A BLOW-UP RESULT FOR DYADIC MODELS OF THE EULER EQUATIONS

IN-JEE JEONG AND DONG LI

ABSTRACT. We partially answer a question raised by Kiselev and Zlatos in [11]; in the generalized dyadic model of the Euler equation, a blow-up of $H^{1/3+\delta}$-norm occurs. We recover a few previous blow-up results for various related dyadic models as corollaries.

1. INTRODUCTION

In this paper, we consider the following infinite system of ordinary differential equations (ODEs):

$$\frac{da_j(t)}{dt} = \alpha\big(\lambda^j a_{j-1}^2(t) - \lambda^{j+1} a_j(t)a_{j+1}(t)\big)$$
$$+ \beta\big(\lambda^j a_{j-1}(t)a_j(t) - \lambda^{j+1} a_{j+1}^2(t)\big), \tag{1.1}$$

for $j \geq 0$ and with the boundary condition $a_{-1}(t) \equiv 0$. The coefficients α and β are usually taken to be nonnegative constants. We will assume $\lambda = 2$ throughout, but our results hold for arbitrary $\lambda > 1$ with proper adjustments of the parameters. The special case $\alpha = 1, \beta = 0$ is often called the KP equations, which have appeared in the literature almost simultaneously in two papers [9, 10]. The opposite extreme $\alpha = 0, \beta = 1$ first appeared in Obukhov's work [12] and was suggested as an alternative to the KP equations in [13]. Hence (1.1) can be viewed as a linear combination of these two models.

These types of infinite system of ODEs are called dyadic models of the Euler equations. For a heuristic derivation of the KP equations from the Euler equations, one can see [10] for an argument based on the wavelet expansion of a scalar function over dyadic cubes. Alternatively, consider the Euler equations in $\mathbb{R}^n$ with periodic boundary conditions and rewrite the equations in terms of the Fourier coefficients of the velocity vector field. Then one obtains an infinite system of ODEs for the evolution of Fourier coefficients which share several structural similarities with (1.1). We will return to this point after Lemma 1, from which equations (1.1) appear naturally from some constitutive relations.

To state blow-up and regularity results for dyadic models, let us first define analogues of the Sobolev norms in the space of sequences. The H^s-norm of a solution $a = (a_0, a_1, ...)$ at time t is defined by the formula

$$||a(t)||_s^2 := \sum_{j=0}^{\infty} 2^{2sj} a_j^2(t).$$

In particular, we define the energy $E(t)$ as the square of H^0-norm (or the usual l^2-norm):

$$E(t) := \sum_{j=0}^{\infty} a_j^2(t).$$

Regarding the KP equations, the following blow-up result have been proved several times (we have listed the references in a more or less chronological order):

Theorem ([9, 10, 13, 11, 1]). *For every nonzero initial data, the H^s-norm of any solution[1] becomes infinite in finite time for all $s > s_{cr} := 1/3$.*

Now for the Obukhov equations, there is the following regularity result:

Theorem ([11]). *If the initial data have finite H^s-norm for some $s > 1$, then the corresponding solution exists globally and has finite H^s-norm for all $t \geq 0$.*

In [11], Kiselev and Zlatos raised the question of whether blow-up can occur in the case $\alpha, \beta > 0$ (or more generally $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$). Corollary 6 of this paper answers this question affirmatively, at least in the case when β is small relative to α. Our proof of blow-up is quite different and seems to be simpler than the previous proofs for the KP equations ([9, 10, 13, 11]). Roughly speaking, all the previous blow-up proofs rest on the intuition that at least a fixed proportion of the energy contained in the jth component must be transferred to the higher components within a time scale of τ^{-j} for some $\tau > 0$. To achieve this, one has to make strong use of the "positivity" of the KP equations; that is, once we have $a_j(t_0) \geq 0$ for some j and t_0, then $a_j(t) \geq 0$ for all future $t > t_0$. This positivity in turn implies that there is no "backward" transfer of energy; to be more precise, if the initial data satisfy $a_{j_0+1}(0) \geq 0$, then $a_{j_0+1}(t) \geq 0$, and

$$\frac{d}{dt} E_{j_0}(t) := \frac{d}{dt} \Big(\sum_{j=0}^{j_0} a_j^2(t) \Big) = -2\lambda^{j_0+1} a_{j_0}^2 a_{j_0+1} \leq 0$$

for all $t \geq 0$. Unfortunately, this mechanism of forward energy transfer seems to break down once we have both $\alpha, \beta > 0$. The proof in [1] still makes use of positivity but it appears to be different from others; we will come back to their proof after Lemma 1.

Next, let us consider the following system of equations, where there is an extra "dissipation" term on the right hand side:

$$\frac{da_j}{dt} = \alpha\big(\lambda^j a_{j-1}^2 - \lambda^{j+1} a_j a_{j+1}\big) + \beta\big(\lambda^j a_{j-1} a_j - \lambda^{j+1} a_{j+1}^2\big) - \nu\lambda^{2\gamma j} a_j, \quad (1.2)$$

again for $j \geq 0$ and with $a_{-1}(t) \equiv 0$. Here, $\alpha = 1, \nu = 1$ can be assumed with appropriate rescaling and $\gamma > 0$ is a parameter representing the intensity of the dissipation. In the special case $\beta = 0$, these equations are often called the dyadic Navier-Stokes Equations (NSEs), and they are already studied quite extensively in the literature. In particular, the following blow-up result has been proved by Cheskidov in [4]:

---

[1]The meaning of solution here requires some clarification. It is known (cf. Proposition 2.1 of [11]) that local wellposedness holds in $C_t^0 H^s$ with $s \geq 1$. Alternatively one can work with finite energy Leray-Hopf type solutions (cf. [2, 4]) for which uniqueness is a subtle issue.

Theorem ([4]). *Consider equations (1.2) with $\alpha = 1$, $\beta = 0$, $\nu = 1$, and $\gamma < 1/3$. For every $\delta > 0$, there exists a constant $M(\delta)$ such that if the initial data satisfy $a_j(0) \geq 0$ for all $j \geq 0$ and $\|a(0)\|_\delta > M(\delta)$, then $\|a(t)\|^3_{1/3+\delta}$ is not locally integrable on $[0, \infty)$.*

In particular, any solution blows up in finite time in $H^{1/3+\delta}$-norm for every $\delta > 0$. Our proof of the main theorem recovers this $H^{1/3+\delta}$-norm blow-up in Corollary 8.

2. RESULTS AND CONJECTURES

To begin, we borrow a lemma from [11] from which the model (1.1) follows naturally. We omit the proof since it is immediate in view of the (formal) energy conservation constraint.

Lemma 1. *Assume that real-valued functions $a_j(t)$ ($j \geq 0$) satisfy a system of ODEs of the form*
$$\frac{da_j(t)}{dt} = F_j(a(t))$$
where

- *for each $j \geq 0$, the map F_j is a quadratic function of $a(t)$;*
- *F_j can involve only $a_{j-1}(t)$, $a_j(t)$, and $a_{j+1}(t)$;*
- *each term for F_j has a factor of λ^j times a constant which is independent of j, i.e.*
$$F_j = \sum_{\substack{\mu_1 = \pm 1, 0 \\ \mu_2 = \pm 1, 0}} C_{\mu_1, \mu_2} \lambda^j a_{j+\mu_1}(t) a_{j+\mu_2}(t),$$
  where C_{μ_1, μ_2} are constants independent of j;
- *and the energy $\sum a_j^2(t)$ is (formally) conserved.*

Then the system is necessarily of the form (1.1).

The Euler equations are, of course, energy conserving (for smooth solutions) and have quadratic nonlinearity. The factor λ^j was inserted so that the dyadic model would share similar functional estimates with the Euler equations. One can argue that to model 3D Euler equations, the choice $\lambda = 2^{5/2}$ is appropriate[2] (see [9]). Lastly, the fact that F_j only consists of a_{j-1} and a_{j+1} certainly does not hold in the case of the Euler equations, but certain "locality of interactions" assumptions are believed to hold in the theory of turbulence. For example one can see [7, 8].

Let us remark on the property of energy conservation. A formal calculation yields that

$$\frac{d}{dt} E(t) = 2 \sum_{j \geq 0} a_j (\lambda^j a_{j-1}^2 - \lambda^{j+1} a_j a_{j+1})$$

$$= 2 \Big(\sum_{j \geq 1} \lambda^j a_{j-1}^2 a_j \Big) - 2 \Big(\sum_{j \geq 0} \lambda^{j+1} a_j^2 a_{j+1} \Big) = 0.$$

But in the above computation, an interchange of the order of summation and differentiation must be justified, and it is sufficient to require that $\|a(t)\|_s < \infty$ for

---

[2]Roughly speaking, this is based on the estimate that (here P_{2^j} is the usual Littlewood-Paley projector adapted to the frequency block $|\xi| \sim 2^j$ and one can think of u as the velocity in Euler) $\|P_{2^j} u \cdot \nabla P_{2^j} u\|_{L^2(\mathbb{R}^d)} \lesssim 2^{(\frac{d}{2}+1)j} \|P_{2^j} u\|^2_{L^2(\mathbb{R}^d)}$. For $d = 3$, the factor is $2^{5/2}$.

$s > s_{cr} = 1/3$. However, for solutions with less regularity, this computation is no longer valid and the dissipation of energy can indeed occur[3]. In [1] it was established that for every initial condition with nonnegative components, the energy dissipates to zero as $t \to \infty$. In particular, it implies finite-time blow-up in every H^s norm for $s > 1/3$.

We are ready to state our main result.

Theorem 2 (The full model with diffusion). *Consider the equations (1.2) with parameters $\lambda = 2$, $\alpha = 1$, $\nu = 1$, and $\beta \geq 0$. For every $s > 1/3$ and $\gamma < 1/3$, there exists a value $\beta_{s,\gamma} > 0$ such that for $\beta \in [0, \beta_{s,\gamma})$, there exists a class of initial data for which the corresponding solutions blow up in finite time in H^s-norm. More precisely, for each solution $a(t)$, $\|a(t)\|_s^2$ is not locally integrable on $[0, \infty)$.*

Remark 3. As will be clear from our proof, the initial data $a(0) = (a_j(0))_{j=0}^\infty$ can even taken to be compactly supported, in the sense that for some integer $j_0 > 0$, $a_j(0) = 0$ for all $j \geq j_0$.

Remark 4. Instead of considering only nearest neighborhood interactions (i.e a_j, a_{j-1}, a_{j+1}), one can generalize the full model (1.2) to arbitrarily finitely many (or even infinitely many with sufficiently fast decay of interaction) neighborhood interactions. It is expected that our method of proof also carriers over to this case.

Remark 5. Although Theorem 2 settles the blow-up of (1.2) more or less satisfactorily, the proof itself (albeit simple) gives little information on the transfer of energy mechanism in the model. On the other hand, the previous proofs on the blow-up of KP model *do* respect the details of the dynamics and give some insight of the cascade mechanism. In light of this, it is still desirable to give a more "dynamic" proof in this flavor. After all, one of the main reasons for studying the dyadic models is to understand energy cascade and even turbulence transport.

Before the proof of Theorem 2, we state two direct corollaries which simply correspond to cases $\nu = 0$ and $\beta = 0$, respectively.

Corollary 6 (The full model with no diffusion). *Consider the equations (1.1) with parameters $\lambda = 2$, $\alpha = 1$, and $\beta \geq 0$, and fix $s > 1/3$ together with $(2^s - 2^{1-2s})/(1 + 2^{1-3s}) > \beta$. Then for every nonnegative initial data (that is, $a_j(0) \geq 0$ for all j), there is finite-time blow up in H^s-norm.*

Remark 7. Here and below (in Corollary 8), the blow-up of H^s-norm is again understood as that $\|a(t)\|_s^2$ is not locally integrable on $[0, \infty)$.

For example, when $\beta < 6/5$, every nonnegative initial data blow-up in the H^1-norm.

Corollary 8 (KP with diffusion). *Consider the equations (1.2) with $\lambda = 2$, $\alpha = 1$, $\beta = 0$, $\nu = 1$, and $\gamma < 1/3$. For every $s > 1/3$, let $\theta = \theta(s, \gamma)$ be a constant such that*

$$-\frac{4}{3} < \theta < 2(s-1),$$
$$-\frac{4}{3} < \theta < -4\gamma.$$

---

[3]This is in some sense connected to the Onsager's conjecture.

There exists a constant $C = C(s, \gamma, \theta) > 0$, such that once the initial data $(a_j(0))_{j=0}^{\infty} \in H^s$ satisfy

$$\sum_{j=0}^{\infty} 2^{j(\theta+1)} a_j(0) > C,$$

there is finite-time blow up in H^s-norm.

Proof of Theorem 2. We fix some $s > 1/3$ and assume towards contradiction that $||a(t)||_s^2$ is locally integrable[4] on $[0, \infty)$. By setting $b_j(t) := \lambda^j a_j(t)$, we simplify the equation as follows:

$$\frac{db_j}{dt} = \left(\lambda^2 b_{j-1}^2 - b_j b_{j+1}\right) + \beta\left(\lambda b_{j-1} b_j - \lambda^{-1} b_{j+1}^2\right) - \lambda^{2\gamma j} b_j. \tag{2.1}$$

Then we consider the sum

$$A(t) := \sum_{j=0}^{\infty} b_j^2 w^{-j}$$

where $w > 1$ is a constant to be optimized later. We observe that if $w^{-1} \leq \lambda^{-2} \cdot 2^{2s}$, then $A(t)$ is also integrable since $A(t) \leq ||a(t)||_s^2$ for all $t \geq 0$. Then we consider the quantities

$$\frac{d}{dt}(b_j w^{-j}) = (\lambda^2 b_{j-1}^2 w^{-j} - b_j b_{j+1} w^{-j}) + \beta(\lambda b_{j-1} b_j w^{-j} - \lambda^{-1} b_{j+1}^2 w^{-j}) - \lambda^{2\gamma j} b_j w^{-j}$$

and sum them over all $j \geq 0$. By the Cauchy-Schwartz inequality, the infinite sum appearing on the right hand side is bounded in absolute value by const $\cdot (A(t) + \sqrt{A(t)})$, and therefore the sum can be rearranged whenever $A(t)$ is finite. We therefore obtain:

$$\frac{d}{dt}(\sum_{j=0}^{\infty} b_j w^{-j}) = \lambda^2 \sum_{j=1}^{\infty} b_{j-1}^2 w^{-j} - \sum_{j=0}^{\infty} b_j b_{j+1} w^{-j} \tag{2.2}$$

$$+ \beta\lambda \sum_{j=1}^{\infty} b_{j-1} b_j w^{-j} - \beta\lambda^{-1} \sum_{j=0}^{\infty} b_{j+1}^2 w^{-j} - \sum_{j=0}^{\infty} \lambda^{2\gamma j} b_j w^{-j}.$$

We note in advance that again by the Cauchy-Schwartz inequality,

$$(\sum_{j=0}^{\infty} b_j w^{-j})^2 \leq (\sum_{j=0}^{\infty} w^{-j})(\sum_{j=0}^{\infty} b_j^2 w^{-j}) = \frac{A(t)}{1 - w^{-1}} \tag{2.3}$$

holds. Then first four terms on the right hand side of (2.2) can be estimated as follows:

$$\lambda^2 w^{-1} \sum_{j=1}^{\infty} b_{j-1}^2 w^{-j-1} - w^{1/2} \sum_{j=0}^{\infty} (b_j w^{-\frac{j}{2}})(b_{j+1} w^{-\frac{j+1}{2}})$$

$$+ \beta\lambda w^{-1/2} \sum_{j=1}^{\infty} (b_{j-1} w^{-\frac{j-1}{2}})(b_j w^{-j}) - \beta\lambda^{-1} w \sum_{j=0}^{\infty} b_{j+1}^2 w^{-(j+1)}$$

$$\geq (\lambda^2 w^{-1} - w^{1/2} - \beta\lambda w^{-1/2} - \beta\lambda^{-1} w) \cdot A(t).$$

---

[4]This in turn would imply that one can freely interchange summation and differentiation in the argument below. Alternatively, one can recast the equations into integral (in time) formulation and justify passing the limit under the integral.

Regarding the last term, we have

$$
\begin{aligned}
-\sum_{j=0}^{\infty} \lambda^{2\gamma j} b_j w^{-j} &= -\sum_{j=0}^{\infty} (b_j w^{-\frac{j}{2}})(\lambda^{2\gamma j} w^{-\frac{j}{2}}) \geq -(\sum_{j=0}^{\infty} b_j^2 w^{-j})^{\frac{1}{2}} (\sum_{j=0}^{\infty} \lambda^{4\gamma j} w^{-j})^{\frac{1}{2}} \\
&= -\frac{1}{1-\lambda^{4\gamma} w^{-1}} \cdot (\sum_{j=0}^{\infty} b_j^2 w^{-j})^{\frac{1}{2}} \\
&\geq -\frac{1}{1-\lambda^{4\gamma} w^{-1}} (\eta A(t) + \frac{1}{4\eta}),
\end{aligned}
$$

where $\eta > 0$ is a constant. Here we have assumed that $\lambda^{4\gamma} w^{-1} < 1$. Adding above two estimates together with (2.3), we conclude for some $C_2 > 0$,

$$
\frac{d}{dt} (\sum_{j=0}^{\infty} b_j w^{-j}) \geq C_1 \cdot (\sum_{j=0}^{\infty} b_j w^{-j})^2 - C_2.
$$

Therefore, if we have $C_1 > 0$ and $\sum b_j(0) w^{-j} > \sqrt{C_2/C_1}$, then $\sum b_j w^{-j}$ will not be locally integrable on $[0, \infty)$, which is a contradiction to the fact that $A(t)$ is locally integrable. To this end, we need

$$
C_1 = (1 - w^{-1}) \cdot \left(\lambda^2 w^{-1} - w^{1/2} - \beta \lambda w^{-1/2} - \beta \lambda^{-1} w - \frac{\eta}{1-\lambda^{4\gamma} w^{-1}} \right) > 0.
$$

But at the expense of choosing η sufficiently small, it is enough to have

$$
\lambda^2 w^{-1} - w^{1/2} - \beta \lambda w^{-1/2} - \beta \lambda^{-1} w > 0.
$$

Therefore, to conclude the proof, we need:

$$
\begin{aligned}
w^{-1} &\leq \lambda^{-2} 2^{2s} \\
w^{-1} &< \lambda^{-4\gamma} \\
\lambda^2 w^{-1} - w^{1/2} &> \beta(\lambda w^{-1/2} + \lambda^{-1} w), \tag{2.4}
\end{aligned}
$$

where $\lambda = 2$. Assuming for the moment that $\beta = 0$, we have $\lambda > w^{3/4}$ from the last inequality which gives restrictions $s > 1/3$, $\gamma < 1/3$. On the other hand, it is clear now that once we have $s > 1/3, \gamma < 1/3$, we can choose w in a way that for small $\beta > 0$, all the above three inequalities are satisfied. $\square$

Proof of Corollary 6. Since $\nu = 0$, the second inequality of (2.4) is not needed. One can just choose $w^{-1} = 2^{2(s-1)}$ and this gives

$$
\frac{2^s - 2^{1-2s}}{1 + 2^{1-3s}} > \beta.
$$

$\square$

Proof of Corollary 8. Since $\beta = 0$, the conditions on w^{-1} in (2.4) take the form

$$
\begin{aligned}
2^{-\frac{4}{3}} &< w^{-1} \leq 2^{2(s-1)}, \\
2^{-\frac{4}{3}} &< w^{-1} < 2^{-4\gamma}.
\end{aligned}
$$

Denoting $w^{-1} = 2^{\theta}$ then yields the result. $\square$

Let us close by presenting a few conjectures which would complement or generalize regularity and blow-up results currently known. We first explain the result of [1]: recall that we have already mentioned their dissipation of energy result. But they also proved the existence of so-called "self-similar solutions", which are natural analogues of the fixed point in the forced case. To be specific, consider the forced KP equations:

$$\frac{d}{dt}a_j(t) = \lambda^j a_{j-1}^2(t) - \lambda^{j+1} a_j(t) a_{j+1}(t), \ \ (j \geq 1)$$
$$\frac{d}{dt}a_0(t) = -\lambda a_0(t) a_1(t) + f_0,$$

where $f_0 > 0$ is a constant. Then it is immediate that there exists a unique fixed point which have finite energy. This fixed point satisfies $\bar{a}_j = \text{const} \cdot \lambda^{-j/3}$ so it has finite H^s-norm precisely for $s < 1/3$. In [5, 6], it was established that this fixed point is the unique global attractor of the dynamics.

When there is no forcing, there does not exist nontrivial fixed points. However, self-similar solutions are the correct analogues; we define a solution self-similar if for every $j \geq 0$, $a_{j+1}(t)/a_j(t)$ is constant in time. From this requirement, it is straightforward to check that the solution must have the form

$$a_j(t) = \frac{c_j}{t - t_0}$$

for some constants c_j and $t_0 > 0$ which satisfy the recurrence (for $\lambda = 2$)

$$c_j c_{j+1} = 2^{-j} c_j + c_{j-1}^2/2$$

for all $j \geq 0$ with $c_{-1} = 0$. The choice of $c_0 > 0$ uniquely determines the whole sequence and the self-similar solution, modulo the choice of $t_0 > 0$ which is independent. The hard part is to show that there exists a value of $c_0 > 0$ (which turns out to be unique) such that the self-similar solution $a(t)$ has finite energy. Then it is not hard to see that the self-similar solution satisfies $c_j \sim \text{const} \cdot \lambda^{-j/3}$. Note this power-law decay in j can already be noticed from our proof; the scale $a_j(t) \sim \lambda^{-j/3}$ roughly corresponds to the case where we have equalities in the Cauchy-Schwartz inequalities used in the proof. Now it is very desirable to show that the self-similar solutions are the global attractors of the unforced dynamics. If we believe in the convergence towards self-similar ones, it is natural to conjecture that in the KP equations, the H^s-norms remain finite for all $s < 1/3$. This finiteness of smaller Sobolev norms are partially obtained in the works [2, 3]. Also, one can revert all inequalities in our proof and try to get some a priori estimates on the solution, which look similar to some regularity results proved in [2, 3]. Finally, it is tempting to believe that such self-similar solutions also exist for our equation, at least when β is small. One can write down the recurrence relation as above but this relation is now more complicated.

Acknowledgements

References

[1] D. Barbato, F. Flandoli, and F. Morandin. Energy dissipation and self-similar solutions for an unforced inviscid dyadic model. *Trans. Amer. Math. Soc.*, 363(4):1925–1946, 2011.

[2] David Barbato and Francesco Morandin. Positive and non-positive solutions for an inviscid dyadic model: well-posedness and regularity. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):1105–1123, 2013.

[3] David Barbato, Francesco Morandin, and Marco Romito. Smooth solutions for the dyadic model. *Nonlinearity*, 24(11):3083–3097, 2011.

[4] Alexey Cheskidov. Blow-up in finite time for the dyadic model of the Navier-Stokes equations. *Trans. Amer. Math. Soc.*, 360(10):5101–5120, 2008.

[5] Alexey Cheskidov, Susan Friedlander, and Nataša Pavlović. Inviscid dyadic model of turbulence: the fixed point and Onsager's conjecture. *J. Math. Phys.*, 48(6):065503, 16, 2007.

[6] Alexey Cheskidov, Susan Friedlander, and Nataša Pavlović. An inviscid dyadic model of turbulence: the global attractor. *Discrete Contin. Dyn. Syst.*, 26(3):781–794, 2010.

[7] E. I. Dinaburg and Ya. G. Sinai. A quasilinear approximation for the three-dimensional Navier-Stokes system. *Mosc. Math. J.*, 1(3):381–388, 471, 2001.

[8] G.L. Eyink. Locality of turbulent cascades. *Physica D Nonlinear Phenomena*, 207:91–116, July 2005.

[9] Susan Friedlander and Nataša Pavlović. Blowup in a three-dimensional vector model for the Euler equations. *Comm. Pure Appl. Math.*, 57(6):705–725, 2004.

[10] Nets Hawk Katz and Nataša Pavlović. Finite time blow-up for a dyadic model of the Euler equations. *Trans. Amer. Math. Soc.*, 357(2):695–708 (electronic), 2005.

[11] Alexander Kiselev and Andrej Zlatoš. On discrete models of the Euler equation. *Int. Math. Res. Not.*, (38):2315–2339, 2005.

[12] A. M. Obukhov. Some general properties of equations describing the dynamics of the atmosphere. *Izv. Akad. Nauk SSSR Ser. Fiz. Atmosfer. i Okeana*, 7:695–704, 1971.

[13] Fabian Waleffe. On some dyadic models of the Euler equations. *Proc. Amer. Math. Soc.*, 134(10):2913–2922 (electronic), 2006.

E-mail address: `ijeong@math.princeton.edu, dli@math.ubc.ca`