Efficient finite dimensional approximations for the bilinear Schrödinger equation with bounded variation controls

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***Abstract*— In the present analysis, we consider the controllability problem of the abstract Schrödinger equation :**

$$\partial_t \psi = A\psi + uB\psi$$

where A is a skew-adjoint operator, B a control potential and u is the control command.

We are interested by approximation of this equation by finite dimensional systems.

Assuming that A has a pure discrete spectrum and B is in some sense regular with respect to A we show that such an approximation is possible. More precisely the solutions are approximated by their projections on finite dimensional subspaces spanned by the eigenvectors of A.

This approximation is uniform in time and in the control u, if this control has bounded variation with a priori bounded total variation. Hence if these finite dimensional systems are controllable with a fixed bound on the total variation of u then the system is approximatively controllable.

The main outcome of our analysis is that we can build solutions for low regular controls u such as bounded variation ones and even Radon measures.

I. Extended abstract

a) The wellposedness: Let $\mathcal{H}$ be a separable Hilbert space (possibly infinite dimensional) with scalar product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the corresponding norm, A, B be two (possibly unbounded) skew-symmetric operators on $\mathcal{H}$. We consider the formal bilinear control system

$$\frac{d}{dt}\psi(t) = A\psi(t) + u(t)B\psi(t), \tag{1}$$

where the scalar control u is to be chosen in a set of real functions.

For any real interval I, we define

$$\Delta_I := \{(s,t) \in I^2 \mid s \le t\}.$$

Definition : Propagator on a Hilbert space
Let I be a real interval. A family $(s,t) \in \Delta_I \mapsto X(s,t)$ of linear contractions, that is Lipschitz maps with Lipschitz constant less than one, on a Hilbert space $\mathcal{H}$, strongly continuous in t and s and such that

1) for any $s < r < t$, $X(t,s) = X(t,r)X(r,s)$,
2) $X(t,t) = I_{\mathcal{H}}$,

is called a contraction propagator on $\mathcal{H}$.

Let us now fix some scalar function $u : I \mapsto \mathbf{R}$ and define $A(t) = A + u(t)B$.

Recall that a family $t \in I \mapsto U(t) \in E$, E a subset of a Banach space X, is in $BV(I,E)$ if there exists $N \geq 0$ such that

$$\sum_{j=1}^{n} \|U(t_j) - U(t_{j-1})\|_X \leq N$$

for any partition $a = t_0 < t_1 < \ldots < t_n = b$ of the interval (a,b). The mapping

$$U \in BV(I,E) \mapsto \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{j=1}^{n} \|U(t_j) - U(t_{j-1})\|_X$$

is a semi-norm on $BV(I,E)$ that we denote with $\|\cdot\|_{BV(I,E)}$. The semi-norm in $BV(I,E)$ is also called total variation.

Assumptions
Let I be a real interval and $\mathcal{D}$ dense subset of $\mathcal{H}$

1) $A(t)$ is a maximal skew-symmetric operator on $\mathcal{H}$ with domain $\mathcal{D}$,
2) $t \mapsto A(t)$ has bounded variation from I to $L(\mathcal{D},\mathcal{H})$, where $\mathcal{D}$ is endowed with the graph topology associated to $A(a)$ for $a = \inf I$[1],
3) $M := \sup_{t\in I} \left\|(1 - A(t))^{-1}\right\|_{L(\mathcal{H},\mathcal{D})} < \infty$,

We do not assume $t \mapsto A(t)$ to be continuous. However as a consequence of Assumption 2 (see [Edw57, Theorem 3]) it admits right and left limit in $L(\mathcal{D},\mathcal{H})$, $A(t-0) = \lim_{\varepsilon\to 0^+} A(t-\varepsilon)$, $A(t+0) = \lim_{\varepsilon\to 0^+} A(t+\varepsilon)$, for all $t \in I$, and $A(t-0) = A(t+0)$ for all $t \in I$ except a countable set.

The the core of our analysis is the following result due to Kato (see [Kat53, Theorem 2 and Theorem 3]). It gives sufficient conditions for the well-posedness of the system (1).

[1] The bounded variation of $t \mapsto A(t)$ ensures that any choice of $s \in I$ will be equivalent.

Theorem 1: If $t \in I \mapsto A(t)$ satisfies the above assumptions, then there exists a unique contraction propagator $X : \Delta_I \to L(\mathcal{H})$ such that if $\psi_0 \in \mathcal{D}$ then $X(t,s)\psi_0 \in \mathcal{D}$ and for $(t,s) \in \Delta_I$

$$\|A(t)X(t,s)\psi_0\| \leq Me^{M\|A\|_{BV(I,L(\mathcal{D},\mathcal{H}))}}\|A(s)\psi_0\|.$$

and in this case $X(t,s)\psi_0$ is strongly left differentiable in t and right differentiable in s with derivative (when $t = s$) $A(t+0)\psi_0$ and $-A(t-0)\psi_0$ respectively.

In the case in which $t \mapsto A(t)$ is continuous and skew-adjoint, if $\psi_0 \in \mathcal{D}$ then $t \in (s, +\infty) \mapsto X(t,s)\psi_0$ is strongly continuously differentiable in $\mathcal{H}$ with derivative $A(t)X(t,s)\psi_0$.

This theorem addresses the problem of existence of solution for the kind of non-autonomous system we consider here under very mild assumptions on the control command u. For instance we consider bounded variation controls. Under some additional assumptions such as the boundedness of the control potential, we can also consider Radon measures.

Let us insist on the quantitative aspect of the theorem as it provides an estimate on the growth of the solution in the norm associated with A. This is quantitative aspect is the starting point of the subsequent comments.

b) Some supplementary regularity: The regularity of the solution with respect to the natural structure of the problem is considered now. In that respect we can adopt two complementary strategies :

1) the regularity of the input-output mapping, the flow of the problem, is obtained by proving that the control potential if it is regular enough do not alter the regularity properties of the uncontrolled problem.
2) the regularity can be added to the functional setting of the wellposedness problem; namely we solve the problem imposing regularity to the constructed solution.

For the first strategy we introduce the following definition.

Definition : Weakly coupled

Let k be a non negative real. A couple of *skew-adjoint* operators (A, B) is k-*weakly coupled* if

1) A is invertible with bounded inverse from $D(A)$ to $\mathcal{H}$,
2) for any real t, $e^{tB}D(|A|^{k/2}) \subset D(|A|^{k/2})$,
3) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B-c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\psi \mapsto \||A|^{k/2}u\|$.

We set, for every positive real k,

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k\psi, \psi\rangle}.$$

The optimal exponential growth is defined by

$$c_k(A,B) := \sup_{t\in\mathbf{R}} \frac{\log \|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2})}}{|t|}.$$

Theorem 2: Let k be a non negative real. Let (A, B) be k-weakly coupled .

For any $u \in BV([0,T],\mathbf{R}) \cap B_{L^\infty([0,T])}(0, 1/\|B\|_A)$, there exists a family of contraction propagators in $\mathcal{H}$ that extends uniquely as contraction propagators to $D(|A|^{k/2})$: $\Upsilon^u : \Delta_{[0,T]} \to L(D(|A|^{k/2}))$ such that

1) for any $t \in [0,T]$, for any $\psi_0 \in D(|A|^{k/2})$

$$\|\Upsilon_t(\psi_0)\|_{k/2} \leq e^{c_k(A,B)\left|\int_0^t u\right|}\|\psi_0\|_{k/2}$$

2) for any $t \in [0,T]$, for any $\psi_0 \in D(|A|^{1+k/2})$ for any $u \in BV([0,T],\mathbf{R}) \cap B_{L^\infty([0,T])}(0, 1/\|B\|_A))$, there exists m (depending only on A, B and $\|u\|_{L^\infty([0,T])}$)

$$\|\Upsilon_t(\psi_0)\|_{1+k/2} \leq me^{m\|u\|_{BV([0,T],\mathbf{R})}} \times \times e^{c_k(A,B)\left|\int_0^t u\right|}\|\psi_0\|_{1+k/2}$$

Moreover, for every ψ_0 in $D(|A|^{k/2})$, the end-point mapping

$$\Upsilon(\psi_0) : BV([0,T],K) \to D(|A|^{k/2}) \quad u \mapsto \Upsilon^u(0,T)(\psi_0)$$

is continuous.

As announced before these theorem the two strategies were used in complement to establish the second point of the theorem.

There is several outcomes to our analysis. First each attainable target from an initial state has to be as regular as the initial state and the control potential allows it to be. For instance if we consider the harmonic potential for the Shrödinger equation and we try to control it by a smooth bounded potential then from any initial eigenvector one cannot attain non-smooth non exponentially decaying states.

c) A negative result: An auxiliary result of our analysis is an immediate generalisation of the famous negative result by Ball, Marsden and Slemrod [BMS82] that the attainable set is included in a countable union of compact sets for the initial Hilbert setting of the problem for integrable control laws. We can show that this still holds for much smaller spaces than the initial Hilbert space, for instance the domains and iterated domains of the uncontrolled problem, for much less regular controls such as bounded variation function or even Radon measures.

Theorem 3: Let k be a non negative real. Let (A, B) be k-weakly coupled . Let $\psi_0 \in D(|A|^{k/2})$.Then

$$\bigcup_{L,T,a>0} \left\{\alpha\Upsilon^u(\psi_0), \|u\|_{BV([0,T],\mathbf{R})} \leq L, t \in [0,T], |\alpha| \leq a\right\}$$

is a meagre set (in the sense of Baire) in $L^\infty(I, D(|A|^{k/2})$ as a union of relatively compact subsets.

d) Gallerkin approximation for bounded control potential: In a practical setting it is clear that this negative result is useless. Nonetheless we consider that such a negative result has a philosophical consequence, natural systems for which regularity of the bounded potential is natural cannot be exactly controllable. such an observation is even more dramatic if one considers systems with continuous spectrum.

Once such a comment is made, practical questions remains, one of them is the question of the approximate controllability (see [BCC13, Definition 1]), we choose here to give a corollary of our analysis that can be helpfull when one want to tackle this issue.

For every Hilbert basis $\Phi = (\phi_k)_{k\in\mathbf{N}}$ of $\mathcal{H}$, we define, for every N in $\mathbf{N}$,

$$\begin{array}{rcll} \pi_N^\Phi & : \mathcal{H} & \to & \mathcal{H} \\ & \psi & \mapsto & \sum_{j\leq N}\langle\phi_j,\psi\rangle\phi_j\,. \end{array}$$

Definition

Let (A,B) be a couple of unbounded operators and $\Phi = (\phi_n)_{n\in\mathbf{N}}$ be an Hilbert basis of $\mathcal{H}$. Let $N\in\mathbf{N}$ and denote $\mathcal{L}_N^\Phi = \mathrm{span}(\phi_1,\ldots,\phi_N)$. The *Galerkin approximation* of order N of system (1), when it makes sense, is the system

$$\dot{x} = (A^{(\Phi,N)} + uB^{(\Phi,N)})x \qquad (\Sigma_N^\Phi)$$

where $A^{(\Phi,N)}$ and $B^{(\Phi,N)}$, defined by

$$A^{(\Phi,N)} = \pi_N^\Phi A_{\restriction\mathcal{L}_N} \quad \text{and} \quad B^{(\Phi,N)} = \pi_N^\Phi B_{\restriction\mathcal{L}_N},$$

are the *compressions* of A and B (respectively) associated with $\mathcal{L}_N$.

Notice that if A is skew-adjoint and Φ an Hilbert basis made of eigenvectors of A then $(A^{(\Phi,N)}, B^{(\Phi,N)})$ satisfies the same assumptions as (A,B). We can therefore define the contraction propagator $X^u_{(\Phi,N)}(t,0)$ of (Σ_N^Φ) associated with bounded variation control u. We can also write that $(A^{(\Phi,N)}, B^{(\Phi,N)})$ is k-weakly coupled for any prositive real k as the weak coupling is actually invariant or at least does not deteriorate by compression with respect to a basis of eigenvectors of A.

In the case of bounded potentials we can state the following proposition. Below we consider B be bounded in $D(|A|^{k/2})$ which implies that (A,B) is k-weakly coupled .

Theorem 4: Let k be a positive real. Let A with domain $D(A)$ be the generator of a contraction semigroup and let B be bounded in $\mathcal{H}$ and $D(|A|^{k/2})$ with $B(1-A)^{-1}$ compact. Let s be non-negative numbers with $0\leq s<k$. Then for every $\varepsilon>0$, $L\geq 0$, $n\in\mathbf{N}$, and $(\psi_j)_{1\leq j\leq n}$ in $D(|A|^{k/2})^n$ there exists $N\in\mathbf{N}$ such that for any $u\in\mathcal{R}((0,T])$,

$$|u|([0,T]) < L \Rightarrow \|\Upsilon^u_t(\psi_j) - X^u_{(N)}(t,0)\pi_N\psi_j\|_{s/2} < \varepsilon,$$

for every $t\geq 0$ and $j=1,\ldots,n$.

Hence if these finite dimensional systems are controllable with a fixed bound on the total variation of u then the system is approximatively controllable.

e) An example Smooth potentials on compact manifolds: This example motivated the present analysis because of its physical importance. We consider Ω a compact Riemannian manifold endowed with the associated Laplace-Beltrami operator Δ and the associated measure μ, $V,W:\Omega\to\mathbf{R}$ two smooth functions and the bilinear quantum system

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \Delta\psi + V\psi + u(t)W\psi. \qquad (2)$$

With the previous notations, $\mathcal{H} = L^2(\Omega,\mathbf{C})$ endowed with the Hilbert product $\langle f,g\rangle = \int_\Omega \bar{f}g\mathrm{d}\mu$, $A = -\mathrm{i}(\Delta+V)$ and $B=-\mathrm{i}W$. For every $r\geq 0$, $D(|A|^r) = H^r(\Omega,\mathbf{C})$. There exists a Hilbert basis $(\phi_k)_{k\in\mathbf{N}}$ of H made of eigenvectors of A. Each eigenvalue of A has finite multiplicity. For every k, there exists λ_k in $\mathbf{R}$ such that $A\phi_k = \mathrm{i}\lambda_k\phi_k$. The sequence $(\lambda_k)_k$ tends to $+\infty$ and, up to a reordering, is non decreasing.

Since B is bounded from $D(|A|^k)$ to $D(|A|^k)$, $(A,B,\mathbf{R})$ satisfies our assumptions and (A,B) is k-weakly coupled for non negative real k.

References

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