IGUSA QUARTIC AND BORCHERDS PRODUCTS

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ABSTRACT. By applying Borcherds' theory of automorphic forms on bounded symmetric domains of type IV, we give a 5-dimensional linear system of automorphic forms of weight 6 on Igusa quartic 3-fold which induces an $\mathfrak{S}_6$-equivariant rational map of degree 16 from Igusa quartic to Segre cubic. In particular we have a rational self-map of Igusa quartic of degree 16.

1. INTRODUCTION

The purpose of this paper is to give an application of the theory of automorphic forms on bounded symmetric domains of type IV due to Borcherds [B1], [B2]. We consider Igusa quartic 3-fold $\mathcal{I}$ given by

$$\sum_i x_i = \Big(\sum_i x_i^2\Big)^2 - 4\Big(\sum_i x_i^4\Big) = 0 \subset \mathbb{P}^5, \tag{1.1}$$

where $(x_1 : \cdots : x_6)$ is a homogenous coordinate of $\mathbb{P}^5$. It is classically known (Baker [Ba], Chap.V, Dolgachev [D]) that Igusa quartic is the dual variety of Segre cubic 3-fold $\mathcal{S}$ defined by

$$\sum_i x_i = \sum_i x_i^3 = 0. \tag{1.2}$$

The symmetry group $\mathfrak{S}_6$ of degree 6 naturally acts on $\mathcal{I}$ and $\mathcal{S}$ as automorphisms. Igusa quartic $\mathcal{I}$ is isomorphic to the Satake compactification $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ of the quotient of the Siegel upper half plane $\mathfrak{H}_2$ of degree two by the 2-congrunce subgroup $\Gamma_2(2)$ of $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$ (Igusa [I], page 397; also see van der Geer [vG]). The natural action of $\mathfrak{S}_6(\cong \Gamma_2/\Gamma_2(2))$ on $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ coincides with the above one on $\mathcal{I}$. On the other hand, let $M = U(2)^{\oplus 2} \oplus A_1(2)$ be the transcendental lattice of a generic Kummer surface associated with a smooth curve of genus 2 and let $\mathcal{D}(M)$ be a bounded symmetric domain of type IV and of dimension 3 associated with M. Then it follows from Gritsenko, Hulek [Gr], [GHu] that $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ is isomorphic to the Baily-Borel compactification $\overline{\mathcal{D}(M)/\Gamma_M}$ where Γ_M is a subgroup of the orthogonal group $\mathrm{O}(M)$ such that $\mathrm{O}(M)/\Gamma_M \cong \mathfrak{S}_6$. This isomorphism is $\mathfrak{S}_6$-equivariant.

In this paper, by applying Borcherds' theory, we present holomorphic automorphic forms $\Psi_{10}, \Psi_{30}, \Psi_{24}$ on $\mathcal{D}(M)$ of weight $10, 30, 24$ which coincide with the Siegel modular forms with Humbert surfaces H_1, H_4, H_5 as zero divisors, respectively (Theorem 8.6). Moreover we give a 5-dimensional linear system of holomorphic automorphic forms on $\mathcal{D}(M)$ of weight 6 which induces the linear system $\mathcal{L}$ of cubics on $\mathcal{I}$ given by

$$(x_i - x_j)(x_k - x_l)(x_m - x_n) \quad (\{i,j,k,l,m,n\} = \{1,\dots,6\}) \tag{1.3}$$

(Theorem 8.9). The linear system $\mathcal{L}$ gives an $\mathfrak{S}_6$-equivariant rational map of degree 16 from Igusa quartic to Segre cubic (Theorem 8.10). Thus we have a rational self-map of Igusa quartic of degree 16 (Corollary 8.11). Note that Mukai [M] recently found a holomorphic self-map of Igusa quartic of degree 8. On the other hand,

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Segre cubic is an arithmetic quotient of a 3-dimensional complex ball (see Hunt [H]). This complex ball can be naturally embedded into a bounded symmetric domain of type IV and of dimension 6. Recently, by applying Borcherds theory, the author [K3] gives a 5-dimensional space of automorphic forms on the complex ball which defines the dual map from Segre cubic to Igusa quartic.

We use an idea of Allcock, Freitag [AF] in which they gave an embedding of the moduli space of marked cubic surfaces into $\mathbb{P}^9$ by applying Borcherds' theory. For a given lattice L of signature $(2,n)$, we consider vector-valued modular forms with respect to the Weil representation of $\mathrm{SL}(2,\mathbb{Z})$ on the group ring $\mathbb{C}[L^*/L]$ where L^* is the dual of L. There are two types of liftings of vector-valued modular forms both of which give automorphic forms on the bounded symmetric domain $\mathcal{D}(L)$ of type IV associated with L. One is called *Borcherds product* or *multiplicative lifting* which is an automorphic form with known zeros and poles. Another one is called *additive lifting* which is an automorphic form with respect to the subgroup of the orthogonal group $\mathrm{O}(L)$ acting trivially on L^*/L. Borcherds gives explicit formulae for the Fourier coefficients of the additive lifting in terms of the Fourier coefficients of the vector-valued modular form.

We apply Borcherds' theory to the lattice $N = U(2)^{\oplus 2} \oplus A_1^{\oplus 2}$ of signature $(2,4)$ instead of M because M has *odd* rank 5 which makes a difficulty of calculations. We can embed M into N as a primitive sublattice which induces an embedding of the domain $\mathcal{D}(M)$ into $\mathcal{D}(N)$. We remark that $\mathcal{D}(N)$ is the period domain of $K3$ surfaces associated with six lines on $\mathbb{P}^2$ (see Matsumoto, Sasaki, Yoshida [MSY]). We construct automorphic forms $\Phi_4, \Phi_{10}, \Phi_{30}, \Phi_{48}$ on $\mathcal{D}(N)$ of weight $4, 10, 30, 48$ as Borcherds products (Theorem 6.3). By restricting them to $\mathcal{D}(M)$, we get Siegel modular forms $\Psi_{10}, \Psi_{30}, \Psi_{24}$ mentioned as above (Φ_4 vanishes on $\mathcal{D}(M)$). For example, Ψ_{10} is the product $\prod \theta_m^2(\tau)$ of the square of even theta constants. We remark that the *Borcherds product* Φ_4 is also obtained by *additive lifting* (Remark 7.3). Such example was already given by the author in the case of the moduli space of Enriques surfaces ([K1], Remark 4.7). On the other hand, to each 2-dimensional isotorpic subspace in $A_N = N^*/N$, we associate an automorphic form of weight 10 as additive lifting. By restricting them, we get fifteen automorphic forms on $\mathcal{D}(M)$ of weight 6 corresponding to fifteen functions given in (1.3) (Theorem 8.9).

The plan of this paper is as follows. In §2, we recall a theory of lattices. Section 3 is devoted to the theory of periods of $K3$ surfaces which are double covers of $\mathbb{P}^2$ branched along six lines, due to Matsumoto, Sasaki, Yoshida [MSY]. In section 4, we recall a description of the Igusa quartic as an arithmetic quotient of a bounded symmetric domain of type IV. Moreover we study the boundary components and Heegner divisors (= Humbert surfaces) on the Igusa quartic. In section 5, we recall the Weil representation of $\mathrm{SL}(2,\mathbb{Z})$ on the group ring $\mathbb{C}[N^*/N]$ and calculate its character. We study the 5-dimensional subspace appeared in the Weil representation. The result will be used to construct additive liftings in §7. In section 6, by using Borcherds products, we show that there exist holomorphic automorphic forms on $\mathcal{D}(N)$ of weight $4, 10, 30, 48$ with known zeros, and in section 7, by using additive liftings, we give a 5-dimensional space of automorphic forms on $\mathcal{D}(N)$. Finally, in §8, we discuss automorphic forms on Igusa quartic.

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2. PRELIMINARIES

A *lattice* is a free abelian group L of finite rank equipped with a non-degenerate symmetric integral bilinear form $\langle , \rangle : L \times L \to \mathbb{Z}$. For $x \in L \otimes \mathbb{Q}$, we call $x^2 = x \cdot x$ the *norm* of x. For a lattice L and a rational number m, we denote by $L(m)$ the free $\mathbb{Z}$-module L with the $\mathbb{Q}$-valued bilinear form obtained from the bilinear form of L by multiplication with m. The signature of a lattice is the signature of the real quadratic space $L \otimes \mathbb{R}$. A lattice is called *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$.

We denote by U the even unimodular lattice of signature $(1, 1)$, and by $A_m,\ D_n$ or $\ E_k$ the even *negative* definite lattice defined by the Cartan matrix of type $A_m,\ D_n$ or $\ E_k$ respectively. For an integer m, we denote by $\langle m\rangle$ the lattice of rank 1 generated by a vector with norm m. We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M, and by $L^{\oplus m}$ the orthogonal direct sum of m-copies of L.

Let L be an even lattice and let $L^* = \mathrm{Hom}(L, \mathbb{Z})$. We denote by A_L the quotient L^*/L which is called the *discriminant group* of L, and define maps

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \quad b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$$

by $q_L(x + L) = \langle x, x\rangle \bmod 2\mathbb{Z}$ and $b_L(x + L, y + L) = \langle x, y\rangle \bmod \mathbb{Z}$. We call q_L the *discriminant quadratic form* of L and b_L the *discriminant bilinear form*. A lattice is called *2-elementary* if its discriminant group is a 2-elementary abelian group. We denote by u, v or $q_\pm$ the discriminant quadratic form of 2-elementary lattice $U(2)$, D_4 or $\langle \pm 2\rangle$ respectively. For any 2-elementary lattice L, the discriminant form q_L is a direct sum of $u, v, q_\pm$. An even 2-elementary lattice L is called type I if q_L is a direct sum of u and v, and type II if otherwise. It is known that the isomorphism class of an even indefinite 2-elementary lattice is determined by its signature, the rank of A_L and its type I or II.

Let $\mathrm{O}(L)$ be the orthogonal group of L, that is, the group of isomorphisms of L preserving the bilinear form. Similarly $\mathrm{O}(q_L)$ denotes the group of isomorphisms of A_L preserving q_L. There is a natural map

$$\mathrm{O}(L) \to \mathrm{O}(q_L) \tag{2.1}$$

whose kernel is denoted by $\widetilde{\mathrm{O}}(L)$. For more details we refer the reader to Nikulin [N1].

3. SIX LINES ON $\mathbb{P}^2$ AND $K3$ SURFACES

Let $\ell_1, \dots, \ell_6$ be ordered six lines on $\mathbb{P}^2$ in general position, that is, no three lines meet at one point. Let $\bar{X}$ be the double cover of $\mathbb{P}^2$ branched along the sextic $\ell_1 + \cdots + \ell_6$. Then $\bar{X}$ has 15 ordinary nodes over the point $p_{ij} = \ell_i \cap \ell_j$. Let X be the minimal resolution of $\bar{X}$ which is a $K3$ surface. Obviously X contains 15 mutually disjoint smooth rational curves E_{ij} which are the exceptional curves over p_{ij}, and six smooth rational curves $\tilde{\ell}_i$ $(1 \le i \le 6)$ which are the proper transforms of ℓ_i. Denote by $\mathrm{Pic}(X)$ the Picard lattice of X and by S_X the smallest primitive sublattice of $\mathrm{Pic}(X)$ containing 21 smooth rational curves $E_{ij},\ \tilde{\ell}_i$.

3.1. **Proposition.** (Matsumoto, Sasaki, Yoshida [MSY]) *The lattice S_X is a primitive sublattice of $H^2(X, \mathbb{Z})$ of signature $(1, 15)$, $S_X^*/S_X \cong (\mathbb{Z}/2\mathbb{Z})^6$ and $q_{S_X} \cong u \oplus u \oplus q_+ \oplus q_+$. The group $O(q_{S_X})$ is isomorphic to $\mathfrak{S}_6 \times \mathbb{Z}/2\mathbb{Z}$, where $\mathfrak{S}_6$ is the symmetric group of degree 6. The natural map $O(S_X) \to O(q_{S_X})$ is surjective.*

Proof. The assertion follows from Corollary 2.1.6 and Proposition 2.8.2 in [MSY]. Here we give an another proof by using Nikulin's lattice theory. First note that S_X is the invariant sublattice of $H^2(X, \mathbb{Z})$ under the action of the covering transformation of $X \to \mathbb{P}^2$. It follows from Nikulin [N2], Theorem 4.2.2 that S_X is an even 2-elementary lattice of signature $(1, 15)$ and with $q_{S_X} = u \oplus u \oplus q_+ \oplus q_+$. Note that there exists a subgroup $F \cong (\mathbb{Z}/2\mathbb{Z})^5$ of A_{S_X} on which the restriction of q_{S_X} has values in $\mathbb{Z}/2\mathbb{Z}$, that is, $q_{S_X}|F$ is a quadratic form of dimension 5 over $\mathbb{F}_2$. Moreover $q_{S_X}|F$ contains a radical $\langle \kappa \rangle \cong \mathbb{Z}/2\mathbb{Z}$, that is, κ is perpendicular to all elements in F with respect to b_{S_X}, and b_{S_X} induces a symplectic form on $F/\langle \kappa \rangle$ of dimension 4 over $\mathbb{F}_2$. Thus the orthogonal group $\mathrm{O}(q_{S_X})$ is isomorphic to $\mathrm{Sp}(4, \mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z} \cong \mathfrak{S}_6 \times \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ is generated by the involution changing two components $q_+ \oplus q_+$ and acting trivially on $u \oplus u$. The surjectivity of the natural map $\mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$ follows from [N1], Theorem 3.6.3. □

We call X *generic* if $S_X = \mathrm{Pic}(X)$. Let T_X be the orthogonal complement of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$ which is called the *transcendental lattice* of X. Also we denote by N_X the orthogonal complement of S_X. It is known that $q_{S_X} = -q_{N_X}$ (e.g. Nikulin [N1], Corollary 1.6.2) and hence $q_{N_X} \cong u \oplus u \oplus q_- \oplus q_-$. Since N_X is a 2-elementary lattice of signature $(2, 4)$, the isomorphism class of N_X is determined by its signature and q_{N_X} (Nikulin [N1], Theorem 3.6.2). Thus we have

$$N_X \cong U(2)^{\oplus 2} \oplus A_1^{\oplus 2} \cong A_1(-1)^{\oplus 2} \oplus A_1^{\oplus 4}.$$

We denote by N an abstract lattice of signature $(2, 4)$ and with $q_N = u \oplus u \oplus q_- \oplus q_-$. If X is generic, then $T_X \cong N$. We denote by $\kappa_N \in A_N$ the radical corresponding to κ (see the proof of Proposition 3.1).

An elementary calculation shows the following Lemma.

3.2. **Lemma.** *The discriminant group A_N consists of the following 64 vectors*:

Type (00) : $\alpha = 0,\ \#\alpha = 1$;

Type (0) : $\alpha \neq 0,\ q_N(\alpha) = 0,\ \#\alpha = 15$;

Type (1) : $\alpha \neq \kappa_N,\ q_N(\alpha) = 1,\ \#\alpha = 15$;

Type (10) : $\alpha = \kappa_N,\ \#\alpha = 1$;

Type $(1/2)$: $q_N(\alpha) = 1/2,\ \#\alpha = 12$;

Type $(3/2)$: $q_N(\alpha) = 3/2,\ \#\alpha = 20$.

Define

$$\mathcal{D}(N) = \{[\omega] \in \mathbb{P}(N \otimes \mathbb{C})\ :\ \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}$$

where $\bar{\omega}$ is the complex conjugate of ω. It is known that $\mathcal{D}(N)$ is a disjoint union of two copies of a bounded symmetric domain of type IV and of dimension 4. We denote by Γ_N the group $\widetilde{\mathrm{O}}(N)$ which acts properly discontinuously on $\mathcal{D}(L)$. It is known that the Baily-Borel compactification of the quotient $\mathcal{D}(N)/\Gamma_N$ is the coarse moduli space of ordered six lines on $\mathbb{P}^2$ ([MSY]).

Now we define the Heegner divisors on $\mathcal{D}(N)$. Fix a vector $\alpha \in A_N$ with $q_N(\alpha) \neq 0$ and a negative rational number n. For $r \in N^*$ with $r^2 < 0$, we denote by $r^\perp$ the hyperplane in $\mathcal{D}(N)$ perpendicular to r. We define a *Heegner divisor* $\mathcal{H}(N)_{\alpha,n}$ by

$$\mathcal{H}(N)_{\alpha,n} = \sum_r r^\perp$$

where r runs through all vectors r in N^* satisfying $r \bmod N = \alpha$ and $\langle r, r \rangle = n$. For simplicity, we denote by $\mathcal{H}(N)_\alpha$ the Heegner divisor $\mathcal{H}(N)_{\alpha,n}$ for $n = -1, -1/2$ or $-3/2$ according to $q_N(\alpha) = 1, 3/2$ or $1/2$, respectively. Also we denote by $\mathcal{H}(N)_1, \mathcal{H}(N)_{3/2}$ or $\mathcal{H}(N)_{1/2}$ the union of all $\mathcal{H}(N)_\alpha$ where α runs through all vectors α with $q_N(\alpha) = 1$ ($\alpha \neq \kappa_N$), $3/2$ or $1/2$ respectively. The geometric meaning of these Heegner divisors is known. For example, A generic point in $\mathcal{H}(N)_1$ corresponds to six lines such that three points p_{ij}, p_{kl}, p_{mn} are collinear where $\{i,j,k,l,m,n\} = \{1,\dots,6\}$. For more details we refer the reader to [LPS], Theorem 3.6.

3.3. **Reflections.** Let r be a (-4)-vector in N with $r/2 \in N^*$. Then the reflection s_r defined by

$$s_r(x) = x - \frac{2\langle x, r\rangle}{\langle r, r\rangle} r = x + \langle x, r/2\rangle r \quad (x \in N) \tag{3.1}$$

is contained in $\mathrm{O}(N)$. The reflection s_r induces a reflection t_α on A_N associated with $\alpha = r/2 \bmod N$ defined by

$$t_\alpha(\beta) = \beta + 2b_N(\beta, \alpha)\alpha \quad (\beta \in A_N). \tag{3.2}$$

4. Igusa quartic and a bounded symmetric domain of type IV

Let $\mathfrak{H}_2$ be the Siegel upper half plane of degree two and let $\Gamma_2(2)$ be the principal 2-congurence subgroup of $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$. We denote by $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ the Satake compactification of the quotient $\mathfrak{H}_2/\Gamma_2(2)$. Igusa [I] showed that $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ can be embedded into $\mathbb{P}^4$ by using theta constants, whose image is a quartic hypersurface $\mathcal{I}$ given by the equation (1.1) called *Igusa quartic* (Igusa gave a different form. See [vG]). The boundary of the compactification consists of fifteen 1-dimensional components and fifteen 0-dimensinal components which correspond to fifteen lines and fifteen points on the Igusa quartic. By definition $\mathfrak{H}_2/\Gamma_2(2)$ is the moduli space of principally polarized abelian surfaces with a level 2-structure.

On the other hand it is known that $\mathfrak{H}_2$ is isomorphic to a bounded symmetric domain of type IV and of dimension 3 as bounded symmetric domains (e.g. [vG]). Put

$$M = U(2)^{\oplus 2} \oplus A_1(2). \tag{4.1}$$

Then M is an even lattice of signature $(2,4)$ and is isomorphic to the transcendental lattice of a generic Kummer surface associated with a smooth curve of genus two. Define

$$\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) \ : \ \langle \omega, \omega\rangle = 0, \ \langle \omega, \bar{\omega}\rangle > 0\}$$

which is a disjoint union of two copies of a bounded symmetric domain of type IV and of dimension 3. The quotient $\mathcal{D}(M)/\mathrm{O}(M)$ is birational to the moduli space of Kummer surfaces associated with a smooth curve of genus 2.

The discriminant group $A_M = M^*/M$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/4\mathbb{Z}$. Let $A_M^{(2)}$ be the 2-elementary subgroup of A_M. Then $A_M^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^5$ and the restriction of the discriminant form q_M on $A_M^{(2)}$ has a radical $\mathbb{Z}/2\mathbb{Z}$. We denote by κ_M the generator of the radical. If a is a generator of the component $A_1(2)$ of the decomposition (4.1) of M, then $\kappa_M = a/2 \bmod M$. The discriminant bilinear form b_M induces a symplectic form on $A_M^{(2)}/\langle \kappa_M\rangle$ of dimension 4 over $\mathbb{F}_2$. Thus the orthogonal group $\mathrm{O}(q_M)$ is isomorphic to $\mathrm{Sp}(4, \mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z} \cong \mathfrak{S}_6 \times \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ is generated by the involution -1_{A_M}. Put

$$\widetilde{\mathrm{O}}(M) = \{\gamma \in \mathrm{O}(M) \ : \ \gamma \mid A_M = 1\}, \quad \Gamma_M = \{\gamma \in \mathrm{O}(M) \ : \ \gamma \mid A_M = \pm 1\}.$$

Obviously we have two exact sequences

$$1 \longrightarrow \Gamma_M \longrightarrow \mathrm{O}(M) \longrightarrow \mathfrak{S}_6 \longrightarrow 1, \quad 1 \longrightarrow \widetilde{\mathrm{O}}(M) \longrightarrow \Gamma_M \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Gritsenko, Hulek [GHu], §1 gave an explicit correspondence between $\mathrm{Sp}_4(\mathbb{Z})$ and $\mathrm{SO}(3,2)_{\mathbb{Z}}$. First we remark that in the paper [GHu] they considered the lattice $M(-1/2) = U^{\oplus 2} \oplus A_1(-1)$ instead of M. However M is obtained from $M(-1/2)$ by multiplying the bilinear form by -2, and hence $\mathrm{O}(M) \cong \mathrm{O}(M(-1/2))$. They gave an isomorphism

$$\Psi : \Gamma_2 \to \mathrm{SO}(M) \cap \mathrm{O}^+(M)$$

explicitly, where $\mathrm{SO}(M)$ is the special orthogonal group and $\mathrm{O}^+(M)$ is the subgroup of $\mathrm{O}(M)$ preserving a component of $\mathcal{D}(M)$. Since M has odd rank, -1_M and $\mathrm{SO}(M)$ generate $\mathrm{O}(M)$. Since -1_M acts trivially on $\mathcal{D}(M)$ and $\mathrm{O}(M)$ interchanges two components of $\mathcal{D}(M)$, we have an isomorphism $\mathfrak{H}_2/\Gamma_2 \cong \mathcal{D}(M)/\mathrm{O}(M)$. Note that $M^*/M \cong M/2M$. By using this fact and the explicit isomorphism given in [GHu], we see that the image of the principal 2-congurence subgroup $\Gamma_2(2)$ is contained in $\widetilde{\mathrm{O}}(M) \cap \mathrm{SO}(M)$. Since $-1_M \in \Gamma_M$ represents -1_{A_M} and acts trivially on $\mathcal{D}(M)$, we have an isomorphism

$$\mathfrak{H}_2/\Gamma_2(2) \cong \mathcal{D}(M)/\Gamma_M.$$

Now we conclude that Igusa quartic $\mathcal{I}$ is isomorphic to the Baily-Borel compactification $\overline{\mathcal{D}(M)/\Gamma_M}$ of the quotient $\mathcal{D}(M)/\Gamma_M$.

Next we study boundaries and Heegner divisors on $\overline{\mathcal{D}(M)/\Gamma_M}$.

4.1. **Lemma.** *The discriminant group* $A_M/\{\pm 1_{A_M}\}$ *consists of the following* 48 *vectors*:

Type (00) : $\alpha = 0,\ \#\alpha = 1$;

Type (0) : $\alpha \neq 0,\ q_M(\alpha) = 0,\ \#\alpha = 15,$

Type (1) : $q_M(\alpha) = 1,\ \alpha \neq \kappa_M,\ \#\alpha = 15,$

Type (10) : $\alpha = \kappa_M,\ \#\alpha = 1,$

Type $(3/4)$: $q_M(\alpha) = 3/4,\ \#\alpha = 6,$

Type $(7/4)$: $q_M(\alpha) = 7/4,\ \#\alpha = 10.$

Proof. The assertion follows from a direct calculation. We remark that the involution -1_{A_M} acts trivially on the 2-elementary subgroup $A_M^{(2)}$ of A_M. In particular -1_{A_M} fixes all vectors of type $(00), (0), (1), (10)$, and

$$-1_{A_M}(\alpha) = \alpha + \kappa_M$$

if α is of type $(3/4), (7/4)$. □

4.2. **Boundary components.** We call a subgroup T of M an *isotoropic sublattice* if the symmetric bilinear form vanishes on T. Since M has the signature $(2,4)$, the rank of an isotoropic sublattice is at most 2. Similary we define an *isotoropic subspace* of A_M as a subgroup on which the discriminant quadratic form q_M vanishes. The dimension of an isotoropic subspace is also at most 2.

It is known that $\overline{\mathcal{D}(M)/\Gamma_M}(\cong \overline{\mathfrak{H}_2/\Gamma_2(2)})$ has fifteen 0-dimensional boundary components and fifteen 1-dimensional boundary components. 0-dimensional (resp. 1-dimensional) boundary components bijectively correspond to primitive isotoropic sublattices of rank 1 (resp. of rank 2) in M modulo Γ_M. A primitive isotoropic sublattice of rank 1 (resp. isotoropic sublattice of rank 2) in M determines a non-zero isotoropic vector (resp. 1-dimensional isotoropic subspace) in A_M. For example, if $\langle e_1, e_2\rangle$ is a primitive isotoropic sublattice of M generated by e_1, e_2, then $\langle e_1/2 \bmod M,\ e_2/2 \bmod M\rangle$ is an isotoropic subspace in A_M.

4.3. **Lemma.** *The* 0*-dimensional* (*resp.* 1*-dimensional*) *boundary components correspond to non-zero isotoropic vectors* (*resp.* 1*-dimensional isotoropic subspaces*) *in* A_M.

Proof. Since Γ_M acts trivially on isotoropic vectors in A_M, it suffices to see that there exists exactly 15 non-zero isotoropic vectors and 15 isotoropic subspaces in A_M. The first assertion follows from Lemma 4.1. Moreover we see that for each non-zero isotoropic vector $\alpha \in A_M$, there are 7 non-zero isotropic vectors (including α) perpendicular to α. This implies that there are three isotoropic subspaces containing α. Since there are 15 non-zero isotoropic vectors and each 1-dimensional isotoropic subspace contains three non-zero isotoropic vectors, the number of 1-dimensional isotoropic subspaces is $(15 \times 3)/3 = 15$. □

4.4. **Remark.** The incidence relation between 15 0-dimensional boundary components and 15 1-dimensional boundary components is called $(15)_3$-*configuration* because each 0-dimensional boundary component is contained in exactly three 1-dimensional boundary components and each 1-dimensional boundary component contains exactly three 0-dimensional boundary componets (e.g. see [vG]).

4.5. **Heegner divisors.** Next we define Heegner divisors on $\mathcal{D}(M)$ as those on $\mathcal{D}(L)$. Let $r \in M^*$ with $r^2 < 0$. Denote by $r^\perp$ the hyperplane in $\mathcal{D}(M)$ orthogonal to r. Fix a vector $\alpha \in A_M$ with $q_M(\alpha) \neq 0$, $\alpha \neq \kappa_M$ and a negative rational number n. We define a *Heegner divisor* $\mathcal{H}(M)_{\alpha,n}$ by

$$\mathcal{H}(M)_{\alpha,n} = \sum_r r^\perp$$

where r runs through all vectors r in M^* satisfying $r \bmod M = \alpha$ and $\langle r, r\rangle = n$. For simplicity we denote by $\mathcal{H}(M)_\alpha$ the Heegner divisor $\mathcal{H}(M)_{\alpha,n}$ for $n = -1, -5/4$ or $-1/4$ according to $q_M(\alpha) = 1, 3/4$ or $7/4$ respectively. We also denote by $\mathcal{H}(M)_1, \mathcal{H}(M)_{3/4}$ or $\mathcal{H}(M)_{7/4}$ the union of all $\mathcal{H}(M)_\alpha$ where α runs through all vectors of type $(1), (3/4)$ or $(7/4)$ respectively. The image of a Heegner divisor in $\mathcal{D}(M)/\Gamma_M$ is also called a *Heegner divisor*.

In Gritsenko, Hulek [GHu], Lemma 3.2, they proved that any two vectors in $M(1/2)^*$ with the same norm and the same image in $A_{M(1/2)}$ are conjugate under the action of $\mathrm{O}(M(1/2))$. It follows that all (-1)-vectors $r \in M^*$ with $r \bmod M$ being of type (1) are conjugate under the action of $\mathrm{O}(M)$. The same statement holds for $(-5/4)$- or $(-1/4)$-vectors $r \in M^*$ with $r \bmod M$ being of type $(3/4)$ or $(7/4)$ respectively. Therefore $\mathcal{H}(M)_1/\Gamma_M, \mathcal{H}(M)_{3/4}/\Gamma_M$ or $\mathcal{H}(M)_{7/4}/\Gamma_M$ has exactly fifteen, six or ten irreducible components $\mathcal{H}(M)_\alpha/\Gamma_M$ where $\alpha \in A_M/\{\pm 1\}$ is of type $(1), (3/4)$ or $(7/4)$ respectively (see Lemma 4.1).

4.6. **Remark.** In the theory of moduli of abelian surfaces, Heegner divisors are called *Humbert surfaces* (e.g. [vG], [GHu]). Let us compare Heegner divisors and Humbert surfaces. Recall that $M = U(2)^{\oplus 2} \oplus A_1(2)$ is obtained from $M(1/2) = U^{\oplus 2} \oplus A_1$ by multiplying the bilinear form by 2. In the notation as in [GHu], the *Humbert surface* H_Δ *of the discriminant* Δ is the image of the Heegner divisor $\mathcal{H}_{-\Delta/2}$ on $\mathcal{D}(M(1/2))$ because they consider the lattice of signature $(3, 2)$ and hence we should take the opposite sign (see the definition on page 476 in [GHu]). The Heegner divisor $\mathcal{H}_{-\Delta/2}$ on $\mathcal{D}(M(1/2))$ corresponds to the Heegner divisor $\mathcal{H}_{-\Delta/4}$ on $\mathcal{D}(M)$. Thus the closure of $\mathcal{H}(M)_{7/4}/\Gamma_M, \mathcal{H}(M)_1/\Gamma_M$ or $\mathcal{H}(M)_{3/4}/\Gamma_M$ in Baily-Borel compactification of $\mathcal{D}(M)/\Gamma_M$ is equal to the Humbert surface H_1, H_4 or H_5 given in [GHu], [vG], respectively.

5. The Weil representation

In this section we recall the Weil representation associated with the lattice $N = U(2) \oplus U(2) \oplus A_1 \oplus A_1$ given in §3, and calculate its character. The following Table 1 means that for each vector $u \in A_N$ of given type, m_j is the number of vectors $v \in A_N$ of given type with $\langle u, v\rangle = j/2$.

u	00	00	00	00	00	00	0	0	0	0	0	0	1	1	1	1	1	1
v	00	0	1	10	3/2	1/2	00	0	1	10	3/2	1/2	00	0	1	10	3/2	1/2
m_0	1	15	15	1	20	12	1	7	7	1	12	4	1	7	7	1	8	8
m_1	0	0	0	0	0	0	0	8	8	0	8	8	0	8	8	0	12	4
u	10	10	10	10	10	10	3/2	3/2	3/2	3/2	3/2	3/2	1/2	1/2	1/2	1/2	1/2	1/2
v	00	0	1	10	3/2	1/2	00	0	1	10	3/2	1/2	00	0	1	10	3/2	1/2
m_0	1	15	15	1	0	0	1	9	6	0	10	6	1	5	10	0	10	6
m_1	0	0	0	0	20	12	0	6	9	1	10	6	0	10	5	1	10	6

Table 1.

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then S and T generate $\mathrm{SL}(2, \mathbb{Z})$. Let ρ be the *Weil representation* of $\mathrm{SL}(2, \mathbb{Z})$ on the group ring $\mathbb{C}[A_N]$ defined by

$$\rho(T)(e_\alpha) = e^{\pi\sqrt{-1}q_N(\alpha)}e_\alpha, \quad \rho(S)(e_\alpha) = \frac{\sqrt{-1}}{8}\sum_{\beta\in A_N} e^{-2\pi\sqrt{-1}b_N(\beta,\alpha)}e_\beta. \tag{5.1}$$

By definition and Table 1, we see that $\rho(S^2)(e_\alpha) = -e_\alpha$. The action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{C}[A_N]$ factorizes to the one of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$. The conjugacy classes of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$ consist of $\pm E, \pm S, \pm T, \pm T^2, ST, (ST)^2$. Let χ_i $(1 \leq i \leq 10)$ be the characters of irreducible representations of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$. One can easily compute the character table of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$ by using GAP [GAP]. For the convenience of the reader we give the character table (Table 2) of $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$.

5.1. **Lemma.** *Let χ be the character of the Weil representation of $SL(2, \mathbb{Z}/4\mathbb{Z})$ on $\mathbb{C}[A_N]$. Let*

$$\chi = \sum_{i=1}^{10} m_i\chi_i$$

be the decomposition of χ into irreducible characters. Then

$$\chi = \chi_3 + 5\chi_4 + 5\chi_6 + 6\chi_9 + 10\chi_{10}.$$

Proof. By definition (5.1) and Table 1, we see that $\mathrm{trace}(E) = 2^6$, $\mathrm{trace}(-E) = -2^6$, $\mathrm{trace}(S) = 0$, $\mathrm{trace}(-S) = 0$, $\mathrm{trace}(T) = -8\sqrt{-1}$, $\mathrm{trace}(-T) = 8\sqrt{-1}$, $\mathrm{trace}(T^2) = 0$, $\mathrm{trace}(-T^2) = 0$, $\mathrm{trace}(ST) = -1$ and $\mathrm{trace}((ST)^2) = 1$. The assertion now follows from the Table 2. □

	E	$-E$	S	$-S$	T	$-T$	T^2	$-T^2$	ST	$(ST)^2$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	-1	1	1	1	1
χ_3	1	-1	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	-1	1	-1	1
χ_4	1	-1	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	-1	1	-1	1
χ_5	2	2	0	0	0	0	2	2	-1	-1
χ_6	2	-2	0	0	0	0	-2	2	1	-1
χ_7	3	3	1	1	-1	-1	-1	-1	0	0
χ_8	3	3	-1	-1	1	1	-1	-1	0	0
χ_9	3	-3	$-\sqrt{-1}$	$\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	1	-1	0	0
χ_{10}	3	-3	$\sqrt{-1}$	$-\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	1	-1	0	0

TABLE 2.

5.2. **Definition.** Let W (resp. W_0) be the subspace in $\mathbb{C}[A_N]$ on which the character of $\mathrm{SL}(2,\mathbb{Z})$ is given by $5\chi_4$ (resp. χ_3). Note that the action of $\mathrm{O}(q_N)$ on $\mathbb{C}[A_N]$ commutes with the action of $\mathrm{SL}(2,\mathbb{Z})$. Therefore $\mathrm{O}(q_N)$ acts on W and W_0. In the section 7 we will construct a 5-dimensional space of automorphic forms on $\mathcal{D}(N)$ associated with W (For W_0, see Remark 7.3).

5.3. **Definition.** Let I be a 2-dimensional isotropic subspace of A_N with respect to q_N. Note that I is a maximal isotoropic subspace. Let V be the subspace of A_N generated by I and κ_N. Take a vector $\alpha_0 \in A_N$ satisfying $q_N(\alpha_0) = 3/2$ and $b_N(\alpha_0, c) = 0$ for any $c \in I$. Note that α_0 is unique modulo V because $I^{\perp}/V = \mathbb{F}_2$. Define

$$M_+ = \{\alpha_0 + c : c \in I\}, \quad M_- = \{\alpha_0 + c + \kappa_N : c \in I\},$$

and

$$\theta_V = \sum_{\beta\in M_+} e_\beta - \sum_{\beta\in M_-} e_\beta \in \mathbb{C}[A_N].$$

This definition is the same as the one given in the case of the moduli space of plane quartic curves in [K2].

5.4. **Lemma.**

(i) $\rho(S)(\theta_V) = -\sqrt{-1}\theta_V$ and $\rho(T)(\theta_V) = -\sqrt{-1}\theta_V$. *In particular* θ_V *is contained in* W.

(ii) *For* $\beta \in V$ *with* $q_N(\beta) = 1$, $t_\beta(\theta_V) = -\theta_V$ *where* t_β *is the reflection associated with* β.

Proof. (i) If $\beta \in M_\pm$, then $q_N(\beta) = 3/2$, and hence $\rho(T)(\theta_V) = -\sqrt{-1}\theta_V$. Next by definition (5.1),

$$\rho(S)(\theta_V) = \frac{\sqrt{-1}}{8}\sum_{\delta}\left(\sum_{\beta\in M_+} e^{-2\pi\sqrt{-1}\, b_N(\delta,\beta)} - \sum_{\beta\in M_-} e^{-2\pi\sqrt{-1}\, b_N(\delta,\beta)}\right) e_\delta. \tag{5.2}$$

We denote by $\frac{\sqrt{-1}}{8}\cdot c_\delta$ the coefficient of e_δ in the equation (5.2). If $\delta \in M_+$, then $b_N(\delta,\beta) = 1/2$ for $\beta \in M_+$ and $b_N(\delta,\beta) \in \mathbb{Z}/2\mathbb{Z}$ for $\beta \in M_-$. Hence $c_\delta = -2^2 - 2^2 = -2^3$. Similary if $\delta \in M_-$, then $c_\delta = 2^3$.

Now assume $\delta \notin M_{\pm}$. If $\delta \in V$, we can easily see that $c_{\delta} = 0$. Hence we assume $\delta \notin V$. First consider the case $b_N(\delta, \kappa_N) \in \mathbb{Z}/2\mathbb{Z}$. Since $V^{\perp} = V$, there exists $\gamma \in V$ such that $b_N(\gamma, \delta) \notin \mathbb{Z}/2\mathbb{Z}$. In this case $I = \delta^{\perp} \cap I \cup \{\gamma + a : a \in \delta^{\perp} \cap I\}$. This implies that

$$\sum_{\beta \in M_+} e^{-2\pi\sqrt{-1}\ b_N(\delta,\beta)} = \sum_{\beta \in M_-} e^{-2\pi\sqrt{-1}\ b_N(\delta,\beta)} = 0.$$

Finally if $b_N(\delta, \kappa_N) \notin \mathbb{Z}/2\mathbb{Z}$, then $\delta = \alpha_0 + \delta'$ and $b_N(\delta', \kappa_N) \in \mathbb{Z}/2\mathbb{Z}$. Then this case reduces to the previous case.

(ii) Let $\beta \in V$. Then $\beta = c + \kappa_N$, $c \in I$. If $c' \in I$, then $\langle \beta, \alpha_0 + c' \rangle = 1/2$. Therefore the reflection t_{β} defined by the equation (3.2) interchanges M_+ and M_- and hence the assertion follows. □

5.5. **Lemma.** *There exist exactly fifteen* 2*-dimensional isotropic subspaces in* A_N*.*

Proof. Recall that each non-zero isotropic vector $\alpha \in A_N$, there exist exactly 7 non-zero isotropic vectors (including α) perpendicular to α (see Table 1). It follows that there are three maximal totally isotropic subspaces containing α. Since the number of non-zero isotropic vectors is 15, the number of maximal totally isotropic subspace is $15 \times 3/3 = 15$. □

Thus we have 15 vectors θ_V in W.

5.6. **Lemma.** *As a* $O(q_N)(\cong \mathfrak{S}_6 \times \mathbb{Z}/2\mathbb{Z})$ *module,* W *is irreducible.*

Proof. It is well known that there are no irreducible representations of $\mathfrak{S}_6$ of degree $2, 3, 4$. If W' is an irreducible representation of $\mathfrak{S}_6$ and dim $W' \geq 2$, then dim $W' \geq 5$. Hence it suffices to see that there are no 1-dimensional invariant subspaces under the action of $\mathfrak{S}_6$. Assume that W is a direct sum of 1-dimensional representations. A direct calculation shows that there exist five linearly independent vectors $\theta_{V_1}, \ldots, \theta_{V_5}$ where V_i $(i = 1, \ldots, 5)$ is a subspace of A_N generated by a 2-dimensional isotropic subspace and κ_N. It follows from Lemma 5.4, (ii) that W is a direct sum of alternating representations. In particular any 1-dimensional subspace is invariant under the action of $\mathfrak{S}_6$. However any vector θ_V as above is not invariant under the action of $\mathfrak{S}_6$. This is a contradiction. □

In the section 7 we will construct a 5-dimensional space of automorphic forms on $\mathcal{D}(N)$ associated with W.

5.7. **Correction.** The proof of Lemma 9 in [K3] is not complete. We should add a sentence "There exist five lineraly independent vectors $\nu_{\alpha_0^1}, \ldots, \nu_{\alpha_0^5}$." Then the same proof as in Lemma 5.6 holds in this case, too.

6. BORCHERDS PRODUCTS

Borcherds products are automorphic forms on $\mathcal{D}(N)$ whose zeros and poles lie on Heegner divisors. First of all we recall the definition of automorphic forms. Define

$$\widetilde{\mathcal{D}}(N) = \{\omega \in N \otimes \mathbf{C} \ : \ \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0\}.$$

Then the canonical map $\widetilde{\mathcal{D}}(N) \to \mathcal{D}(N)$ is a $\mathbf{C}^*$-bundle. Let Γ be a subgroup of $\mathrm{O}(N)$ of finite index. A holomorphic function

$$\Phi : \widetilde{\mathcal{D}}(N) \to \mathbf{C}$$

is called a holomorphic automorphic form of weight k with respect to Γ on $\mathcal{D}(N)$ if Φ is homogeneous of degree $-k$, that is, $\Phi(c\cdot\omega) = c^{-k}\Phi(\omega)$ for $c \in \mathbf{C}^*$, and is invariant under Γ.

Let γ be a representation of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{C}[A_N]$. A *vector-valued modular form of weight k and of type γ* is a holomorphic map

$$f = \{f_\alpha\}_{\alpha\in A_N} : H^+ \to \mathbb{C}[A_N]$$

satisfying

$$f(A\tau) = (c\tau+d)^k\gamma(A)f(\tau)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. We assume that f is holomorphic at cusps.

In this section, we will show that there exists a holomorphic automorphic form of weight 4, 10, 30 or 48 whose zero divisor is the Heegner divisor $\mathcal{H}(N)_{\kappa_N}$, $\mathcal{H}(N)_{3/2}$, $\mathcal{H}(N)_1$ or $\mathcal{H}(N)_{1/2}$ respectively. To show the existence of such Borcherds products, we introduce the *obstraction space* consisting of all vector-valued modular forms $\{f_\alpha\}_{\alpha\in A_N}$ of weight $3 = (\mathrm{rank}(N)/2)$ with respect to the dual representation ρ^* of the Weil representation ρ given in (5.1):

$$\rho^*(T)(e_\alpha) = e^{-\pi\sqrt{-1}\langle\alpha,\alpha\rangle}e_\alpha, \quad \rho^*(S)(e_\alpha) = -\frac{\sqrt{-1}}{8}\sum_{\beta\in A_N} e^{2\pi\sqrt{-1}\langle\beta,\alpha\rangle}e_\beta. \tag{6.1}$$

In other words,

$$f_\alpha(\tau+1) = e^{-\pi\sqrt{-1}\,\langle\alpha,\alpha\rangle}f_\alpha(\tau), \quad f_\alpha(-1/\tau) = -\frac{\sqrt{-1}\tau^3}{8}\sum_\beta e^{2\pi\sqrt{-1}\,\langle\alpha,\beta\rangle}f_\beta(\tau). \tag{6.2}$$

We will apply the next theorem to show the existence of such Borcherds products.

6.1. **Theorem.** (Borcherds [B2], Freitag [F], Theorem 5.2) *A linear combination*

$$\sum_{\alpha\in A_N,\, n<0} c_{\alpha,n}\mathcal{H}(N)_{\alpha,n} \quad (c_{\alpha,n}\in\mathbb{Z})$$

of Heegner divisors is the divisor of an automorphic form on $\mathcal{D}(N)$ of weight k if for every cusp form

$$f = \{f_\alpha(\tau)\}_{\alpha\in A_N}, \quad f_\alpha(\tau) = \sum_{n\in\mathbb{Q}} a_{\alpha,n}e^{2\pi\sqrt{-1}n\tau}$$

in the obstruction space, the relation

$$\sum_{\alpha\in A_N,\, n<0} a_{\alpha,-n/2}c_{\alpha,n} = 0$$

holds. In this case the weight k is given by

$$k = \sum_{\alpha\in A_N,\, n\in\mathbb{Z}} b_{\alpha,n/2}c_{\alpha,-n}$$

where $b_{\alpha,n}$ are the Fourier coefficients of the Eisenstein series with the constant term $b_{0,0} = -1/2$ and $b_{\alpha,0} = 0$ for $\alpha \neq 0$.

In the following we study the divisors $\sum_{\alpha\in A_N, n<0} c_{\alpha,n}\mathcal{H}(N)_{\alpha,n}$ where $c_{\alpha,n}$ depends only on the type of α. Recall that there are 1, 15, 15, 1, 20 and 12 elements in A_N of types (00), (0), (1), (10), $(3/2)$ and $(1/2)$, respectively (see Lemma 3.2). We consider vector valued modular forms

$$f_{00},\ f_0,\ f_1,\ f_{10},\ f_{3/2},\ f_{1/2} \tag{6.3}$$

where each f_t is the sum of the f_α as α runs through the elements of A_N of type t. Then we obtain a 6-dimensional representation

$$\rho^* : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(V)$$

induced by (6.1) where V is a 6-dimensional vector space (for simplicity, we use the same notation ρ^*). Then a direct calculation shows that ρ^* is given by

$$\rho^*(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} \end{pmatrix}. \tag{6.4}$$

$$\rho^*(S) = \frac{-\sqrt{-1}}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 15 & -1 & -1 & 15 & 3 & -5 \\ 15 & -1 & -1 & 15 & -3 & 5 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 20 & 4 & -4 & -20 & 0 & 0 \\ 12 & -4 & 4 & -12 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

6.2. **Lemma.** *The dimension of the space of modular forms of weight* 3 *and of type* ρ^* *is* 2. *The dimension of the space of Eisenstein forms of weight* 3 *and of type* ρ^* *is also* 2. *In particular there are no non-zero cusp forms in the obstraction space.*

Proof. The dimension of the space of modular forms of weight 3 and of type ρ^* is given by

$$d + dk/12 - \alpha(e^{\pi\sqrt{-1}k/2}\rho^*(S)) - \alpha((e^{\pi\sqrt{-1}k/3}\rho^*(ST))^{-1}) - \alpha(\rho^*(T))$$

([B2], section 4, [F], Proposition 2.1). Here $k=3$ is the weight,

$$d = \dim\{x \in V : \rho^*(-E)x = (-1)^k x\}$$

and

$$\alpha(A) = \sum_{\lambda} \alpha$$

where λ runs through all eigenvalues of A and $\lambda = e^{2\pi\sqrt{-1}\alpha}$, $0 \le \alpha < 1$. Since $\rho^*(S^2) = \rho^*(-E)$ and $\rho^*(S^2)(e_\alpha) = -e_\alpha$ for any α, we have $d = 6$. A direct calculation shows that $\alpha(e^{\pi\sqrt{-1}k/2}\rho^*(S)) = 3/2$,

$\alpha((e^{\pi\sqrt{-1}k/3}\rho^*(ST))^{-1}) = 2$ and $\alpha(\rho^*(T)) = 2$. Thus we have proved the first assertion. It follows from [F], Remark 2.2 that the space of Eisenstein forms of weight 3 and of type ρ^* is isomorphic to

$$\{x \in V \ : \ \rho^*(T)(x) = x, \ \rho^*(-E) = (-1)^k x\}$$

which has dimension 2. Therefore the second and hence the third assertions hold. □

We need Fourier coefficients of Eisenstein series of weight 3 with respect to ρ^*. Since ρ^* is trivial on the principal congruence subgroup of level 4, these Eisenstein series are linear combiantions of the standard ones. For $(a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, let $G_k^{(a_1,a_2)}(\tau, N)$ be the Eisenstein series of weight k and level N corresponding to (a_1, a_2) (e.g. see [F]). Then

$$(c\tau + d)^{-k} G_k^{(a_1,a_2)}((a\tau + b)/(c\tau + d), N) = G_k^{(a_1,a_2)A}(\tau, N) \tag{6.6}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Note that

$$G_k^{(-a_1,-a_2)}(\tau, N) = (-1)^k G_k^{(a_1,a_2)}(\tau, N).$$

Put $E_1 = G_3^{(0,1)}(\tau, 4)$, $E_2 = G_3^{(1,0)}(\tau, 4)$, $E_3 = G_3^{(1,1)}(\tau, 4)$, $E_4 = G_3^{(1,2)}(\tau, 4)$, $E_5 = G_3^{(1,3)}(\tau, 4)$, $E_6 = G_3^{(2,1)}(\tau, 4)$. It follows from the equation (6.6) that the actions of T and S on these forms are as follows: T fixes E_1 and sends

$$E_2 \to E_3 \to E_4 \to E_5 \to E_2, \quad E_6 \to -E_6,$$

and S sends, up to τ^3,

$$E_1 \to E_2 \to -E_1, \quad E_3 \to E_5 \to -E_3, \quad E_4 \to -E_6 \to -E_4.$$

An elementary calculation shows that Eisenstein forms of weight 3 and of type ρ^* are given by

$$\begin{aligned}
f_{00} &= aE_1 + \frac{i(a+b)}{8}(E_2 + E_3 + E_4 + E_5), \\
f_0 &= bE_1 + \frac{i(15a-b)}{8}(E_2 + E_3 + E_4 + E_5), \\
f_1 &= \frac{i(15a-b)}{8}(E_2 - E_3 + E_4 - E_5) + bE_6, \\
f_{10} &= \frac{i(a+b)}{8}(E_2 - E_3 + E_4 - E_5) + aE_6, \\
f_{3/2} &= \frac{i(20a+4b)}{8}(E_2 - iE_3 - E_4 + iE_5), \\
f_{1/2} &= \frac{i(12a-4b)}{8}(E_2 + iE_3 - E_4 - iE_5),
\end{aligned}$$

where a, b are parameters. On the other hand, the Fourier series of $G_k^{(a_1,a_2)}(\tau, N)$ are known (e.g. Freitag [F], §1) . It follows that the Fourier series of E_i are given by :

$$E_1 = \frac{i(2\pi)^3}{2^8}\{-i + 4iq + \cdots\},$$

$$E_2 = \frac{i(2\pi)^3}{2^7}\{q^{1/4} + 4q^{1/2} + 8q^{3/4} + 16q + \cdots\},$$

$$E_3 = \frac{i(2\pi)^3}{2^7}\{iq^{1/4} - 4q^{1/2} - 8iq^{3/4} + 16q + \cdots\},$$

$$E_4 = \frac{i(2\pi)^3}{2^7}\{-q^{1/4} + 4q^{1/2} - 8q^{3/4} + 16q + \cdots\},$$

$$E_5 = \frac{i(2\pi)^3}{2^7}\{-iq^{1/4} - 4q^{1/2} + 8iq^{3/4} + 16q + \cdots\},$$

$$E_6 = \frac{i(2\pi)^3}{2^7}\{2iq^{1/2} + 0 \cdot q + \cdots\}.$$

Put $a = -\frac{2^7}{(2\pi)^3}$ and $b = 0$. Then we have

$$f_{00} = -1/2 + 10q + \cdots, \quad f_0 = 120q + \cdots, \quad f_1 = 30q^{1/2} + \cdots,$$

$$f_{10} = 4q^{1/2} + \cdots, \quad f_{3/2} = 10q^{1/4} + \cdots, \quad f_{1/2} = 48q^{3/4} + \cdots.$$

Combining Lemma 6.2 and Theorem 6.1, we have

6.3. **Theorem.** *There exists a holomorphic automorphic form Φ_k on $\mathcal{D}(N)$ with respect to a subgroup of $O(N)$ of finite index whose weight k is 4, 10, 30 or 48, and whose zero divisor is the Heegner divisor $\mathcal{H}(N)_{\kappa_N}$, $\mathcal{H}(N)_{3/2}$, $\mathcal{H}(N)_1$ or $\mathcal{H}(N)_{1/2}$ respectively.*

7. ADDITIVE LIFTINGS

For an even lattice L of signature $(2, n)$, the *additive lifting* is a correspondence from vector-valued modular forms of weight k and type ρ to automorphic forms of weight $k + n/2 - 1$ on $\mathcal{D}(L)$ with respect to $\widetilde{\mathrm{O}}(L)$. By using additive liftings, we construct a 5-dimensional space of automorphic forms on $\mathcal{D}(N)$ with respect to $\Gamma_N = \widetilde{\mathrm{O}}(N)$.

Let W be the subspace in $\mathbb{C}[A_N]$ defined in (5.2). Recall that $\mathrm{O}(q_N)$ acts on $\mathbb{C}[A_N]$ which commutes with the action of $\mathrm{SL}(2, \mathbb{Z})$. Hence $\mathrm{O}(q_N)$ acts on W. First we consider the following special vectors in W. Let $\eta(\tau)$ be the Dedekind eta function. Then

$$\eta(\tau + 1)^{18} = -\sqrt{-1} \cdot \eta(\tau)^{18}, \quad \eta(-1/\tau)^{18} = -\sqrt{-1}\tau^9 \cdot \eta(\tau)^{18}.$$

Therefore, for $\theta \in W$, $\eta(\tau)^{18} \cdot \theta$ is a vector-valued modular form of weight 9 and of type ρ. By applying additive lifting ([B1], Theorem 14.3), we have an automorphic form F_θ on $\mathcal{D}(N)$ of weight 10 associated to $\eta(\tau)^{18}\theta$. Let $A_k(\mathcal{D}(N), \Gamma_N)$ be the space of automorphic forms of $\mathcal{D}(N)$ of weight k with respect to Γ_N. Recall that $\mathrm{O}(q_N)$ acts on W, and also acts naturally on $\mathcal{D}(N)$. Additive lifting is an $\mathrm{O}(q_N)$-equivariant map

$$W \to A_{10}(\mathcal{D}(N), \Gamma_N) \tag{7.1}$$

by sending θ to F_θ. We denote by $\widetilde{W}$ the image of the map (7.1).

7.1. **Lemma.** *$\widetilde{W}$ is a 5-dimensional space on which O(q_N) acts irreducibly.*

Proof. It suffices to see that the map (7.1) is non-zero. Then the assertion follows from the Schur's lemma. We use Theorem 14.3 in [B1]. We fix an orthogonal decomposition $N = U(2) \oplus U(2) \oplus A_1 \oplus A_1$. Let e_1, f_1 be a basis of the first factor $U(2)$ with $e_1^2 = f_1^2 = 0, \langle e_1, f_1 \rangle = 2$, and let e_2, f_2 be a basis of the second $U(2)$ with $e_2^2 = f_2^2 = 0, \langle e_2, f_2 \rangle = 2$. Let a_1 be a basis of the third factor A_1 and a_2 a basis of the fourth factor. Put $z = e_1$ and $z' = f_1/2$ and let $K = z^{\perp}/\mathbb{Z}z = U(2) \oplus A_1 \oplus A_1 \subset N$. Obviously $A_N = U(2)^*/U(2) \oplus K^*/K$. For simplicity, we denote by $\overline{x}$ for x mod $N \in A_N$. We consider the Fourier expansion around z. Since

$$\eta(\tau)^{18} = q^{3/4} + \cdots ,$$

for $\theta = (c_\alpha) \in W$, the initial term of

$$\eta(\tau)^{18}\theta = \sum_{\alpha \in A_N} e_\alpha \sum_{n \in \mathbb{Q}} c_\alpha(n) e^{2\pi\sqrt{-1}n\tau}$$

is

$$\sum_{\alpha \in A_N,\ q_N(\alpha)=3/2} e_\alpha c_\alpha q^{3/4}.$$

Now we consider a special vector $\lambda = e_2/2 + f_2 + a_1/2$. Since $\langle \lambda, \lambda \rangle = 3/2 > 0$, λ has positive inner products with all elements in the interior of the Weyl chamber. Also note that $\overline{\lambda}$ is of type $(3/2)$. We choose $\theta = (c_\alpha) \in W$ satisfying $c_{\overline{\lambda}} \neq 0$ and $c_{\overline{e_1/2+\lambda}} = 0$ (for example, we take the isotoropic subspace I generated by $\overline{f_1/2}$ and $\overline{e_2/2}$, and consider V generated by I and $\kappa_N = \overline{a_1/2 + a_2/2}$. Then the support of θ_V is

$$\{\ \overline{(f_1 + a_i)/2}, \quad \overline{(e_2 + a_i)/2}, \quad \overline{(f_1 + e_2 + a_i)/2}\ \}_{i=1,2},$$

and hence θ_V satisfies the condition). Now it follows from [B1], Theorem 14.3 that the Fourier coefficient of $e^{2\pi\sqrt{-1}\langle \lambda, Z\rangle}$ in the lifting F_θ of $\eta(\tau)^{18}\theta$ is equal to

$$c_{\overline{\lambda}}(\lambda^2/2) \cdot e^{2\pi\sqrt{-1}\langle \lambda, z'\rangle} + c_{\overline{e_1/2+\lambda}}((e_1/2+\lambda)^2/2) \cdot e^{2\pi\sqrt{-1}\langle e_1/2+\lambda, z'\rangle} = c_{\overline{\lambda}}(3/4) = c_{\overline{\lambda}} \neq 0.$$

Hence the lifting of $\eta(\tau)^{18}\theta$ is non-zero. □

7.2. **Theorem.** *Let $\theta_V \in W$ be as in Lemma 5.4. Let F_V be the additive lifting of $\eta(\tau)^{18} \cdot \theta_V$. Then F_V is an automorphic form on $\mathcal{D}(N)$ of weight 10 with respect to Γ_N. Moreover F_V vanishes exactly along*

$$\sum_{\alpha \in V,\ q_N(\alpha)=1} \mathcal{H}(N)_\alpha$$

with multiplicity one.

Proof. Let $\alpha \in V$ with $q_N(\alpha) = 1$. Recall that the reflection t_α is represented by a reflection s_r acting on $\mathcal{D}(N)$ where $r \in N$ with $r^2 = -4$ and $r/2$ mod $N = \alpha$ (see the equations (3.1), (3.2)). It now follows from Lemma 5.4 and the O(q_N)-equivariance of the additive lifting (7.1) that F_V vanishes along $\mathcal{H}(N)_\alpha$ where $\alpha \in V$ with $q_N(\alpha) = 1$. Therefore the product of fifteen F_V has weight 150 and vanishes along Heegner divisors $\mathcal{H}(N)_\alpha$ ($\alpha \in V, q_N(\alpha) = 1, \alpha \neq \kappa_N$) with at least multiplicity 3 and along Heegner divisor $\mathcal{H}(N)_{\kappa_N}$ with at least multiplicity 15.

On the other hand, consider the automorphic forms Φ_4, Φ_{30} of weight 4, 30 in Theorem 6.3. Then $\Phi_4^{15} \cdot \Phi_{30}^3$ has weight 150 and vanishes along Heegner divisors $\mathcal{H}(N)_\alpha$ ($\alpha \in V, q_N(\alpha) = 1, \alpha \neq \kappa_N$) with exactly multiplicity 3 and along the Heegner divisor $\mathcal{H}(N)_{\kappa_N}$ with exactly multiplicity 15. Then the ratio

$$\prod_V F_V / (\Phi_4^{15} \cdot \Phi_{30}^3)$$

is a holomorphic automorphic form of weight 0, and hence it is constant by Koecher principle. □

7.3. **Remark.** Let W_0 be the 1-dimensional subspace in $\mathbb{C}[A_N]$ on which the character of $\mathrm{SL}(2, \mathbb{Z})$ is given by χ_3 (Lemma 5.1). Let θ_0 be a generator of W_0. Then by definition,

$$\rho(S)(\theta_0) = \sqrt{-1}\theta_0, \quad \rho(T)(\theta_0) = \sqrt{-1}\theta_0.$$

Hence $\eta(\tau)^6 \cdot \theta_0$ is a vector-valued modular form of weight 3 and of type ρ. Moreover we see that $t_{\kappa_V}(\theta_0) = -\theta_0$ where t_{κ_V} is the reflection associated with κ_V. Similar argument as above shows that the additive lifting F_0 of $\eta(\tau)^6 \cdot \theta_0$ is an automorphic form of weight 4 and vanishes along $\mathcal{H}(N)_{\kappa_N}$. Moreover by considering the ratio F_0/Φ_4, the *additive lifting* F_0 coincides with the *Borcherds product* Φ_4 in Theorem 6.3.

8. Automorphic forms on $\mathcal{D}(M)$

In this section we show the existence of some automorphic forms on $\mathcal{D}(M)$ by restricting the automorphic forms on $\mathcal{D}(N)$ obtained in the previous sections. First we fix an embedding of M into N as follows. Fix a decomposition

$$N = U(2) \oplus U(2) \oplus A_1 \oplus A_1.$$

Let a_1 (resp. a_2) be a basis of the first (resp. second) component A_1 in the above decomposition. Then we consider M as a sublattice of N generated by $U(2) \oplus U(2)$ and $a_1 - a_2$. Note that M is the orthogonal complement of $a_1 + a_2$ and is primitive in N. This embedding induces an embedding $\mathcal{D}(M) \subset \mathcal{D}(N)$.

8.1. **Lemma.** *Any $\gamma \in \widetilde{O}(M)$ can be extended to an isometry of N acting trivially on A_N. In other words, $\widetilde{O}(M) \subset \widetilde{O}(N)$.*

Proof. This follows from Proposition 1.5.1 in Nikulin [N1]. □

8.2. **Lemma.** *The restriction of $\mathcal{H}(N)_1$, $\mathcal{H}(N)_{3/2}$ or $\mathcal{H}(N)_{1/2}$ to $\mathcal{D}(M)$ is $\mathcal{H}(M)_1$, $2\mathcal{H}(M)_{7/4}$ or $2\mathcal{H}(M)_{3/4}$ respectively.*

Proof. For $r \in N$, write $r = r_1 + \frac{m}{2}(a_1 + a_2)$ where $r_1 \in M^*$ and $m \in \mathbb{Z}$. If $r/2 \in N^*$, then $r_1/2 \in M^*$. Now assume $r^2 = -4, r/2 \in N^*$ and $r \bmod N \neq \kappa_N$. Then if $(r_1)^2 \geq 0$, then the hyperplane $r_1^\perp$ does not meet $\mathcal{D}(M)$. If $(r_1)^2 < 0$, then $(r_1)^2 = -4 + m^2 = -4, -3$. The case $m = 1$ ($(r_1/2)^2 = -3/4$) does not occur because the values of q_M are $0, 1, 3/4, 7/4 \bmod 2\mathbb{Z}$. Hence we have $(r_1)^2 = -4$ and $m = 0$. Thus the restriction of $\mathcal{H}(N)_1$ to $\mathcal{D}(M)$ is $\mathcal{H}(M)_1$. Similarly in case that $r/2 \in N^*$ and $r^2 = -2$ or $r^2 = -6$, then $m \neq 0$ because the norm of any vector in M is divided by 4. If $r^2 = -6$ and $m = 2$, then $r_1 \in M$ with $r_1^2 = -2$. This is a contradiction. Hence $(r_1)^2 = -1$ if $r^2 = -2$ and $(r_1)^2 = -5$ if $r^2 = -6$. Note that the hyperplanes $a_1^\perp$ and $a_2^\perp$ on $\mathcal{D}(N)$ cut the same hyperplane $(a_1 - a_2)^\perp$ on $\mathcal{D}(M)$. Hence the restriction of $\mathcal{H}(N)_{3/2}$ or $\mathcal{H}(N)_{1/2}$ to $\mathcal{D}(M)$ is $2\mathcal{H}(M)_{7/4}$ or $2\mathcal{H}(M)_{3/4}$ respectively. □

Recall that the 2-elementary subgroup $A_M^{(2)}$ of A_M together with q_M is a 5-dimensional quadratic space over $\mathbb{F}_2$ with the radical κ_M.

8.3. **Lemma.** (1) *Let I be a 2-dimensional isotoropic subspace of A_N and let V be the subspace generated by I and κ_N. Then the restriction of the Heegner divisor*

$$\sum_{\alpha \in V,\ q_N(\alpha)=1,\ \alpha \neq \kappa_N} \mathcal{H}(N)_\alpha$$

to $\mathcal{D}(M)$ is of the form

$$\sum_{\beta \in V_1,\ q_M(\beta)=1,\ \beta \neq \kappa_M} \mathcal{H}(M)_\beta \tag{8.1}$$

where V_1 is a 3-dimensional subspace of $A_M^{(2)}$ such that $b_M|V_1 \equiv 0$.

(2) *The Heegner divisor (8.1) contains exactly 7 lines through three 0-dimensional components in a line.*

Proof. (1) The same argument in the proof of Lemma 8.2 shows that the projections of three non-isotoropic vectors in V not equal to κ_N generate a 3-dimensional subspace $V_1 \subset A_M^{(2)}$ satisfying $b_M|V_1 \equiv 0$.

(2) Let $\beta_1, \beta_2, \beta_3$ be non-isotoropic vectors in V_1 not equal to κ_M and let I be the maximal isotoropic subspace of V_1. Since dim$V_1 = 3$ and $b_M|V_1 \equiv 0$, κ_M is contained in V_1, that is, $\beta_1 + \beta_2 + \beta_3 = \kappa_M$. For each β_i $(i = 1, 2, 3)$, there are exactly 7 non-zero isotoropic vectors perpendicular to β_i, and hence there exist exactly three 2-dimensional isotorpic subspaces I, I_i', I_i'' perpendicular to β_i. Note that I, I_i', I_i'' contains a non-zero isotoropic vector $\beta_i + \kappa_M$. Denote by ℓ, ℓ_i', ℓ_i'' the corresponding lines respectively. Then three lines meet at one point corresponding to $\beta_i + \kappa_M$ and the Heegner divisor $\mathcal{H}(M)_{\beta_i}$ contains ℓ, ℓ_i', ℓ_i''. □

Now, by restricting the automorphic forms $\Phi_{30}, \Phi_{10}, \Phi_{48}$ in Theorem 6.3 to $\mathcal{D}(M)$, we have the following:

8.4. **Corollary.** *There exists automorphic forms Ψ_{30}, Ψ_{10} or Ψ_{48} on $\mathcal{D}(M)$ of weight 30, 10 or 48 whose zero divisor on $\mathcal{D}(M)$ is $\mathcal{H}(M)_1$, $2\mathcal{H}(M)_{7/4}$ or $2\mathcal{H}(M)_{3/4}$ respectively.*

Proof. The restriction to $\mathcal{D}(M)$ is not identically zero, and hence it is an automorphic form with respect to a subgroup of $\mathrm{O}(M)$ of finite index by Lemma 8.1. Now the assertion follows from Lemmas 8.2. □

8.5. **Remark.** The restriction of Φ_4 to $\mathcal{D}(M)$ is identically zero.

Let $\alpha \in A_M$ be of type $(7/4)$ and let r be in M with $\langle r, r\rangle = -4$ and $r/4 \bmod M = \alpha$. Then the reflection s_r defined by

$$s_r(x) = x - \frac{2\langle x, r\rangle}{\langle r, r\rangle} r = x + \langle x, r/2\rangle r$$

is contained in $\mathrm{O}(M)$. Since $r/2 \bmod M = \kappa_M$, the action of s_r on A_M is equal to -1_{A_M} (see the proof of Lemma 4.1). Therefore s_r is contained in Γ_M. Moreover the set of fixed points of s_r is the hyperplane $r^\perp$. Hence the projection $\mathcal{D}(M) \to \mathcal{D}(M)/\Gamma_M$ is ramified along the Heegner divisor of type $(7/4)$ with ramification degree two. Finally let Ψ_{24} be the square root of Ψ_{48}. Thus we have proved the following Theorem.

8.6. **Theorem.** *The zero divisors of Ψ_{30}, Ψ_{10} or Ψ_{24} on $\mathcal{D}(M)/\Gamma_M$ is $\mathcal{H}(M)_1/\Gamma_M$, $\mathcal{H}(M)_{7/4}/\Gamma_M$ or $\mathcal{H}(M)_{3/4}/\Gamma_M$ respectively.*

8.7. **Remark.** The automorphic froms Ψ_{30}, Ψ_{10} or Ψ_{24} are known as Siegel modular forms (see [vG]). Ψ_{30}, Ψ_{10} or Ψ_{24} coincides with the Siegel modular form with the same Humbert surface as its zero divisor. This follows from Koecher principle. For example, Ψ_{10} coincides with the product $\prod \theta_m^2(\tau)$ of the square of even theta constants. The divisor $\mathcal{H}(M)_1/\Gamma_M$, $\mathcal{H}(M)_{7/4}/\Gamma_M$ or $\mathcal{H}(M)_{3/4}/\Gamma_M$ on $\overline{\mathcal{D}(M)/\Gamma_M}$ consists of fifteen, ten or six irreducible components, and coincides with the Humbert surface H_4, H_1 or H_5, respectively (see Remark 4.6).

Finally we consider the restriction of the 5-dimensional space $\widetilde{W}$ on $\mathcal{D}(N)$ to $\mathcal{D}(M)$. Let $(x_1 : \cdots : x_6)$ be a homogenous coordinate of $\mathbb{P}^5$. Then Igusa quartic $\mathcal{I}$ and Segre cubic $\mathcal{S}$ are given by the equations (1.1), (1.2), respectively. We note that Igusa quartic is also called Castelnuovo-Richimond quartic (see Dolgachev [D], page 478).

The symmetric group $\mathfrak{S}_6$ naturally acts on $\mathcal{I}$ and $\mathcal{S}$ as automorphisms. It is classically known that the dual variety of $\mathcal{I}$ is $\mathcal{S}$ ([Ba], Chap.V).

Recall that the boundary of Satake's compactification $\overline{\mathfrak{H}_2/\Gamma_2(2)}$ consists of fifteen 1-dimensional and 15 0-dimensional components. The fifteen 1-dimensional componets are the fifteen lines on $\mathcal{I}$ defined by

(8.2) $$(a : a : b : b : -a-b : -a-b)$$

and its permutations (see [vG], Theorem 4.1). Each line contains three 0-dimensional components. For example, the line defined by (8.2) contains three 0-dimensional components

$$(1:1:1:1:-2:-2),\ (1:1:-2:-2:1:1),\ (-2:-2:1:1:1:1).$$

The singular locus of $\mathcal{I}$ is nothing but the union of fifteen lines.

On the other hand, the fifteen differences $x_i - x_j$ $(i \neq j)$ are modular forms of weight 2 each of which defines an irreducible component of H_4 ([vG], §8). Each divisor defined by $x_i - x_j$ contains three lines meeting at one point. For example, the divisor defined by $x_1 - x_2$ contains three lines

$$(a:a:b:b:-a-b:-a-b),\ (a:a:b:-a-b:b:-a-b),\ (a:a:b:-a-b:-a-b:b)$$

which meet at $(2:2:-1:-1:-1:-1)$. Moreover the divisor defined by the following modular form of weight 6

(8.3) $$(x_i - x_j)(x_k - x_l)(x_m - x_n) \quad (\{i,j,k,l,m,n\} = \{1,\ldots,6\}).$$

contains 7 lines containing one of the three 0-dimensional components on a line. For example, $(x_1 - x_2)(x_3 - x_4)(x_5 - x_6)$ contains 7 lines passing a 0-dimensional component on the line $(a:a:b:b:-a-b:-a-b)$. Combining this with Lemma 8.3, we have the following Lemma.

8.8. **Lemma.** *There exists a bijective correspondence between the fifteen Heegner divisors given in* (8.1) *and the fifteen divisors defined by* (8.3).

Let Φ_4 be the automorphic form of weight 4 with the Heegner divisor $\mathcal{H}(N)_{\kappa_N}$ (Theorem 6.3). The ratio F_V/Φ_4 is a holomorphic automorphic form of weight 6 whose zero divisor is

$$\sum_{\alpha \in V,\ q_N(\alpha)=1, \alpha \neq \kappa_N} \mathcal{H}(N)_\alpha$$

(see Theorem 7.2).

8.9. **Theorem.** *The linear system obtained by the restriction of fifteen* F_V/Φ_4 *to* $\mathcal{D}(M)$ *coincide with the one defined by the modular forms given in* (8.3)

Proof. Note that the restriction of each F_V/Φ_4 to $\mathcal{D}(M)$ is an automorphic form of weight 6 with respect to a subgroup Γ of $\mathrm{O}(M)$ of finite index (Theorem 6.3, Theorem 7.2, Lemma 8.1). Under the above identification given in Lemma 8.8, this form and the corresponding automorphic form (8.3) have the same weight 6 and the same zero divisor (Theorem 6.3, Theorem 7.2). Hence they coinside by Koecher principle. □

Finally we discuss the geometric meaning of the linear system $\mathcal{L}$ generated by 15 cubics given in (8.3). Let $\mathbb{P}^4$ be the subspace of $\mathbb{P}^5$ defined by $\sum x_i = 0$. Consider $\mathcal{L}$ as a linear system on $\mathbb{P}^4$. Its base locus consists of the 15 lines defined by $x_i = x_j = x_k = x_l$. For $(x_i) \in \mathbb{P}^5$, if we consider x_i as an affine coordinate of $\mathbb{P}^1$, then fifteen functions given in (8.3) induce an $\mathfrak{S}_6$-equivariant isomorphism from the moduli space P_1^6 of ordered six points of $\mathbb{P}^1$ onto Segre cubic $\mathcal{S}$ (see [D], Theorem 9.4.10, [DO], page 15). This implies that the linear system $\mathcal{L}$ on $\mathbb{P}^4$ defines an $\mathfrak{S}_6$-equivariant rational map Φ from $\mathbb{P}^4$ to $\mathcal{S}$ whose general fiber is a rational curve. The proof of the following theorem is due to I. Dolgachev.

8.10. **Theorem.** *The linear system* $\mathcal{L}$ *gives an* $\mathfrak{S}_6$*-equivariant rational map* ϕ *from* $\mathcal{I}$ *to* $\mathcal{S}$ *of degree* 16.

Proof. The base locus of the rational map $\Phi : \mathbb{P}^4 \to \mathcal{S}$ consists of 15 lines through two points in the set of six points

(8.4) $$\{p_i = (x_1 : \cdots : x_6) \ : \ x_i = -5, \ x_j = 1 \ (j \neq i)\}_{i=1,\ldots,6}.$$

Note that these six points are in general position. Take a point $p \in \mathbb{P}^4$ such that 7 points $p, p_1, \ldots, p_6$ are in general position. Then there is a unique rational normal curve C of degree 4 passing through $p, p_1, \ldots, p_6$ ([GHa], p.179, p.530). Each cubic in $\mathcal{L}$ has singularities at six points p_i. Therefore if p is contained in a cubic in $\mathcal{L}$, then C is contained in this cubic. This implies that C is a fiber of the map Φ. Recall that the singular locus of $\mathcal{I}$ is the union of fifteen lines ([vG], Theorem 4.1). Since the six points $p_1, \ldots, p_6$ do not lie on Igusa quartic $\mathcal{I}$, a general C intersects $\mathcal{I}$ at $4 \times 4 = 16$ points. Hence the resrtiction ϕ of Φ to $\mathcal{I}$ has degree 16. □

Since the dual variety of Segre cubic is Igusa quartic ([Ba]), we have the following Corollary.

8.11. **Corollary.** *The rational map* $\phi : \mathcal{I} \to \mathcal{S}$ *induces a rational self-map of* $\mathcal{I}$ *of degree* 16.

8.12. **Remark.** The author [K3] gave a 5-dimensional linear system of holomorphic automorphic forms on a 3-dimensional complex ball by applying Borcherds theory of automorphic forms. This linear system gives the dual map from Segre cubic to Igusa quartic. The author does not know the geometric meaning of the space $\widetilde{W}$ of automorphic forms on $\mathcal{D}(N)$.

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