EFFECTIVE CURVES ON $\overline{\mathrm{M}}_{0,n}$ FROM GROUP ACTIONS

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ABSTRACT. We study new effective curve classes on the moduli space of stable pointed rational curves given by the fixed loci of subgroups of the permutation group action. We compute their numerical classes and provide a strategy for writing them as effective linear combinations of F-curves, using Losev-Manin spaces and toric degeneration of curve classes.

1. INTRODUCTION

One of the central problems in the birational geometry of a projective variety X is determining its cone of effective curves $\overline{\mathrm{NE}}_1(X)$. In the minimal model program, this is the first step toward understanding and classifying all the contractions of X, that is, projective morphisms from X to other varieties.

Let $\overline{\mathrm{M}}_{0,n}$ be the moduli space of stable n-pointed rational curves. Because Kapranov's construction ([Kap93, Theorem 4.3.3]) is very similar to a blow-up construction of a toric variety, many people wondered if the birational geometry of $\overline{\mathrm{M}}_{0,n}$ might be similar to that of toric varieties. For instance, the cone of effective cycles of a toric variety is generated by its torus invariant boundaries; $\overline{\mathrm{M}}_{0,n}$ was conjectured to have a similar property.

Conjecture 1.1 ([KM96, Question 1.1])**.** *The cone of k-dimensional effective cycles is generated by k-dimensional intersections of boundary divisors, for $1 \leq k \leq \dim \overline{\mathrm{M}}_{0,n} - 1$.*

But in the last decade, there have been several striking results showing that the divisor theory of $\overline{\mathrm{M}}_{0,n}$ is more complicated than that of toric varieties. For instance, there are non-boundary type extremal effective divisors on $\overline{\mathrm{M}}_{0,n}$ ([Ver02, CT13, DGJ14]). Furthermore, very recently Castravet and Tevelev showed that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space for large n ([CT13b]).

On the other hand, for effective curve classes, there are few known results in the literature. In [KM96], Keel and McKernan proved Conjecture 1.1 for cycles of dimension $k = 1$ and $n \leq 7$, but it is still unknown for $n > 7$. Conjecture 1.1 for $k = 1$ is now widely known as the *F-Conjecture*, and one-dimensional intersections of boundary divisors are called *F-curves*. Keel and McKernan also showed that if there is any other extremal ray R of the curve cone $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$ and $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$ is not round at R, then R is generated by a rigid curve intersecting the interior, $\mathrm{M}_{0,n}$ (See [CT12, Theorem 2.2] for a proof). This has motivated several researchers to search for rigid curves on $\overline{\mathrm{M}}_{0,n}$ to find a potential counterexample

to the F-Conjecture. Castravet and Tevelev constructed rigid curves by applying their hypergraph construction in [CT12]. There is a slightly weaker notion of the rigidity of curves (so-called rigid maps). In [CT12] and [Che11], Castravet, Tevelev, and Chen constructed two types of examples of rigid maps. But all of these examples are numerically equivalent to effective linear combinations of F-curves, thus they do not give counterexamples to the F-conjecture. To our knowledge, these examples, the F-curves, and some curve classes that arise from obvious families of point configurations are the only explicit examples of effective curves on $\overline{M}_{0,n}$ in the literature.

1.1. **Aim of this paper.** This project began because we wanted to study the geometric and numerical properties of some new effective curves on $\overline{M}_{0,n}$ that arise from a finite group action.

There is a natural S_n-action on $\overline{M}_{0,n}$ permuting the marked points. Let G be a subgroup of S_n. Let $\overline{M}_{0,n}^G$ be the union of irreducible components of the G-fixed locus that intersect the interior $M_{0,n}$. If we impose certain numerical conditions on G, then $\overline{M}_{0,n}^G$ becomes an irreducible curve on $\overline{M}_{0,n}$.

The computation of the numerical class of $\overline{M}_{0,n}^G$ (or equivalently, its intersection with boundary divisors) is elementary. But surprisingly, there has been no study of these curve classes on $\overline{M}_{0,n}$. We believe that the reason is that even though it is straightforward to compute the numerical class of such a curve, it is difficult to determine whether the curve is numerically equivalent to an *effective* linear combination of F-curves. Indeed, it is a difficult computation to find an actual effective linear combination of F-curves, and that led to the main result of our paper.

1.2. **Main result.** The main result of this paper is not a single theorem, but a method to approach this computational problem using Losev-Manin spaces $\overline{L}_n$ ([LM00]) and toric degenerations. Losev-Manin spaces $\overline{L}_n$ are special cases of Hassett's moduli spaces of stable weighted pointed rational curves ([Has03]). As moduli spaces, they parametrize pointed chains of rational curves, and they are contractions of $\overline{M}_{0,n+2}$. A significant geometric property of $\overline{L}_n$ is that it is the closest toric variety to $\overline{M}_{0,n+2}$ among Hassett's spaces. $\overline{L}_n$ is a toric variety whose corresponding polytope is the permutohedron of dimension $n-1$ ([GKZ08, Section 7.3]). We give a method to compute a toric degeneration of an effective curve class on $\overline{L}_n$. The computation is a result of an interesting interaction between the moduli theoretic interpretation of $\overline{L}_n$ and the combinatorial structure of the permutohedron.

For any effective curve class C on $\overline{M}_{0,n}$, we are able to compute the numerical class of its image $\rho(C)$ for $\rho : \overline{M}_{0,n} \to \overline{L}_{n-2}$. By using the toric degeneration method, we can find an effective linear combination of one dimensional toric boundaries representing $\rho(C)$. Since each toric boundary component is the image of a unique F-curve, it is an "approximation" of the effective linear combination for C. By taking the proper transform, we have a (not necessarily effective) linear combination of F-curves for C on $\overline{M}_{0,n}$. To find an effective linear combination, we use a computational strategy described in Section 7.

It is significant to note that our computational strategy doesn't use any special properties of the finite group action used to define the curves $\overline{\mathrm{M}}_{0,n}^{G}$, and thus we believe that the approach using Losev-Manin spaces and toric degenerations may also be applicable to other curve classes not of the form $\overline{\mathrm{M}}_{0,n}^{G}$. For example, this could give a second approach to analyzing hypergraph curves whose classes are computed in [CT12] using the technique of arithmetic breaks.

In this paper, we study $\overline{\mathrm{M}}_{0,n}^{G}$ when G is either a cyclic group or a dihedral group.

1.3. Cyclic group case.

1.3. **Cyclic group case.** If G is a cyclic group, the geometric and numerical properties of $\overline{\mathrm{M}}_{0,n}^{G}$ are relatively easy to prove without using technical tools such as toric degeneration. We prove the following theorem for arbitrary n.

Theorem 1.2. *Let $G = \langle \sigma \rangle$ be a cyclic group.*

(1) *(Lemma 2.4) The invariant subvariety $\overline{\mathrm{M}}_{0,n}^{G}$ is nonempty if and only if σ is balanced (see Definition 2.5).*

(2) *(Lemmas 3.1, 3.4) The invariant subvariety $\overline{\mathrm{M}}_{0,n}^{G}$ is irreducible and when it is a curve, it is isomorphic to $\mathbb{P}^1$.*

(3) *(Theorem 4.8) In this case, $\overline{\mathrm{M}}_{0,n}^{G}$ is numerically equivalent to a linear combination of F-curves such that all coefficients are one.*

(4) *(Theorem 4.17) $\overline{\mathrm{M}}_{0,n}^{G}$ is movable.*

1.4. **Dihedral group case.** If the group G is not cyclic, then in general the intersection number with the canonical divisor is positive, so it is possible that $\overline{\mathrm{M}}_{0,n}^{G}$ is rigid (Section 5). Thus by using this idea, we might find a new extremal ray of $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$. But we checked all such curves for $n \leq 12$, and none of them is a counterexample to the F-conjecture. In the paper we include two concrete examples on $\overline{\mathrm{M}}_{0,9}$ and $\overline{\mathrm{M}}_{0,12}$. These are introduced in Examples 5.7 and 5.8 and completed in sections 7 and 8.

1.5. **The M0nbar package for Macaulay2.** We have written a great deal of code in `Macaulay2` ([GS]) over a period of many years. For this project, we collected our work in a package called `M0nbar` for `Macaulay2`. The package code is available at the second author's website:

http://faculty.fordham.edu/dswinarski/M0nbar/

This package was used to check many of the calculations in Sections 7 and 8. We have posted code samples for these calculations and some others on the website:

http://faculty.fordham.edu/dswinarski/invariant-curves/

1.6. **Structure of the paper.** Here is an outline of this paper. In Section 2, we introduce several definitions we will use in this paper. In Section 3, we prove several geometric and numerical properties for invariant curves for cyclic groups. In Section 4, we prove Theorem 1.2. The dihedral group case is explained in Section 5. The main method of this paper, using Losev-Manin spaces, is described in Section 6. In Section 7, we give a computational strategy to find an effective $\mathbb{Z}$-linear combination from a non-effective combination. In Section 8, we compute an example on $\overline{\mathrm{M}}_{0,12}$.

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2. LOCI IN $\overline{\mathrm{M}}_{0,n}$ FIXED BY A FINITE GROUP

We work over the complex numbers $\mathbb{C}$ throughout the paper.

Consider the natural S_n-action on $\overline{\mathrm{M}}_{0,n}$ permuting the n marked points.

Definition 2.1. Fix a subgroup $G \leq S_n$ and let $\overline{\mathrm{M}}_{0,n}^G$ be the union of those irreducible components of the G-fixed locus for the induced G-action on $\overline{\mathrm{M}}_{0,n}$ that intersect the interior $\mathrm{M}_{0,n}$. Then an irreducible component of $\overline{\mathrm{M}}_{0,n}^G$ is a subvariety of $\overline{\mathrm{M}}_{0,n}$.

Remark 2.2. One motivation for studying these loci is the following. There are several different possible descriptions of Keel-Vermeire divisors ([Ver02]). One of them is using $\overline{\mathrm{M}}_{0,n}^G$: a Keel-Vermeire divisor is the case that G is a cyclic group of order two generated by $(12)(34)(56)$.

Remark 2.3. In general, there are several irreducible components of the G-fixed loci which are contained in the boundary. For example, see Example 3.8. That is why in the definition we choose only those components that meet the interior.

Let $n \geq 3$. For a subgroup $G \subset S_n$, consider $(\mathbb{P}^1, x_1, \cdots, x_n) \in \overline{\mathrm{M}}_{0,n}^G$. Then for each $\sigma \in G$, there is $\phi_\sigma \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ such that $\phi_\sigma(x_i) = x_{\sigma(i)}$. It defines a group representation $\phi : G \to \mathrm{PGL}_2$. This representation is faithful, because if $\phi_\sigma = \mathrm{id} \in \mathrm{PGL}_2$, $x_i = \phi_\sigma(x_i) = x_{\sigma(i)}$ thus $\sigma = \mathrm{id} \in S_n$. We can conclude that $\overline{\mathrm{M}}_{0,n}^G$ is nonempty only if there exists a faithful representation $\phi : G \to \mathrm{PGL}_2$.

A finite subgroup of PGL_2 is one of following.

- A finite cyclic group C_k.
- A dihedral group D_k.
- A_4, S_4, A_5.

Moreover, any $\psi \in \mathrm{PGL}_2$ with finite order r is, up to conjugation, a rotation along a pivotal axis on $\mathbb{P}^1 \cong S^2$ by the angle $\frac{2\pi}{r}$. Therefore it has two fixed points, and except them, all other orbits have length r. Thus, to obtain a faithful representation, for all $\sigma \in G - \{e\}$, the number of elements in $\mathrm{Stab}_\sigma := \{i \in [n] | \sigma(i) = i\}$ must be at most two.

These restrictions already give all possible $\sigma \in S_n$ with nonempty fixed locus $\overline{\mathrm{M}}_{0,n}^{\langle \sigma \rangle}$.

Lemma 2.4. Let $\sigma \in S_n$ and let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ where the right hand side is a product of disjoint nontrivial cycles. Suppose that $\overline{\mathrm{M}}_{0,n}^{\langle \sigma \rangle}$ is nonempty. If we denote the length of σ_i by ℓ_i, then $\ell_1 = \ell_2 = \cdots = \ell_k$ and $n - \sum_{i=1}^k \ell_i \leq 2$.

Conversely, for a $\sigma \in S_n$ satisfying the conditions in Lemma 2.4, it is easy to find $(\mathbb{P}^1, x_1, \cdots, x_n) \in \overline{\mathrm{M}}_{0,n}^{\langle \sigma \rangle}$.

Definition 2.5. A permutation $\sigma \in S_n$ is called *balanced* if we can write σ as a product of disjoint nontrivial cycles $\sigma_1 \sigma_2 \cdots \sigma_k$ such that

(1) the length of all σ_i's are equal to a fixed ℓ;
(2) $n - k\ell \leq 2$.

3. Cyclic group cases

In this section, we will consider cyclic group cases.

Lemma 3.1. *Let $G = \langle \sigma \rangle$ be a cyclic group of order r such that σ is balanced. Let j be the number of trivial (length 1) cycles in σ. Then:*

(1) *The dimension of $\overline{\mathrm{M}}_{0,n}^G$ is $\frac{n-j}{r} - 1$.*
(2) *$\overline{\mathrm{M}}_{0,n}^G$ is irreducible.*

Proof. Note that $n - j$ marked points of order r can be decomposed into $\frac{n-j}{r}$ orbits and each orbit is determined by a choice of a point of $\mathbb{P}^1$. Thus a point of $\overline{\mathrm{M}}_{0,n}^G$ is determined by isomorphism classes of $\frac{n-j}{r} + 2$ distinct points on $\mathbb{P}^1$, where the last two points are σ-fixed points. Hence the dimension of $\overline{\mathrm{M}}_{0,n}^G$ is $\frac{n-j}{r} + 2 - 3 = \frac{n-j}{r} - 1$. This proves (1).

Moreover, by the above description, there is a dominant rational map $(\mathbb{P}^1)^{\frac{n-j}{r}+2} - \Delta \dashrightarrow \overline{\mathrm{M}}_{0,n}^G$. Therefore $\overline{\mathrm{M}}_{0,n}^G$ is irreducible. $\qquad\square$

Example 3.2. (1) For $n = 6$, there are three types of positive-dimensional subvarieties. One is codimension one, which is the case $j = 0$ and $r = 2$. So the group G is generated by $(12)(34)(56)$ or one of its S_6 conjugates. Thus in this case $\overline{\mathrm{M}}_{0,n}^G$ is a Keel-Vermeire divisor. If $j = 2$ and $r = 2$, G is generated by $(12)(34)$ or one of its S_6-conjugates. Finally, if $j = 0$ and $r = 3$, G is generated by $(123)(456)$ or one of its S_6-conjugates.

(2) When $n = 7$, there are two positive dimensional subvarieties. If $j = 1$ and $r = 2$, $\overline{\mathrm{M}}_{0,n}^G$ is two-dimensional. If $j = 1$ and $r = 3$, it is a curve.

Remark 3.3. A simple consequence is that $\overline{\mathrm{M}}_{0,n}^G$ has codimension at least two if $n \geq 7$. Indeed, $\frac{n-j}{r} - 1 = n - 4$ has integer solutions with $0 \leq j \leq 2$ and $r \geq 2$ only if $n \leq 6$. Similarly, $\overline{\mathrm{M}}_{0,n}^G$ has codimension two only if $n \leq 8$. So in general, we are only able to obtain subvarieties with large codimension.

Lemma 3.4. *Let $G = \langle \sigma \rangle$ be a cyclic group where σ is a balanced permutation. If the dimension of $\overline{\mathrm{M}}_{0,n}^G$ is one, then $\overline{\mathrm{M}}_{0,n}^G \cong \mathbb{P}^1$.*

Proof. Since the $n = 4$ case is obvious, suppose that $n \geq 5$. Pick one element from each cycle, and let S be the set of them. Also, if there are non-marked σ-fixed points, then enlarge S to include these σ-fixed points, too. When $n \geq 5$, it is easy to see that $|S| = 4$. Then there is a morphism $\pi : \overline{\mathrm{M}}_{0,n}^G \to \overline{\mathrm{M}}_{0,S} \cong \mathbb{P}^1$. Then π is a regular birational morphism. A birational morphism from a complete curve to a nonsingular complete curve is an isomorphism (see for instance [Mum99, Proposition III.9.1]). $\qquad\square$

Remark 3.5. In general, $\overline{\mathrm{M}}_{0,n}^G$ is a rational variety because there is a birational map $(\mathbb{P}^1)^k \dashrightarrow \overline{\mathrm{M}}_{0,n}^G$. It would be interesting if one can describe the geometry of $\overline{\mathrm{M}}_{0,n}^G$ in terms of concrete blow-ups and blow-downs of $(\mathbb{P}^1)^k$.

Definition 3.6. For $G = \langle \sigma \rangle$ with a balanced σ, if $\dim \overline{\mathrm{M}}_{0,n}^G = 1$, we will denote $\overline{\mathrm{M}}_{0,n}^G$ by C^σ.

In this case, there are exactly two nontrivial disjoint cycles of length r. Let j, r be two integers satisfying $0 \le j \le 2$, $r > 1$, $\frac{n-j}{r} = 2$. So j refers the number of fixed marked points, and r is the length of a general orbit on $\mathbb{P}^1$. We will say that σ (or G) is *of type* (j, r).

For any $n \ge 5$, there is a cyclic group G such that $\dim \overline{\mathrm{M}}_{0,n}^G = 1$. Indeed, by taking an appropriate $0 \le j \le 2$, $n - j$ can be an even number $2r$ with $r > 1$. Thus $\frac{n-j}{r} - 1 = 1$ has an integer solution and we are able to find G.

In the next lemma, we show that if $j > 0$, then C^σ comes from a curve on $\overline{\mathrm{M}}_{0,n-1}$.

Lemma 3.7. *Suppose that for a balanced σ with $G = \langle \sigma \rangle$, $\sigma(i) = i$. Let $G' = \langle \sigma' \rangle$ be the cyclic subgroup of S_{n-1} generated by $\sigma' := \sigma|_{[n]-\{i\}}$. Consider $C^{\sigma'} \subset \overline{\mathrm{M}}_{0,[n]-\{i\}} \cong \overline{\mathrm{M}}_{0,n-1}$. For the forgetful map $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1}$, $\pi|_{C^\sigma} : C^\sigma \to C^{\sigma'}$ is an isomorphism.*

Proof. It is straightforward to check that $\pi(C^\sigma) = C^{\sigma'}$ and that π restricts to a birational map on C^σ. Thus $C^\sigma \to C^{\sigma'}$ is an isomorphism. $\square$

Example 3.8. Let $G = \langle (12)(34) \rangle$ and $n = 6$. Consider $X_1 = \mathbb{P}^1$ with 4 marked points x_1, x_2, x_3, and x_4 such that G-invariant. Then there are two fixed points p, q on X_1. Let $X_2 = \mathbb{P}^1$ have three marked points x_5, x_6, r. Consider the gluing $X = X_1 \cup X_2$ along p and r. Then this is a G-invariant curve, and there are 1-dimensional moduli $C \subset \overline{\mathrm{M}}_{0,6} - \mathrm{M}_{0,6}$ of these curves. C is disjoint from $C^{(12)(34)}$, because for every degenerated curve in $C^{(12)(34)}$, two fixed marked points x_5 and x_6 are on the spine (for instance when x_1 approaches x_3 or x_4) or on distinct tails (for example if x_1 approaches one of σ-fixed points on $\mathbb{P}^1$).

This example shows that, in general, the fixed point locus of G has extra irreducible components contained in the boundary of $\overline{\mathrm{M}}_{0,n}$.

Since the classes of boundary divisors span $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n}, \mathbb{Q})$ ([Kee92, p.550]), to obtain the numerical class of C^σ, it suffices to know the intersection numbers of C^σ with boundary divisors (consult [Moo13, Section 2] for notations).

Proposition 3.9. *Let $\sigma \in S_n$ be a balanced permutation of type (j, r) with two nontrivial orbits and let σ_1, σ_2 be two non-trivial disjoint cycles in σ. Let F be the set of σ-invariant marked points.*

(1) Let $I = \{h, i\}$ where $h \in \sigma_1$ and $i \in \sigma_2$. Then $C^\sigma \cdot D_I = 1$.
(2) Suppose that $F = \emptyset$. For $I = \sigma_\ell$, $C^\sigma \cdot D_I = 2$.
(3) Suppose that $F = \{a\}$. For $I = \sigma_\ell$ or $I = \sigma_\ell \cup \{a\}$, $C^\sigma \cdot D_I = 1$.
(4) Suppose that $F = \{a, b\}$. For $I = \sigma_\ell \cup \{a\}$, $C^\sigma \cdot D_I = 1$.

(5) *Except for the above cases, all other intersection numbers are zero.*

(6) *If $n \geq 6$, $C^\sigma \cdot D_2 = r^2$ and $C^\sigma \cdot D_{\lfloor \frac{n}{2} \rfloor} = 2$. All other intersections are zero.*

(7) *If $n = 5$ (so $j = 1$ and $r = 2$), then $C^\sigma \cdot D_2 = 6$.*

Proof. We count all points on C^σ that parametrize singular curves. They arise when some of the marked points (equivalently, some of the orbits) collide. There are two marked orbits (orbits consisting of marked points) of order r and j marked orbits of order one.

First of all, consider the case that two marked orbits collide. In particular, suppose that x_h for $h \in \sigma_1$ approaches x_i for $i \in \sigma_2$, where x_h (resp. x_i) is the h-th (resp. i-th) marked point. Then simultaneously $x_{\sigma^k(h)} = x_{\sigma_1^k(h)}$ approaches $x_{\sigma^k(i)} = x_{\sigma_2^k(i)}$, with a constant rate of speed. Thus the stable limit is on $\cap_{0 \leq k < r} D_{\{\sigma^k(h), \sigma^k(i)\}}$, which parametrizes a curve with r tails such that on each tail there are two marked points $x_{\sigma^k(h)}$ and $x_{\sigma^k(i)}$. This proves (1).

For $i \in \sigma_1$, if x_i approaches one of two σ-fixed points on $\mathbb{P}^1$, then we have another singular stable limit. If x_i goes to a non-marked point, then the limit is on D_I where $I = \sigma_1$. If x_i goes to a fixed marked point x_a, then the limit is on $D_{I \cup \{a\}}$. So we obtain parts (2), (3), and (4).

Except those cases above, all other points on C^σ parametrize smooth curves. So we have part (5).

If $n \geq 7$, then $r \geq 3$. In this case, item (1) is the only case where $C^\sigma \cdot D_I > 0$ with $|I| = 2$. Because for a fixed i, there are r different possible choices of j, we have $C^\sigma \cdot D_2 = r^2$. There are two additional degenerations corresponding to the case that x_i approaches one of two fixed points. In any case of (2), (3), and (4), it is straightforward to check that $|I| = \lfloor \frac{n}{2} \rfloor$. Therefore $C^\sigma \cdot D_{\lfloor \frac{n}{2} \rfloor} = 2$. The case of $n = 5, 6$ are obtained by a simple case by case analysis with the same idea. So we have (6) and (7). $\qquad\square$

Corollary 3.10. *Let $\sigma \in S_n$ be a balanced permutation of type (j, r). Then*

$$C^\sigma \cdot K_{\overline{\mathrm{M}}_{0,n}} = -4 + j.$$

Proof. The numerical class of the canonical divisor $K_{\overline{\mathrm{M}}_{0,n}}$ is given by

$$(1) \qquad\qquad K_{\overline{\mathrm{M}}_{0,n}} = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} \left(-2 + \frac{k(n-k)}{n-1} \right) D_k$$

([Pan97, Proposition 1]). By using Proposition 3.9 and formula (1), we can compute the intersection number. $\qquad\square$

Since $\psi \equiv K_{\overline{\mathrm{M}}_{0,n}} + 2D$ ([Moo13, Lemma 2.9]), it is immediate to get the following result.

Corollary 3.11. *Let $\sigma \in S_n$ be a balanced permutation of type (j, r). Then*

$$C^\sigma \cdot \psi = 2r^2 + j.$$

4. THE CURVE CLASS C^σ AS AN EFFECTIVE LINEAR COMBINATION OF F-CURVES

In this section, we show two facts. First, the curve class C^σ in Definition 3.6 is an effective $\mathbb{Z}$-linear combination of F-curves. Second, it is movable, so it is in the dual cone of the effective cone of $\overline{M}_{0,n}$.

4.1. **The nonadjacent basis and its dual basis.** For $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}} \cong \mathrm{H}^2(\overline{M}_{0,n}, \mathbb{Q})$, there is a basis due to Keel and Gibney called the *nonadjacent* basis. Let G_n be a cyclic graph with n vertices $[n] := \{1, 2, \cdots, n\}$ labeled in that order. For a subset $I \subset [n]$, let $t(I)$ be the number of connected components of the subgraph generated by vertices in I. A subset I is called *adjacent* if $t(I) = 1$. Since G_n is cyclic, if $t(I) = k$, then $t(I^c) = k$.

Proposition 4.1 ([Car09, Proposition 1.7]). *Let $\mathcal{B}$ be the set of boundary divisors D_I for $I \subset [n]$ with $t(I) \geq 2$. Then $\mathcal{B}$ forms a basis of $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$.*

Definition 4.2. The set $\mathcal{B}$ in Proposition 4.1 is called the *nonadjacent basis*.

Since there is an intersection pairing $\mathrm{H}^2(\overline{M}_{0,n}, \mathbb{Q}) \times \mathrm{H}_2(\overline{M}_{0,n}, \mathbb{Q}) \to \mathbb{Q}$, we obtain a basis of $\mathrm{H}_2(\overline{M}_{0,n}, \mathbb{Q})$ dual to the nonadjacent basis. Indeed, many vectors in this basis are F-curves.

Example 4.3. On $\overline{M}_{0,5}$, the nonadjacent basis is

$$\{D_{\{1,3\}}, D_{\{1,4\}}, D_{\{2,4\}}, D_{\{2,5\}}, D_{\{3,5\}}\}.$$

It is straightforward to check that the dual basis is (in the corresponding order)

$$\{F_{\{1,2,3,45\}}, F_{\{1,4,5,23\}}, F_{\{2,3,4,15\}}, F_{\{1,2,5,34\}}, F_{\{3,4,5,12\}}\}.$$

So all of the dual elements are F-curves.

Example 4.4. On $\overline{M}_{0,6}$, the nonadjacent basis is

$$\{D_{\{1,3\}}, D_{\{1,4\}}, D_{\{1,5\}}, D_{\{2,4\}}, D_{\{2,5\}}, D_{\{2,6\}}, D_{\{3,5\}}, D_{\{3,6\}}, D_{\{4,6\}},$$

$$D_{\{1,2,4\}}, D_{\{1,2,5\}}, D_{\{1,3,4\}}, D_{\{1,3,5\}}, D_{\{1,3,6\}}, D_{\{1,4,5\}}, D_{\{1,4,6\}}\}.$$

The dual basis is

$$\{F_{\{1,2,3,456\}}, F_{\{1,4,23,56\}}, F_{\{1,5,6,234\}}, F_{\{2,3,4,156\}}, F_{\{2,5,16,34\}}, F_{\{1,2,6,345\}},$$

$$F_{\{3,4,5,126\}}, F_{\{3,6,12,45\}}, F_{\{4,5,6,123\}}, F_{\{3,4,12,56\}}, F_{\{5,6,12,34\}}, F_{\{1,2,34,56\}},$$

$$F_{\{5,6,13,24\}} + F_{\{1,2,3,456\}} + F_{\{2,3,4,156\}} - F_{\{2,3,16,45\}},$$

$$F_{\{2,3,16,45\}}, F_{\{1,6,23,45\}}, F_{\{4,5,16,23\}}\}.$$

Except for the curve dual to $D_{\{1,3,5\}}$, all other vectors in the dual basis are F-curves.

Proposition 4.5. *Let D_I be a boundary divisor in the nonadjacent basis. The dual of D_I is an F-curve if and only if $t(I) = 2$. In this case, if $I_1 \sqcup I_2 = I$ and $J_1 \sqcup J_2 = I^c$ are the decompositions of I and I^c into two connected sets, then the dual of D_I is F_{I_1, I_2, J_1, J_2}.*

Proof. Since I is not connected, $t(I) \geq 2$. For $t(I) > 2$, let $I_1, I_2, \cdots, I_{t(I)}$ be the connected components of I. To obtain $D_I \cdot F_{A_1,A_2,A_3,A_4} = 1$, we need $A_i \sqcup A_j = I$ for two distinct $i, j \in \{1, 2, 3, 4\}$. Then at least one of A_i must be disconnected. So D_{A_i} is in the nonadjacent basis and $D_{A_i} \cdot F_{A_1,A_2,A_3,A_4} = -1$. Therefore the dual element is not an F-curve.

Now suppose that $t(I) = 2$. Let $I = I_1 \sqcup I_2$ and $I^c = J_1 \sqcup J_2$ be the decompositions of I and I^c into connected components. Then for $D_K \in \mathcal{B}$, we have a nonzero intersection $D_K \cdot F_{I_1,I_2,J_1,J_2}$ only if K is one of $I_1, I_2, J_1, J_2, I_1 \sqcup I_2, I_1 \sqcup J_1, I_1 \sqcup J_2$ or their complements. But except $I = I_1 \sqcup I_2$ (equivalently, its complement $I^c = J_1 \sqcup J_2$), all of them are connected so only D_I is in $\mathcal{B}$. Thus F_{I_1,I_2,J_1,J_2} is the dual element of D_I. $\qquad\square$

In general, the dual curve for D_I with $t(I) > 2$ is a complicated non-effective $\mathbb{Z}$-linear combination of F-curves. We give an inductive algorithm to find the combination.

Proposition 4.6. *For D_I with $t(I) = t \geq 3$, let $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_t$ and $I^c = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_t$ be the decompositions into connected components, in the circular order of $I_1, J_1, I_2, J_2, \cdots, I_t, J_t$. Then the dual basis of D_I is a $\mathbb{Z}$-linear combination of F-curves. The linear combination can be computed inductively.*

Proof. If $t(I) = 3$, it is straightforward to check that the dual curve for D_I is

$$F_{I_1 \sqcup I_2, J_1 \sqcup J_2, I_3, J_3} + F_{I_1, J_1, I_2, J_2 \sqcup I_3 \sqcup J_3} + F_{J_1, I_2, J_2, I_1 \sqcup I_3 \sqcup J_3} - F_{J_1, I_2, I_1 \sqcup J_3, J_2 \sqcup I_3}.$$

Consider the F-curve $F_{I_1 \sqcup I_2, J_1 \sqcup J_2, I_3 \sqcup I_4 \sqcup \cdots \sqcup I_t, J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$. It intersects positively with $D_{I_1 \sqcup J_1 \sqcup I_2 \sqcup J_2}$, D_I, $D_{I_1 \sqcup I_2 \sqcup J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$, and negatively with $D_{I_1 \sqcup I_2}$, $D_{J_1 \sqcup J_2}$, $D_{I_3 \sqcup I_4 \sqcup \cdots \sqcup I_t}$, and $D_{J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$. Except them, all other intersection numbers are zero. Note that $I_1 \sqcup J_1 \sqcup I_2 \sqcup J_2$ is connected so $D_{I_1 \sqcup J_1 \sqcup I_2 \sqcup J_2} \notin \mathcal{B}$. For all other boundary divisors above with nonzero intersection numbers, the numbers of connected components of corresponding subsets of $[n]$ are strictly less than t, because $I_1 \sqcup J_t$ is a connected set. Let E_I, $E_{I_1 \sqcup I_2 \sqcup J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$, $E_{I_1 \sqcup I_2}$, $E_{J_1 \sqcup J_2}$, $E_{I_3 \sqcup I_4 \sqcup \cdots \sqcup I_t}$, and $E_{J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$ be the dual elements which are explicit $\mathbb{Z}$-linear combinations of F-curves by the induction hypothesis. Then

$$F_{I_1 \sqcup I_2, J_1 \sqcup J_2, I_3 \sqcup I_4 \sqcup \cdots \sqcup I_t, J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t} - E_{I_1 \sqcup I_2 \sqcup J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$$

$$+ E_{I_1 \sqcup I_2} + E_{J_1 \sqcup J_2} + E_{I_3 \sqcup I_4 \sqcup \cdots \sqcup I_t} + E_{J_3 \sqcup J_4 \sqcup \cdots \sqcup J_t}$$

is the dual basis of D_I. $\qquad\square$

Remark 4.7. The rank of $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$ is $2^{n-1} - \binom{n}{2} - 1$. On the other hand, the number of boundary divisors D_I with $t(I) = 2$ is $\binom{n}{4}$. Therefore if n is large, most dual vectors are not F-curves.

4.2. Writing C^σ as an effective linear combination of F-curves.

Theorem 4.8. *Let $\sigma \in S_n$ be a balanced permutation which is the product of two nontrivial disjoint cycles σ_1 and σ_2. Then C^σ can be written as an effective sum of F-curves whose nonzero coefficients are all one.*

Proof. By applying the left S_n-action to S_n, we may assume that $\sigma = \sigma_1\sigma_2 = (1, 2, \cdots, r)(r+1, r+2, \cdots, 2r)$. Let S be the set of boundary divisors that intersect C^σ nontrivially. We claim that for any $D_I \in S$, $t(I) \leq 2$. Indeed, for $D_I \in S$ with $|I| = 2$, then $t(I) \leq 2$ is obvious. If $|I| > 2$, then I is one of σ_i, which is a connected set, or possibly the union of σ_i and an element of $[n] - (\sigma_1 \sqcup \sigma_2)$. Since $[n] - (\sigma_1 \sqcup \sigma_2)$ has at most two elements, the union also has at most two connected components.

For each nonadjacent $D_I \in S$, the dual element is an F-curve by Lemma 4.5. Thus C^σ is the effective linear combination of dual elements for nonadjacent $D_I \in S$. Finally, the intersection number $C^\sigma \cdot D_I$ is greater than one only if $j = 0$ and $I = \sigma_i$. But in this case I is connected so it does not affect the linear combination. $\square$

Corollary 4.9. *The number of F-curves appearing in the effective linear combination of C^σ described by Theorem 4.8 is $r^2 - 2 + j$.*

Proof. We may assume that $G = \langle (1, 2, \cdots, r)(r+1, r+2, \cdots, 2r) \rangle$. Note that C^σ intersects $r^2 + 2$ boundary divisors by Proposition 3.9. If $j = 0$ (so $n = 2r$), then $C^\sigma \cdot D_{\{1,2,\cdots,r\}} = 2$, $C^\sigma \cdot D_{\{1,2r\}} = C^\sigma \cdot D_{\{r,r+1\}} = 1$ and other intersecting boundaries are all nonadjacent. Thus on the effective linear combination above, $r^2 - 2$ F-curves appear. If $j = 1$ and $n = 2r + 1$, then $C^\sigma \cdot D_{\{1,2,\cdots,r\}} = C^\sigma \cdot D_{\{r+1,r+2,\cdots,2r\}} = C^\sigma \cdot D_{\{r,r+1\}} = 1$ and other intersecting boundaries are nonadjacent so there are $r^2 - 1$ F-curves on the effective linear combination. The case of $j = 2$ is similar. $\square$

Example 4.10. Let $\sigma = (12)(34) \in S_6$. By Proposition 3.9, C^σ intersects nontrivially

$$D_{\{1,3\}}, D_{\{1,4\}}, D_{\{2,3\}}, D_{\{2,4\}}, D_{\{1,2,5\}}, D_{\{1,2,6\}}$$

and the intersection numbers are all one. Among these divisors, $D_{\{2,3\}}$ and $D_{\{1,2,6\}}$ are adjacent divisors. Thus

$$C^\sigma \equiv F_{\{1,2,3,456\}} + F_{\{1,4,23,56\}} + F_{\{2,3,4,156\}} + F_{\{5,6,12,34\}}.$$

Example 4.11. Let $\sigma = (123)(456) \in S_6$. Then by using Proposition 3.9 and the dual basis from Example 4.4, we obtain

$$C^\sigma \equiv F_{\{1,4,23,56\}} + F_{\{1,5,6,234\}} + F_{\{2,3,4,156\}}$$

$$+ F_{\{2,5,16,34\}} + F_{\{1,2,6,345\}} + F_{\{3,4,5,126\}} + F_{\{3,6,12,45\}}.$$

Example 4.12. Let $n = 7$ and $\sigma = (123)(456) \in S_7$. Then C^σ intersects

$$D_{\{1,2,3\}}, D_{\{4,5,6\}}, D_{\{1,4\}}, D_{\{1,5\}}, D_{\{1,6\}}, D_{\{2,4\}}, D_{\{2,5\}}, D_{\{2,6\}}, D_{\{3,4\}}, D_{\{3,5\}}, D_{\{3,6\}}$$

and all intersection numbers are one. Among them, only $D_{\{1,2,3\}}, D_{\{4,5,6\}}, D_{\{3,4\}}$ are adjacent divisors. Thus

$$C^\sigma \equiv F_{\{1,4,23,567\}} + F_{\{1,5,67,234\}} + F_{\{1,6,7,2345\}}$$

$$+ F_{\{2,3,4,1567\}} + F_{\{2,5,34,167\}} + F_{\{2,6,17,345\}} + F_{\{3,4,5,1267\}} + F_{\{3,6,45,127\}}.$$

Remark 4.13. The numerical class of $\overline{\mathrm{M}}_{0,n}^G$ often has more symmetry beyond the input group $G \subset S_n$. For instance, for the curve above, we computed that the stabilizer of $[C^\sigma]$ has order 72. See the code samples on the second author's website for more details.

4.3. **Cone of F-curves and C^σ.** We can ask several natural questions about the curve classes C^σ which have implications for the birational geometry of $\overline{\mathrm{M}}_{0,n}$. Is C^σ movable? Or as another extremal case, is C^σ rigid? A movable curve can be used to compute a facet of the effective cone. On the other hand, a rigid rational curve on $\overline{\mathrm{M}}_{0,n}$ could be a candidate for a possible counterexample to the F-conjecture ([CT12, Section 2]).

In this section, we show that C^σ is always movable and it is on the boundary of the Mori cone $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$.

Lemma 4.14. *Let $\sigma \in S_n$ be a balanced permutation with two nontrivial disjoint cycles. Pick four indexes $S := \{i, j, k, l\} \subset [n]$ and let $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,S} \cong \mathbb{P}^1$ be the forgetful map. If $S \cap \sigma_1 = \emptyset$ or $S \cap \sigma_2 = \emptyset$, then $\deg \pi_*[C^\sigma] = 0$.*

Proof. Note that for a forgetful map $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,\{i,j,k,l\}}$, the locus of smooth curves maps to the locus of smooth curves. It is straightforward to check that if $S \cap \sigma_1 = \emptyset$ or $S \cap \sigma_2 = \emptyset$, then each image of $(X, x_1, x_2, \cdots, x_n) \in C^\sigma$ is a smooth curve in $\overline{\mathrm{M}}_{0,\{i,j,k,l\}}$. Therefore $C^\sigma \to \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,S}$ is not surjective and $\deg \pi_*[C^\sigma] = 0$. $\qquad\square$

Proposition 4.15. *For $n \geq 7$, all C^σ are on the boundary of the Mori cone $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$.*

Proof. Suppose that $\sigma = \sigma_1\sigma_2$. Then one of $\sigma_1, [n] - \sigma_1$ has four or more elements. If you take four of them and denote the set of them by $S = \{i, j, k, l\}$, then by Lemma 4.14, $\pi_*[C^\sigma] = 0$ for the projection $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,S}$.

Now assume that there is an effective linear combination

$$C^\sigma \equiv \sum_I x_I E_I$$

for some effective curves E_I and $x_I \in \mathbb{Q}_+$. For the projection $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,S}$, an F-curve maps to a point if at least two of i, j, k, l are on the same partition. Also an F-curve maps to isomorphically onto $\overline{\mathrm{M}}_{0,S}$ if none of i, j, k, l are on the same partition. Thus if we take an F-curve F_J with a partition splits S into four singleton sets, then $\pi_*[F_J] = [\overline{\mathrm{M}}_{0,S}]$. Therefore x_J must be zero since $\pi_*[C^\sigma] = 0$. In other words, for *any* effective linear combination of effective-curves, F_J does not appear. This implies that C^σ is on a facet which is disjoint from the ray generated by F_J. Therefore C^σ is on the boundary of $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,n})$. $\qquad\square$

Remark 4.16. By using the same argument, we can show that for $\sigma = (12)(34) \in S_6$, C^σ is on the boundary of $\overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,6})$.

Theorem 4.17. *For $n \geq 7$, all C^σ are movable.*

Proof. Let $\sigma = \sigma_1\sigma_2$ and $G = \langle\sigma\rangle$. Also Suppose that the length of σ_i is r. By Lemma 4.14, for two projections $\pi_1 : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,[n]-\sigma_1}$ and $\pi_2 : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,[n]-\sigma_2}$, C^σ is contained in a fiber. Moreover, $\pi : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,[n]-\sigma_1} \times \overline{\mathrm{M}}_{0,[n]-\sigma_2}$ is surjective, because $|([n] - \sigma_1) \cap ([n] - \sigma_2)| = j \leq 2$. Note that the dimension of a general fiber is $(n-3) - 2(n-r-3) = 2r - n + 3 = 3 - j$.

On the interior of $\overline{M}_{0,[n]-\sigma_1}$, there exists a unique point p which is invariant with respect to $\langle\sigma_2\rangle$-action. Indeed, for a $\mathbb{P}^1$ with specific coordinates, choose two pivotal points determining the rotational axis, and another point for one of marked points in σ_2. Then all other marked points for the curve parametrized by p are determined by the σ_2-action. Thus there is a three dimensional moduli and if we apply PGL_2-action, then we obtain a point. Similarly, we have a $\langle\sigma_1\rangle$-invariant point q on $\overline{M}_{0,[n]-\sigma_2}$. If we regard the induced $G = \langle\sigma\rangle$-action on $\overline{M}_{0,[n]-\sigma_1} \times \overline{M}_{0,[n]-\sigma_2}$, then $\pi : \overline{M}_{0,n} \to \overline{M}_{0,[n]-\sigma_1} \times \overline{M}_{0,[n]-\sigma_2}$ is G-equivariant and C^σ should be in the fiber $\pi^{-1}(p, q)$.

If $j = 2$, a general fiber is a curve. And the special fiber $\pi^{-1}(p,q)$ is an irreducible curve, because a general point of it is determined by the cross ratio of four marked points x_1, x_2, x_3, x_4 where $x_1 \in \sigma_1$, $x_2 \in \sigma_2$ and x_3, x_4 are two fixed points. So $C^\sigma = \pi^{-1}(p,q)$. Therefore C^σ is movable.

Finally, if $j < 2$, then C^σ is the image of $C^{\sigma'}$ for $\rho : \overline{M}_{0,n+2-j} \to \overline{M}_{0,n}$ by Lemma 3.7. Note that $C^{\sigma'}$ is in an algebraic family $\mathcal{C}$ covers $\overline{M}_{0,n+2-j}$, by above observation. By composing with ρ, we obtain a family $\mathcal{C}'$ containing C^σ and covers $\overline{M}_{0,n}$. Thus C^σ is movable, too. $\qquad\square$

5. Dihedral group cases

Next we will discuss the case where G is a dihedral group. Surprisingly, the geometry of $\overline{M}_{0,n}^G$ when G is a dihedral group is very different from the geometry of $\overline{M}_{0,n}^G$ when G is a cyclic group.

Let G be a subgroup of S_n which is isomorphic to D_k with $k \geq 3$. Then, up to conjugation, G acts on $\mathbb{P}^1 \cong S^2$, as the symmetry group of a bipyramid over a regular k-gon. There is a unique orbit (corresponding to two pivotal points) of order two, there are two orbits of order k, and all other orbits have order $2k$.

Since G permutes marked points, if a point x on $\mathbb{P}^1$ is a marked point, then all points in $G \cdot x$ are marked points, too. So if we have $(\mathbb{P}^1, x_1, x_2, \cdots, x_n) \in \overline{M}_{0,n}^G$, we have a partition of $[n]$ into subsets of size 2, k, and $2k$. The number of size 2 orbits is 0 or 1. The number of size k orbits is at most two. The dimension of the stratum is the number of orbits of order $2k$ because to fix a D_k-action on $\mathbb{P}^1$, we need to pick three points already, which are two pivotal points and a point of order k.

Define G-invariant families of n-pointed $\mathbb{P}^1$ as following. Fix a coordinate on $\mathbb{P}^1$ and fix a G-action on $\mathbb{P}^1 \cong S^2$ as a rotation group of a bipyramid over a regular k-gon. Take a generic orbit O of order $2k$, specify the number of *special orbits* (orbits of order 2 and k), and assign marked points to some of these orbits. Then we have an element of $M_{0,n}$. By varying O, we obtain a rational map $f : \mathbb{P}^1 \dashrightarrow \overline{M}_{0,n}$. Since $\mathbb{P}^1$ is a curve, this rational map can be extended to all of $\mathbb{P}^1$. $\overline{M}_{0,n}^G$ is the image of f.

Remark 5.1. If there are no special marked orbits, i.e., if we choose a general orbit only, then the one-dimensional family over $\mathbb{P}^1$ obtained by varying the orbit of order $2k$ is a $(2 : 1)$ cover of a rational curve $\overline{M}_{0,n}^G$ on $\overline{M}_{0,n}$.

In summary, for a dihedral group $G \cong D_k$, $\overline{\mathrm{M}}_{0,n}^G$ is a curve if there exists a unique G-orbit of order $2k$. There might be an extra marked orbit of order 2 and at most two marked orbits of order k. By an argument similar to that of Lemma 3.4, we obtain that $\overline{\mathrm{M}}_{0,n}^G$ is isomorphic to $\mathbb{P}^1$. Thus we get the following result.

Lemma 5.2. *Let $G \cong D_k$. $\overline{\mathrm{M}}_{0,n}^G$ is a curve on $\overline{\mathrm{M}}_{0,n}$ only if $n = 2k, 2k+2, 3k, 3k+ 2, 4k,$ and $4k + 2$.*

Definition 5.3. For a dihedral group $G \cong D_k \subset S_n$, suppose that $\overline{\mathrm{M}}_{0,n}^G$ is a curve. We say that the G action is *of type (a, b)* if a is the number of order two marked orbits and b is the number of order k special orbits. So $0 \leq a \leq 1$ and $0 \leq b \leq 2$.

If there is no orbit of order k consisting of marked points, then $\overline{\mathrm{M}}_{0,n}^G$ is a curve we have already described:

Lemma 5.4. *Let $G \cong D_k$ such that $\overline{\mathrm{M}}_{0,n}^G$ is a curve. Let $\sigma \in G$ have order k. If the G action is of type $(a, 0)$, then $\overline{\mathrm{M}}_{0,n}^G = C^\sigma$.*

Proof. It is clear that $\overline{\mathrm{M}}_{0,n}^G \subset \overline{\mathrm{M}}_{0,n}^{\langle \sigma \rangle} = C^\sigma$, and both of them are irreducible curves, so $\overline{\mathrm{M}}_{0,n}^G = C^\sigma$. $\qquad\square$

Example 5.5. The simplest case is $G \cong D_3$ and $n = 6$. Let $G = \langle (123)(456), (14)(26)(35) \rangle$. Then G is of type $(0, 0)$ and it is the rotation group of a triangular prism inscribed in S^2 whose top vertices are $1, 2, 3$ and whose bottom vertices are $4, 5, 6$ in the same order.

To compute the numerical class of $\overline{\mathrm{M}}_{0,n}^G$, we need to compute the intersection numbers with boundary divisors. A point configuration on a general point of $\overline{\mathrm{M}}_{0,n}^G$ degenerates if the 'moving' orbit of order $2k$ approaches a special orbit (an orbit of order 2 or k). Note that special orbits might not consist of marked points.

Proposition 5.6. *Let $G \subset S_n$ be a finite group isomorphic to D_k for $k \geq 3$. Suppose that $\overline{\mathrm{M}}_{0,n}^G$ is a curve. Let $\sigma \in G$ be an order k element.*

(1) *If the G action is of type $(a, 0)$, then σ is a product of two disjoint cycles σ_1 and σ_2 of length k.*

   (a) *If $a = 0$,*

   $$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 2, & I = \sigma_1 \text{ or } I = \sigma_2, \\ 1, & I = \{i, j\} \text{ where } i \in \sigma_1, j \in \sigma_2, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) *If $a = 1$, let ℓ be one of two marked points in the orbit of order two. Then:*

   $$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 1, & I = \sigma_1 \cup \{\ell\} \text{ or } \sigma_2 \cup \{\ell\}, \\ 1, & I = \{i, j\}, i \in \sigma_1, j \in \sigma_2, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If the G action is of type $(a, 1)$, then σ is a product of three disjoint cycles $\sigma_1, \sigma_2, \sigma_3$ of order k. If we take a reflection $\tau \in G - \langle \sigma \rangle$, then exactly one of them (say σ_3) is invariant for τ-action. Furthermore, we are able to take τ, such that there is $m \in \sigma_3$ so that $\tau(m) = m$.*

(a) *If $a = 0$ and k is odd, then:*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 2, & I = \sigma_1 \text{ or } I = \sigma_2, \\ 1, & I = \{i, j\} \text{ where } i \in \sigma_1, j \in \sigma_2, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau(i)), \sigma^t(m)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

(b) *If $a = 0$ and k is even, then:*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 2, & I = \sigma_1 \text{ or } I = \sigma_2, \\ 2, & I = \{i, \sigma^{2t+1}(\tau(i))\} \text{ where } i \in \sigma_1, 0 \le t < k/2, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau(i)), \sigma^t(m)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

(c) *If $a = 1$ and k is odd, pick ℓ with order two. Then we have:*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 1, & I = \sigma_1 \cup \{\ell\} \text{ or } \sigma_2 \cup \{\ell\}, \\ 1, & I = \{i, j\} \text{ where } i \in \sigma_1, j \in \sigma_2, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau(i)), \sigma^t(m)\}, i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

(d) *If $a = 1$ and k is even, pick ℓ with order two. Then we have:*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 1, & I = \sigma_1 \cup \{\ell\} \text{ or } \sigma_2 \cup \{\ell\}, \\ 2, & I = \{i, \sigma^{2t+1}(\tau(i))\} \text{ where } i \in \sigma_1, 0 \le t < k/2, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau(i)), \sigma^t(m)\}, i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If the G action is of type $(a, 2)$, then σ is a product of four disjoint cycles σ_a, $1 \le a \le 4$ of order k. For any reflection $\tau \in G - \langle \sigma \rangle$, two of σ_a's (say σ_3 and σ_4) are τ-invariant. Furthermore, by taking appropriate reflections τ_1, τ_2, we may assume that there is $m \in \sigma_3$ and $n \in \sigma_4$ such that $\tau_1(m) = m$ and $\tau_2(n) = n$.*

(a) *If $a = 0$,*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 2, & I = \sigma_1 \text{ or } I = \sigma_2, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau_1(i)), \sigma^t(m)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau_2(i)), \sigma^t(n)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

(b) *If $a = 1$ and if ℓ is one of two marked points of order two,*

$$D_I \cdot \overline{\mathrm{M}}_{0,n}^G = \begin{cases} 1, & I = \sigma_1 \cup \{\ell\} \text{ or } I = \sigma_2 \cup \{\ell\}, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau_1(i)), \sigma^t(m)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 1, & I = \{\sigma^t(i), \sigma^t(\tau_2(i)), \sigma^t(n)\} \text{ where } i \in \sigma_1, 0 \le t < k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Item (1) is just a restatement of Proposition 3.9. Items (2) and (3) can be obtained by considering when a general orbit $\sigma_1 \cup \sigma_2$ can collide with a point in special orbits $(\sigma_3, \sigma_4, \{\ell, \tau(\ell)\})$. We leave the proof as an exercise for the reader. □

Example 5.7. Let $G = \langle (123)(456)(789), (14)(26)(35)(89) \rangle \cong D_3$ and $C = \overline{\mathrm{M}}_{0,9}^G$ on $\overline{\mathrm{M}}_{0,9}$. In this case, G is of type $(0,1)$. Then the marked points $x_1, \cdots, x_6$ form an orbit of order 6 and the marked points x_7, x_8, x_9 form an orbit of order 3. $C \cdot D_{\{1,2,3\}} = C \cdot D_{\{4,5,6\}} = 2$ and $C \cdot D_{\{i,j\}} = 1$ when $1 \le i \le 3$ and $4 \le j \le 6$. Also $C \cdot D_I = 1$ if D_I is one of

$$D_{\{1,4,7\}}, D_{\{2,5,8\}}, D_{\{3,6,9\}}, D_{\{1,6,8\}}, D_{\{2,4,9\}}, D_{\{3,5,7\}}, D_{\{1,5,9\}}, D_{\{2,6,7\}}, D_{\{3,4,8\}}.$$

All other intersection numbers are zero.

An interesting fact about C is that for *every* projection $\pi : \overline{\mathrm{M}}_{0,9} \to \overline{\mathrm{M}}_{0,4}$, C is not contracted, because for any four of nine points, their cross ratio is not a constant. In particular, C is not a fiber of a hypergraph morphism ([CT12, Definition 4.4]), thus it is not a curve constructed in [CT12].

Proposition 5.6 allows us to write the class of C as a (non-effective) linear combination of F-curves. We create a vector of intersection numbers with the nonadjacent basis of divisors, and use the command `writeCurveInSingletonSpineBasis` in the `M0nbar` package for `Macaulay2` to obtain the coefficients of C in the so-called "singleton spine basis" of F-curves. The resulting expression is supported on 37 F-curves. 26 curves in this expression have positive coefficients, and 11 curves in this expression have negative coefficients.

In Example 7.1 we will express the class of this curve as an effective linear combination of F curves.

Example 5.8. For notational simplicity, write $[12] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c\}$. Let

$$G = \langle (123)(456)(789)(abc), (14)(26)(35)(89)(ac) \rangle \cong D_3$$

and $C := \overline{\mathrm{M}}_{0,12}^G$ on $\overline{\mathrm{M}}_{0,12}$. Then $x_1, x_2, \cdots, x_6$ form an orbit of order 6; x_7, x_8, x_9 form an orbit of order 3; and x_a, x_b, x_c form an orbit of order 3. $D_{\{1,2,3\}} \cdot C = D_{\{4,5,6\}} \cdot C = 2$ and $D_I \cdot C = 1$ for the following 18 irreducible components of D_3,

$$D_{\{1,4,7\}}, D_{\{2,5,8\}}, D_{\{3,6,9\}}, D_{\{1,6,8\}}, D_{\{2,4,9\}}, D_{\{3,5,7\}},$$

$$D_{\{1,5,9\}}, D_{\{2,6,7\}}, D_{\{3,4,8\}}, D_{\{1,5,c\}}, D_{\{2,6,a\}}, D_{\{3,4,b\}},$$

$$D_{\{1,4,a\}}, D_{\{2,5,b\}}, D_{\{3,6,c\}}, D_{\{1,6,b\}}, D_{\{2,4,c\}}, D_{\{3,5,a\}}.$$

Proposition 5.6 allows us to write the class of C as a (non-effective) linear combination of F-curves. We create a vector of intersection numbers with the nonadjacent basis of divisors, and use the command `writeCurveInSingletonSpineBasis`

in the MOnbar package for Macaulay2 to obtain the coefficients of C in the so-called "singleton spine basis" of F-curves. The resulting expression is supported on 103 F-curves. 69 curves in this expression have positive coefficients, and 34 curves in this expression have negative coefficients.

In Section 8 we will express the class of this curve as an effective linear combination of F-curves.

5.1. **Rigidity.** For a rational curve $f : \mathbb{P}^1 \to X$ to a smooth projective variety X of dimension d,

$$(2) \qquad \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X) \geq -K_X \cdot f_*\mathbb{P}^1 + d\chi(\mathcal{O}_{\mathbb{P}^1}) = -K_X \cdot f_*\mathbb{P}^1 + d$$

by [Kol96, Theorem 1.2]. If f is rigid and $X = \overline{M}_{0,n}$, then $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, \overline{M}_{0,n}) \leq \mathrm{Aut}(\mathbb{P}^1) = 3$, thus $K_{\overline{M}_{0,n}} \cdot f_*\mathbb{P}^1 \geq n - 6$. In particular, for $n \geq 7$, the intersection must be positive.

On the other hand,

$$(3) \qquad K_{\overline{M}_{0,n}} = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} \left(-2 + \frac{k(n-k)}{n-1} \right) D_k$$

so for $n \geq 7$, except the coefficient of D_2, all other coefficients are nonnegative. Thus if we want to find an example of a rigid curve, then its intersection with D_k for $k \geq 3$ should be large compared to the intersection with D_2.

Let $G \subset S_n$ be isomorphic to D_k with odd k and $n = 4k$. Example 5.8 is the case of $k = 3$. In this case, $\overline{M}_{0,n}^G \cdot D_3 = 2k^2$, $\overline{M}_{0,n}^G \cdot D_k = 4$ and $\overline{M}_{0,n}^G \cdot D_i = 0$ for all $i \neq 3, k$ by Proposition 5.6. So

$$\overline{M}_{0,n}^G \cdot K_{\overline{M}_{0,n}} = 2k^2 - 8.$$

Therefore for $k \geq 3$, it is a large positive number, and we may have a rigid curve. When $k = 3$, it gives a curve on $\overline{M}_{0,12}$ (Example 5.8). This example is different from the rigid curve of [CT12, Section 4], because it does not intersect D_2 or D_4.

To show equality in (2), we would need to evaluate the normal bundle to the rational curve. The computation of the blow-up formula for the normal bundle is not easy unless the blow-up center is contained in the curve.

6. Toric degenerations on Losev-Manin spaces

Let $C = \overline{M}_{0,n}^G$ be a curve of the type described in the previous section. We would like to compute the numerical class of a curve C and find an effective linear combination of F-curves which is numerically equivalent to C. In the case of a dihedral group, the idea of using the basis dual to the nonadjacent basis in Section 4 does not work anymore. To do this computation, instead of computing the class directly, we will use Losev-Manin space ([LM00]) to find an approximation of it first.

6.1. **Background on Losev-Manin spaces.** In this section, we give a brief review on Losev-Manin spaces.

Fix a sequence of positive rational numbers $A = (a_1, a_2, \cdots, a_n)$ where $0 < a_i \leq 1$ and $\sum a_i > 2$. Then the moduli space $\overline{\mathrm{M}}_{0,A}$ of weighted pointed stable curves ([Has03]) is the moduli space of pairs $(X, x_1, x_2, \cdots, x_n)$ such that:

- X is a connected, reduced projective curve of $p_a(X) = 0$,
- $(X, \sum a_i x_i)$ is a semi-log canonical pair,
- $\omega_X + \sum a_i x_i$ is ample.

It is well known that $\overline{\mathrm{M}}_{0,A}$ is a fine moduli space of such pairs, and there exists a divisorial contraction $\rho : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,A}$ which preserves the interior $\mathrm{M}_{0,n}$. For more details, see [Has03].

Definition 6.1. [LM00], [Has03, Section 6.4] The *Losev-Manin space* $\overline{\mathrm{L}}_n$ is a special case of Hassett's weighted $(n + 2)$-pointed stable curves such that two of weights are 1 and the rest of them are $\frac{1}{n}$. For example, if the first and second points are weight 1 points, then $\overline{\mathrm{L}}_n = \overline{\mathrm{M}}_{0,\left(1,1,\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)}$.

The Losev-Manin space $\overline{\mathrm{L}}_n$ is a projective toric variety, and among toric $\overline{\mathrm{M}}_{0,A}$, it is the closest one to $\overline{\mathrm{M}}_{0,n+2}$. Indeed, $\overline{\mathrm{L}}_n$ is a toric variety whose corresponding polytope is the permutohedron of dimension $n - 1$ ([GKZ08, Section 7.3]), which is obtained from an $(n-1)$-dimensional simplex by carving smaller dimensional faces. So $\overline{\mathrm{L}}_n$ can be obtained by successive blow-ups of a projective space as following. Let $p_1, p_2, \cdots, p_n$ be the standard coordinate points of $\mathbb{P}^{n-1}$. For a nonempty subset $I \subset [n]$, let L_I be the linear subspace of $\mathbb{P}^{n-1}$ spanned by $\{p_i\}_{i \in I}$. Blow-up $\mathbb{P}^{n-1}$ along the n coordinates points. Next, blow-up along the proper transforms of L_I with $|I| = 2$. After that, blow-up along the proper transforms of L_I with $|I| = 3$. If we perform these blow-ups along all L_I up to $|I| = n - 3$, we obtain $\overline{\mathrm{L}}_n$.

Note that there is a dominant reduction morphism $\rho : \overline{\mathrm{M}}_{0,n+2} \to \overline{\mathrm{L}}_n$ ([Has03, Theorem 4.1]), because it is a special case of Hassett's space. If i and j-th points are weight 1 points on $\overline{\mathrm{L}}_n$, we will use the notation $\rho_{i,j}$ for the reduction map ρ, to indicate which points are weight 1 points on $\overline{\mathrm{L}}_n$.

As a moduli space, $\overline{\mathrm{L}}_n$ is a fine moduli space of chains of rational curves. With this weight distribution, all stable rational curves are chains of $\mathbb{P}^1$. Moreover, if p, q are two points with weight 1 and $x_1, x_2, \cdots, x_n$ are points with weight $\frac{1}{n}$, then one of p, q is on one of end components and the other one is on the other end component. We can pick one of them (say p) as the 0-point. The other point becomes the ∞-point. (This notation will be justified soon.) So each boundary stratum corresponds to an *ordered* partition of $[n] := \{1, 2, \cdots, n\}$ by reading the subset of marked points on each irreducible component (from 0-point to ∞-point). For example, the trivial partition $[n]$ corresponds to the big cell. The partition $I|J$ corresponds a toric divisor $D_{I \cup \{0\}} = D_{J \cup \{\infty\}}$. A partition $I_1|I_2|\cdots|I_{n-1}$ of length $n - 1$ (so only one of I_i has two elements and the others are singleton sets) corresponds to a toric boundary curve.

Definition 6.2. For an ordered partition $I_1|I_2|\cdots|I_k$ of $[n]$, we say a rational chain

$$(X, x_1, x_2, \cdots, x_n, 0, \infty) \in \overline{L}_n$$

is *of type* $I_1|I_2|\cdots|I_k$ if

- X is a rational chain of k projective lines $X_1, X_2, \cdots, X_k$;
- $0 \in X_1$, $\infty \in X_k$;
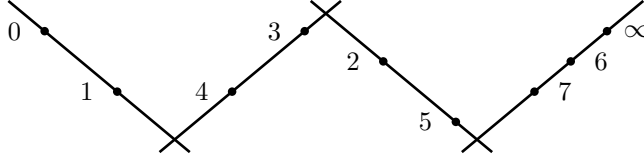- $x_i \in X_j$ if and only if $i \in I_j$.

FIGURE 1. A rational chain of type $1|34|25|67$

For each marked point x_i with weight $1/n$, we are able to take a one parameter subgroup T_i of $(\mathbb{C}^*)^{n-1} \subset \overline{L}_n$, which moves x_i only. T_i acts on $(X, x_1, x_2, \cdots, x_n, 0, \infty)$ as a multiplication of $\mathbb{C}^*$ on the component containing x_i. Note that every component of X has two special points (singular points or points with weight 1). Let X_i be the irreducible component containing x_i, and let y be the special point of X_i which is closer to 0-point. For $t \in T_i$, the limit

$$\lim_{t \to 0} t \cdot (X, x_1, x_2, \cdots, x_n, 0, \infty)$$

is the $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, 1, 1)$-stable curve obtained by first making a bubble at y, then putting x_i on the bubble, and then stabilizing. If x_i is the only marked point with weight $1/n$ on X_i, then $(X, x_1, \cdots, x_n, 0, \infty)$ is T_i-invariant.
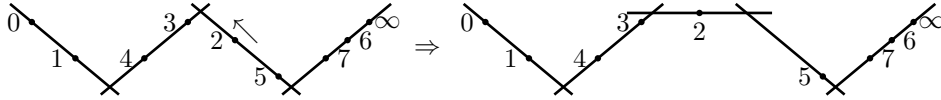
FIGURE 2. A description of $\lim_{t \to 0} t \cdot (X, x_1, x_2, \cdots, x_n, 0, \infty)$ for $t \in T_2$

On $\overline{L}_n$, torus invariant divisors are all the images of boundary divisors on $\overline{M}_{0,n+2}$ of the form D_I where $0 \in I$ and $\infty \notin I$. Thus all toric boundary cycles (which are intersections of toric divisors) are images of F-strata. In particular, all 1-dimensional toric boundary cycles are images of F-curves.

6.2. **Computing limit cycles.** In this section, we describe a method to compute a numerically equivalent effective linear combination of toric boundary curves for a given effective curve $\overline{C} \subset \overline{L}_n$. Because $\overline{L}_n$ is a toric variety, the Mori cone is generated by torus invariant curves. Thus, there exists an effective linear combination of toric boundary cycles representing $[\overline{C}]$. We will apply this idea to the image $\rho(C)$ for an invariant curve C on $\overline{M}_{0,n+2}$. Thus this effective linear combination is an

approximation of F-curve linear combination of C on $\overline{\mathrm{M}}_{0,n+2}$. Later, in Section 7, we will discuss a strategy that uses this approximation to find an effective linear combination of F-curves for C on $\overline{\mathrm{M}}_{0,n+2}$.

The basic idea of computing limit cycles on $\overline{\mathrm{L}}_n$ is the following. For each marked point x_i with weight $1/n$, we have a one parameter subgroup $T_i \subset (\mathbb{C}^*)^{n-1}$ which moves x_i only. So if we choose an ordering of the marked points $x_1, x_2, \cdots, x_n$, then we have a sequence of one parameter subgroups $T_1, T_2, \cdots, T_{n-1}$, where T_i is the one parameter subgroup moving x_i. Note that they generate the big torus $T := (\mathbb{C}^*)^{n-1} \subset \overline{\mathrm{L}}_n$.

Let $\overline{C} \subset \overline{\mathrm{L}}_n$ be an effective curve. Let $\mathrm{Chow}_1(\overline{\mathrm{L}}_n)$ be the Chow variety parameterizing algebraic cycles of dimension one. Let $C^0 := \overline{C}$ and consider $[C^0] \in \mathrm{Chow}_1(\overline{\mathrm{L}}_n)$. Obviously $T \subset \overline{\mathrm{L}}_n$ acts on $\mathrm{Chow}_1(\overline{\mathrm{L}}_n)$. We will compute the limit $[C^1] := \lim_{t \to 0} t \cdot [C^0]$ for $t \in T_1$. Then $[C^1]$ is a T_1-invariant point on $\mathrm{Chow}_1(\overline{\mathrm{L}}_n)$ and $[C^1]$ is an effective cycle on $\overline{\mathrm{L}}_n$. In Section 6.3 we describe a way to find the irreducible components of $[C^1]$, and in Section 6.4 we explain how to compute the multiplicity of each irreducible component.

For the next step, we can compute $[C^2] := \lim_{t \to 0} t \cdot [C^1]$ for $t \in T_2$, componentwise. Note that each irreducible component of $[C^1]$ is contained in an irreducible component of boundary, which is isomorphic to the product $\prod \overline{\mathrm{L}}_k$ for small k. Also T_2 acts on exactly one of $\overline{\mathrm{L}}_k$. Hence the limit computation is very similar to the previous computation. If we perform the limit computations successively for T_i with $1 \leq i \leq n-1$, then the limit cycle is invariant for all T_i. Therefore it is invariant for T, and the limit $[C^{n-1}]$ is a linear combination

$$[C^{n-1}] = \sum b_i [B_i]$$

of torus invariant curves B_i. Then $[C^{n-1}]$ is numerically equivalent to $[C^0] = [\overline{C}]$. Moreover, each torus invariant curve B_i is the image of a unique F-curve F_{I_i} on $\overline{\mathrm{M}}_{0,n+2}$.

We summarize this process in Section 6.6.

Remark 6.3. The method described above for computing a numerically equivalent effective linear combination of boundaries works for any toric variety and any curve on it. Of course, if the given toric variety is complicated, the actual computation is hopeless. But in the case of $\overline{\mathrm{L}}_n$, even though as a toric variety it is very complicated, this computation is doable because of its beautiful modular interpretation and inductive structure.

6.3. **Limit components.** In this section, we explain how to find irreducible components appearing in the limit of given curve on $\overline{\mathrm{L}}_n$. We will describe our method with an example $\overline{C} = \rho_{8,9}(C)$, where C is the D_3-invariant curve on $\overline{\mathrm{M}}_{0,9}$ from Example 5.5. We will use the reduction map $\rho_{8,9} : \overline{\mathrm{M}}_{0,9} \to \overline{\mathrm{L}}_7$ and take the 8th point as our 0-point and 9th marked point as the ∞-point.

A general point of $C \cap \mathrm{M}_{0,9}$ can be written as

$$\left(\mathbb{P}^1, z, \omega z, \omega^2 z, \frac{1}{z}, \frac{\omega}{z}, \frac{\omega^2}{z}, 1, \omega, \omega^2 \right)$$

where z is a coordinate function and ω is a cubic root of unity. By using a Möbius transform $x \mapsto \frac{1-\omega^2}{1-\omega} \cdot \frac{x-\omega}{x-\omega^2}$, we obtain new coordinates

$$C(z) := \left(\mathbb{P}^1, \alpha\frac{z-\omega}{z-\omega^2}, \alpha\frac{\omega z - \omega}{\omega z - \omega^2}, \alpha\frac{\omega^2 z - \omega}{\omega^2 z - \omega^2}, \alpha\frac{1-\omega z}{1-\omega^2 z}, \alpha\frac{\omega - \omega z}{\omega - \omega^2 z}, \alpha\frac{\omega^2 - \omega z}{\omega^2 - \omega^2 z}, 1, 0, \infty \right)$$

where $\alpha = \frac{1-\omega^2}{1-\omega}$.

From this description of the general point of C, we can recover the special points on C, which correspond to singular curves via stable reduction. For example, $\lim_{z \to \omega} C(z)$ on $\overline{M}_{0,9}$ is the point corresponding to the following rational curve with four irreducible components.

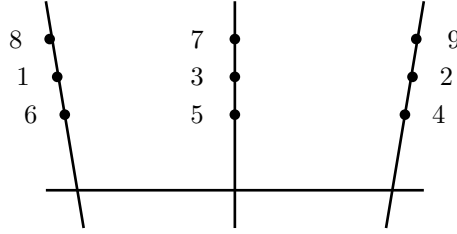FIGURE 3. The stable curve corresponds to $\lim_{z \to \omega} C(z)$ on $\overline{M}_{0,9}$

Let $C^0 := \overline{C} = \rho(C)$ be the image of C on $\overline{L}_{0,7}$. Then a general point of C^0 has the same coordinates as C, but the limit curve is different. For example, $\lim_{z \to \omega} C^0(z)$ is obtained by contracting the tail containing x_7 in $\lim_{z \to \omega} C(z)$ on $\overline{M}_{0,9}$. Let $T_1 = \langle t \rangle$ be the one parameter subgroup corresponding to the first marked point. Then T_1-action is given by

$$t \cdot C^0(z) = \left(\mathbb{P}^1, t \cdot \alpha\frac{z-\omega}{z-\omega^2}, \alpha\frac{\omega z - \omega}{\omega z - \omega^2}, \alpha\frac{\omega^2 z - \omega}{\omega^2 z - \omega^2}, \alpha\frac{1-\omega z}{1-\omega^2 z}, \alpha\frac{\omega - \omega z}{\omega - \omega^2 z}, \alpha\frac{\omega^2 - \omega z}{\omega^2 - \omega^2 z}, 1, 0, \infty \right)$$

and $\lim_{t \to 0} t \cdot C^0(z)$ for general z is a rational curve with two irreducible components X_0 and X_∞ such that X_0 contains x_1 and x_8, and X_∞ contains the rest of them with the same coordinates. Therefore on $\lim_{t \to 0} t \cdot [C^0]$ on $\mathrm{Chow}_1(\overline{L}_7)$, there is an irreducible component C_m (the so-called main component) containing all limits of the form $\lim_{t \to 0} t \cdot C^0(z)$. We see that C_m is contained in $D_{\{1,8\}}$.
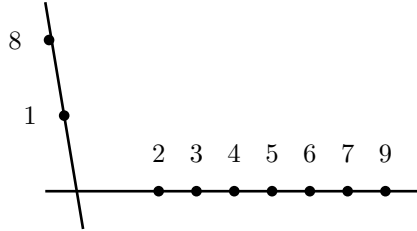

FIGURE 4. The $(\frac{1}{7}, \frac{1}{7}, \cdots, \frac{1}{7}, 1, 1)$-stable curve corresponds to $\lim_{z \to \omega} C^0(z)$ for general z on $\overline{L}_7$

On the other hand, there are three points on C^0 that are already contained in the toric boundary $\cup D_I$. These three points are the cases where $z \to 1$, $z \to \omega$,

and $z \to \omega^2$. If $p \in C^0 \cap D_I$ and $D_I \cap D_{\{1,8\}} = \emptyset$, then $p^1 := \lim_{t \to 0} t \cdot p \notin D_{\{1,8\}}$. Therefore there must be an extra component connecting p^1 and C_m. For example, if $p := \lim_{z \to 1} C^0(z)$, then p corresponds to a chain of rational curves $X_0 \cup X_1 \cup X_\infty$ such that $x_2, x_5, x_8 \in X_0$, $x_1, x_4, x_7 \in X_1$, and $x_3, x_6, x_9 \in X_\infty$, or equivalently, an ordered partition $25|147|36$. Then $p^1 = \lim_{t \to 0} t \cdot p$ corresponds to a chain corresponds to $25|1|47|36$. Similarly, $q := \lim_{z \to \omega} C^0(z)$ is a curve of partition type $16|357|24$ and $r := \lim_{z \to \omega^2} C^0(z)$ is a curve of partition type $34|267|15$. Hence q^1 corresponds to a curve of type $1|6|357|24$ and r^1 corresponds to a curve of type $34|267|1|5$. Note that q^1 is already on $D_{\{1,8\}}$.

Let $z(t)$ be a holomorphic function such that $z(t) - 1$ has a simple zero at $t = 0$. Then on $\lim_{t \to 0} t \cdot C^0(z(t))$, the marked points x_1, x_2, and x_5 approach x_8 at a constant rate, and x_3 and x_6 approach x_9 at a constant rate, too. Therefore, the limit corresponds to a rational chain $X_0 \cup X_1 \cup X_\infty$ such that $x_1, x_2, x_5, x_8 \in X_0$, $x_4, x_7 \in X_1$, and $x_3, x_6, x_9 \in X_\infty$. Thus there is a new component C_1 of $\lim_{t \to 0} t \cdot [C^0]$ such that a general point of C_1 corresponds to a partition $125|47|36$. Moreover, because the limit curve $\lim_{t \to 0} t \cdot C^0$ is T_1-invariant, C_1 is T_1-invariant. So two T_1-limits of the general point, which correspond to curves of type $1|25|47|36$ and $25|1|47|36$, are on C^1.

If $z(t)$ is another holomorphic function such that $z(t) - \omega^2$ has a simple zero at $t = 0$, then on $\lim_{t \to 0} t \cdot C^0(z(t))$, the marked points x_3 and x_4 approach x_8 at a constant rate and x_5 approaches x_9 at a constant rate, too. So $\lim_{t \to 0} t \cdot C^0(z(t))$ is a curve of partition type $34|1267|5$. Therefore, there is a new component C_{ω^2} whose general point parameterizes a curve of type $34|1267|5$. The two special points on C_{ω^2} parametrize curves of type $34|1|267|5$ and $34|267|1|5$. So C_{ω^2} does not intersect the main component C_m.

Note that $\lim_{t \to 0} t \cdot C^0(z) = \lim_{t \to 0} t^2 \cdot C^0(z)$ for a general fixed z. But for $z(t)$ as in the previous paragraph (that is, $z(t) - \omega^2$ has a simple zero at $t = 0$), as $t \to 0$, on $t^2 \cdot C^0(z(t))$, we have $x_1, x_3, x_4 \to x_8$ and $x_5 \to x_9$. So $\lim_{t \to 0} t \cdot C^0(z(t))$ is a curve of type $134|267|5$. Thus there is another component $C_{2 \cdot \omega^2}$ of C^1 whose general point parametrizes a curve of type $134|267|5$, and its two special points parametrize curves of type $1|34|267|5$ and $34|1|267|5$, respectively. The point corresponding to the curve of type $1|34|267|5$ is on $C_m \subset D_{\{1,8\}}$.

For notational convenience, we will use following notation. Let E be an irreducible curve on $\overline{\mathrm{L}}_n$, which is contained in a toric boundary $\cap D_I$ whose open dense subset is parameterizing curves of type $I_1|I_2|\cdots|I_k$. Then we say E is *of type* $I_1|I_2|\cdots|I_k$.

In summary, the limit cycle $[C^1] := \lim_{t \to 0} t \cdot [C^0]$ for $t \in T_1$ has four irreducible components C_m, C_1, C_ω^2, and $C_{2 \cdot \omega^2}$ whose types are $1|234567$, $125|47|36$, $34|1267|5$, and $134|267|5$ respectively.

6.4. **Multiplicities.** The extra irreducible components appearing on $\lim_{t \to 0} t \cdot [C^0]$ may have multiplicities greater than 1. We are able to evaluate the multiplicity of each irreducible component by computing the number of preimages of a general point $p \in \lim_{t \to 0} t \cdot [C^0]$, on $\epsilon \cdot [C^0]$ for small ϵ. This can be done by finding an explicit analytic germ $z(t)$ which gives the same $\lim_{t \to 0} t^k C^0(z(t))$.

Example 6.4. For $C_{2 \cdot \omega^2}$ in Section 6.3, if we take $z(t) = \omega^2 + \beta t + \cdots$, then on the limit cycle $X_0 \cup X_1 \cup X_\infty = \lim_{t \to 0} t^2 C^0(z(t))$, the coordinates of x_1, x_2, and x_4 on X_0 are:

$$x_1 = \frac{\alpha(\omega^2 - \omega)}{\beta}, x_2 = \alpha\omega\beta, x_4 = -\alpha\omega\beta.$$

Since a nonzero scalar multiple gives the same cross ratio of x_1, x_2, x_4 and 0, this configuration is equivalent to

$$x_1 = \alpha(\omega^2 - \omega), x_2 = \alpha\omega\beta^2, x_4 = -\alpha\omega\beta^2.$$

Obviously $\pm\beta$ give the same limit. Thus the multiplicity is two.

By using a similar idea, we find that the components C_m, C_1, C_ω^2 have multiplicity 1, and $C_{2 \cdot \omega^2}$ has multiplicity 2.

6.5. **Remaining steps of the computation.** In this section, for reader's convenience, we give the computation of $\overline{C} = \rho_{8,9}(C)$ for C in Example 5.5.

Set $C^0 := \overline{C}$. The first limit $[C^1] := \lim_{t \to 0} t \cdot [C^0]$ for $t \in T_1$ has four irreducible components,

$$C_m, C_1, C_{\omega^2}, \text{ and } C_{2 \cdot \omega^2}$$

which are of type 1|234567, 125|47|36, 34|1267|5, and 134|267|5 respectively. The component 134|267|5 has multiplicity two. All other multiplicities are one.

Topologically, C^1 is a tree of rational curves on $\overline{L}_7$. The main component C_m is the spine and there are three tails, C_1, $C_{\omega^2} \cup C_{2 \cdot \omega^2}$, and a 'tail point' whose partition is 1|6|357|24. (This is the point q^1 in Section 6.3.) For the main component C_m (which is already in $D_{\{1,8\}}$), there are three special points lying on the boundary of $D_{\{1,8\}}$. If we take the limit $\lim_{t \to 0} t \cdot [C_m]$ for $t \in T_2$, then except possibly for these three special points, all other points go to a unique boundary stratum, $D_{\{1,8\}} \cap D_{\{1,2,8\}}$. So if the limits of three special points are not contained in $D_{\{1,8\}} \cap D_{\{1,2,8\}}$, then there must be new rational curves connect the limit of general points on C_m and the limits of special points.

For a smooth point $[(X, x_1, \cdots, x_9)] \in C_1$, let Y_2 be the irreducible component of X containing x_2. If the 0-point is not on Y_2, let Y_0 be the *connected* component of $\overline{X - Y_2}$ containing 0-point. Similarly, Y_∞ is the connected component of $\overline{X - Y_2}$ containing ∞-point, if ∞ point is not on Y_2. Then we are able to evaluate limit $[C^2] := \lim_{t \to 0} t \cdot [C^1]$ for $t \in T_2$ in the same way after replacing Y_0 by a 0-point, and Y_∞ by an ∞-point because T_2 only acts nontrivially on Y_2. $[C^2]$ has a main component (by abuse of notation, call it C_m) and three rational chains whose components are of type

$$2|15|47|36, \ 12|5|47|36,$$

$$1|6|2357|4, \ 1|26|357|4 \ (2),$$

$$34|2|167|5, \ 34|12|67|5, \ 134|2|67|5 \ (2), \ 1|234|67|5$$

respectively. The integers in parentheses refer the multiplicity of each irreducible component. The main component is on the boundary stratum $D_{\{1,8\}} \cap D_{\{1,2,8\}}$, so it is of type 1|2|34567.

The limit $[C^3] := \lim_{t \to 0} t \cdot [C^2]$ for $t \in T_3$ has a main component of type $1|2|3|4567$ and three chains

$$2|15|47|3|6, \ 12|5|47|3|6, \ 1|2|5|347|6, \ 1|2|35|47|6 \ (2),$$

$$1|6|3|257|4, \ 1|6|23|57|4, \ 1|26|3|57|4 \ (2), 1|2|36|57|4,$$

$$3|4|2|167|5, \ 3|4|12|67|5, \ 3|14|2|67|5 \ (2), 13|4|2|67|5 \ (2), 1|3|24|67|5, \ 1|23|4|67|5.$$

The next limit $[C^4]$ has a main component and three chains

$$2|15|4|7|3|6, \ 12|5|4|7|3|6, \ 1|2|5|4|37|69, \ 1|2|5|34|7|69, \ 1|2|35|4|7|6 \ (2), 1|2|3|45|7|6,$$

$$1|6|3|257|4, \ 1|6|23|57|4, \ 1|26|3|57|4 \ (2), \ 1|2|36|57|4, \ 1|2|3|6|457, \ 1|2|3|46|57 \ (2),$$

and

$$3|4|2|167|5, \ 3|4|12|67|5, \ 3|14|2|67|5 \ (2), 13|4|2|67|5 \ (2), 1|3|24|67|5,$$

$$1|23|4|67|5, \ 1|2|3|4|567.$$

Similarly, $[C^5]$ has a main component and three chains

$$2|15|4|7|3|6, \ 12|5|4|7|3|6, \ 1|2|5|4|37|69, \ 1|2|5|34|7|69, \ 1|2|35|4|7|6 \ (2),$$

$$1|2|3|45|7|6, \ 1|2|3|4|5|67$$

and

$$1|6|3|5|27|4, \ 1|6|3|25|7|4, \ 1|6|23|5|7|4, \ 1|26|3|5|7|4 \ (2), \ 1|2|36|5|7|4,$$

$$1|2|3|6|5|47, \ 1|2|3|6|45|7, \ 1|2|3|46|5|7 \ (2), 1|2|3|4|56|7,$$

and

$$3|4|2|167|5, \ 3|4|12|67|5, \ 3|14|2|67|5 \ (2), 13|4|2|67|5 \ (2), 1|3|24|67|5,$$

$$1|23|4|67|5, \ 1|2|3|4|567.$$

Finally, the main component of $[C^6]$ becomes a point, which is a torus invariant point of type $1|2|3|4|5|6|7$. Three tails are

$$2|15|4|7|3|6, \ 12|5|4|7|3|6, \ 1|2|5|4|37|6|9, \ 1|2|5|34|7|6|9, \ 1|2|35|4|7|6 \ (2),$$

$$1|2|3|45|7|6, \ 1|2|3|4|5|67$$

and

$$1|6|3|5|27|4, \ 1|6|3|25|7|4, \ 1|6|23|5|7|4, \ 1|26|3|5|7|4 \ (2), \ 1|2|36|5|7|4,$$

$$1|2|3|6|5|47, \ 1|2|3|6|45|7, \ 1|2|3|46|5|7 \ (2), 1|2|3|4|56|7,$$

and

$$3|4|2|6|17|5, \ 3|4|2|16|7|5, \ 3|4|12|6|7|5, \ 3|14|2|6|7|5 \ (2), 13|4|2|6|7|5 \ (2),$$

$$1|3|24|6|7|5, \ 1|23|4|6|7|5, \ 1|2|3|4|6|57, \ 1|2|3|4|56|7.$$

Now we are able to describe each one dimensional boundary stratum as the image of an F-curve on $\overline{M}_{0,9}$. By abuse of notation, let F_{I_1,I_2,I_3,I_4} be the image of an F-curve on $\overline{L}_7 \cong \overline{M}_{0,((\frac{1}{7})^n,1,1)}$ of F_{I_1,I_2,I_3,I_4}. The image F_{I_1,I_2,I_3,I_4} is a torus invariant curve if and only if $8 \in I_1$, $9 \in I_4$, $|I_2| = |I_3| = 1$. It is contracted if

and only if $8, 9 \in I_1$ and otherwise the image is not a torus invariant curve. For example, the torus invariant stratum $2|15|4|7|3|6$ is $F_{\{82,1,5,34679\}}$. Thus we obtain

$$
\begin{aligned}
\overline{C} = C^0 \quad \equiv \quad & F_{\{82,1,5,34679\}} + F_{\{8,1,2,345679\}} + F_{\{81245,3,7,69\}} + F_{\{8125,3,4,679\}} \\
& + 2F_{\{812,3,5,4679\}} + F_{\{8123,4,5,679\}} + F_{\{812345,6,7,9\}} + F_{\{81356,2,7,49\}} \\
& + F_{\{8136,2,5,479\}} + F_{\{816,2,3,4579\}} + 2F_{\{81,2,6,34579\}} + F_{\{812,3,6,4579\}} \\
& + F_{\{812356,4,7,9\}} + F_{\{81236,4,5,79\}} + 2F_{\{8123,4,6,579\}} + F_{\{81234,5,6,79\}} \\
& + F_{\{82346,1,7,59\}} + F_{\{8234,1,6,579\}} + F_{\{834,1,2,5679\}} + 2F_{\{83,1,4,25679\}} \\
& + 2F_{\{8,1,3,245679\}} + F_{\{813,2,4,5679\}} + F_{\{81,2,3,45679\}} + F_{\{812346,5,7,9\}} \\
& + F_{\{81234,5,6,79\}}.
\end{aligned}
$$

on $\overline{L}_7$.

6.6. **Summary of the computation.** Here we summarize the strategy used in this section. Let C be an irreducible curve on $\overline{L}_n$, which intersects the big cell of $\overline{L}_n$. Let T_i be the one parameter subgroup moving x_i only. Let $L(C, n, i)$ denote the procedure to evaluate the limit cycle $\lim_{t \to 0} t \cdot C$ for $t \in T_i$ on $\overline{L}_n$.

Algorithm 6.5 ($L(C, n, i)$). Let C be an irreducible curve on $\overline{L}_n$.

(1) Write coordinates $(\mathbb{P}^1, x_1(z), x_2(z), \cdots, x_n(z))$ of a general point on C, such that the 0-point is 0 and the ∞-point is ∞.
(2) Find all special points $p_1, p_2, \cdots, p_k$ on $C \cap (\overline{L}_n - (\mathbb{C}^*)^{n-1})$. Suppose that p_j occurs when $z = z_j$.
(3) Take $\lim_{t \to 0} t \cdot C(z)$ for $t \in T_i$ and general $z \in C$. The closure is the main component C_m.
(4) For each p_j, find all limits of the form $\lim_{t \to 0} t \cdot C(z(t))$ where $z(t)$ is a holomorphic function such that $z(t) - z_j$ has a pole of order r. Take the closure of all such limits and obtain irreducible components $C_{r \cdot z_j}$ connecting $\lim_{t \to 0} t \cdot p_j$ and C_m.
(5) Evaluate the multiplicity of each irreducible component $C_{r \cdot z_j}$, by counting the number of preimages of a general point $p \in C_{r \cdot z_j}$ on $\epsilon \cdot C_{r \cdot z_j}$.

Then we can evaluate the toric degeneration by applying the algorithm $L(C, n, i)$ several times.

Algorithm 6.6 (Evaluation of the limit cycle). Set $C^0 = C$ and $i = 1$.

(1) Write $C^{i-1} = \sum m_j C_j$ as a linear combination of irreducible components.
(2) Each irreducible component C_j lies on a boundary stratum, which is isomorphic to $\prod \overline{L}_k$. Furthermore, there is a unique $\overline{L}_k$ where x_{i+1} is not forgotten.
(3) Apply $L(C_j, k, i)$ to each irreducible component and set C^i as the formal sum of all limits of irreducible components.
(4) Set $i = i + 1$ and repeat (1) and (2) to evaluate C^i for $2 \le i \le n - 1$.
(5) C^{n-1} is the desired toric degeneration.

7. Finding an effective linear combination

Let C be an effective rational curve on $\overline{\mathrm{M}}_{0,n+2}$ and $\overline{C}$ be its image in $\overline{\mathrm{L}}_n$. Using the techniques of the previous section, we are able to compute the numerical class of $\overline{C}$ on $\overline{\mathrm{L}}_n$ as an effective linear combination for toric boundary curves. We can regard it as a first approximation of an effective linear combination of F-curves of C. In this section, we will discuss some computational ideas to write C as an effective $\mathbb{Z}$-linear combination of F-curves.

On $\overline{\mathrm{L}}_n$, suppose that $\overline{C} \equiv \sum b_I F_I$ where $b_I > 0$ and F_I is a torus invariant curve which is the image of an F-curve, which, abusing notation, we also denote F_I. Consider $\sum b_I F_I$ on $\overline{\mathrm{M}}_{0,n+2}$. In general, it is not numerically equivalent to C because C passes through several exceptional loci of $\rho : \overline{\mathrm{M}}_{0,n+2} \to \overline{\mathrm{L}}_n$. Since ρ is a composition of smooth blow-downs, we are able to compute the curve classes on the Mori cone of exceptional fiber passing through C, which we need to subtract from $\sum b_I F_I$. This yields a numerical class of C of the form

$$C \equiv \sum b_I F_I - \sum c_J F_J$$

where $b_I, c_J > 0$.

Example 7.1. Consider the curve C in Example 5.7. Note that for $\rho_{8,9} : \overline{\mathrm{M}}_{0,9} \to \overline{\mathrm{L}}_7$, all the exceptional loci intersecting C are F-curves. By considering the proper transform, on $\overline{\mathrm{M}}_{0,9}$, we have

$$
\begin{aligned}
C \equiv\ & F_{\{82,1,5,34679\}} + F_{\{8,1,2,345679\}} + F_{\{81245,3,7,69\}} + F_{\{8125,3,4,679\}} \\
& + 2F_{\{812,3,5,4679\}} + F_{\{8123,4,5,679\}} + F_{\{812345,6,7,9\}} + F_{\{81356,2,7,49\}} \\
& + F_{\{8136,2,5,479\}} + F_{\{816,2,3,4579\}} + 2F_{\{81,2,6,34579\}} + F_{\{812,3,6,4579\}} \\
& + F_{\{812356,4,7,9\}} + F_{\{81236,4,5,79\}} + 2F_{\{8123,4,6,579\}} + F_{\{81234,5,6,79\}} \\
& + F_{\{82346,1,7,59\}} + F_{\{8234,1,6,579\}} + F_{\{834,1,2,5679\}} + 2F_{\{83,1,4,25679\}} \\
& + 2F_{\{8,1,3,245679\}} + F_{\{813,2,4,5679\}} + F_{\{81,2,3,45679\}} + F_{\{812346,5,7,9\}} \\
& + F_{\{81234,5,6,79\}} \\
& - 2F_{\{1,2,3,456789\}} - 2F_{\{4,5,6,123789\}} - F_{\{1,4,7,235689\}} - F_{\{2,6,7,134589\}} \\
& - F_{\{3,5,7,124689\}}.
\end{aligned}
$$

To make the given linear combination of F-curves effective, we need to add some numerically trivial linear combination of F-curves. By [KM94, Theorem 7.3], the vector space of numerically trivial curve classes on $\overline{\mathrm{M}}_{0,n}$ is generated by Keel relations.

Definition 7.2. [KM94, Lemma 7.2.1] Let $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 \sqcup I_5$ be a partition of $[n]$. Then the following linear combination of F-curves is numerically trivial:

$$F_{I_1,I_2,I_3,I_4 \sqcup I_5} + F_{I_1 \sqcup I_2,I_3,I_4,I_5} - F_{I_1,I_4,I_3,I_2 \sqcup I_5} - F_{I_1 \sqcup I_4,I_3,I_2,I_5}$$

We call relations of this form *Keel relations* among F-curves.

Note that in a Keel relation, all F-curves share a common set I_3. Moreover, two F-curves with the same sign share exactly one common set, and two F-curves

with different sign share exactly two common sets. (For instance, $F_{I_1,I_2,I_3,I_4 \sqcup I_5}$ and $F_{I_1,I_4,I_3,I_2 \sqcup I_5}$ have common sets I_1 and I_3.). Conversely, this is a necessary and sufficient condition for the existence of a Keel relation containing certain F-curves.

Let F_I and F_J be two F-curves on $\overline{\mathrm{M}}_{0,n}$, where $I := \{I_1, I_2, I_3, I_4\}$ and $J := \{J_1, J_2, J_3, J_4\}$. The *common refinement* $R_{I,J}$ of two partitions I and J is the set of nonempty subsets of $[n]$ of the form $I_i \cap J_j$ for $1 \leq i, j \leq 4$. And the *intersection* $S_{I,J}$ of I and J is the set of all nonempty subsets $K \subset [n]$ such that $K = I_i$ for some i and $K = J_j$ for some j, too.

Definition 7.3. We say two F-curves F_I, F_J on $\overline{\mathrm{M}}_{0,n}$ are *adjacent* if $|R_{I,J}| = 5$.

(The motivation for this terminology will become clear below.)

Lemma 7.4. *Let F_I and F_J be two F-curves on $\overline{\mathrm{M}}_{0,n}$.*

(1) *There is a Keel relation containing F_I and F_J if and only if F_I and F_J are adjacent.*

(2) *In this case, the number of Keel relations containing both F_I and F_J is two.*

(3) *If $|S_{I,J}| = 2$, then the signs of F_I and F_J in a Keel relation containing them are different.*

(4) *If $|S_{I,J}| = 1$, then the signs of F_I and F_J in a Keel relation containing them are same.*

(5) *Suppose F_I, F_J, F_K are pairwise adjacent and $|S_{I,J}| = 2$, $|S_{I,K}| = 2$, and $|S_{J,K}| = 1$. Then there is a unique Keel relation containing all of them.*

Proof. Straightforward computation. $\square$

Next, we describe two families of graphs.

Definition 7.5. Let E be a $\mathbb{Z}$-linear combination of F-curves. We define a rooted infinite graph $G(E)$ as follows.

(1) The set of vertices of $G(E)$ is the infinite set of expressions equivalent to E.

(2) The root is the vertex E.

(3) Two vertices E' and E'' are connected by an edge if $E'' = E' + R$ for some Keel relation R.

For each nonnegative integer l, let $G(E, l)$ be the subgraph of $G(E)$ consisting of vertices that are connected to the root by a path of length less than or equal to l.

We will restrict our attention to a smaller graph $\widetilde{G}(E)$. It has the same vertices as $G(E)$, but fewer edges:

Definition 7.6. Let E be a $\mathbb{Z}$-linear combination of F-curves. Write $E = \sum_{I \in \mathcal{I}} b_I F_I - \sum_{J \in \mathcal{J}} c_J F_J$ with $b_J, c_J > 0$ for all $I \in \mathcal{I}$ and $J \in \mathcal{J}$. We define a rooted graph $\widetilde{G}(E)$ as follows.

(1) The set of vertices of $\widetilde{G}(E)$ is the (infinite) set of expressions equivalent to E.

(2) The root is the vertex E.

(3) Two vertices E' and E'' are connected by an edge if $E'' = E' + R$, where R is a Keel relation containing at least one positive curve F_I from E' and at least one negative curve F_J from E'.

For each nonnegative integer l, let $\widetilde{G}(E, l)$ be the subgraph of $\widetilde{G}(E)$ consisting of vertices that are connected to the root by a path of length less than or equal to l.

We are now ready to describe our strategy for finding effective expressions for curve classes:

Strategy 7.7. Let $E = \sum_{I \in \mathcal{I}} b_I F_I - \sum_{J \in \mathcal{J}} c_J F_J$ be a $\mathbb{Z}$-linear combination of F-curves with $b_I, c_J > 0$ for all $I \in \mathcal{I}$ and $J \in \mathcal{J}$.

(1) Let $m(E) = \sum_{J \in \mathcal{J}} c_J$. We use the integer $m(E)$ as a measure of how far the E expression is from being effective.
(2) Beginning with $l = 1$, compute $\widetilde{G}(E, l)$. If $\widetilde{G}(E, l)$ contains a vertex E' corresponding to an expression $E' = \sum_{I \in \mathcal{I}'} b_I' F_I - \sum_{J \in \mathcal{J}'} c_J' F_J$ with

$$m(E') = \sum_{J \in \mathcal{J}'} c_J' < m(E) = \sum_{J \in \mathcal{J}} c_J,$$

start over again replacing E by E'. If $m(E') = m(E)$ for all $E' \in \widetilde{G}(E, l)$, repeat this step with $l = l + 1$.
(3) Continue until an effective expression ($m(E') = 0$) is found.

This strategy is implemented in the `MOnbar` package for `Macaulay2` in the command `seekEffectiveExpression`.

Strategy 7.7 is not an algorithm because it is not guaranteed to produce an effective expression, even if an effective expression is known to exist; for an example where the strategy fails, see the calculations for the D_4 fixed curve on $\overline{\mathrm{M}}_{0,12}$ at the link below to the second author's webpage. Nevertheless, although our strategy is not an algorithm, we were still able to use it successfully to check that all the curves $\overline{\mathrm{M}}_{0,n}^{G}$ for G dihedral and $n \leq 12$ are effective linear combinations of F-curves.

By Lemmas 5.2 and 5.4, the curves $\overline{\mathrm{M}}_{0,n}^{D_k}$ which are not of the form C^σ are:

$$
\begin{array}{cc}
n = 9 & k = 3 \\
n = 11 & k = 3 \\
n = 12 & k = 3 \\
n = 12 & k = 4.
\end{array}
$$

Moreover, when $n = 12$, there is also a curve of the form $\overline{\mathrm{M}}_{0,n}^{A_4}$.

We present our calculations for two examples ($n = 9$ and $k = 3$, and $n = 12$ and $k = 3$) here in the paper. The remaining calculations can be found on our website for this project:

http://faculty.fordham.edu/dswinarski/invariant-curves/

Example 7.8. Let C be the curve in Example 5.5. We found a noneffective $\mathbb{Z}$-linear combination in Example 7.1. By using Strategy 7.7, we can find Keel relations which make the linear combination into an effective one. In the following expression, each

term in parentheses is a Keel relation.

$$
\begin{aligned}
C \quad \equiv \quad & F_{\{82,1,5,34679\}} + F_{\{8,1,2,345679\}} + F_{\{81245,3,7,69\}} + F_{\{8125,3,4,679\}} \\
& + 2F_{\{812,3,5,4679\}} + F_{\{8123,4,5,679\}} + F_{\{812345,6,7,9\}} + F_{\{81356,2,7,49\}} \\
& + F_{\{8136,2,5,479\}} + F_{\{816,2,3,4579\}} + 2F_{\{81,2,6,34579\}} + F_{\{812,3,6,4579\}} \\
& + F_{\{812356,4,7,9\}} + F_{\{81236,4,5,79\}} + 2F_{\{8123,4,6,579\}} + F_{\{81234,5,6,79\}} \\
& + F_{\{82346,1,7,59\}} + F_{\{8234,1,6,579\}} + F_{\{834,1,2,5679\}} + 2F_{\{83,1,4,25679\}} \\
& + 2F_{\{8,1,3,245679\}} + F_{\{813,2,4,5679\}} + F_{\{81,2,3,45679\}} + F_{\{812346,5,7,9\}} \\
& + F_{\{81234,5,6,79\}} \\
& - 2F_{\{1,2,3,456789\}} - 2F_{\{4,5,6,123789\}} - F_{\{1,4,7,235689\}} - F_{\{2,6,7,134589\}} \\
& - F_{\{3,5,7,124689\}} \\
& + \left(F_{\{1,2,3,456789\}} + F_{\{13,2,8,45679\}} - F_{\{1,2,8,345679\}} - F_{\{81,2,3,45679\}} \right) \\
& + \left(F_{\{4,5,6,123789\}} + F_{\{8123,45,6,79\}} - F_{\{8123,4,6,579\}} - F_{\{81234,5,6,79\}} \right) \\
& + \left(F_{\{4,5,6,123789\}} + F_{\{4,56,79,8123\}} - F_{\{81236,4,5,79\}} - F_{\{8123,4,6,579\}} \right) \\
& + \left(F_{\{812,4,5,3679\}} + F_{\{8124,3,5,679\}} - F_{\{812,3,5,4679\}} - F_{\{8123,4,5,679\}} \right) \\
& + \left(F_{\{8124,5,6,379\}} + F_{\{81246,3,5,79\}} - F_{\{8124,3,5,679\}} - F_{\{81234,5,6,79\}} \right) \\
& + \left(F_{\{812469,3,5,7\}} + F_{\{81246,5,9,37\}} - F_{\{81246,3,5,79\}} - F_{\{812346,5,7,9\}} \right) \\
& + \left(F_{\{81,3,6,24579\}} + F_{\{813,2,6,4579\}} - F_{\{81,2,6,34579\}} - F_{\{812,3,6,4579\}} \right) \\
& + \left(F_{\{813,6,45,279\}} + F_{\{81345,2,6,79\}} - F_{\{813,2,6,4579\}} - F_{\{8123,6,45,79\}} \right) \\
& + \left(F_{\{81345,6,9,27\}} + F_{\{813459,2,6,7\}} - F_{\{81345,2,6,79\}} - F_{\{812345,6,7,9\}} \right) \\
& + \left(F_{\{823,1,4,5679\}} + F_{\{83,1,2,45679\}} - F_{\{834,1,2,5679\}} - F_{\{83,1,4,25679\}} \right) \\
& + \left(F_{\{1,8,23,45679\}} + F_{\{1,2,3,456789\}} - F_{\{83,1,2,45679\}} - F_{\{1,3,8,245679\}} \right) \\
& + \left(F_{\{83,2,4,15679\}} + F_{\{823,1,4,5679\}} - F_{\{83,1,4,25679\}} - F_{\{813,2,4,5679\}} \right) \\
& + \left(F_{\{823,4,56,179\}} + F_{\{82356,1,4,79\}} - F_{\{823,1,4,5679\}} - F_{\{8123,4,56,79\}} \right) \\
& + \left(F_{\{82356,4,17,9\}} + F_{\{1,4,7,235689\}} - F_{\{82356,1,4,79\}} - F_{\{812356,4,7,9\}} \right) \\[6pt]
\equiv \quad & F_{\{82,1,5,34679\}} + F_{\{81245,3,7,69\}} + F_{\{8125,3,4,679\}} + F_{\{812,3,5,4679\}} \\
& + F_{\{81356,2,7,49\}} + F_{\{8136,2,5,479\}} + F_{\{816,2,3,4579\}} + F_{\{81,2,6,34579\}} \\
& + F_{\{82346,1,7,59\}} + F_{\{8234,1,6,579\}} + F_{\{8,1,3,245679\}} + F_{\{13,2,8,45679\}} \\
& + F_{\{812,4,5,3679\}} + F_{\{8124,5,6,379\}} + F_{\{81246,5,9,37\}} + F_{\{81,3,6,24579\}} \\
& + F_{\{813,6,45,279\}} + F_{\{81345,6,9,27\}} + F_{\{823,1,4,5679\}} + F_{\{8,1,23,45679\}} \\
& + F_{\{83,2,4,15679\}} + F_{\{823,4,56,179\}} + F_{\{82356,4,17,9\}}.
\end{aligned}
$$

8. An example on $\overline{\mathrm{M}}_{0,12}$

In this section, we compute a numerically equivalent effective linear combination of F-curves for Example 5.8. Recall that in the example,

$$
G = \langle (123)(456)(789)(abc), (14)(26)(35)(89)(bc) \rangle \cong D_3,
$$

G is of type $(0,2)$. Let $C = \overline{\mathrm{M}}_{0,12}^G \subset \overline{\mathrm{M}}_{0,12}$.

By using the method of Section 6, on $\overline{\mathrm{L}}_{10}$, we have

$$
\begin{aligned}
\overline{C} \quad \equiv \quad & F_{\{82,1,5,3467abc9\}} + F_{\{8,1,2,34567abc9\}} + F_{\{812457ab,3,c,69\}} + F_{\{812457a,3,b,6c9\}} \\
& + F_{\{812457,3,a,6bc9\}} + F_{\{81245,3,7,6abc9\}} + F_{\{8125,3,4,67abc9\}} + 2F_{\{812,3,5,467abc9\}} \\
& + F_{\{8123,4,5,67abc9\}} + F_{\{8123457ab,6,c,9\}} + F_{\{8123457a,6,b,c9\}} + F_{\{8123457,6,a,bc9\}} \\
& + F_{\{812345,6,7,abc9\}} \\
& + F_{\{813567ab,2,c,49\}} + F_{\{813567a,2,b,4c9\}} + F_{\{813567,2,a,4bc9\}} + F_{\{81356,2,7,4abc9\}} \\
& + F_{\{8136,2,5,47abc9\}} + F_{\{816,2,3,457abc9\}} + 2F_{\{81,2,6,3457abc9\}} + F_{\{812,3,6,457abc9\}} \\
& + F_{\{8123567ab,4,c,9\}} + F_{\{8123567a,4,b,c9\}} + F_{\{8123567,4,a,bc9\}} + F_{\{812356,4,7,abc9\}} \\
& + F_{\{81236,4,5,7abc9\}} + 2F_{\{8123,4,6,57abc9\}} + F_{\{81234,5,6,7abc9\}} \\
& + F_{\{823467ab,1,c,59\}} + F_{\{823467a,1,b,5c9\}} + F_{\{823467,1,a,5bc9\}} + F_{\{82346,1,7,5abc9\}} \\
& + F_{\{8234,1,6,57abc9\}} + F_{\{834,1,2,567abc9\}} + 2F_{\{83,1,4,2567abc9\}} + 2F_{\{8,1,3,24567abc9\}} \\
& + F_{\{813,2,4,567abc9\}} + F_{\{81,2,3,4567abc9\}} + F_{\{8123467ab,5,c,9\}} + F_{\{8123467a,5,b,c9\}} \\
& + F_{\{8123467,5,a,bc9\}} + F_{\{812346,5,7,abc9\}} + F_{\{81234,5,6,7abc9\}}.
\end{aligned}
$$

In this degeneration, there are three rational tails. Rows 1–4 in the expression above form the first tail, rows 4–8 form a second tail, and rows 9–12 form the third tail.

By considering the proper transform, we deduce that on $\overline{\mathrm{M}}_{0,12}$,

$$
\begin{aligned}
\overline{C} \quad \equiv \quad & F_{\{82,1,5,3467abc9\}} + F_{\{8,1,2,34567abc9\}} + F_{\{812457ab,3,c,69\}} + F_{\{812457a,3,b,6c9\}} \\
& + F_{\{812457,3,a,6bc9\}} + F_{\{81245,3,7,6abc9\}} + F_{\{8125,3,4,67abc9\}} + 2F_{\{812,3,5,467abc9\}} \\
& + F_{\{8123,4,5,67abc9\}} + F_{\{8123457ab,6,c,9\}} + F_{\{8123457a,6,b,c9\}} + F_{\{8123457,6,a,bc9\}} \\
& + F_{\{812345,6,7,abc9\}} \\
& + F_{\{813567ab,2,c,49\}} + F_{\{813567a,2,b,4c9\}} + F_{\{813567,2,a,4bc9\}} + F_{\{81356,2,7,4abc9\}} \\
& + F_{\{8136,2,5,47abc9\}} + F_{\{816,2,3,457abc9\}} + 2F_{\{81,2,6,3457abc9\}} + F_{\{812,3,6,457abc9\}} \\
& + F_{\{8123567ab,4,c,9\}} + F_{\{8123567a,4,b,c9\}} + F_{\{8123567,4,a,bc9\}} + F_{\{812356,4,7,abc9\}} \\
& + F_{\{81236,4,5,7abc9\}} + 2F_{\{8123,4,6,57abc9\}} + F_{\{81234,5,6,7abc9\}} \\
& + F_{\{823467ab,1,c,59\}} + F_{\{823467a,1,b,5c9\}} + F_{\{823467,1,a,5bc9\}} + F_{\{82346,1,7,5abc9\}} \\
& + F_{\{8234,1,6,57abc9\}} + F_{\{834,1,2,567abc9\}} + 2F_{\{83,1,4,2567abc9\}} + 2F_{\{8,1,3,24567abc9\}} \\
& + F_{\{813,2,4,567abc9\}} + F_{\{81,2,3,4567abc9\}} + F_{\{8123467ab,5,c,9\}} + F_{\{8123467a,5,b,c9\}} \\
& + F_{\{8123467,5,a,bc9\}} + F_{\{812346,5,7,abc9\}} + F_{\{81234,5,6,7abc9\}} \\
& - 2F_{\{1,2,3,456789abc\}} - 2F_{\{4,5,6,123789abc\}} - F_{\{1,4,7,235689abc\}} - F_{\{3,5,7,124689abc\}} \\
& - F_{\{2,6,7,134589abc\}} - F_{\{1,5,c,2346789ab\}} - F_{\{2,6,a,1345789bc\}} - F_{\{3,4,b,1256789ac\}} \\
& - F_{\{1,4,a,2356789bc\}} - F_{\{2,5,b,1346789ac\}} - F_{\{3,6,c,1245789ab\}} - F_{\{1,6,b,2345789ac\}} \\
& - F_{\{2,4,c,1356789ab\}} - F_{\{3,5,a,1246789bc\}}.
\end{aligned}
$$

Using the `seekEffectiveExpression` command in the `MOnbar` package for `Macaulay2`, we obtain

$$
\begin{aligned}
\overline{C} \equiv\ & F_{\{82,1,5,3467abc9\}} + F_{\{8,1,2,34567abc9\}} + F_{\{812457ab,3,c,69\}} + F_{\{812457a,3,b,6c9\}} \\
& + F_{\{812457,3,a,6bc9\}} + F_{\{81245,3,7,6abc9\}} + F_{\{8125,3,4,67abc9\}} + 2F_{\{812,3,5,467abc9\}} \\
& + F_{\{8123,4,5,67abc9\}} + F_{\{8123457ab,6,c,9\}} + F_{\{8123457a,6,b,c9\}} + F_{\{8123457,6,a,bc9\}} \\
& + F_{\{812345,6,7,abc9\}} \\
& + F_{\{813567ab,2,c,49\}} + F_{\{813567a,2,b,4c9\}} + F_{\{813567,2,a,4bc9\}} + F_{\{81356,2,7,4abc9\}} \\
& + F_{\{8136,2,5,47abc9\}} + F_{\{816,2,3,457abc9\}} + 2F_{\{81,2,6,3457abc9\}} + F_{\{812,3,6,457abc9\}} \\
& + F_{\{8123567ab,4,c,9\}} + F_{\{8123567a,4,b,c9\}} + F_{\{8123567,4,a,bc9\}} + F_{\{812356,4,7,abc9\}} \\
& + F_{\{81236,4,5,7abc9\}} + 2F_{\{8123,4,6,57abc9\}} + F_{\{81234,5,6,7abc9\}} \\
& + F_{\{823467ab,1,c,59\}} + F_{\{823467a,1,b,5c9\}} + F_{\{823467,1,a,5bc9\}} + F_{\{82346,1,7,5abc9\}} \\
& + F_{\{8234,1,6,57abc9\}} + F_{\{834,1,2,567abc9\}} + 2F_{\{83,1,4,2567abc9\}} + 2F_{\{8,1,3,24567abc9\}} \\
& + F_{\{813,2,4,567abc9\}} + F_{\{81,2,3,4567abc9\}} + F_{\{8123467ab,5,c,9\}} + F_{\{8123467a,5,b,c9\}} \\
& + F_{\{8123467,5,a,bc9\}} + F_{\{812346,5,7,abc9\}} + F_{\{81234,5,6,7abc9\}} \\
& - 2F_{\{1,2,3,456789abc\}} - 2F_{\{4,5,6,123789abc\}} - F_{\{1,4,7,235689abc\}} - F_{\{3,5,7,124689abc\}} \\
& - F_{\{2,6,7,134589abc\}} - F_{\{1,5,c,2346789ab\}} - F_{\{2,6,a,1345789bc\}} - F_{\{3,4,b,1256789ac\}} \\
& - F_{\{1,4,a,2356789bc\}} - F_{\{2,5,b,1346789ac\}} - F_{\{3,6,c,1245789ab\}} - F_{\{1,6,b,2345789ac\}} \\
& - F_{\{2,4,c,1356789ab\}} - F_{\{3,5,a,1246789bc\}} \\
& + \left(F_{\{1238,4,56,79abc\}} - F_{\{1238,4,579abc,6\}} - F_{\{12368,4,5,79abc\}} + F_{\{123789abc,4,5,6\}} \right) \\
& + \left(F_{\{1238,45,6,79abc\}} - F_{\{1238,4,579abc,6\}} - F_{\{12348,5,6,79abc\}} + F_{\{123789abc,4,5,6\}} \right) \\
& + \left(F_{\{124578ab,36,9,c\}} - F_{\{124578ab,3,69,c\}} - F_{\{1234578ab,6,9,c\}} + F_{\{1245789ab,3,6,c\}} \right) \\
& + \left(F_{\{12,3,45679abc,8\}} - F_{\{1,245679abc,3,8\}} - F_{\{18,2,3,45679abc\}} + F_{\{1,2,3,456789abc\}} \right) \\
& + \left(F_{\{135678ab,24,9,c\}} - F_{\{1235678ab,4,9,c\}} - F_{\{135678ab,2,49,c\}} + F_{\{1356789ab,2,4,c\}} \right) \\
& + \left(F_{\{15,234678ab,9,c\}} - F_{\{1,234678ab,59,c\}} - F_{\{1234678ab,5,9,c\}} + F_{\{1,2346789ab,5,c\}} \right) \\
& + \left(F_{\{1248,3,57,69abc\}} - F_{\{12458,3,69abc,7\}} - F_{\{1248,3,5,679abc\}} + F_{\{124689abc,3,5,7\}} \right) \\
& + \left(F_{\{128,3679abc,4,5\}} - F_{\{1238,4,5,679abc\}} - F_{\{128,3,4679abc,5\}} + F_{\{1248,3,5,679abc\}} \right) \\
& + \left(F_{\{1,23,45679abc,8\}} - F_{\{1,245679abc,3,8\}} - F_{\{1,2,38,45679abc\}} + F_{\{1,2,3,456789abc\}} \right) \\
& + \left(F_{\{1,238,4,5679abc\}} - F_{\{1,25679abc,38,4\}} - F_{\{1,2,348,5679abc\}} + F_{\{1,2,38,45679abc\}} \right) \\
& + \left(F_{\{1,2368,47,59abc\}} - F_{\{1,23468,59abc,7\}} - F_{\{1,2368,4,579abc\}} + F_{\{1,235689abc,4,7\}} \right) \\
& + \left(F_{\{1,238,4579abc,6\}} - F_{\{1,2348,579abc,6\}} - F_{\{1,238,4,5679abc\}} + F_{\{1,2368,4,579abc\}} \right) \\
& + \left(F_{\{1368,2,479ac,5b\}} - F_{\{1368,2,479abc,5\}} - F_{\{13568,2,479ac,b\}} + F_{\{1346789ac,2,5,b\}} \right) \\
& + \left(F_{\{13568b,2,49c,7a\}} - F_{\{135678a,2,49c,b\}} - F_{\{13568,2,49bc,7a\}} + F_{\{13568,2,479ac,b\}} \right) \\
& + \left(F_{\{1345689bc,2,7,a\}} - F_{\{13568,2,49abc,7\}} - F_{\{135678,2,49bc,a\}} + F_{\{13568,2,49bc,7a\}} \right) \\
& + \left(F_{\{1,238,4579ac,6b\}} - F_{\{1,238,4579abc,6\}} - F_{\{1,2368,4579ac,b\}} + F_{\{1,2345789ac,6,b\}} \right) \\
& + \left(F_{\{1,2368b,47,59ac\}} - F_{\{1,2368,47,59abc\}} - F_{\{1,234678,59ac,b\}} + F_{\{1,2368,4579ac,b\}} \right) \\
& + \left(F_{\{1,234678b,59c,a\}} - F_{\{1,234678,59bc,a\}} - F_{\{1,234678a,59c,b\}} + F_{\{1,234678,59ac,b\}} \right) \\
& + \left(F_{\{134589bc,2,67,a\}} - F_{\{1345689bc,2,7,a\}} - F_{\{134589bc,2,6,7a\}} + F_{\{134589abc,2,6,7\}} \right)
\end{aligned}
$$

$$+ \left(F_{\{134589bc,26,7,a\}} - F_{\{134589bc,2,67,a\}} - F_{\{1234589bc,6,7,a\}} + F_{\{1345789bc,2,6,a\}} \right)$$
$$+ \left(F_{\{123458,6,7a,9bc\}} - F_{\{123458,6,7,9abc\}} - F_{\{1234578,6,9bc,a\}} + F_{\{1234589bc,6,7,a\}} \right)$$
$$+ \left(F_{\{1358b,2,49c,67a\}} - F_{\{13568b,2,49c,7a\}} - F_{\{1358b,2,479ac,6\}} + F_{\{134589bc,2,6,7a\}} \right)$$
$$+ \left(F_{\{18,2,35b,4679ac\}} - F_{\{18,2,34579abc,6\}} - F_{\{168,2,35b,479ac\}} + F_{\{1358b,2,479ac,6\}} \right)$$
$$+ \left(F_{\{146789ac,2,3,5b\}} - F_{\{168,2,3,4579abc\}} - F_{\{1368,2,479ac,5b\}} + F_{\{168,2,35b,479ac\}} \right)$$
$$+ \left(F_{\{12478,3,5a,69bc\}} - F_{\{124578,3,69bc,a\}} - F_{\{12478,3,5,69abc\}} + F_{\{1246789bc,3,5,a\}} \right)$$
$$+ \left(F_{\{1248,35,69abc,7\}} - F_{\{1248,3,57,69abc\}} - F_{\{12348,5,69abc,7\}} + F_{\{12478,3,5,69abc\}} \right)$$
$$+ \left(F_{\{123478,5,6,9abc\}} - F_{\{12348,5,6,79abc\}} - F_{\{123468,5,7,9abc\}} + F_{\{12348,5,69abc,7\}} \right)$$
$$+ \left(F_{\{1a,25679bc,38,4\}} - F_{\{1,25679abc,38,4\}} - F_{\{138,25679bc,4,a\}} + F_{\{1,2356789bc,4,a\}} \right)$$
$$+ \left(F_{\{138a,2567,4,9bc\}} - F_{\{1235678,4,9bc,a\}} - F_{\{138,2567,4,9abc\}} + F_{\{138,25679bc,4,a\}} \right)$$
$$+ \left(F_{\{1389abc,2,4,567\}} - F_{\{138,2,4,5679abc\}} - F_{\{1238,4,567,9abc\}} + F_{\{138,2567,4,9abc\}} \right)$$
$$+ \left(F_{\{12389abc,4,56,7\}} - F_{\{1238,4,56,79abc\}} - F_{\{123568,4,7,9abc\}} + F_{\{1238,4,567,9abc\}} \right)$$
$$+ \left(F_{\{125678a,3b,4,9c\}} - F_{\{1235678a,4,9c,b\}} - F_{\{125678a,3,4,9bc\}} + F_{\{1256789ac,3,4,b\}} \right)$$
$$+ \left(F_{\{18a,2567,39bc,4\}} - F_{\{138a,2567,4,9bc\}} - F_{\{18a,25679bc,3,4\}} + F_{\{125678a,3,4,9bc\}} \right)$$
$$+ \left(F_{\{1a,235679bc,4,8\}} - F_{\{1a,25679bc,38,4\}} - F_{\{125679abc,3,4,8\}} + F_{\{18a,25679bc,3,4\}} \right)$$
$$+ \left(F_{\{12,3,48,5679abc\}} - F_{\{12,3,45679abc,8\}} - F_{\{128,3,4,5679abc\}} + F_{\{125679abc,3,4,8\}} \right)$$
$$+ \left(F_{\{1238,4,5,679abc\}} - F_{\{128,3679abc,4,5\}} - F_{\{1258,3,4,679abc\}} + F_{\{128,3,4,5679abc\}} \right)$$

$$\begin{aligned}
\equiv \quad & F_{\{124578ab,36,9,c\}} + F_{\{1,28,34679abc,5\}} + F_{\{1,2,345679abc,8\}} + F_{\{1a,235679bc,4,8\}} \\
& + F_{\{1234678,5,9bc,a\}} + F_{\{1238,45,6,79abc\}} + F_{\{134589bc,26,7,a\}} + F_{\{1,2368b,47,59ac\}} \\
& + F_{\{1,234678b,59c,a\}} + F_{\{125678a,3b,4,9c\}} + F_{\{1248,35,69abc,7\}} + F_{\{1238,4,5,679abc\}} \\
& + F_{\{1,23,45679abc,8\}} + F_{\{1234678a,5,9c,b\}} + F_{\{15,234678ab,9,c\}} + F_{\{12389abc,4,56,7\}} \\
& + F_{\{12478,3,5a,69bc\}} + F_{\{146789ac,2,3,5b\}} + F_{\{1358b,2,49c,67a\}} + F_{\{128,3,4579abc,6\}} \\
& + F_{\{18,2,35b,4679ac\}} + F_{\{18a,2567,39bc,4\}} + F_{\{12,3,48,5679abc\}} + F_{\{123478,5,6,9abc\}} \\
& + F_{\{128,3,4679abc,5\}} + F_{\{1,238,4579ac,6b\}} + F_{\{124578a,3,69c,b\}} + F_{\{123458,6,7a,9bc\}} \\
& + F_{\{18,2,34579abc,6\}} + F_{\{1234578a,6,9c,b\}} + F_{\{135678ab,24,9,c\}} + F_{\{1389abc,2,4,567\}}.
\end{aligned}$$

The calculation took 162 seconds.

Recall that in Example 5.8 we described a noneffective expression for this curve class with 103 terms that was obtained using only simple linear algebra techniques (as opposed to the toric degeneration techniques used here). The `seekEffectiveExpression` command was also able to find an effective expression starting from the 103 term expression, but the calculation took 21210 seconds. We consider this example as evidence that the toric degeneration method is superior to the simple linear algebra approach.

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