle^{1/q} & \\ \leq C \langle \hat{\Phi}_0^{4q} \rangle^{\frac{1}{2q}} \langle \hat{\Phi}_0^{4q} \rangle^{\frac{1}{2q}} \Big\langle \Big(\int_{B_\tau(j)} |\nabla \tilde{w}_k|^2 \, dx \Big)^p \Big\rangle^{1/p} L^{-2d} \langle \Phi_0^{4q} \rangle^{1/q} & \quad (4.83) \end{aligned}$$

which implies (4.77). □

With this modification of Lemma 4.6, we proceed exactly as in the proof of (4.74) to obtain the bound

$$\sum_k \mathbb{E} \left[\left| \sum_{j \in D_L} (\Delta_j \Gamma)^k(Z))^k \Delta_k \overline{\Delta_j \Gamma(Z^A)} \right|^2 \right] \leq C \langle \Phi_0^{8q} \rangle^{1/q} L^d L^{-4d} (\log L)^2. \quad (4.84)$$

By combining Theorem 2.4 with (4.52), (4.74), and (4.84) we conclude that

$$d_{\mathcal{W}} \left(\frac{\Gamma_{L,\beta} - m_{L,\beta}}{\sigma_{L,\beta}}, Z \right) \leq C \frac{L^{-2d}}{\sigma^3} \mathbb{E}[\Phi_0^6] + C \frac{L^{-3d/2} \log(L)}{\sigma^2} \mathbb{E}[\Phi_0^{8q}]^{\frac{1}{2q}}. \quad (4.85)$$

for all $d \geq 2$. The exponent $q > 2$ is the Hölder conjugate of p, where $2p$ is the exponent from Meyers' estimate. This concludes the proof of Theorem 1.1.

5 Estimates for the periodic Green's function

$d \geq 3$: Proof of Lemma 4.4

Here we follow ideas used to prove a uniform decay estimate for Green's functions in $\mathbb{R}^d$, as in Theorem 1.1 of [20] and Lemma 2.8 of [17]; the difference here is the periodicity, so we include a proof for completeness. Let $y \in D_L$ and let $u(x) = G(x, y)$ be the periodic Green's function, which satisfies

$$-\nabla \cdot (a \nabla u) + \beta u = \delta_y - |D_L|^{-1}, \quad x \in D_L. \quad (5.86)$$

in the weak sense. Suppose that u also satisfies

$$|\{x \in D_L \mid u(x) > 0\}| \leq \frac{1}{2} |D_L|. \quad (5.87)$$

(If this is not the case, then we could apply the same argument to the function $-u$ instead.) Then, for any $k > 0$, the function $u_k(x) = \max(0, \min(u, k))$ satisfies

$$|\{x \in D_L \mid u_k(x) \neq 0\}| = |\{x \in D_L \mid u(x) > 0\}| \leq \frac{1}{2} |D_L|. \quad (5.88)$$

Since $\|u_k\|_\infty \leq k$, we observe that u_k satisfies

$$\int_{D_L} \nabla u_k \cdot a \nabla u_k \, dx = \int_{D_L} \nabla u \cdot a \nabla u_k \, dx \leq -\beta \int_{D_L} u u_k \, dx + 2k \leq 2k.$$

Therefore,

$$\int_{D_L} |\nabla u_k|^2 \, dx \leq 2k/a_*. \tag{5.89}$$

Considering (5.88), we know there is a constant C, independent of k, L and β, such that

$$\left(\int_{D_L} |u_k|^q \, dx \right)^{1/q} \leq C \left(\int_{D_L} |\nabla u_k|^2 \, dx \right)^{1/2} \tag{5.90}$$

where $q = 2d/(d-2)$ is the critical Sobolev exponent. By scaling, this is a consequence of the Sobolev imbedding theorem and the Poincaré inequality for functions $v \in H^1_{per}(D_1)$ which also satisfy $|\{x \in D_1 \mid v(x) = 0\}| \geq 1/2$ (for example, see Lemma 4.8 of [21]). By applying Chebychev's inequality, then (5.90) and (5.89), we obtain the estimate

$$|\{x \in D_L \mid u(x) \geq k\}| = |\{x \in D_L \mid u_k(x) \geq k\}| \quad \leq \quad k^{-q} \int_{D_L} |u_k|^q \, dx \leq C k^{-q/2}. \tag{5.91}$$

This is a weak-$L^p(D_L)$ estimate on $u^+ = \max(u, 0)$, for $p = q/2 = d/(d-2)$:

$$\|u^+\|_{L^p_W(D_L)} = \sup_{t>0} \;\; t \, |\{x \in D_L \mid |u^+(x)| > t\}|^{1/p} \leq C, \tag{5.92}$$

where the constant C is independent of L and $\beta \geq 0$.

Now let $\alpha \in (1, p)$, $x_0 \in D_L$, $R < dist(x_0, y)$. The weak bound (5.92) implies that $u^+ \in L^\alpha(B_R(x_0))$. By using the identity

$$\int_{B_R} |u^+|^\alpha \, dx = \alpha \int_0^\infty t^{\alpha-1} |\{x \in B_R \mid u^+(x) \geq t\}| \, dt \leq |B_R| s^\alpha + \alpha \int_s^\infty t^{\alpha-1} |\{x \in B_R \mid u^+(x) \geq t\}| \, dt$$

and optimizing in s, we see that

$$\|u^+\|_{L^\alpha(B_R)} \leq C \left(\frac{p}{p-\alpha} \right)^{1/\alpha} |B_R|^{\frac{p-\alpha}{p\alpha}}, \tag{5.93}$$

where the constant C depends on α and p, but not on L or R or $\beta \geq 0$. Since $-\nabla \cdot (a \nabla u) + \beta u = -|D_L|^{-1}$ in B_R, the estimates of De Giorgi and Moser give us a bound on $u^+(x)$ in terms of $\|u^+\|_{L^\alpha(B_R(x_0))}$. Specifically, Theorem 4.1 of [21] (or Theorem 8.17 of [13]) implies that u is locally bounded and satisfies:

$$\sup_{x \in B_{R/2}(x_0)} u^+(x) \quad \leq \quad C R^{-d/\alpha} \left(\int_{B_R} (u^+(y))^\alpha \, dy \right)^{1/\alpha} + C R^2 |D_L|^{-1}, \tag{5.94}$$

with a constant C that depends only on d, a_*, a^*, and α. Note that in Theorem 4.1 of [21], the constant depends on $|\beta| R^2$. However, it is easy to see from the proof (method 1) that if β is known to be non-negative, then the bound is independent of β, so the same bound holds under rescaling (as in Theorem 4.14 of [21]).

By combining (5.93) and (5.94) we have

$$\sup_{x \in B_{R/2}(x_0)} u^+(x) \leq C R^{-d/\alpha} |B_R|^{\frac{p-\alpha}{p\alpha}} + C R^2 |D_L|^{-1} \leq C R^{-d/p} + C R^2 L^{-d} \leq C R^{2-d},$$

where the constant C depends on the dimension, but not on L, $\beta \geq 0$, R. In particular,

$$u^+(x) \leq C\,(dist(x,y))^{2-d}. \tag{5.95}$$

Now, assuming (5.87) holds for u (otherwise, replace u by $-u$), let us choose $r \leq 0$ such that both

$$|\{x \in D_L \mid u(x) > r\}| \leq \frac{1}{2}|D_L|$$

and

$$|\{x \in D_L \mid u(x) < r\}| \leq \frac{1}{2}|D_L|$$

hold. Consider the function $\bar{u} = r - u$ which satisfies

$$-\nabla \cdot (a\nabla \bar{u}) + \beta \bar{u} = -\delta_y + |D_L|^{-1} - \beta|r|$$

and

$$|\{x \in D_L \mid \bar{u}(x) > 0\}| \leq \frac{1}{2}|D_L|.$$

To the functions $\bar{u}_k = \max(0, \min(\bar{u}, k))$ and $\bar{u}^+ = \max(0, \bar{u})$ we apply the same argument used to obtain (5.95). The result is:

$$\bar{u}^+(x) \leq C\,(dist(x,y))^{2-d}. \tag{5.96}$$

In deriving (5.94) for $\bar{u}^+$, we must use the fact that $\bar{u}$ is a subsolution of $-\nabla \cdot (a\nabla \bar{u}) + \beta \bar{u} = |D_L|^{-1}$ away from y, since $-\beta|r| \leq 0$. That is,

$$\int_{B_R} \nabla \varphi \cdot a\nabla \bar{u} + \beta \bar{u}\varphi\, dx \leq |D_L|^{-1} \int_{B_R} \varphi\, dx$$

holds for all $\varphi \in H_0^1(B_R)$ which satisfy $\varphi \geq 0$. Thus, Theorem 4.1 of [21] (or Theorem 8.17 of [13]) still applies. Apart from this detail, the argument is identical. By combining (5.95) and (5.96) we obtain

$$r - C\,(dist(x,y))^{2-d} \leq r - \bar{u}^+(x) \leq u(x) \leq u^+(x) \leq C\,(dist(x,y))^{2-d} \tag{5.97}$$

On the other hand, (5.95) implies that

$$\int_{D_L} u^+(x)\, dx \leq CL^2.$$

We combine this with the fact that $\int_{D_L} u\, dx = 0$ to conclude that

$$\begin{aligned} 0 &= \int_{D_L} u^+(x)\, dx + \int_{\{r<u\leq 0\}} u(x)\, dx + \int_{\{u \leq r\}} u(x)\, dx \\ &\leq CL^2 + r|\{x \in D_L \mid u(x) \leq r\}| \\ &= CL^2 + r|\{x \in D_L \mid \bar{u}(x) \geq 0\}| \leq CL^2 + rL^d/2. \end{aligned}$$

Hence $|r| \leq 2CL^{2-d}$. Combining this with (5.97) we obtain $|u(x)| \leq C dist(x,y)^{2-d}$, as desired. □

$d = 2$: Proof of Lemma 4.5

Lemma 4.5 relies on the following oscillation estimate, which is a version of Lemma 2.8(i) of [17]:

Lemma 5.1 *Let $d = 2$. For any $q \geq 1$, there is a constant $C > 0$ such that*

$$R^{-2} \int_{B_R(x_0)} |G(x,y) - \bar{G}_R(y)|^q \, dx \leq C$$

holds for all $x_0 \in D_L$, $y \in D_L \setminus B_{2R}(x_0)$, $R > 0$, $L > 1$ and $\beta \geq 0$, where $\bar{G}_R(y)$ is the average of $G(\cdot, y)$ over the ball $B_R(x_0)$.

Proof of Lemma 5.1: This is proved as in Lemma 2.8 of [17] (see part (i), Step 2) for the free-space Green's function (see Step 2 in the proof therein); here we include the proof for completeness. Fix $y \in D_L$. Let $u(x) = G(x,y)$, which satisfies

$$-\nabla \cdot (a \nabla u) + \beta u = \delta_y - |D_L|^{-1}.$$

Let $\overline{u_R}$ be the average of u over the ball B_R. Without loss of generality, suppose $\overline{u_R} \geq 0$. For $k \geq 0$, define

$$u_k = \max(\min(u, \overline{u_R} + k), \overline{u_R} - k).$$

We claim that

$$\int_{D_L} |\nabla u_k|^2 \, dx \leq \frac{2k}{a_*}. \tag{5.98}$$

To see this, observe that for any constant $c \in \mathbb{R}$,

$$\begin{aligned} \int_{D_L} \nabla u_k \cdot a \nabla u_k \, dx &= \int_{D_L} \nabla (u_k + c) \cdot a \nabla u \, dx \\ &= u_k(y) - |D_L|^{-1} \int_{D_L} u_k(x) \, dx - \beta \int_{D_L} u(u_k + c) \, dx. \end{aligned} \tag{5.99}$$

If $\overline{u_R} \in [0,k]$, let $c = 0$. Then $u(x)(u_k(x) + c) \geq 0$ at every point $x \in D_L$. Hence

$$\beta \int_{D_L} u(u_k + c) \, dx \geq 0. \tag{5.100}$$

Therefore, (5.98) follows from (5.99). If $\overline{u_R} > k$, let $c = k - \overline{u_R}$. Then $u_k + c \geq 0$. Also, $u(x) > \overline{u_R} - k > 0$ must hold wherever $(u_k(x) + c) > 0$. Hence (5.100) still holds. Moreover, $0 \leq u_k(x) + c \leq 2k$, so again (5.98) follows from (5.99).

Now let $v(x) = u(x) - \overline{u_R}$. Let $v_k(x) = \max(\min(v(x), k), -k) = u_k(x) - \overline{u_R}$. Let $\overline{v_R}$ and $\overline{v_{k,R}}$ be the average of v and v_k over B_R, respectively. Hence $\overline{v_R} = 0$. Then the goal is to bound

$$\begin{aligned} \left(R^{-2} \int_{B_R} |v|^q \, dx \right)^{1/q} &= \left(R^{-2} \int_{B_R \cap \{|v| \leq k\}} |v_k|^q \, dx + R^{-2} \int_{B_R \cap \{|v| > k\}} |v|^q \, dx \right)^{1/q} \\ &\leq C \left(R^{-2} \int_{B_R \cap \{|v| \leq k\}} |v_k - \overline{v_{k,R}}|^q \, dx \right)^{1/q} + C |\overline{v_{k,R}}| \\ &\quad + C \left(R^{-2} \int_{B_R \cap \{|v| > k\}} |v|^q \, dx \right)^{1/q}. \end{aligned}$$

Since $\overline{v_R} = 0$, we have

$$|\overline{v_{k,R}}| \leq 2 \left(R^{-2} \int_{B_R \cap \{|v| \geq k\}} |v|^q \, dx \right)^{1/q}.$$

Therefore,

$$\begin{aligned} \left(R^{-2} \int_{B_R} |v|^q \, dx \right)^{1/q} &\leq C \left(R^{-2} \int_{B_R} |v_k - \overline{v_{k,R}}|^q \, dx \right)^{1/q} \\ &\quad + C \left(R^{-2} \int_{B_R \cap \{|v| > k\}} |v|^q \, dx \right)^{1/q}. \end{aligned} \tag{5.101}$$

By the Sobolev inequality and then (5.98), we know that for any $s \in [1, \infty)$ there is a constant C_s (depending only on s) such that

$$\left(R^{-2} \int_{B_R} |v_k - \overline{v_{k,R}}|^s \, dx \right)^{1/s} \leq C_s \left(\int_{B_R} |\nabla v_k|^2 \, dx \right)^{1/2} = C_s \left(\int_{B_R} |\nabla u_k|^2 \, dx \right)^{1/2} \leq C k^{1/2}.$$

To estimate the last integral appearing in (5.101) we use

$$\int_{B_R \cap \{|v| > k\}} |v|^q \, dx = \int_0^\infty q t^{q-1} |\{|v| \geq \max(t, k)\}| \, dt \leq |\{|v| \geq k\}| k^q + q \int_k^\infty t^{q-1} |\{|v| \geq t\}| \, dt,$$

and

$$|\{|v| \geq k\}| \leq |\{|v_k| \geq k\}| \leq k^{-s} \int_{B_R} |v_k|^s \, dx.$$

Let $s > 2q$. Then

$$\begin{aligned} \int_{B_R} |v_k|^s \, dx &\leq C \int_{B_R} |v_k - \overline{v_{k,R}}|^s \, dx + C R^2 (\overline{v_{k,R}})^s \\ &\leq C \int_{B_R} |v_k - \overline{v_{k,R}}|^s \, dx + C R^2 \left(R^{-2} \int_{B_R} |v|^q \, dx \right)^{s/q} \\ &\leq C R^2 k^{s/2} + C R^2 \left(R^{-2} \int_{B_R} |v|^q \, dx \right)^{s/q}. \end{aligned}$$

So, if $I_q = (R^{-2} \int_{B_R} |v|^q \, dx)^{1/q}$, we have

$$\int_{B_R} |v_k|^s \, dx \leq C R^2 k^{s/2} + C R^2 I_q^s$$

and $|\{|v| \geq k\}| \leq R^2 k^{-s/2} + C k^{-s} R^2 I_q^s$.

Combining these bounds and returning to (5.101), we obtain

$$
\begin{aligned}
I_q &\le Ck^{1/2} + CR^{-2/q}\left(|\{|v| \ge k\}|k^q + q\int_k^\infty t^{q-1}|\{|v| \ge t\}|\,dt\right)^{1/q} \\
&\le Ck^{1/2} + CR^{-2/q}\left(R^2 k^{-s/2}k^q + Ck^{-s}R^2 I_q^s k^q\right)^{1/q} \\
&\quad + CR^{-2/q}\left(q\int_k^\infty t^{q-1}|\{|v| \ge t\}|\,dt\right)^{1/q} \\
&\le Ck^{1/2} + CR^{-2/q}\left(R^2 k^{-s/2}k^q + Ck^{-s}R^2 I_q^s k^q\right)^{1/q} \\
&\quad + CR^{-2/q}\left(q\int_k^\infty t^{q-1}(R^2 t^{-s/2} + Ct^{-s}R^2 I_q^s)\,dt\right)^{1/q} \\
&\le Ck^{1/2} + Ck^{1-s/(2q)} + CI_q^{s/q}k^{1-s/q}. \qquad (5.102)
\end{aligned}
$$

By choosing $k = \alpha I_q$ with $\alpha > 0$ sufficiently large, we see that this implies $I_q \le C$. □

Now we continue with the proof of Lemma 4.5. By assumption, $dist(x_0, y) > 2R$. Let φ be a smooth function supported in $B_{2R}(x_0)$ and satisfying: $0 \le \varphi(x) \le 1$ for all x, $\varphi(x) = 1$ for $x \in B_R(x_0)$, and $|\nabla\varphi| \le C/R$. Applying Lemma 3.2 to $u(x) = G(x, y)$ with this choice of φ, we conclude

$$
\begin{aligned}
\int_{B_R} |\nabla u|^2\,dx &\le K_1|D_L|^{-1}\int_{B_{2R}} (u-b)\varphi^2\,dx - K_1\beta\int_{B_{2R}} u(u-b)\varphi^2\,dx \\
&\quad + K_2\int_{B_{2R}} |\nabla\varphi|^2(u-b)^2\,dx. \qquad (5.103)
\end{aligned}
$$

If we choose

$$
b = \left(\int_{B_{2R}} \varphi^2\,dx\right)^{-1}\int_{B_{2R}} u\varphi^2\,dx,
$$

then Jensen's inequality implies

$$
\int_{B_{2R}} u(u-b)\varphi^2\,dx \ge 0.
$$

Therefore, since $\beta \ge 0$,

$$
\begin{aligned}
\int_{B_R} |\nabla u|^2\,dx &\le K_1|D_L|^{-1}\int_{B_{2R}} (u-b)\varphi^2\,dx + K_2\int_{B_{2R}} |\nabla\varphi|^2(u-b)^2\,dx \\
&\le K_1R^{-2}\int_{B_{2R}} |u-b|\,dx + CR^{-2}\int_{B_{2R}} (u-b)^2\,dx. \qquad (5.104)
\end{aligned}
$$

On the other hand, if $\bar{u}$ denotes the average of $u(x)$ over $B_{2R}(x_0)$, we know from Lemma 5.1 that

$$
R^{-2}\int_{B_{2R}} |u-\bar{u}|\,dx \le C.
$$

Hence

$$
|\bar{u}-b| \le \left(\int_{B_{2R}} \varphi^2\,dx\right)^{-1}\int_{B_{2R}} |\bar{u}-u(x)|\varphi^2\,dx \le CR^{-2}\int_{B_{2R}} |\bar{u}-u(x)|\varphi^2\,dx \le C.
$$

Applying Lemma 5.1 again, we obtain

$$R^{-2}\int_{B_{2R}}(u-b)^2\,dx \le CR^{-2}\int_{B_{2R}}(u-\bar{u})^2\,dx + CR^{-2}\int_{B_{2R}}(\bar{u}-b)^2\,dx \le C.$$

Similarly,

$$R^{-2}\int_{B_{2R}}|u-b|\,dx \le CR^{-2}\int_{B_{2R}}|u-\bar{u}|\,dx + CR^{-2}\int_{B_{2R}}(\bar{u}-b)\,dx \le C.$$

In view of (5.104) and the fact that C is independent of R, L and $\beta \ge 0$, we have proved the desired result. □

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