On complex H-type Lie algebras

Nathaniel Eldredge *

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Abstract

H-type Lie algebras were introduced by Kaplan as a class of real Lie algebras generalizing the familiar Heisenberg Lie algebra $\mathfrak{h}^3$. The H-type property depends on a choice of inner product on the Lie algebra $\mathfrak{g}$. Among the H-type Lie algebras are the complex Heisenberg Lie algebras $\mathfrak{h}^{2n+1}_{\mathbb{C}}$, for which the standard Euclidean inner product not only satisfies the H-type condition, but is also compatible with the complex structure, in that it is Hermitian. We show that, up to isometric isomorphism, these are the only complex Lie algebras with an inner product satisfying both conditions. In other words, the family $\mathfrak{h}^{2n+1}_{\mathbb{C}}$ comprises all of the complex H-type Lie algebras.

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1 Introduction

Since their introduction by Kaplan [8], H-type Lie algebras, and their corresponding nilpotent Lie groups, have attracted interest as a natural generalization of the classical real Heisenberg Lie algebra $\mathfrak{h}^3$ of dimension 3 and the corresponding real Heisenberg group $\mathbb{H}^3$. The Heisenberg group is a motivating example in many areas of mathematics, and in many cases, facts about the Heisenberg group carry over into the H-type setting. For instance, H-type groups carry a natural structure as sub-Riemannian manifolds, and the analysis of their sub-Laplacians has attracted considerable interest. As a sampling, we mention [1, 3, 5, 6, 7, 9].

The H-type condition for a (real) Lie algebra $\mathfrak{g}$ is dependent on a choice of inner product $\langle\cdot,\cdot\rangle$ (i.e. a positive definite, symmetric, bilinear form) on

*School of Mathematical Sciences, University of Northern Colorado, 501 20th St., Campus Box 122, Greeley, CO, 80639, USA. Email: `neldredge@unco.edu`. This work was supported by a grant from the Simons Foundation (#355659, Nathaniel Eldredge).

$\mathfrak{g}$, so it is really a property of the pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. For example, in $\mathfrak{h}^3$, the natural Euclidean inner product will do.

In this note, we focus on the complex Heisenberg (or Heisenberg–Weyl) Lie algebras $\mathfrak{h}_{\mathbb{C}}^{2n+1}$, which, when considered as real Lie algebras and equipped with their natural Euclidean inner products, are likewise H-type. But these Lie algebras also carry a complex structure, and the Euclidean inner product is Hermitian with respect to this structure, which is a natural compatibility condition. As such, analysis on the complex Heisenberg groups $\mathbb{H}_{\mathbb{C}}^{2n+1}$ can take advantage of all the tools of complex geometry, together with the many results for H-type groups mentioned above. However, the purpose of this note is to show that this harmonious relationship between these structures is essentially unique to these specific Lie algebras (and their respective Lie groups).

As an application, we refer to [4], in which we studied a property known as strong hypercontractivity for the hypoelliptic heat kernel on a stratified complex Lie group. An essential hypothesis for this result was that the heat kernel should satisfy a logarithmic Sobolev inequality. For most Lie groups, it remains an open problem to determine whether this inequality holds, but it follows from the results of [3, 6] that the inequality holds in every H-type Lie group. Thus, the strong hypercontractivity theorem proved in [4] holds in particular for any complex Lie group which, when considered as a real Lie group, is also H-type. The result of the present note implies that these Lie groups are precisely the family $\mathbb{H}_{\mathbb{C}}^{2n+1}$. As this is a relatively limited class of examples, we see this as further motivation to try to extend the logarithmic Sobolev inequality beyond the H-type case.

2 Definitions and examples

We begin by recalling the definition of an H-type Lie algebra, as formulated in [2, Definition 18.1.1]. (Kaplan's original definition [8] is equivalent, but slightly less convenient for our purposes.) Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and let $\mathfrak{v} = \mathfrak{z}^{\perp}$. For $z \in \mathfrak{z}$ and $u \in \mathfrak{v}$, define $J_z u$ as the unique element of $\mathfrak{v}$ satisfying

$$\langle J_z u, v\rangle = \langle z, [u, v]\rangle \quad \text{for all } v \in \mathfrak{v}. \tag{1}$$

It is clear that each $J_z : \mathfrak{v} \to \mathfrak{v}$ is a linear map, and moreover $z \mapsto J_z$ is linear in z.

Definition 1. We say that $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is **H-type** if the following two conditions hold:

1. $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$
2. For each $z \in \mathfrak{z}$ with $\|z\| = 1$, $J_z : \mathfrak{v} \to \mathfrak{v}$ is an isometry with respect to $\langle\cdot,\cdot\rangle$.

We observe that an H-type Lie algebra is necessarily nilpotent of step 2. A simply-connected Lie group is said to be H-type if its Lie algebra is H-type in the above sense.

Now suppose that $\mathfrak{g}$ is a complex Lie algebra, whose complex structure we denote by i. If we wish to equip $\mathfrak{g}$ with a real inner product, it is natural to demand some compatibility with the complex structure. Specifically, we would like the inner product to be **Hermitian**, i.e., for $x, y \in \mathfrak{g}$ we have $\langle ix, iy\rangle = \langle x, y\rangle$. We may then define J in terms of this inner product by (1). We observe for later use that, as a consequence of the Hermitian property of the inner product, we have for $\alpha, \beta \in \mathbb{C}$ and $u, z \in \mathfrak{g}$,

$$J_{\alpha z}(\beta u) = \alpha\bar{\beta} J_z u. \tag{2}$$

That is, $J_z u$ is complex linear in z and conjugate linear in u.

The question of interest in this note is when both of the above properties hold, motivating the following definition.

Definition 2. A **complex H-type Lie algebra** is a pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, where $\mathfrak{g}$ is a complex Lie algebra and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{g}$, such that the following two conditions hold:

- The inner product $\langle\cdot,\cdot\rangle$ is Hermitian with respect to the complex structure of $\mathfrak{g}$.
- Forgetting the complex structure on $\mathfrak{g}$, the pair $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is H-type in the sense of Definition 1.

We can likewise define a **complex H-type Lie group** as a connected and simply connected complex Lie group G equipped with a Hermitian left-invariant Riemannian metric g which, when viewed as an inner product on the Lie algebra of G, satisfies the above conditions.

Example 3. The **complex Heisenberg Lie algebra** of complex dimension $2n + 1$ is the complex Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ generated (over $\mathbb{C}$) by the basis of the $2n + 1$ vectors $\{x_1, y_1, \ldots, x_n, y_n, z\}$, with the bracket defined by $[x_k, y_k] = z$, and for $j \neq k$, $[x_j, y_k] = [x_j, z] = [y_j, z] = 0$. We may equip $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with the real inner product $\langle \cdot, \cdot \rangle$ that makes all of $x_k, ix_k, y_k, iy_k, z, iz$ orthonormal; it is clear that this inner product is Hermitian. The center $\mathfrak{z}$ of $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ is spanned (over $\mathbb{C}$) by z, so we clearly have $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. Defining J as above, it is easy to compute

$$J_z x_k = y_k \quad J_z y_k = -x_k \quad J_z i x_k = -i y_k \quad J_z i y_k = i x_k$$

so that J_z is an isometry. Moreover, every element $w \in \mathfrak{z}$ is of the form $w = \alpha z$ for some $\alpha \in \mathbb{C}$, and $\|w\| = |\alpha|$, so using (2) we see that J_w is an isometry whenever $\|w\| = 1$. Thus $(\mathfrak{h}_{\mathbb{C}}^{2n+1}, \langle \cdot, \cdot \rangle)$ is a complex H-type Lie algebra.

Of course, the complex Heisenberg Lie algebras are a very special family within the far larger class of all complex Lie algebras. Likewise, the class of H-type Lie algebras, although fairly restrictive, is still much broader than this specific family. For instance, there exist H-type Lie algebras having centers of any given real dimension [8], while the complex Heisenberg Lie algebras all have centers of real dimension 2.

Nevertheless, we shall now prove that the complex Heisenberg Lie algebras are, up to isometric isomorphism, the only complex H-type Lie algebras.

3 Main result

Theorem. *Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be a complex H-type Lie algebra as defined above. Then for some n, $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is isometrically isomorphic to $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ with its standard Hermitian inner product.*

In particular, complex H-type Lie algebras are completely classified by their dimension. We also immediately obtain the analogous classification of complex H-type Lie groups.

Proof. Suppose $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is complex H-type, and let $\mathfrak{v}$, $\mathfrak{z}$ and J be defined as above.

We recall the well-known Clifford algebra identity for H-type Lie algebras:

$$J_z J_w + J_w J_z = -2 \langle z, w \rangle I, \quad z, w \in \mathfrak{z}. \tag{3}$$

To prove this, first consider the case when $w = z$ and $\|z\| = 1$. Then for any $u, v \in \mathfrak{v}$, we have

$$\left\langle J_z^2 u, v \right\rangle = \langle z, [J_z u, v] \rangle = -\langle z, [v, J_z u] \rangle = -\langle J_z v, J_z u \rangle = -\langle v, u \rangle$$

since J_z is an isometry. So $J_z^2 = -I$. The general case follows by scaling and polarization.

We begin by showing that $\mathfrak{z}$ must have complex dimension 1. If not, then we can find $z, w \in \mathfrak{z}$ with $\|z\| = \|w\| = 1$ and $\langle z, w \rangle = \langle iz, w \rangle = 0$. Then by (3) and (2) we have

$$\begin{aligned} 0 &= -2\langle z, w \rangle I = J_z J_w + J_w J_z \\ 0 &= -2\langle iz, w \rangle I = J_{iz} J_w + J_w J_{iz} = i J_z J_w + J_w i J_z = i(J_z J_w - J_w J_z). \end{aligned}$$

Thus $J_w J_z = J_z J_w = 0$, contradicting the requirement that J_z, J_w be isometries.

Therefore, $\mathfrak{z}$ is the complex span of a single unit vector z. We recursively construct an orthonormal basis for $\mathfrak{v}$ over $\mathbb{R}$, of the form $\{x_k, ix_k, y_k, iy_k : k = 1, \ldots, n\}$. Suppose $\{x_k, ix_k, y_k, iy_k : k = 1, \ldots, m-1\}$ have been constructed and do not span $\mathfrak{v}$. Let x_m be any unit vector orthogonal to all of x_k, ix_k, y_k, iy_k for $k = 1, \ldots, m$. Then set $y_m = J_z x_m$. We have $\|y_m\| = 1$, and a few straightforward computations verify that $\{x_k, ix_k, y_k, iy_k : k = 1, \ldots, m\}$ are now orthogonal. When the process terminates, we have the desired orthonormal basis.

To compute brackets, for $j \neq k$ we have

$$\begin{aligned} \langle z, [x_k, y_k] \rangle &= \langle J_z x_k, y_k \rangle = \langle y_k, y_k \rangle = 1 \\ \langle z, [x_k, x_j] \rangle &= \langle J_z x_k, x_j \rangle = \langle y_k, x_j \rangle = 0 \\ \langle z, [y_k, y_j] \rangle &= \langle J_z y_k, y_j \rangle = \langle J_z y_k, J_z x_j \rangle = \langle y_k, x_j \rangle = 0 \\ \langle z, [x_k, y_j] \rangle &= \langle J_z x_k, y_j \rangle = \langle y_k, y_j \rangle = 0. \end{aligned}$$

Similar computations show that if z is replaced by iz, all of the above expressions vanish. Each bracket is in $\mathfrak{z}$ and hence a complex scalar multiple of z, so we have

$$[x_k, y_k] = z, \quad [x_k, x_j] = [y_k, y_j] = [x_k, y_j] = 0.$$

The corresponding brackets for ix_k, iy_k, etc, follow from the complex bilinearity of the bracket. These are precisely the same relations as for the complex Heisenberg Lie algebra $\mathfrak{h}_{\mathbb{C}}^{2n+1}$, and the basis is orthonormal, just as for the standard inner product on $\mathfrak{h}_{\mathbb{C}}^{2n+1}$. Therefore, the unique complex linear map $\mathfrak{g} \to \mathfrak{h}_{\mathbb{C}}^{2n+1}$ sending $x_1, y_1, \ldots, x_n, y_n, z \in \mathfrak{g}$ to the standard basis for $\mathfrak{h}_{\mathbb{C}}^{2n+1}$ (described in Example 3) is an isometric isomorphism of complex Lie algebras. $\square$

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