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# OBSERVATIONS FROM MEASURABLE SETS AND APPLICATIONS

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Abstract. We find new quantitative estimates on the space-time analyticity of solutions to linear parabolic equations with time-independent coefficients and apply them to obtain observability inequalities for its solutions over measurable sets.

## 1. Introduction

Mixing up ideas developed in [35], [2] and [32], it was shown in [3] that the heat equation over bounded domains $\Omega$ in $\mathbb{R}^n$ can be null controlled at all times $T > 0$ with interior and bounded controls acting over space-time measurable sets $\mathcal{D} \subset \Omega \times (0, T)$ with positive Lebesgue measure, when $\Omega$ is a Lipchitz polyhedron or a $C^1$ domain in $\mathbb{R}^n$. [3] also established the boundary null-controllability with bounded controls over measurable sets $\mathcal{J} \subset \partial\Omega \times (0, T)$ with positive surface measure.

In this work we explain the techniques necessary to apply the same methods in [3] in order to obtain the interior and boundary null controllability of some higher order or non self-adjoint parabolic evolutions with time-independent analytic coefficients over analytic domains $\Omega$ of $\mathbb{R}^n$ and with bounded controls acting over measurable sets with positive measure. We also show the null-controllability with controls acting over possibly different measurable regions over each component of the Dirichlet data of higher order parabolic equations or over each component of the solution to second order parabolic systems; both at the interior and at the boundary. Finally, we show that the same methods imply the null-controllability of some not completely uncoupled parabolic systems with bounded interior controls acting over only one of the components of the system and on measurable regions.

We explain the technical details for parabolic higher order equations with constant coefficients and for second order systems with time independent analytic coefficients. We believe that this set of examples will make it clear to the experts that the combination of the methods in [35], [2], [32] with others here imply analog results to those in [3] for parabolic evolutions associated to possibly non self-adjoint higher order elliptic equations or second order systems with time independent analytic coefficients over analytic domains: existence of bounded null-controls acting over measurable sets and the uniqueness and bang-bang property of certain optimal controls.

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Throughout the work $0 < T \leq 1$ denotes a positive time, $\Omega \subset \mathbb{R}^n$ a is bounded domain with analytic boundary $\partial\Omega$, $\nu$ is the exterior unit normal vector to the boundary of $\Omega$ and $d\sigma$ denotes surface measure on $\partial\Omega$, $B_R(x_0)$ stands for the ball centered at $x_0$ and of radius $R$, $B_R = B_R(0)$. For measurable sets $\omega \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n \times (0,T)$, $|\omega|$ and $|\mathcal{D}|$ stand for the Lebesgue measures of the sets; for measurable sets $\Gamma \subset \partial\Omega$ and $\mathcal{J}$ in $\partial\Omega \times (0,T)$, $|\Gamma|$ and $|\mathcal{J}|$ denote respectively their surface measures in $\partial\Omega$ and $\partial\Omega \times \mathbb{R}$. $|\alpha| = \alpha_1 + \cdots + \alpha_\ell$, when $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is a $\ell$-tuple in $\mathbb{N}^\ell$, $\ell \geq 1$.

To describe the analyticity of the boundary of $\Omega$ we assume that there is some $\delta > 0$ such that for each $x_0$ in $\partial\Omega$ there is, after a translation and rotations, a new coordinate system (where $x_0 = 0$) and a real analytic function $\varphi : B'_\delta \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ verifying

$$\begin{aligned} &\varphi(0') = 0,\ |\partial^\alpha_{x'}\varphi(x')| \leq |\alpha|!\,\delta^{-|\alpha|-1},\ \text{when } x' \in B'_\delta,\ \alpha \in \mathbb{N}^{n-1}, \\ &B_\delta \cap \Omega = B_\delta \cap \{(x', x_n) : x' \in B'_\delta,\ x_n > \varphi(x')\}, \\ &B_\delta \cap \partial\Omega = B_\delta \cap \{(x', x_n) : x' \in B'_\delta,\ x_n = \varphi(x')\}. \end{aligned} \tag{1.1}$$

The existence of the bounded null-controls acting over the measurable sets for the set of examples follows by standard duality arguments (cf. [5] or [20]) from the following list of observability inequalities.

**Theorem 1.** *Let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure and $m \geq 1$. Then, there is a constant $N = N(\Omega, T, m, \mathcal{D}, \delta)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |u(x,t)|\, dxdt$$

*holds for all solutions $u$ to*

$$\begin{cases} \partial_t u + (-1)^m \Delta^m u = 0, & \text{in } \Omega \times (0,T), \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.2}$$

*with $u_0$ in $L^2(\Omega)$.*

*Remark* 1. The constant $N$ in Theorem 1 is of the form $e^{N/T^{1/(2m-1)}}$ with $N = N(\Omega, |\omega|, \delta)$, when $\mathcal{D} = \omega \times (0,T)$, $0 < T \leq 1$ and $\omega \subset \Omega$ is a measurable set. The later is consistent with the case of the heat equation [8].

The second and third are two boundary observability inequalities over measurable sets for the higher order evolution (1.2). The first over a general measurable set and the second over two possibly different measurable sets with the same projection over the time $t$-axis. To simplify, we give the details only for the evolution associated to $\Delta^2$.

**Theorem 2.** *Assume that $\mathcal{J} \subset \partial\Omega \times (0,T)$ is a measurable set with positive surface measure in $\partial\Omega \times (0,T)$. Then, there is $N = N(\Omega, \mathcal{J}, T, \delta)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{J}} |\tfrac{\partial \Delta u}{\partial \nu}(x,t)| + |\Delta u(x,t)|\, d\sigma dt, \tag{1.3}$$

*holds for all solutions $u$ to*

$$\begin{cases} \partial_t u + \Delta^2 u = 0, & \text{in } \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

*with $u_0$ in $L^2(\Omega)$.*

*Remark* 2. When $\mathcal{J} = \Gamma \times (0,T)$, $0 < T \leq 1$ and $\Gamma \subset \partial\Omega$, the constant $N$ in Theorem 2 is of the form $e^{N/T^{1/3}}$ with $N = N(\Omega, |\Gamma|, \delta)$.

**Theorem 3.** *Assume that $E \subset (0,T)$ is a measurable set with positive measure and that $\Gamma_i \subset \partial\Omega$, $i = 1,2$, are measurable sets with positive surface measure. Then, there is $N = N(\Omega, |\Gamma_1|, |\Gamma_2|, E, \delta)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \leq N \int_E \|\tfrac{\partial \Delta u}{\partial \nu}(t)\|_{L^1(\Gamma_1)} + \|\Delta u(t)\|_{L^1(\Gamma_2)}\, dt,$$

*holds for all solutions $u$ to* (1.4).

*Remark* 3. We do not know if the sets $\Gamma_1 \times E$ and $\Gamma_2 \times E$ can be replaced by general measurable sets $\mathcal{J}_i \subset \partial\Omega \times (0,T)$, $i = 1,2$.

Now we consider the evolutions associated with strongly coupled second order *time independent* parabolic systems with a possible non self-adjoint structure, as the second order system

$$\begin{cases} \partial_t \mathbf{u} - \mathbf{L}\mathbf{u} = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{u} = 0, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \qquad \text{with } \mathbf{L} = (L^1, \dots, L^\ell), \tag{1.5}$$

with

$$L^\xi \mathbf{u} = \partial_{x_i}(a_{ij}^{\xi\eta}(x)\partial_{x_j} u^\eta) + b_j^{\xi\eta}(x)\partial_{x_j} u^\eta + c^{\xi\eta}(x)u^\eta, \;\; \xi = 1, \dots, \ell,$$

and $\mathbf{u}_0$ in $L^2(\Omega)^\ell$. Here, $\mathbf{u}$ denotes the vector-valued function $(u^1, \dots, u^\ell)$ and the summation convention of repeated indices is understood. We assume that $a_{ij}^{\xi\eta}$, $b_j^{\xi\eta}$ and $c^{\xi\eta}$ are analytic functions over $\overline{\Omega}$, i.e., there is $\delta > 0$ such that

$$|\partial_x^\gamma a_{ij}^{\xi\eta}(x)| + |\partial_x^\gamma b_j^{\xi\eta}(x)| + |\partial_x^\gamma c^{\xi\eta}(x)| \leq \delta^{-|\gamma|-1}|\gamma|!, \text{ for all } \gamma \in \mathbb{N}^n \text{ and } x \in \overline{\Omega}, \tag{1.6}$$

and only requires that the higher order terms of the system (1.5) have a self-adjoint structure; i.e.

$$a_{ij}^{\xi\eta}(x) = a_{ji}^{\eta\xi}(x), \text{ for all } x \in \overline{\Omega}, \; \xi, \eta = 1, \dots, \ell, \; i, j = 1, \dots, n, \tag{1.7}$$

together with the strong ellipticity condition

$$\sum_{\xi,\eta,i,j} a_{ij}^{\xi\eta}(x)\zeta_i^\xi \zeta_j^\eta \geq \delta \sum_{i,\xi} |\zeta_i^\xi|^2, \text{ for all } \zeta = (\zeta_i^\xi) \text{ in } \mathbb{R}^{n\ell} \text{ and } x \in \overline{\Omega}. \tag{1.8}$$

The results described below also hold when the higher order coefficients of the system verify (1.7) and the weaker Legendre-Hadamard condition [13, p. 76],

$$\sum_{i,j,\xi,\eta} a_{ij}^{\xi\eta}(x)\varsigma_i\varsigma_j\vartheta^\xi\vartheta^\eta \geq \delta|\varsigma|^2|\vartheta|^2, \text{ when } \varsigma \in \mathbb{R}^n, \vartheta \in \mathbb{R}^\ell, \; x \in \mathbb{R}^n, \tag{1.9}$$

in place of (1.8). Recall that the Lamé system of elasticity

$$\nabla \cdot \left(\mu(x)\left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right)\right) + \nabla\left(\lambda(x)\nabla \cdot \mathbf{u}\right),$$

with $\mu \geq \delta$, $\mu + \lambda \geq 0$ in $\mathbb{R}^n$, $\ell = n$ and

$$a_{ij}^{\xi\eta}(x) = \mu(x)(\delta_{\xi\eta}\delta_{ij} + \delta_{i\eta}\delta_{j\xi}) + \lambda(x)\delta_{j\eta}\delta_{\xi i},$$

are examples of systems verifying (1.9).

The observability inequalities related to parabolic second order systems are as follows. The first is an interior observability inequality with possibly different measurable interior observation regions for each component of the system but with the same projection over the time $t$-axis.

**Theorem 4.** *Let $E \subset (0,T)$ be a measurable, $|E| > 0$ and $\omega_\eta \subset \Omega$, $\eta = 1, \dots, \ell$, be measurable with $|\omega_\eta| \geq \omega_0$, $\eta = 1, \dots, \ell$, for some $\omega_0 > 0$. Then, there is $N = N(\Omega, T, E, \omega_0, \delta)$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \leq N \int_E \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)}\, dt$$

*holds for all solutions $\mathbf{u}$ to (1.5).*

*Remark* 4. We do not know if the sets $\omega_\eta \times E$, $\eta = 1, \dots, \ell$, can be replaced by different and more general measurable sets $\mathcal{D}_\eta \subset \Omega \times (0,T)$.

The second is a boundary observability inequality over possibly different measurable sets with the same projection over the time $t$-axis for each component of the system and the third, a boundary observability over a general measurable subset of $\partial\Omega \times (0,T)$.

**Theorem 5.** *Let $E \subset (0,T)$ be a measurable set with a positive measure and $\gamma_\eta \subset \partial\Omega$, $\eta = 1, \dots, \ell$, be measurable sets with $\min_{\eta=1,\dots,\ell} |\gamma_\eta| \geq \gamma_0$, for some $\gamma_0 > 0$. Then, there is $N = N(\Omega, E, T, \gamma_0, \delta) \geq 1$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \leq N \int_E \sum_{\eta=1}^{\ell} \|\tfrac{\partial u^\eta}{\partial \nu}(t)\|_{L^1(\gamma_\eta)}\, dt$$

*holds for all solutions $\mathbf{u}$ to (1.5). Here $\frac{\partial \mathbf{u}}{\partial \nu} = \left(\frac{\partial u^1}{\partial \nu}, \dots, \frac{\partial u^\ell}{\partial \nu}\right)$ with $\frac{\partial u^\eta}{\partial \nu} \triangleq a_{ij}^{\eta\xi} \partial_{x_j} u^\xi \nu_i$, for $\eta = 1, \dots, \ell$.*

*Remark* 5. We do not know if the sets $\gamma_\eta \times E$, $\eta = 1, \dots, \ell$, can be replaced by different and general measurable sets $\mathcal{J}_\eta \subset \partial\Omega \times (0,T)$, $\eta = 1, \dots, \ell$.

**Theorem 6.** *Let $\mathcal{J}$ be measurable subset of $\partial\Omega \times (0,T)$ with positive measure. Then, there is $N = N(\Omega, \delta, \mathcal{J}, T)$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \leq N \int_{\mathcal{J}} |\tfrac{\partial \mathbf{u}}{\partial \nu}(q,t)|\, d\sigma dt.$$

*holds for all solutions $\mathbf{u}$ to (1.5),*

With the same methods as for Theorem 1 one can also get an observability inequality for (1.5) with observations over general interior measurable sets.

**Theorem 7.** *Let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure. Then there is $N = N(\Omega, T, \mathcal{D}, \delta) \geq 1$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \leq N \int_{\mathcal{D}} |\mathbf{u}(x,t)|\, dxdt,$$

*holds for all solutions $\mathbf{u}$ to (1.5).*

*Remark* 6. The constant in Theorem 7 is of the form $e^{N/T}$ with $N = N(\Omega, \omega, \delta)$, when $\mathcal{D} = \omega \times (0,T)$, $0 < T \leq 1$ and $\omega \subset \Omega$.

Finally, the last observability inequality deals with the observation of only one interior component of two coupled parabolic equations over a measurable set (See [36] for the case of open sets). In particular, we consider the *time independent* not completely uncoupled parabolic system

$$\begin{cases} \partial_t u - \Delta u + a(x)u + b(x)v = 0, & \text{in } \Omega \times (0,T), \\ \partial_t v - \Delta v + c(x)u + d(x)v = 0, & \text{in } \Omega \times (0,T), \\ u = 0, \;\; v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \;\; v(0) = v_0, & \text{in } \Omega, \end{cases} \tag{1.10}$$

with $a$, $b$, $c$ and $d$ analytic in $\overline{\Omega}$, $b(\cdot) \neq 0$, somewhere in $\overline{\Omega}$ and with

$$|\partial_x^\gamma a(x)| + |\partial_x^\gamma b(x)| + |\partial_x^\gamma c(x)| + |\partial_x^\gamma d(x)| \leq \delta^{-|\gamma|-1}|\gamma|!, \text{ for all } \gamma \in \mathbb{N}^n \text{ and } x \in \overline{\Omega},$$

for some $\delta > 0$. Then, we get the following bound.

**Theorem 8.** *Let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure. Then there is $N = N(\Omega, \mathcal{D}, T, \delta)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} + \|v(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |u(x,t)|\, dxdt,$$

*holds for all solutions $(u,v)$ to* (1.10)

*Remark* 7. Theorem 8 is still valid when the Laplace operator $\Delta$ in (1.10) is replaced by two second elliptic operators $\nabla \cdot (\mathbf{A}_i(x)\nabla\cdot)$, $i = 1, 2$, with matrices $\mathbf{A}_i$ real-analytic, symmetric and positive-definite over $\overline{\Omega}$. Here, we must make sure that the higher order terms of the system remain uncoupled: a diagonal principal part. Otherwise, we do not know if such kind of observability estimates are possible. We believe that generally they are not.

As far as we know, the observability inequalities for the evolutions (1.2) for $m \geq 2$ and (1.5) have not been proved with Carleman methods; not even when $\mathcal{D}$, $\mathcal{J}$, $\Gamma_1$, $\Gamma_2$, $\omega_\eta$, $\gamma_\eta$ are open sets and $E = (0,T)$, cases where the standard techniques to prove Carleman inequalities should make it more feasible. The reasons for these are the difficulties that one confronts when dealing with the calculation and test of the positivity of the commutators associated to the Carleman methods for higher order equations and second order systems.

The method we use relies on the *telescoping series method* - built with ideas borrowed from [24] and first used in [32] - and on local observability inequalities for analytic functions over measurable sets: the Lemma 1 as in [2, 3] and a new extension of Lemma 1, the Lemma 2 below. We use Lemma 2 in the proof of Theorem 8.

**Lemma 1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\omega \subset \Omega$ be a measurable set of positive measure. Let $f$ be an analytic function in $\Omega$ satisfying*

$$|\partial_x^\alpha f(x)| \leq M\rho^{-|\alpha|}|\alpha|!, \text{ for } x \in \Omega \text{ and } \alpha \in \mathbb{N}^n,$$

*for some numbers $M$ and $\rho$. Then, there are $N = N(\Omega, \rho, |\omega|)$ and $\theta = \theta(\Omega, \rho, |\omega|)$, $0 < \theta < 1$, such that*

$$\|f\|_{L^\infty(\Omega)} \leq NM^{1-\theta}\Big(\fint_\omega |f|\, dx\Big)^\theta.$$

Lemma 1 was first derived in [34]. See also [27] and [28] for close results. The reader can find a simpler proof of Lemma 1 in [2, §3]. The proof there is built with ideas from [21], [27] and [34].

**Lemma 2.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\omega \subset \Omega$ be a measurable set with positive Lebesgue measure. Let $f$ be an analytic function in $\Omega$ satisfying*

$$|\partial_x^\alpha f(x)| \leq M|\alpha|!\,\rho^{-|\alpha|}, \text{ for } \alpha \in \mathbb{N}^n \text{ and } x \in \Omega,$$

*for some $M > 0$ and $0 < \rho \leq 1$. Then, there are constants $N = N(\Omega, \rho, |\omega|, n)$ and $\theta = \theta(\Omega, \rho, |\omega|)$, $0 < \theta < 1$, such that*

$$\|\partial_x^\alpha f\|_{L^\infty(\Omega)} \leq |\alpha|!\,(\rho/N)^{-|\alpha|-1} M^{1-\frac{\theta}{2^{|\alpha|}}} \Big(\fint_\omega |f|\,dx\Big)^{\frac{\theta}{2^{|\alpha|}}}, \text{ when } \alpha \in \mathbb{N}^n.$$

To the best of our knowledge, the works studying the space-time analyticity of solutions to linear parabolic equations or systems with space-time analytic coefficients over analytic domains with zero Dirichlet lateral data or with other types of zero lateral data [9, 30, 10, 6, 15, 17] do not in general state clearly the quantitative estimates on the analyticity of the solutions derived from the methods they use. Likely, the authors were mostly interested in the qualitative behavior.

As far as we understand, the best quantitative bound that one can get for solutions to (1.2), (1.4), (1.5) and (1.10) with initial data in $L^2(\Omega)$ from the works [9, 30, 10, 6, 15, 17] is the following:

*There is $0 < \rho \leq 1$, $\rho = \rho(m, n, \delta)$ such that*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}-p}|\alpha|!\; p!\, t^{-\frac{|\alpha|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)}, \forall \alpha \in \mathbb{N}^n, p \in \mathbb{N}, \tag{1.11}$$

*where $2m$ is the order of the evolution.*

The arguments in [9, 30, 10, 6, 15, 17] show that (1.11) holds when the coefficients of the underlying linear parabolic equation or system are time dependent and satisfy bounds like

$$|\partial_x^\alpha \partial_t^p A(x,t)| \leq \delta^{-1-|\alpha|-p}|\alpha|!\, p!\,, \text{ for all } \alpha \in \mathbb{N}^n,\ p \in \mathbb{N},\ x \in \overline{\Omega} \text{ and } t > 0, \tag{1.12}$$

for some $0 < \delta \leq 1$. On the other hand, there is $\rho = \rho(n, m)$, $0 < \rho \leq 1$, such that the solution to

$$\begin{cases} \partial_t u + (-\Delta)^m u = 0, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(0) = u_0, & \text{in } \mathbb{R}^n, \end{cases}$$

verifies

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}-p}|\alpha|!^{\frac{1}{2m}}\; p!\, t^{-\frac{|\alpha|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{1.13}$$

when $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$. Thus, the radius of convergence of the Taylor series expansion of $u(\cdot, t)$ around points in $\mathbb{R}^n$ is $+\infty$ at all times $t > 0$. The same holds when $(-\Delta)^m$ is replaced above by other elliptic operators or systems of order $2m$ with constant coefficients. These estimates follow from upper bounds of the holomorphic extension to $\mathbb{C}^n$ of the fundamental solutions of higher order parabolic equations or systems with constant coefficients [6, p.15 (15); p.47-48 Theorem 1.1 (3)] and the fact that a function $f$ in $C^\infty(\mathbb{R}^n)$ verifies

$$|\partial_x^\alpha f(0)| \leq |\alpha|!^{\frac{1}{2m}} \rho^{-1-|\alpha|}, \text{ for all } \alpha \in \mathbb{N}^n, \text{ for some } 0 < \rho \leq 1,$$

if and only if $f$ is a holomorphic in $\mathbb{C}^n$ and

$$|f(z)| \leq e^{N|z|^{\frac{2m}{2m-1}}}, \text{ for all } z \in \mathbb{C}^n \text{ and for some } N \geq 1.$$

To prove the observability inequalities in Theorems 1 through 8 we apply Lemmas 1 or 2 to $u(t)$ over $\Omega$ and to $u(x,\cdot)$ over roughly $(\frac{t}{2},t)$ for $x$ in $\Omega$ and $0<t\leq T$, with $u$ a solution to one of the above systems. To get the result of these applications compatible with the *telescoping series* method - to make sure that a certain telescoping series converges - we need better quantifications of the space-time analyticity of the solutions to (1.2), (1.4), (1.5) and (1.10) than the ones in (1.11) or within the available literature [9, 30, 10, 6, 15, 17], where the Taylor series expansion of $u(\cdot,t)$ around a point $x_0$ in $\overline{\Omega}$ is known to converge absolutely only at points whose distance from $x_0$ is less than a fixed constant multiple of $\sqrt[2m]{t}$.

For our purpose, we need to find a quantification of the space-time analyticity which implies that the space-time Taylor series expansion of solutions converge absolutely over $B_\rho(x)\times((1-\rho)t,(1+\rho)t)$, for some $0<\rho\leq 1$, when $(x,t)$ is in $\overline{\Omega}\times(0,1]$. Thus, independently of $0<t\leq 1$ in the space variable.

E. M. Landis and O. A. Oleinik developed in [18] a reasoning which reduces the study of the strong unique continuation property within characteristic hyperplanes for solutions to *time independent* parabolic evolutions to its elliptic counterpart. They informed their readers [18, p. 190] that their argument implies the space-analyticity of solutions to *time-independent* linear parabolic equations from its corresponding elliptic counterpart though they did not bother to quantify their claim. Here, we quantify each step of their reasonings and get the following quantitative estimate.

**Lemma 3.** *There is $\rho=\rho(m,n,\delta)$, $0<\rho\leq 1$, such that*

$$|\partial_x^\alpha\partial_t^p u(x,t)|\leq e^{1/\rho t^{1/(2m-1)}}\rho^{-|\alpha|-p}\,|\alpha|!\,p!\,t^{-p}\|u_0\|_{L^2(\Omega)},$$

*when $x\in\overline{\Omega}$, $0<t\leq 1$, $\alpha\in\mathbb{N}^n$, $p\geq 0$, $2m$ is the order of the evolution and $u$ verifies* (1.2), (1.4), (1.5) *or* (1.10).

It provides a better bound than (1.11) in [9, 30, 10, 6, 15, 17] and it is good, as described above, for our applications to Control Theory. Also observe that Lemma 3 is somehow in between (1.11) and (1.13), since

$$\sup_{t>0}\,t^{-\frac{|\alpha|}{2m}}e^{1/\rho t^{1/(2m-1)}}\lesssim|\alpha|!^{1-\frac{1}{2m}},\ \text{for }\alpha\in\mathbb{N}^n.$$

Lemma 3 also holds for solutions to *time independent* linear parabolic equations associated to elliptic and possibly non self-adjoint equations of order $2m$ with analytic coefficients. We do not complete the details here. The readers can obtain such quantitative estimates from [18] and with arguments similar to those in Section 2.

We believe that Lemma 3 holds when the coefficients of the parabolic evolution are time dependent and verify (1.12) but so far we do not know how to prove it.

The paper is organized as follows: Section 2 proves Lemma 3; Section 3 shows the results related to higher order parabolic equatons, Section 4 verifies the ones for systems and Section 5 recalls some applications of Theorems 1, 2, 4 and 8 to Control Theory. One can find analogous applications of Theorems 3, 5, 6 and 7.

## 2. Proof of Lemma 3

We first prove Lemma 3 for solutions to (1.2). Other time-independent parabolic evolutions associated to *self-adjoint* elliptic scalar operators or systems with analytic coefficients are treated similarly.

*Proof of Lemma 3 for* (1.2). Let $\{e_j\}_{j\geq 1}$ and $\{w_j^{2m}\}_{j\geq 1}$ be respectively the sets of $L^2(\Omega)$-normalized eigenfunctions and eigenvalues for $(-\Delta)^m$ with zero lateral Dirichlet boundary conditions; i.e.,

$$\begin{cases} (-1)^m \Delta^m e_j - w_j^{2m} e_j = 0, & \text{in } \Omega, \\ e_j = \nabla e_j = \cdots = \nabla^{m-1} e_j = 0, & \text{on } \partial\Omega. \end{cases}$$

Take $u_0 = \sum_{j\geq 1} a_j e_j$, with $\sum_{j\geq 1} a_j^2 < +\infty$ and define

$$u(x,y,t) = \sum_{j\geq 1} a_j e^{-tw_j^{2m}} e_j(x) X_j(y), \text{ for } x \in \overline{\Omega},\ y \in \mathbb{R} \text{ and } t > 0,$$

with

$$X_j(y) = \begin{cases} e^{w_j y}, & \text{when } m \text{ is odd}, \\ e^{w_j y e^{\frac{\pi i}{2m}}}, & \text{when } m \text{ is even}, \end{cases} \tag{2.1}$$

where $i = \sqrt{-1}$. Then, $u(x,t) = u(x,0,t)$, solves (1.2) with initial datum $u_0$ and

$$\partial_t^p u(x,y,t) = \sum_{j\geq 1} (-1)^p\, a_j\, w_j^{2mp} e^{-tw_j^{2m}} e_j(x) X_j(y), \quad x \in \overline{\Omega},\ y \in \mathbb{R}. \tag{2.2}$$

Moreover,

$$\begin{cases} (\partial_y^{2m} + \Delta_x^m)(\partial_t^p u(\cdot,\cdot,t)) = 0, & \text{in } \Omega \times \mathbb{R}, \\ \partial_t^p u(\cdot,\cdot,t) = \nabla(\partial_t^p u(\cdot,\cdot,t)) = \cdots = \nabla^{m-1}(\partial_t^p u(\cdot,\cdot,t)) = 0, & \text{on } \partial\Omega \times \mathbb{R}. \end{cases}$$

Because $\partial\Omega$ is analytic, the quantitative estimates on the analyticity up to the boundary for solutions to elliptic equations with analytic coefficients and null-Dirichlet data over nearby analytic boundaries (See [25, Ch. 5] or [11, Ch. 3]), show that there is $\rho = \rho(\Omega)$, $0 < \rho \leq 1$, such that for $x_0$ in $\overline{\Omega}$ and $0 < R \leq 1$

$$\begin{aligned} &\|\partial_x^\alpha \partial_t^p u(\cdot,\cdot,t)\|_{L^\infty(B_{R/2}(x_0,0)\cap\Omega\times\mathbb{R})} \\ &\qquad \leq |\alpha|!\, \rho^{-1-|\alpha|} R^{-|\alpha|} \left( \fint_{B_R(x_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^p u(x,y,t)|^2\, dxdy \right)^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Because

$$\int_{B_R(x_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^p u(x,y,t)|^2\, dxdy \leq \int_{-R}^{R} \int_\Omega |\partial_t^p u(x,y,t)|^2\, dxdy, \tag{2.4}$$

we have from (2.1), (2.2) and the orthogonality of $\{e_j\}_{j\geq 1}$ in $L^2(\Omega)$ that

$$\begin{aligned} &\int_\Omega |\partial_t^p u(x,y,t)|^2\, dx = \int_\Omega \Big| \sum_{j\geq 1} (-1)^p\, a_j\, w_j^{2mp} e^{-tw_j^{2m}} e_j(x) X_j(y) \Big|^2\, dx \\ &= \sum_{j\geq 1} a_j^2\, w_j^{4mp} e^{-2tw_j^{2m}} |X_j(y)|^2 \leq \sum_{j\geq 1} a_j^2\, w_j^{4mp} e^{-2tw_j^{2m}} e^{2w_j|y|} \\ &\leq \max_{j\geq 1} \left\{ w_j^{4mp} e^{-tw_j^{2m}} \right\} \max_{j\geq 1} \left\{ e^{-tw_j^{2m} + 2w_j|y|} \right\} \sum_{j\geq 1} a_j^2. \end{aligned}$$

Next, from Stirling's formula

$$\max_{x\geq 0} x^{2p} e^{-xt} = t^{-2p}\, (2p)^{2p}\, e^{-2p} \lesssim \left(\frac{2}{t}\right)^{2p} p!^2, \text{ when } t > 0 \text{ and } p \geq 0,$$

and the fact that

$$\max_{x\geq 0} e^{-tx^{2m}+2x|y|} = e^{\left(2-\frac{1}{m}\right)\left(\frac{|y|}{mt}\right)^{\frac{1}{2m-1}}}, \text{ when } t>0,\ m\geq 1,$$

we get that

$$\int_\Omega |\partial_t^p u(x,y,t)|^2\,dx \lesssim \left(\frac{2}{t}\right)^{2p} p!^2 e^{2|y|\left(\frac{|y|}{mt}\right)^{\frac{1}{2m-1}}} \sum_{j\geq 1} a_j^2.$$

This, along with (2.3), (2.4) and the choice of $R=1$ show that

$$\|\partial_x^\alpha \partial_t^p u(\cdot,\cdot,t)\|_{L^\infty(B_{1/2}(x_0,0)\cap\Omega\times\mathbb{R})} \leq N|\alpha|!\,p!\,\rho^{-|\alpha|}\left(\frac{2}{t}\right)^p e^{Nt^{-\frac{1}{2m-1}}}\left(\sum_{j\geq 1} a_j^2\right)^{1/2}.$$

In particular,

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}}\rho^{-|\alpha|-p}|\alpha|!\,p!\,t^{-p}\|u_0\|_{L^2(\Omega)}.$$

□

*Remark* 8. The last proof extends to the case $m\geq 2$ its analog for $m=1$ in [3, Lemma 6]. There the authors used that the Green's function over $\Omega$ for $\Delta-\partial_t$ with zero lateral Dirichlet conditions has Gaussian upper bounds. The later shows that one can derive [3, Lemma 6] without knowledge of upper bounds for the Green's function with lateral Dirichlet conditions of the parabolic evolution.

We now give a proof of Lemma 3 for solutions to the systems (1.5) and (1.10). Other time-independent parabolic evolutions associated to possibly *non self-adjoint* elliptic scalar equations with analytic coefficients over $\overline{\Omega}$ are treated similarly.

*Proof of Lemma 3 for* (1.5). The proof of Lemma 3 requires first global bounds on the time-analyticity of the solutions, Lemma 4 below. Of course, there is plenty of literature on the time-analyticity of solutions to abstract evolutions [14, 16, 22, 31] but we give here a proof of Lemma 4 because it serves better our purpose.

**Lemma 4.** *There is $\rho=\rho(\delta)$, $0<\rho\leq 1$, such that*

$$t^p\|\partial_t^p \mathbf{u}(t)\|_{L^2(\Omega)} + t^{p+\frac{1}{2}}\|\nabla\partial_t^p \mathbf{u}(t)\|_{L^2(\Omega)} \leq \rho^{-1-p}p!\,\|\mathbf{u}_0\|_{L^2(\Omega)},$$

*when $p\geq 0$, $0<t\leq 2$ and $\mathbf{u}$ verifies* (1.5) *or* (1.10).

*Proof of Lemma 4.* Let $\mathbf{u}$ solve (1.5). When $\mathbf{u}_0$ is in $C_0^\infty(\Omega)$, the solution $\mathbf{u}$ to (1.5) is in $C^\infty(\overline{\Omega}\times[0,+\infty))$ [10]. By the local energy inequality for (1.5) there is $\rho=\rho(\delta)>0$ such that

$$\sup_{0\leq t\leq 2}\|\mathbf{u}(t)\|_{L^2(\Omega)} \leq \rho^{-1}\|\mathbf{u}_0\|_{L^2(\Omega)}.$$

Multiply first the equation satisfied by $\partial_t^p\mathbf{u}$,

$$\begin{cases} \partial_t^{p+1}\mathbf{u} - \mathbf{L}\partial_t^p\mathbf{u} = 0, & \text{in } \Omega\times(0,+\infty),\\ \partial_t^p\mathbf{u} = 0, & \text{in } \partial\Omega\times(0,+\infty), \end{cases} \tag{2.5}$$

by $t^{2p+2}\partial_t^{p+1}\mathbf{u}$, after by $t^{2p+1}\partial_t^p\mathbf{u}$ and integrate by parts over $\Omega_T = \Omega \times (0,T)$, $0 < T \leq 2$, the two resulting identities. These, standard energy methods, Hölder's inequality together with (1.6) (1.7) and (1.8) imply that

$$T^{p+1}\|\nabla\partial_t^p\mathbf{u}(T)\|_{L^2(\Omega)} + \|t^{p+1}\partial_t^{p+1}\mathbf{u}\|_{L^2(\Omega_T)} \lesssim \|t^p\partial_t^p\mathbf{u}\|_{L^2(\Omega_T)} + (p+1)^{\frac{1}{2}}\,\|t^{p+\frac{1}{2}}\partial_t^p\nabla\mathbf{u}\|_{L^2(\Omega_T)}, \tag{2.6}$$

$$T^{p+\frac{1}{2}}\|\partial_t^p\mathbf{u}(T)\|_{L^2(\Omega)} + \|t^{p+\frac{1}{2}}\partial_t^p\nabla\mathbf{u}\|_{L^2(\Omega_T)} \lesssim (p+1)^{\frac{1}{2}}\,\|t^p\partial_t^p\mathbf{u}\|_{L^2(\Omega_T)}. \tag{2.7}$$

Thus,

$$\|t^{p+1}\partial_t^{p+1}\mathbf{u}\|_{L^2(\Omega_T)} \leq \rho^{-1}\,(p+1)\,\|t^p\partial_t^p\mathbf{u}\|_{L^2(\Omega_T)},\ \text{for } p \geq 0 \tag{2.8}$$

and the iteration of (2.8) and the local energy inequality show that

$$\|t^p\partial_t^p\mathbf{u}(t)\|_{L^2(\Omega_T)} \leq \rho^{-1-p}p!\,\sqrt{T}\,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)},\ \text{for } p \geq 0.$$

This combined with (2.6) and (2.7) implies Lemma 4.

□

The next step is to show that we can realize $\mathbf{u}(x,t)$ and all its partial derivatives with respect to time as functions with one more space variable, say $x_{n+1}$, which satisfy in the $(X,t) = (x, x_{n+1}, t)$ coordinates a *time-independent* parabolic evolution associated to a *self-adjoint* elliptic system with analytic coefficients over $\Omega \times (-1,1) \times (0,+\infty)$ and with zero boundary values over $\partial\Omega \times (-1,1) \times (0,+\infty)$. To accomplish it, consider the system $\mathbf{S} = \left(S^1, \ldots, S^\ell\right)$, which acts on functions $\mathbf{w}$ in $C^\infty(\Omega \times \mathbb{R}, \mathbb{R}^\ell)$, $\mathbf{w} = \left(w^1, \ldots, w^\ell\right)$, as

$$S^\xi\mathbf{w} = \sum_{i,j=1}^{n+1}\sum_{\eta=1}^{\ell}\partial_{x_i}\left(\tilde{a}_{ij}^{\xi\eta}(X)\partial_{x_j}w^\eta\right) + \sum_{\eta=1}^{\ell}\left[\partial_{x_{n+1}}\left(x_{n+1}c^{\xi\eta}(x)w^\eta\right) - x_{n+1}c^{\eta\xi}(x)\partial_{x_{n+1}}w^\eta\right],$$

for $\xi = 1, \ldots, \ell$, where for $\xi, \eta = 1, \ldots, \ell$,

$$\tilde{a}_{ij}^{\xi\eta}(X) = \begin{cases} a_{ij}^{\xi\eta}(x), & \text{for } i,j = 1,\ldots,n, \\ x_{n+1}b_j^{\xi\eta}(x), & \text{for } i = n+1,\ j = 1,\ldots,n \\ x_{n+1}b_i^{\eta\xi}(x), & \text{for } i = 1,\ldots,n,\ j = n+1, \\ M\delta_{\xi\eta}, & \text{for } i = j = n+1. \end{cases}$$

Set $Q_R = \Omega \times (-R,R)$ and $\partial_l Q_R = \partial\Omega \times (-R,R)$, the "lateral"boundary of $Q_R$. From (1.7), $\mathbf{S}$ is a self-adjoint system and for large $M = M(\delta)$, the matrices of coefficients $\tilde{a}_{ij}^{\xi\eta}$ verify one the ellipticity conditions (1.8) or (1.9) with $\delta$ replaced by $\frac{\delta}{2}$ over $Q_1$ when the original coefficients $a_{ij}^{\xi\eta}$ verify respectively (1.8) or (1.9). Choosing $M$ larger if it is necessary, we may assume that

$$\tfrac{\delta}{2}\|\nabla_X\boldsymbol{\varphi}\|_{L^2(Q_1)}^2 \leq -\int_{Q_1}\mathbf{S}\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}\,dX \leq \tfrac{2}{\delta}\|\nabla_X\boldsymbol{\varphi}\|_{L^2(Q_1)}^2, \tag{2.9}$$

when $\boldsymbol{\varphi}$ is in $W_0^{1,2}(Q_1)$ and $\nabla_X = \left(\nabla_x, \partial_{x_{n+1}}\right)$. Also, $\mathbf{S}\boldsymbol{\varphi}(X) = \mathbf{L}\mathbf{v}(x)$, when $\boldsymbol{\varphi}(X) = \mathbf{v}(x)$ and for $\mathbf{w}(X,t) = \partial_t^p \mathbf{u}(x,t)$, $p \geq 0$, we have

$$\begin{cases} \partial_t \mathbf{w} - \mathbf{S}\mathbf{w} = 0, & \text{in } Q_1 \times (0, +\infty), \\ \mathbf{w} = 0, & \text{in } \partial_l Q_1 \times (0, +\infty). \end{cases}$$

The symmetry, coerciveness and compactness of the operator mapping functions $\mathbf{f}$ in $L^2(Q_1)^m$ into the unique solution $\boldsymbol{\varphi}$ in $W_0^{1,2}(Q_1)^m$ to

$$\begin{cases} \mathbf{S}\boldsymbol{\varphi} = \mathbf{f}, & \text{in } Q_1, \\ \boldsymbol{\varphi} = 0, & \text{in } \partial Q_1 \end{cases}$$

[13, Prop. 2.1] gives the existence of a complete orthogonal system $\{\mathbf{e}_k\}$ in $L^2(Q_1)^m$ of eigenfunctions, $\mathbf{e}_k = (e_k^1, \dots, e_k^m)$, satisfying

$$\begin{cases} \mathbf{S}\mathbf{e}_k + \omega_k^2 \, \mathbf{e}_k = 0, & \text{in } Q_1, \\ \mathbf{e}_k = 0, & \text{in } \partial Q_1, \end{cases}$$

with eigenvalues $0 < \omega_1^2 \leq \dots \omega_k^2 \leq \dots$. Fix $0 < T \leq 1$ and for $(X,t)$ in $Q_1 \times (\frac{T}{2}, +\infty)$ consider

$$\mathbf{w}_1(X,t) = \sum_{j \geq 1} a_j e^{-w_j^2 (t - T/2)} \mathbf{e}_j(X),$$

with

$$a_j = \int_{Q_1} \partial_t^p \mathbf{u}(x, \tfrac{T}{2}) \mathbf{e}_j(X) \, dX. \tag{2.10}$$

Clearly, $\mathbf{w}_1(X, \frac{T}{2}) = \partial_t^p \mathbf{u}(x, \frac{T}{2})$ in $Q_1$ and by the multiplications of the equation verified by $\mathbf{w}_1$, first by $\mathbf{w}_1$, after by $\partial_t \mathbf{w}_1$ and the integration by parts of the resulting identities over $Q_1 \times (\frac{T}{2}, \tau)$, for $\frac{T}{2} < \tau \leq 2T$, we get

$$\begin{aligned} \|\mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} + \sqrt{T} \, \|\nabla_X \mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} \\ \lesssim \|\partial_t^p \mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)} + \sqrt{T} \, \|\nabla \partial_t^p \mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)}. \end{aligned}$$

From Lemma 4

$$\|\mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} + \sqrt{T} \, \|\nabla_X \mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} \leq \sqrt{T} \, H(p, T, \rho), \tag{2.11}$$

with

$$H(p, T, \rho) = \rho^{-1-p} p! \, T^{-p-\frac{1}{2}} \|\mathbf{u}_0\|_{L^2(\Omega)}, \ 0 < \rho \leq 1, \ \rho = \rho(\delta). \tag{2.12}$$

Let $\mathbf{w}_2$ be the solution to

$$\begin{cases} \partial_t \mathbf{w}_2 - \mathbf{S}\mathbf{w}_2 = 0, & \text{in } Q_1 \times (\frac{T}{2}, +\infty), \\ \mathbf{w}_2 = \eta(t) \left(\partial_t^p \mathbf{u} - \mathbf{w}_1\right), & \text{on } \partial Q_1 \times (\frac{T}{2}, +\infty), \\ \mathbf{w}_2(0) = \mathbf{0}, & \text{in } Q_1, \end{cases}$$

where $0 \leq \eta \leq 1$ verifies $\eta = 1$, for $-\infty < t \leq T$, $\eta = 0$, for $\frac{3T}{2} \leq t < +\infty$ and $|\partial_t \eta| \leq \frac{1}{T}$. Observe that because $\partial_t^p \mathbf{u} = 0$ on $\partial\Omega \times (0, +\infty)$, $\partial_l Q_1 \subset \partial Q_1$ and $\mathbf{w}_1 = 0$ on $\partial Q_1$, then $\mathbf{w}_2 = 0$ on $\partial_l Q_1$.

The auxiliary function, $\mathbf{v} = \mathbf{w}_2 - \eta(t)(\partial_t^p \mathbf{u} - \mathbf{w}_1)$ satisfies

$$\begin{cases} \partial_t \mathbf{v} - \mathbf{S}\mathbf{v} = -(\partial_t^p \mathbf{u} - \mathbf{w}_1)\partial_t \eta & \text{in } Q_1 \times (T/2, +\infty), \\ \mathbf{v} = 0 & \text{on } \partial Q_1 \times (T/2, +\infty), \\ \mathbf{v}(T/2) = 0 & \text{in } Q_1 \end{cases}$$

and clearly $v \equiv 0$ in $Q_1 \times [\frac{T}{2}, T]$. In particular,

$$\partial_t^p \mathbf{u}(x,T) = \mathbf{w}_1(X,T) + \mathbf{w}_2(X,T), \text{ for } X \text{ in } Q_1. \tag{2.13}$$

By the parabolic regularity

$$\|\mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} + \|\nabla_X \mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} \lesssim \|(\partial_t^p \mathbf{u} - \mathbf{w}_1)\partial_t \eta\|_{L^2(\frac{T}{2},2T;L^2(Q_1))}$$

and from Lemma 4 and (2.11)

$$\|\mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} + \|\nabla_X \mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} \lesssim H(p,T,\rho).$$

Because $\mathbf{w}_2 = \mathbf{v} + \eta(t)\,(\partial_t^p \mathbf{u} - \mathbf{w}_1)$, we get from the latter, Lemma 4 and (2.11)

$$\|\mathbf{w}_2\|_{L^\infty(\frac{T}{2},2T;L^2(Q_1))} + \|\nabla_X \mathbf{w}_2\|_{L^\infty(\frac{T}{2},2T;L^2(Q_1))} \lesssim H(p,T,\rho). \tag{2.14}$$

By separation of variables,

$$\mathbf{w}_2(X,t) = \sum_{j=1}^{+\infty} c_j e^{-\omega_j^2(t-2T)} \mathbf{e}_j(X), \text{ with } c_j = \int_{Q_1} \mathbf{w}_2(X,2T)\mathbf{e}_j(X)\,dX,$$

for $t \geq 2T$. From (2.9), $\omega_1^2 \geq \frac{\delta}{2}$ and

$$\|\mathbf{w}_2(t)\|_{L^2(Q_1)} \leq e^{-\frac{\delta}{2}(t-2T)} \|\mathbf{w}_2(2T)\|_{L^2(Q_1)}, \text{ when } t \geq 2T. \tag{2.15}$$

Also,

$$-\int_{Q_1} \mathbf{S}\mathbf{w}_2(t) \cdot \mathbf{w}_2(t)\,dX = -\int_{Q_1} \partial_t \mathbf{w}_2(t) \cdot \mathbf{w}_2(t)\,dX = \sum_{j=1}^{+\infty} c_j^2 \omega_j^2 e^{-2\omega_j^2(t-2T)},$$

for $t \geq 2T$ and the last identity and (2.9) imply that

$$\|\nabla_X \mathbf{w}_2(t)\|_{L^2(Q_1)} \leq e^{-\frac{\delta}{2}(t-2T)} \|\nabla_X \mathbf{w}_2(2T)\|_{L^2(Q_1)}, \text{ when } t \geq 2T.$$

From (2.14), (2.15) and the last inequality

$$\|\mathbf{w}_2(t)\|_{L^2(Q_1)} + \|\nabla_X \mathbf{w}_2(t)\|_{L^2(Q_1)} \lesssim e^{-\frac{\delta}{2}(t-2T)^+} H(p,T,\rho) \tag{2.16}$$

and we may extend $\mathbf{w}_2$ as zero for $t \leq \frac{T}{2}$. Set

$$\widehat{\mathbf{w}}_2(X,\mu) = \frac{1}{\sqrt{2\pi}} \int_{\frac{T}{2}}^{+\infty} e^{-i\mu t} \mathbf{w}_2(X,t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\mu t} \mathbf{w}_2(X,t)\,dt,$$

for $X$ in $Q_1$ and $\mu$ in $\mathbb{R}$. From (2.16)

$$\|\widehat{\mathbf{w}}_2(\mu)\|_{L^2(Q_1)} + \|\nabla_X \widehat{\mathbf{w}}_2(\mu)\|_{L^2(Q_1)} \lesssim H(p,T,\rho), \text{ for all } \mu \in \mathbb{R}. \tag{2.17}$$

Moreover,

$$\begin{cases} \mathbf{S}\widehat{\mathbf{w}}_2(X,\mu) - i\mu \widehat{\mathbf{w}}_2(X,\mu) = 0, & \text{in } Q_1, \\ \widehat{\mathbf{w}}_2(X,\mu) = 0, & \text{in } \partial_l Q_1, \end{cases} \quad \text{for each } \mu \in \mathbb{R}.$$

For $\mu \neq 0$, define

$$\mathbf{v}_2(X,\zeta,\mu) = e^{i\zeta\sqrt{|\mu|}} \widehat{\mathbf{w}}_2(X,\mu), \quad \zeta \in \mathbb{R}. \tag{2.18}$$

Then,

$$\begin{cases} \mathbf{S}\mathbf{v}_2(X,\zeta,\mu) + i\,\mathrm{sgn}\,(\mu)\,\partial_\zeta^2\mathbf{v}_2(X,\zeta,\mu) = 0, & \text{in } Q_1\times\mathbb{R},\\ \mathbf{v}_2(X,\zeta,\mu) = 0, & \text{in } \partial_l Q_1\times\mathbb{R}. \end{cases}$$

As for the equation verified by $\mathbf{v}_2$, it is elliptic with complex coefficients and its coefficients are independent of the $\zeta$-variable. These and the fact that $\partial_\zeta^k\mathbf{v}_2 = 0$ on $\partial_l Q_1\times\mathbb{R}$ imply by energy methods [26] ($k$ times localized Cacciopoli's inequalities) that

$$\|\partial_\zeta^{j+1}\mathbf{v}_2\|_{L^2(Q_{1-\frac{j+1}{2k}}\times(-1+\frac{j+1}{2k}),1-\frac{j+1}{2k}))} \le \tfrac{k}{\rho}\,\|\partial_\zeta^{j}\mathbf{v}_2\|_{L^2(Q_{1-\frac{j}{2k}}\times(-1+\frac{j}{2k},1-\frac{j}{2k}))},$$

for $j = 0,\dots,k-1$, $k\ge 1$, and for some $0<\rho\le 1$, $\rho=\rho(\delta)$. Its iteration gives

$$\|\partial_\zeta^k\mathbf{v}_2\|_{L^2(Q_{\frac12}\times(-\frac12,\frac12))} \le k!\,\rho^{-k}\|\mathbf{v}_2\|_{L^2(Q_1\times(-1,1))},\ \text{for } k\ge 1,$$

and from (2.17) and (2.18)

$$\|\partial_\zeta^k\mathbf{v}_2\|_{L^2(Q_{\frac12}\times(-\frac12,\frac12))} \lesssim k!\,\rho^{-k}H(T,p,\rho),\ \text{for } k\ge 1. \tag{2.19}$$

For $\psi$ in $L^2(Q_{\frac12})$, set $\gamma(\zeta) = \int_{Q_{\frac12}}\mathbf{v}_2(X,\zeta,\mu)\overline{\psi}(X)\,dX$. Then, from (2.17), (2.18) and (2.19)

$$\|\gamma^{(k)}\|_{L^\infty(-\frac12,\frac12)} \lesssim \rho^{-k}k!\,H(T,p,\rho)\,\|\psi\|_{L^2(Q_{\frac12})},\ \text{for } k\ge 0.$$

Thus, $\gamma(-\frac{i\rho}{2})$ can be calculated via the Taylor series expansion of $\gamma$ around $\zeta=0$ and after adding a geometric series

$$|\gamma(-\tfrac{i\rho}{2})| = \Big|\int_{Q_{\frac12}} e^{\rho\sqrt{|\mu|}/2}\widehat{\mathbf{w}}_2(X,\mu)\overline{\psi}(X)\,dX\Big| \lesssim \|\psi\|_{L^2(Q_{\frac12})}\,H(T,p,\rho).$$

All together,

$$\|\widehat{\mathbf{w}}_2(\cdot,\mu)\|_{L^2(Q_{\frac12})} \lesssim e^{-\rho\sqrt{|\mu|}/2}H(T,p,\rho),\ \text{when } \mu\in\mathbb{R}. \tag{2.20}$$

Define then,

$$\mathbf{U}_2(X,y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\mu T}\widehat{\mathbf{w}}_2(X,\mu)\cosh\Big(y\sqrt{-i\mu}\Big)\,d\mu,$$

for $(X,y)$ in $Q_{\frac12}\times\mathbb{R}$, with $\sqrt{-i\mu} = \sqrt{|\mu|}\,e^{-\frac{i\pi}{4}\,\mathrm{sgn}\,\mu}$. From (2.20),

$$\|\mathbf{U}_2(\cdot,y)\|_{L^2(Q_{\frac12})} \lesssim H(T,p,\rho),\ \text{for } |y|\le\frac{\rho}{4}. \tag{2.21}$$

Observe that $\mathbf{U}_2$ is in $C^\infty(\overline{Q}_{\frac12}\times[-\frac{\rho}{4},\frac{\rho}{4}])$ and that one may derive similar bounds for higher derivatives of $\mathbf{U}_2$. Also,

$$\begin{cases} \mathbf{S}\mathbf{U}_2 + \partial_y^2\mathbf{U}_2 = 0, & \text{in } Q_{\frac12}\times(-\frac{\rho}{4},\frac{\rho}{4}),\\ \mathbf{U}_2 = 0, & \text{in } \partial_l Q_{\frac12}\times(-\frac{\rho}{4},\frac{\rho}{4}) \end{cases} \tag{2.22}$$

and

$$\mathbf{U}_2(X,0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\mu T}\widehat{\mathbf{w}}_2(X,\mu)\,d\mu = \mathbf{w}_2(X,T),\ \text{in } Q_{\frac12}. \tag{2.23}$$

Next,

$$\mathbf{U}_1(X,y) = \sum_{j=1}^{+\infty} e^{-\omega_j^2T/2}a_j\mathbf{e}_j(X)\cosh(\omega_j y),$$

with $a_j$ as in (2.10) satisfies

$$\mathbf{U}_1(X,0)=\mathbf{w}_1(X,T), \text{in } Q_1, \qquad \begin{cases}\mathbf{S}\mathbf{U}_1+\partial_y^2\mathbf{U}_1=0, & \text{in } Q_1\times\mathbb{R},\\ \mathbf{U}_1=0, & \text{in } \partial Q_1\times\mathbb{R},\end{cases} \tag{2.24}$$

and

$$\sup_{|y|\leq 1}\|\mathbf{U}_1(\cdot,y)\|_{L^2(Q_1)}\lesssim e^{1/T}\|\partial_t^p\mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)}\lesssim e^{1/T}H(T,p,\rho). \tag{2.25}$$

Set then, $\mathbf{U}=\mathbf{U}_1+\mathbf{U}_2$. From (2.22), (2.23), (2.24) and (2.13) we have

$$\begin{cases}\mathbf{S}\mathbf{U}+\partial_y^2\mathbf{U}=0, & \text{in } Q_{\frac12}\times(-\frac{\rho}{4},\frac{\rho}{4}),\\ \mathbf{U}=0, & \text{in } \partial_l Q_{\frac12}\times(-\frac{\rho}{4},\frac{\rho}{4}),\\ U(X,0)=\partial_t^p\mathbf{u}(x,T), & \text{in } Q_{\frac12},\end{cases}$$

while (2.21) and (2.25) show that

$$\sup_{|y|\leq\frac{\rho}{4}}\|\mathbf{U}(\cdot,y)\|_{L^2(Q_{\frac12})}\lesssim e^{1/T}H(T,p,\rho),\ \text{with } \rho=\rho(\delta),\ 0<\rho\leq 1. \tag{2.26}$$

Now, $\mathbf{S}+\partial_y^2$ is an elliptic system with analytic coefficients. This, (2.26), the fact that $\mathbf{U}(X,y)=0$, for $(X,y)=(x,x_{n+1},y)$ in $\partial\Omega\times(-\frac12,\frac12)\times(-\frac{\rho}{4},\frac{\rho}{4})$ and that $\partial\Omega$ is analytic imply that there is $\rho=\rho(\delta)$, $0<\rho\leq 1$ (See [26] or [13, Ch. II]) such that

$$\|\partial_X^\gamma\partial_y^q\mathbf{U}(X,y)\|_{L^\infty(Q_{\frac14}\times(-\frac{\rho}{4},\frac{\rho}{4}))}\leq\rho^{-|\gamma|-q}|\gamma|!\,q!\,e^{1/T}H(T,p,\rho),\ \text{for } \gamma\in\mathbb{N}^{n+1},\ q\in\mathbb{N}.$$

Finally, $\mathbf{U}(X,0)=\partial_t^p\mathbf{u}(x,T)$ in $\overline{\Omega}$ and Lemma 3 follows from the latter and (2.12). □

*Remark* 9. Observe that we did not use quantitatively the smoothness of $\partial\Omega$ in the proof of Lemma 4 and that to get the quantitative estimate of Lemma 3 over only $B_{\frac{\delta}{2}}(x_0)\cap\overline{\Omega}\times(0,T]$, with $x_0$ in $\overline{\Omega}$ and $\delta$ as in (1.1), it suffices to know that either $B_\delta(x_0)\subset\Omega$ or that $\partial\Omega\cap B_\delta(x_0)$ is real-analytic.

## 3. Observability for higher order elliptic equations

We can now explain the proof of Theorem 1 by making use of Lemmas 1 and 3 .

*Proof of Theorem 1.* From Lemma 3

$$|\partial_x^\alpha u(x,L)|\leq e^{1/\rho L^{1/(2m-1)}}|\alpha|!\,\rho^{-|\alpha|}\|u(0)\|_{L^2(\Omega)},\ \text{ for } x\in\overline{\Omega}\text{ and } 0<L\leq T$$

and from Lemma 1 there are $N=N(\Omega,|\omega|,\rho)$ and $\theta=\theta(\Omega,|\omega|,\rho)$ in $(0,1)$ such that

$$\|u(L)\|_{L^2(\Omega)}\leq N\|u(L)\|_{L^1(\omega)}^\theta M^{1-\theta},\quad\text{with } M=Ne^{N/L}\|u(0)\|_{L^2(\Omega)}, \tag{3.1}$$

when $\omega\subset\Omega$ is a measurable set with a positive measure. Set for each $t\in(0,T)$,

$$\mathcal{D}_t=\{x\in\Omega:(x,t)\in\mathcal{D}\}\quad\text{and}\quad E=\{t\in(0,T):|\mathcal{D}_t|\geq|\mathcal{D}|/(2T)\}.$$

By Fubini's theorem, $\mathcal{D}_t$ is measurable for a.e. $t\in(0,T)$, $E$ is measurable in $(0,T)$ with $|E|\geq|\mathcal{D}|/(2|\Omega|)$ and $\chi_E(t)\chi_{\mathcal{D}_t}(x)\leq\chi_{\mathcal{D}}(x,t)$ over $\Omega\times(0,T)$. Next, let $q\in(0,1)$ be a constant to be determined later and $l$ be a Lebesgue point of

$E$. Then, from [3, Lemma 2] there is a monotone decreasing sequence $\{l_k\}_{k\geq 1}$ satisfying $\lim_{k\to\infty} l_k = l$, $l < l_1 \leq T$,

$$l_{k+1} - l_{k+2} = q(l_k - l_{k+1}) \quad \text{and} \quad |(l_{k+1}, l_k) \cap E| \geq \frac{l_k - l_{k+1}}{3}, \ \ k \in \mathbb{N}. \tag{3.2}$$

Define

$$\tau_k = l_{k+1} + (l_k - l_{k+1})/6, \quad k \in \mathbb{N}.$$

From (3.1) there are $N = N(\Omega, |\mathcal{D}|, T, \rho)$ and $\theta = \theta(\Omega, |\mathcal{D}|, T, \rho)$, $0 < \theta < 1$, such that

$$\|u(t)\|_{L^2(\Omega)} \leq \left( N e^{\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(t)\|_{L^1(\mathcal{D}_t)} \right)^{\theta} \|u(l_{k+1})\|^{1-\theta}_{L^2(\Omega)},$$

when $t \in [\tau_k, l_k] \cap E$. Integrating the above inequality over $(\tau_k, l_k) \cap E$, from Young's inequality and the standard energy estimate for the solutions to (1.2), we have that for each $\epsilon > 0$,

$$\begin{aligned} \|u(l_k)\|_{L^2(\Omega)} &\leq \epsilon \|u(l_{k+1})\|_{L^2(\Omega)} \\ &\quad + \epsilon^{-\frac{1-\theta}{\theta}} N e^{\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}} \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} \, dt. \end{aligned}$$

Multiplying the above inequality by $\epsilon^{\frac{1-\theta}{\theta}} e^{-\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}}$, replacing $\epsilon$ by $\epsilon^{\theta}$ and finally choosing $\epsilon = e^{-\frac{1}{(l_k - l_{k+1})^{1/(2m-1)}}}$ in the resulting inequality, we obtain that

$$\begin{aligned} & e^{-\frac{N+1-\theta}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_{k+1})\|_{L^2(\Omega)} \\ & \quad \leq N \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} dt. \end{aligned}$$

Therefore, fixing $q$ in (3.2) as $q = \left( \frac{N+1-\theta}{N+1} \right)^{2m-1}$, we have

$$\begin{aligned} & e^{-\frac{N+1-\theta}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(l_{k+1} - l_{k+2})^{1/(2m-1)}}} \|u(l_{k+1})\|_{L^2(\Omega)} \\ & \quad \leq N \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} dt. \end{aligned} \tag{3.3}$$

Summing (3.3) from $k = 1$ to $+\infty$ completes the proof (the telescoping series method). □

To deal with the boundary observability inequalities for the fourth order parabolic evolution, let $\Omega_\rho = \{x \in \mathbb{R}^n : d(x, \overline{\Omega}) < \rho\}$, with $\rho > 0$ sufficiently small. By the inverse function theorem for analytic functions, $\Omega_\rho$ is a domain with analytic boundary (cf. [2, p. 249]) and by standard extension arguments (cf. [12, Chapter I, Theorem 2.3]), the interior null controllability of the system

$$\begin{cases} \partial_t u + \Delta^2 u = \chi_{\Omega_\rho \setminus \Omega} f, & \text{in } \Omega_\rho \times (0, T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_\rho \times (0, T), \\ u(0) = u_0, & \text{in } \Omega_\rho, \end{cases}$$

with initial datum $u_0$ in $L^2(\Omega)$ is a consequence of Theorem 1 (See also Remark 1) by standard duality arguments (HUM method) [20]. The later implies that there are controls $g_1$ and $g_2$ in $L^2(\partial\Omega \times (0, T))$ with

$$\|g_k\|_{L^2(\partial\Omega \times (0,T))} \leq N e^{\frac{N}{T^{1/3}}} \|u_0\|_{L^2(\Omega)}, \quad k = 1, 2,$$

such that the solution $u$ to

$$\begin{cases} \partial_t u + \Delta^2 u = 0, & \text{in } \Omega \times (0,T), \\ u = g_1, \ \ \frac{\partial u}{\partial \nu} = g_2, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

verifies $u(T) \equiv 0$. By a standard duality argument, this full boundary null controllability in turn implies the observability inequality

$$\|\varphi(0)\|_{L^2(\Omega)} \leq e^{N/T^{1/3}} \left[ \|\tfrac{\partial \Delta \varphi}{\partial \nu}\|_{L^2(\partial\Omega\times(0,T))} + \|\Delta\varphi\|_{L^2(\partial\Omega\times(0,T))} \right],$$

for solutions $\varphi$ to the dual equation

$$\begin{cases} -\partial_t \varphi + \Delta^2 \varphi = 0, & \text{in } \Omega \times (0,T), \\ \varphi = \frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

with initial datum $\varphi(T) = \varphi_T$ in $L^2(\Omega)$. Thus, we can derive from the above lines and from the decay of the energy the following result.

**Lemma 5.** *There is $N = N(\Omega, \delta)$ such that the interpolation inequality*

$$\begin{aligned} &\|u(T)\|_{L^2(\Omega)} \\ &\leq \left( e^{N/[(\epsilon_2-\epsilon_1)T^{\frac{1}{3}}]} \left[ \|\tfrac{\partial \Delta u}{\partial \nu}\|_{L^2(\partial\Omega\times[\epsilon_1 T,\epsilon_2 T])} + \|\Delta u\|_{L^2(\partial\Omega\times[\epsilon_1 T,\epsilon_2 T])} \right] \right)^{\frac{1}{2}} \|u_0\|_{L^2(\Omega)}^{\frac{1}{2}}, \end{aligned}$$

*holds for all solutions $u$ to (1.4) and $0 \leq \epsilon_1 < \epsilon_2 \leq 1$.*

Lemmas 3 and 5 imply in a similar way to the reasonings in [3, Theorem 11] the following result.

**Lemma 6.** *Assume that $E \subset (0,T)$ is a measurable set of positive measure and that $\Gamma_i \subset \partial\Omega$, $i = 1, 2$, are measurable subsets with $|\Gamma_1|, |\Gamma_2| \geq \gamma_0 > 0$. Then, for each $\eta \in (0,1)$ there are $N = N(\Omega, \eta, \gamma_0, \delta) \geq 1$ and $\theta = \theta(\Omega, \eta, \gamma_0, \delta)$, $0 < \theta < 1$, such that the inequality*

(3.4)

$$\begin{aligned} &\|u(t_2)\|_{L^2(\Omega)} \leq \\ &\left( e^{N/(t_2-t_1)^{1/3}} \int_{t_1}^{t_2} \chi_E(t) \left[ \|\tfrac{\partial \Delta u(t)}{\partial \nu}\|_{L^1(\Gamma_1)} + \|\Delta u(t)\|_{L^1(\Gamma_2)} \right] dt \right)^{\theta} \|u(t_1)\|_{L^2(\Omega)}^{1-\theta}, \end{aligned}$$

*holds for all solutions $u$ to (1.4), when $0 \leq t_1 < t_2 \leq T$ and $|(t_1,t_2) \cap E| \geq \eta(t_2 - t_1)$. Moreover,*

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(t_2-t_1)^{1/3}}} \|u(t_2)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(q(t_2-t_1))^{1/3}}} \|u(t_1)\|_{L^2(\Omega)} \\ &\leq N \int_{t_1}^{t_2} \chi_E(t) \left[ \|\tfrac{\partial \Delta u(t)}{\partial \nu}\|_{L^1(\Gamma_1)} + \|\Delta u(t)\|_{L^1(\Gamma_2)} \right] dt, \quad \text{when} \quad q \geq \left( \tfrac{N+1-\theta}{N+1} \right)^3. \end{aligned}$$

*Proof.* Suppose that $0 < \eta < 1$ satisfies $|(t_1,t_2) \cap E| \geq \eta(t_2 - t_1)$. Set

$$\tau = t_1 + \frac{\eta}{20}(t_2 - t_1), \ \tilde{t}_1 = t_1 + \frac{\eta}{8}(t_2 - t_1),$$
$$\tilde{t}_2 = t_2 - \frac{\eta}{8}(t_2 - t_1), \ \tilde{\tau} = t_2 - \frac{\eta}{20}(t_2 - t_1).$$

Then, $t_1 < \tau < \tilde{t}_1 < \tilde{t}_2 < \tilde{\tau} < t_2$ and $|E \cap (\tilde{t}_1, \tilde{t}_2)| \geq \frac{3\eta}{4}(t_2 - t_1)$ and it follows from Lemma 5 that there is $N = N(\Omega, \eta, \delta)$ such that

$$\|u(t_2)\|_{L^2(\Omega)} \leq e^{N/(t_2-t_1)^{1/3}} \big[\|\tfrac{\partial \Delta u}{\partial \nu}\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))} + \|\Delta u\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))}\big]^{1/2} \|u(t_1)\|^{1/2}_{L^2(\Omega)}.$$

Next, the inequality

$$\|\tfrac{\partial \Delta u}{\partial \nu}\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))} \leq \|\tfrac{\partial \Delta u}{\partial \nu}\|^{1/2}_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))} \|\tfrac{\partial \Delta u}{\partial \nu}\|^{1/2}_{L^\infty(\partial\Omega\times(\tau,\tilde{\tau}))}$$

and Lemma 3 show that

$$\|\tfrac{\partial \Delta u}{\partial \nu}\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))} \leq N e^{\frac{N}{(t_2-t_1)^{1/3}}} \|u(t_1)\|^{1/2}_{L^2(\Omega)} \|\tfrac{\partial \Delta u}{\partial \nu}\|^{1/2}_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))}. \tag{3.5}$$

Set $v(x,t) = \frac{\partial \Delta u}{\partial \nu}(x,t)$, for $x$ in $\partial\Omega$ and $t > 0$. Then,

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))} \leq (\tilde{\tau} - \tau) \int_{\partial\Omega} \|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})}\, d\sigma\,. \tag{3.6}$$

Denote the interval $[\tau, \tilde{\tau}]$ as $[a, a+L]$, with $a = \tau$ and $L = \tilde{\tau} - \tau = (1 - \frac{\eta}{10})(t_2 - t_1)$. Then, Lemma 3 shows that there is $N = N(\Omega, \eta, \delta)$ such that for each fixed $x$ in $\partial\Omega$, $\tau \leq t \leq \tilde{\tau}$ and $p \geq 0$,

$$|\partial_t^p v(x,t)| \leq \frac{e^{N/(t_2-t_1)^{1/3}} p!}{(\eta(t_2-t_1)/40)^p} \|u(t_1)\|_{L^2(\Omega)} \triangleq \frac{Mp!}{(2\rho L)^\beta}\,, \tag{3.7}$$

with

$$M = e^{N/(t_2-t_1)^{1/3}} \|u(t_1)\|_{L^2(\Omega)} \quad \text{and} \quad \rho = \frac{\eta}{8\,(10-\eta)}\,.$$

Hence it follows from (3.7) and [3, Lemma 13] that

$$\|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})} \leq \Big(\fint_{E\cap(\tilde{t}_1,\tilde{t}_2)} |v(x,t)|\, dt\Big)^\gamma \Big(N e^{N/(t_2-t_1)^{1/3}} \|u(t_1)\|_{L^2(\Omega)}\Big)^{1-\gamma},$$

for all $x$ in $\partial\Omega$, with $N = N(\Omega, \eta, \delta)$ and $\gamma = \gamma(\eta)$ in $(0,1)$. This, along with (3.6) and Hölder's inequality leads to

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))} \leq e^{\frac{N}{(t_2-t_1)^{1/3}}} \Big(\int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\partial\Omega} |v(x,t)|\, d\sigma dt\Big)^\gamma \|u(t_1)\|^{1-\gamma}_{L^2(\Omega)}, \tag{3.8}$$

with some new $N$ and $\gamma$ as above. Because, $t - t_1 \geq \tilde{t}_1 - t_1 = \frac{\eta}{8}\,(t_2 - t_1)$, when $t \in (\tilde{t}_1, \tilde{t}_2)$, we get from Lemma 3 that

$$\|\partial^\alpha_{x'} v(t)\|_{L^\infty(\partial\Omega)} \leq \frac{e^{N/(t_2-t_1)^{1/3}} |\alpha|!}{\rho^{|\alpha|}}\, \|u(t_1)\|_{L^2(\Omega)}, \text{ for } \alpha \in \mathbb{N}^{n-1}$$

and for some new constants $N = N(\Omega, \eta, \delta)$ and $\rho = \rho(\Omega, \delta)$. By the obvious generalization of Lemma 1 to the case of real-analytic functions defined over analytic hypersurfaces in $\mathbb{R}^n$, there are $N = N\,(\Omega, \eta, |\Gamma_1|, \delta)$ and $\vartheta = \vartheta\,(\Omega, |\Gamma_1|, \delta)$, $0 < \vartheta < 1$, such that

$$\int_{\partial\Omega} |v(x,t)|\, d\sigma \leq \left(\int_{\Gamma_1} |v(x,t)|\, d\sigma\right)^\vartheta \left(e^{N/(t_2-t_1)^{1/3}} \|u(t_1)\|_{L^2(\Omega)}\right)^{1-\vartheta}, \tag{3.9}$$

when $t \in E \cap (\tilde{t}_1, \tilde{t}_2)$, and it follows from (3.8), (3.9) as well as Hölder's inequality that

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))} \leq \Big(e^{N/(t_2-t_1)^{1/3}} \int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\Gamma_1} |v(x,t)|\, d\sigma dt\Big)^{\vartheta\gamma} \|u(t_1)\|^{1-\vartheta\gamma}_{L^2(\Omega)}.$$

This, together with (3.5) and the definition of $v$ leads to

$$\|\tfrac{\partial\Delta u}{\partial\nu}\|_{L^2(\partial\Omega\times(\tau,\tilde\tau))}\leq\Big(e^{\frac{N}{(t_2-t_1)^{1/3}}}\int_{E\cap(\tilde t_1,\tilde t_2)}\int_{\Gamma_1}|\tfrac{\partial\Delta u}{\partial\nu}(x,t)|\,d\sigma dt\Big)^{\theta_1}\|u(t_1)\|^{1-\theta_1}_{L^2(\Omega)}.$$

Similarly, we can get that

$$\|\Delta u\|_{L^2(\partial\Omega\times(\tau,\tilde\tau))}\leq\Big(e^{\frac{N}{(t_2-t_1)^{1/3}}}\int_{E\cap(\tilde t_1,\tilde t_2)}\int_{\Gamma_2}\big|\Delta u(x,t)\big|\,d\sigma dt\Big)^{\theta_2}\|u(t_1)\|^{1-\theta_2}_{L^2(\Omega)}.$$

These last two inequalities, as well as the fact that

$$\frac{a^\theta+b^\theta}{2}\leq\Big(\frac{a+b}{2}\Big)^\theta,\ \text{ when }\ a,b>0,\ 0<\theta<1,$$

lead to the first desired estimate (3.4). Next, applying Young's inequality to (3.4), we obtain that for each $\varepsilon>0$,

$$\begin{aligned}\|u(t_2)\|_{L^2(\Omega)}&\leq\varepsilon\|u(t_1)\|_{L^2(\Omega)}\\&\quad+\varepsilon^{-\frac{1-\theta}{\theta}}Ne^{\frac{N}{(t_2-t_1)^{1/3}}}\int_{t_1}^{t_2}\chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\Gamma_1)}+\|\Delta u(t)\|_{L^1(\Gamma_2)}\big]\,dt.\end{aligned}$$

Hence, after some computations, we may get that

$$\begin{aligned}&\varepsilon^{1-\theta}e^{-\frac{N}{(t_2-t_1)^{1/3}}}\|u(t_2)\|_{L^2(\Omega)}-\varepsilon e^{-\frac{N}{(t_2-t_1)^{1/3}}}\|u(t_1)\|_{L^2(\Omega)}\\&\qquad\leq\int_{t_1}^{t_2}\chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\Gamma_1)}+\|\Delta u(t)\|_{L^1(\Gamma_2)}\big]\,dt,\ \text{ for all }\epsilon>0.\end{aligned}$$

Choosing now $\varepsilon=e^{-\frac{1}{(t_2-t_1)^{1/3}}}$ implies the second estimate in the Lemma. □

We now complete the proof of Theorems 2 and 3.

*Proof of Theorems 2 and 3.* Set for each $t\in(0,T)$

$$\mathcal{J}_t=\{x\in\partial\Omega:(x,t)\in\mathcal{J}\}\quad\text{and}\quad E=\{t\in(0,T):|\mathcal{J}_t|\geq|\mathcal{J}|/(2T)\}.$$

By Fubini's theorem, $\mathcal{J}_t$ is measurable for a.e. $t\in(0,T)$, $E$ is measurable in $(0,T)$ with $|E|\geq|\mathcal{J}|/(2|\partial\Omega|)$ and $\chi_E(t)\chi_{\mathcal{J}_t}(x)\leq\chi_{\mathcal{J}}(x,t)$ over $\partial\Omega\times(0,T)$. Then, with similar arguments as the ones in the proof of Lemma 6, we can get that for each $0<\eta<1$, there are $N=N(\Omega,\eta,|\mathcal{J}|,T,\delta)$ and $\theta=\theta(\Omega,\eta,|\mathcal{J}|,T,\delta)$ with $0<\theta<1$, such that

$$\begin{aligned}&\|u(t_2)\|_{L^2(\Omega)}\leq\\&\Big(Ne^{N/(t_2-t_1)^{1/3}}\int_{t_1}^{t_2}\chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)}+\|\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt\Big)^{\theta}\|u(t_1)\|^{1-\theta}_{L^2(\Omega)},\end{aligned}$$

holds for all solutions $u$ to (1.4), when $0\leq t_1<t_2\leq T$ and $|(t_1,t_2)\cap E|\geq\eta(t_2-t_1)$. Moreover,

(3.10)

$$\begin{aligned}&e^{-\frac{N+1-\theta}{(t_2-t_1)^{1/3}}}\|u(t_2)\|_{L^2(\Omega)}-e^{-\frac{N+1-\theta}{(q(t_2-t_1))^{1/3}}}\|u(t_1)\|_{L^2(\Omega)}\\&\leq N\int_{t_1}^{t_2}\chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)}+\|\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt,\ \text{ when }\ q\geq\Big(\tfrac{N+1-\theta}{N+1}\Big)^3.\end{aligned}$$

Now, let $\eta=1/3$ and $q=(N+1-\theta)^3/(N+1)^3$ with $N$ and $\theta$ as above. Assume that $l$ is a Lebesgue point of $E$. By [3, Lemma 2], there is a monotone decreasing

sequence $\{l_k\}_{k\geq 1}$ in $(0,T)$ satisfying $\lim_{k\to\infty} l_k = l$, $l < l_1 \leq T$ and (3.2). These, together with (3.10), imply that

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(l_k-l_{k+1})^{1/3}}}\|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(l_{k+1}-l_{k+2})^{1/3}}}\|u(l_{k+1})\|_{L^2(\Omega)} \\ &\leq N\int_{l_{k+1}}^{l_k} \chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt,\ k\in\mathbb{N}. \end{aligned} \tag{3.11}$$

Finally, adding up (3.11) from $k=1$ to $+\infty$ (the telescoping series) we get that

$$\begin{aligned} \|u(l_1)\|_{L^2(\Omega)} &\leq Ne^{\frac{N+1-\theta}{(l_1-l_2)^{1/3}}}\int_l^{l_1} \chi_E(t)\big[\|\tfrac{\partial\Delta u}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt \\ &\leq N\int_{\mathcal{J}} |\tfrac{\partial\Delta u}{\partial\nu}(x,t)| + |\Delta u(x,t)|\,d\sigma dt, \end{aligned}$$

which completes the proof of Theorem 2.

The previous reasonings show that Lemma 6, as well as [3, Lemma 2] and the telescoping series method imply the observability inequality from two possibly distinct measurable subsets of $\partial\Omega\times(0,T)$ in Theorem 3. □

## 4. Observability for second order systems

Now, we can apply Lemmas 3, 1 and the telescoping series method to sketch a proof Theorem 4.

*Proof of Theorem 4.* From Lemma 3,

$$|\partial_x^\alpha \mathbf{u}(x,L)| \leq e^{1/\rho L}|\alpha|!\,\rho^{-|\alpha|}\|\mathbf{u}(0)\|_{L^2(\Omega)^\ell},\ \text{ for all }\ x\in\overline{\Omega},\ \alpha\in\mathbb{N}^n.$$

Hence, for each $\eta = 1,\dots,\ell$, it holds that

$$|\partial_x^\alpha u^\eta(x,L)| \leq M|\alpha|!\,\rho^{-|\alpha|},\ \text{for all } \alpha\in\mathbb{N}^n,\ x\in\overline{\Omega},\ \text{with } M = e^{1/\rho L}\|\mathbf{u}(0)\|_{L^2(\Omega)^\ell}.$$

From the propagation of smallness for real-analytic functions from measurable sets (cf. Lemma 1), we get that for each $\eta = 1,\dots,\ell$, there are $N_\eta = N_\eta(\Omega,\omega_0,\delta)$ and $\theta_\eta = \theta_\eta(\Omega,\omega_0,\delta)$, $0<\theta_\eta<1$, such that

$$\|u^\eta(L)\|_{L^2(\Omega)} \leq N_\eta\|u^\eta(L)\|_{L^1(\omega_\eta)}^{\theta_\eta} M^{1-\theta_\eta}.$$

Let $N = \max_{1\leq\eta\leq\ell}\{N_\eta\}$ and $\theta = \min_{1\leq\eta\leq\ell}\{\theta_\eta\}$. Then, we get the following interpolation inequality with $\ell$ different observations:

$$\begin{aligned} \|\mathbf{u}(L)\|_{L^2(\Omega)^\ell} &\leq N\Big(\sum_{\eta=1}^{\ell}\|u^\eta(L)\|_{L^1(\omega_\eta)}^{\theta}\Big)M^{1-\theta} \\ &\leq N\Big(\sum_{\eta=1}^{\ell}\|u^\eta(L)\|_{L^1(\omega_\eta)}\Big)^{\theta}\Big(Ne^{N/L}\|\mathbf{u}(0)\|_{L^2(\Omega)^\ell}\Big)^{1-\theta}. \end{aligned} \tag{4.1}$$

Next, let $q\in(0,1)$ be a constant to be determined later and $l$ be a Lebesgue point of $E$. Then, by [3, Lemma 2] there is a decreasing sequence $\{l_m\}_{m\geq 1}$ satisfying $\lim_{m\to\infty} l_m = l$, $l<l_1\leq T$ and (3.2). Define as before for each $m\in\mathbb{N}$,

$$\tau_m = l_{m+1} + (l_m - l_{m+1})/6.$$

Then, by the decay of the energy of the solutions $\mathbf{u}$ to (1.5),

$$\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} \leq \|\mathbf{u}(t)\|_{L^2(\Omega)^\ell},\ \text{for all } t\in(\tau_m,l_m). \tag{4.2}$$

Moreover, it follows from (4.1) that

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^\ell} \leq \Big(Ne^{\frac{N}{l_m-l_{m+1}}}\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}\Big)^\theta \|\mathbf{u}(l_{m+1})\|^{1-\theta}_{L^2(\Omega)^\ell}, \text{ for } \tau_m \leq t < l_m.$$

Applying the Young inequality, we get that for each $\epsilon > 0$,

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^\ell} \leq \epsilon\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} + \epsilon^{-\frac{1-\theta}{\theta}}Ne^{\frac{N}{l_m-l_{m+1}}}\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)},$$

for $\tau_m \leq t < l_m$. Integrating the above inequality over $(\tau_m, l_m)\cap E$, we have by (4.2) that for each $\epsilon > 0$,

$$\begin{aligned}\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} &\leq \epsilon\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell}\\ &\quad + \epsilon^{-\frac{1-\theta}{\theta}}Ne^{\frac{N}{l_m-l_{m+1}}}\int_{l_{m+1}}^{l_m}\chi_E\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}\,dt.\end{aligned}$$

Multiplying the above inequality by $\epsilon^{\frac{1-\theta}{\theta}}e^{-\frac{N}{l_m-l_{m+1}}}$ and replacing $\epsilon$ by $\epsilon^\theta$, we get

$$\begin{aligned}\epsilon^{1-\theta}e^{-\frac{N}{l_m-l_{m+1}}}\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} &\leq \epsilon e^{-\frac{N}{l_m-l_{m+1}}}\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell}\\ &\quad + N\int_{l_{m+1}}^{l_m}\chi_E\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}dt.\end{aligned}$$

Choosse then $\epsilon = e^{-\frac{1}{l_m-l_{m+1}}}$ to obtain that

$$\begin{aligned}&e^{-\frac{N+1-\theta}{l_m-l_{m+1}}}\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} - e^{-\frac{N+1}{l_m-l_{m+1}}}\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell}\\ &\leq N\int_{l_{m+1}}^{l_m}\chi_E\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}dt, \quad \text{when } m \geq 0.\end{aligned} \tag{4.3}$$

Finally, we take $q = \frac{N+1-\theta}{N+1}$. Clearly, $0 < q < 1$ and from (4.3) and (3.2)

$$\begin{aligned}&e^{-\frac{N+1-\theta}{l_m-l_{m+1}}}\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} - e^{-\frac{N+1-\theta}{l_{m+1}-l_{m+2}}}\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell}\\ &\leq N\int_{l_{m+1}}^{l_m}\chi_E\sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}dt\,.\end{aligned} \tag{4.4}$$

Summing (4.4) from $m = 1$ to $+\infty$ completes the proof. □

Because the full boundary $\partial\Omega$ is analytic, we can use the global internal null controllability for the system (1.5) (a known consequence of Theorem 7 by duality) and the standard extension method (cf. [2, p. 249]) to get the following boundary null controllability: for each $\mathbf{u}_0$ in $L^2(\Omega)^\ell$, there is $\mathbf{g} \in L^2(\partial\Omega\times(0,T))^\ell$, with

$$\|\mathbf{g}\|_{L^2(\partial\Omega\times(0,T))^\ell} \leq Ne^{N/T}\|\mathbf{u}_0\|_{L^2(\Omega)^\ell},$$

such that the solution $\mathbf{u}$ to

$$\begin{cases}\partial_t\mathbf{u} - \mathbf{L}^*\mathbf{u} = 0, & \text{in } \Omega\times(0,T),\\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega\times(0,T),\\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega.\end{cases}$$

verifies $\mathbf{u}(T)=0$. Also, by the standard duality argument [20], this boundary null controllability in turn implies the observability inequality:

$$\|\mathbf{w}(0)\|_{L^2(\Omega)^\ell}\leq Ne^{N/T}\|\tfrac{\partial \mathbf{w}}{\partial \nu}\|_{L^2(\partial\Omega\times(0,T))},\quad \big(\tfrac{\partial \mathbf{w}}{\partial \nu}\big)^\xi=a_{ij}^{\xi\eta}\partial_{x_j}w^\eta\nu_i,\quad \xi=1,\dots,\ell,$$

for all solutions $\mathbf{w}$ to

$$\begin{cases}\partial_t\mathbf{w}+\mathbf{L}\mathbf{w}=0, & \text{in } \Omega\times(0,T),\\ \mathbf{w}=0, & \text{on } \partial\Omega\times(0,T),\\ \mathbf{w}(T)=\mathbf{w}_T, & \text{in } \Omega.\end{cases}$$

with $\mathbf{w}_T$ in $L^2(\Omega)^\ell$. Hence, from the latter and the local energy bound for the system (1.5), we can derive the following.

**Lemma 7.** *There is $N=N(\Omega,\varrho,\delta)\geq 1$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell}\leq\Big(e^{N/[(\epsilon_2-\epsilon_1)T]}\|\tfrac{\partial \mathbf{u}}{\partial \nu}\|_{L^2(\partial\Omega\times(\epsilon_1T,\epsilon_2T))}\Big)^{1/2}\|\mathbf{u}_0\|^{1/2}_{L^2(\Omega)^\ell},$$

*holds for any $0\leq\epsilon_1<\epsilon_2\leq 1$ and for all solutions $\mathbf{u}$ to (1.5).*

The Lemmas 3 and 7 imply now with similar reasonings to the ones we used in [3, Theorem 11], in the proof of Lemma 6, as well as in the proofs of Theorem 3 and Theorem 4, that Theorems 5 and 6 hold.

To prove Theorem 8 we need to complete first the proof of Lemma 2. With this purpose, we begin with the following lemma.

**Lemma 8.** *Let $f:[0,1]\to\mathbb{R}$ be an analytic function verifying*

$$\|f^{(m)}\|_{L^\infty(0,1)}\leq M\rho^{-m}m!,\ \text{when } m\geq 0, \tag{4.5}$$

*for some $M>0$ and $0<\rho\leq 1/2$. Then*

$$\|f^{(j)}\|_{L^\infty(0,1)}\leq\big(8M(j+1)!\rho^{-j-1}\big)^{1-\frac{1}{2^j}}\|f\|^{\frac{1}{2^j}}_{L^\infty(0,1)},\quad \text{when } j\geq 0. \tag{4.6}$$

*Proof.* We prove it by induction and we assume that (4.6) holds for $(k-1)$, i.e.,

$$\|f^{(k-1)}\|_{L^\infty(0,1)}\leq(8Mk!\rho^{-k})^{1-\frac{1}{2^{k-1}}}\|f\|^{\frac{1}{2^{k-1}}}_{L^\infty(0,1)} \tag{4.7}$$

and we show that it is valid for $k$. Let then $x\in[0,1]$. For $0<\varepsilon\leq 1/2$ take either $I=[x,x+\varepsilon]$ or $[x-\varepsilon,x]$, so that always $I\subset[0,1]$. Then,

$$f^{(k)}(x)=f^{(k)}(y)+\int_y^x f^{(k+1)}(s)\,ds,\ \text{ for all } y\in I.$$

Integrating the above identity with respect to $y$ over the interval $I$, by (4.5) and the arbitrariness of $x$ in $[0,1]$, we obtain that

$$\|f^{(k)}\|_{L^\infty(0,1)}\leq\varepsilon M(k+1)!\rho^{-k-1}+\frac{2}{\varepsilon}\|f^{(k-1)}\|_{L^\infty(0,1)}, \tag{4.8}$$

when $k\geq 1$ and $0<\varepsilon\leq 1/2$. Choose now

$$\varepsilon=\Big(\frac{2\|f^{(k-1)}\|_{L^\infty(0,1)}}{M(k+1)!\rho^{-k-1}}\Big)^{1/2}.$$

It can be checked by (4.5) that $\varepsilon\leq 1/2$. Hence, it follows from (4.8) that

$$\|f^{(k)}\|_{L^\infty(0,1)}\leq\big(8M(k+1)!\rho^{-k-1}\big)^{1/2}\|f^{(k-1)}\|^{1/2}_{L^\infty(0,1)}.$$

This, together with (4.7), leads to (4.6) and completes the proof. □

The rescaled and translated version of Lemma 8, along with [3, Lemma 13], imply the following.

**Lemma 9.** *Let $f$ be real-analytic in $[a, a+L]$ with $a$ in $\mathbb{R}$, $L > 0$ and $E \subset [a, a+L]$ be a measurable set with positive measure. Assume there are constants $M > 0$ and $0 < \rho \leq 1/2$ such that*

$$|f^{(m)}(x)| \leq M(2\rho L)^{-m} m!, \text{ for } m \geq 0 \text{ and } a \leq x \leq a + L.$$

*Then, there are $N = N(\rho, |E|/L)$ and $\theta = \theta(\rho, |E|/L)$ with $0 < \theta < 1$, such that*

$$\|f^{(k)}\|_{L^\infty(a,a+L)} \leq N\big(8(k+1)!(\rho L)^{-(k+1)}\big) M^{1-\frac{\theta}{2^k}} \Big(\fint_E |f|\, dx\Big)^{\frac{\theta}{2^k}}, \quad \text{when } k \geq 0.$$

Next, we derive the multi-dimensional analogs of Lemmas 8 and 9.

**Lemma 10.** *Let $n \geq 1$ and $f : Q \subset \mathbb{R}^n \to \mathbb{R}$, with $Q = [0,1] \times \cdots \times [0,1]$, be a real-analytic function verifying*

$$\|\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} f\|_{L^\infty(Q)} \leq M\rho^{-|\beta|}\beta_1! \cdots \beta_n!, \ \ \forall \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n, \tag{4.9}$$

*for some $M > 0$ and $0 < \rho \leq 1/2$. Then,*

$$\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f\|_{L^\infty(Q)} \leq \Big(8M\rho^{-|\alpha|-1} \prod_{i=1}^{n} (\alpha_i + 1)!\Big)^{1-\frac{1}{2^{|\alpha|}}} \|f\|_{L^\infty(Q)}^{\frac{1}{2^{|\alpha|}}}. \tag{4.10}$$

*holds for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.*

*Proof.* First, notice that Lemma 8 corresponds to Lemma 10, when $n = 1$. Let now $n \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be in $\mathbb{N}^n$. For $(x_1, \ldots, x_{n-1})$ in $[0,1] \times \cdots \times [0,1]$, define the function $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x_n) \triangleq \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f(x_1, \cdots, x_{n-1}, x_n).$$

It follows from (4.9) that

$$\|\partial_{x_n}^{\beta_n} g_n\|_{L^\infty([0,1])} \leq \Big(M\alpha_1! \cdots \alpha_{n-1}! \rho^{-\sum_{j=1}^{n-1} \alpha_j}\Big) \beta_n! \rho^{-\beta_n}, \text{ for all } \beta_n \geq 0,$$

and Lemma 8 yields that

$$\begin{aligned} &\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f\|_{L^\infty(Q)} \\ &\leq \Big(8M\alpha_1! \cdots \alpha_{n-1}! \rho^{-\sum_{j=1}^{n-1} \alpha_j} (\alpha_n + 1)! \rho^{-\alpha_n - 1}\Big)^{1-\frac{1}{2^{\alpha_n}}} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f\|_{L^\infty(Q)}^{\frac{1}{2^{\alpha_n}}}. \end{aligned}$$

Similarly, we can show that $\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f\|_{L^\infty(Q)}$ is less or equal than

$$\Big(8M\alpha_1! \cdots \alpha_{n-2}! \rho^{-\sum_{j=1}^{n-2} \alpha_j} (\alpha_{n-1} + 1)! \rho^{-\alpha_{n-1} - 1}\Big)^{1-\frac{1}{2^{\alpha_{n-1}}}} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-2}}^{\alpha_{n-2}} f\|_{L^\infty(Q)}^{\frac{1}{2^{\alpha_{n-1}}}}.$$

The iteration of the above arguments $n$ times leads to the desired estimates in (4.10). □

The rescaled and translated versions of Lemma 10 and of Lemma 1 (when $\Omega$ is the unit ball or cube in $\mathbb{R}^n$) and the fact that a ball in $\mathbb{R}^n$ contains a cube of comparable diameter and vice versa are seen to impy Lemma 2 .

Finally, we give the proof of Theorem 8, where we use Lemma 9 with $k = 1$ and Lemma 2 with $|\alpha| \leq 2$.

*Proof of Theorem 8.* Since $b(\cdot) \not\equiv 0$ in $\Omega$ and $b$ is real-analytic in $\overline{\Omega}$, we may assume without loss of generality, that $|b(x)| \geq 1$ over some ball $B_R(x_0) \subset \Omega$ and that $\mathcal{D} \subset B_R(x_0) \times (0,T)$. By Lemma 3, for $x$ in $\overline{\Omega}$ and $0 \leq s < t$,

$$
\begin{aligned}
(4.11)\quad & |\partial_x^\alpha \partial_t^p u(x,t)| + |\partial_x^\alpha \partial_t^p v(x,t)| \\
& \qquad \leq e^{1/\rho(t-s)} |\alpha|!\, p!\, \rho^{-|\alpha|-p}\, (t-s)^{-p} \left[ \|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \right],
\end{aligned}
$$

for all $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$, with $\rho = \rho(\delta)$, $0 < \rho \leq 1$. Hence, we can get from (4.1) that

$$
\begin{aligned}
& \|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \leq \\
& \Big( \int_{B_R(x_0)} |u(x,t)| + |v(x,t)|\, dx \Big)^{\theta} \Big( N e^{N/(t-s)} \big( \|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \big) \Big)^{1-\theta},
\end{aligned}
$$

with $N = N(\Omega, \rho, R)$ and $\theta = \theta(\Omega, \rho, R)$, $0 < \theta < 1$. This, together with the fact that $|b(x)| \geq 1$ over $B_R(x_0)$ and the first equation in (1.10), yield that

$$
\begin{aligned}
(4.12)\quad & \|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \\
& \qquad \leq \Big( \int_{B_R(x_0)} |u(x,t)| + |\partial_t u(x,t)| + |\Delta u(x,t)|\, dx \Big)^{\theta} \\
& \qquad \times \Big( N e^{N/(t-s)} \big( \|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \big) \Big)^{1-\theta}, \text{ when } 0 \leq s < t.
\end{aligned}
$$

Next, let $\eta \in (0,1)$ and $0 \leq t_1 < t_2$. Also, assume that $E \subset (0,T)$ is a measurable set with $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$, for some $\eta \in (0,1)$, and that for each $t \in E$, $|\mathcal{D}_t| \triangleq |\{x \in \Omega : (x,t) \in \mathcal{D}\}| \geq \gamma |\mathcal{D}|$, for some $\gamma > 0$. Set then

$$
\tau = t_1 + \frac{\eta}{10}(t_2 - t_1) \text{ and } F = [\tau, t_2] \cap E.
$$

Clearly, $|F| \geq \frac{\eta}{2}(t_2 - t_1)$. Hence, it follows from (4.11) that when $t \in [\tau, t_2]$ and $x$ is in $\Omega$

$$
|\partial_t^p u(x,t)| \leq \frac{p! N e^{N/\eta(t_2 - t_1)}}{(\eta(t_2 - t_1)/20)^p} \big( \|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)} \big), \text{ for all } p \in \mathbb{N},
$$

with $N = N(\Omega, \rho)$. By Lemma 9, we have that for each $x$ in $\Omega$

$$
\begin{aligned}
& \|\partial_t u(x,\cdot)\|_{L^\infty([\tau,t_2])} \leq \\
& \qquad \Big( \int_F |u(x,s)|\, ds \Big)^{\theta} \Big( N e^{N/(t_2 - t_1)} \big( \|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)} \big) \Big)^{1-\theta},
\end{aligned}
$$

with $N = N(\Omega, \rho, \eta)$ and $\theta = \theta(\Omega, \rho, \eta)$, $0 < \theta < 1$. Hence, by Hölder's inequality

$$
\begin{aligned}
(4.13)\quad & \int_{B_R(x_0)} |\partial_t u(x,t)|\, dx \leq \\
& \Big( N e^{N/(t_2 - t_1)} \big( \|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)} \big) \Big)^{1-\theta} \Big( \int_F \int_{B_R(x_0)} |u(x,s)|\, dx ds \Big)^{\theta}
\end{aligned}
$$

when $\tau \leq t \leq t_2$. It also follows from (4.11) that when $\tau \leq t \leq t_2$ and $x$ is in $\Omega$, we have

$$
|\partial_x^\alpha u(x,t)| \leq |\alpha|! \rho^{-|\alpha|} N e^{N/(t_2 - t_1)} \big( \|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \big), \text{ for all } \alpha \in \mathbb{N}^n,
$$

with $N = N(\Omega, \rho, \eta)$. Now, it holds that for each $t \in F$, $|\mathcal{D}_t| \geq \gamma |\mathcal{D}|$, and it follows from Theorem 2 that

$$\int_{B_R(x_0)} |u(x,t)|\,dx \leq \Big(\int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta} \Big(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\Big)^{1-\theta} \tag{4.14}$$

and

$$\int_{B_R(x_0)} |\Delta u(x,t)|\,dx \leq \Big(\int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta} \Big(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\Big)^{1-\theta}. \tag{4.15}$$

with $N = N(\Omega, |\mathcal{D}|, R, \rho, \eta)$ and $\theta = \theta(\Omega, |\mathcal{D}|, R, \rho, \eta)$, $0 < \theta < 1$. Hence, (4.13) and (4.14), as well as Hölder's inequality imply that

$$\int_{B_R(x_0)} |\partial_t u(x,t)|\,dx \leq \Big(\int_{t_1}^{t_2} \chi_E(s)\|u(s)\|_{L^1(\mathcal{D}_s)}\,ds\Big)^{\theta} \Big(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\Big)^{1-\theta},$$

when $t \in F$. This, together with the inequalities (4.12), (4.14), (4.15) and Hölder's inequality, yield that the inequality

$$\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \leq \Big(\int_{t_1}^{t_2} \chi_E(s)\|u(s)\|_{L^1(\mathcal{D}_s)}\,ds + \int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta} \times \Big(Ne^{N/(t_2-t_1)}\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\Big)^{1-\theta},$$

holds for $t \in F$. Integrating the above inequality with respect to time over the set $F$, recalling that $|F| \geq \frac{\eta}{2}(t_2 - t_1)$, using the energy estimate for solutions to the equations (1.10) and Hölder's inequality, we find that

$$\|u(t_2)\|_{L^2(\Omega)} + \|v(t_2))\|_{L^2(\Omega)} \leq \Big(\int_{t_1}^{t_2} \chi_E(t)\|u(t)\|_{L^1(\mathcal{D}_t)}\,dt\Big)^{\theta} \Big(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1))\|_{L^2(\Omega)}\big)\Big)^{1-\theta},$$

with $N = N(\Omega, |\mathcal{D}|, R, \rho, \eta)$ and $\theta = \theta(\Omega, |\mathcal{D}|, R, \rho, \eta)$, $0 < \theta < 1$.

Finally, by Fubini's theorem and following the reasonings within the second part of the proof of Theorem 2 (i.e., the telescoping series method) we can also derive the desired observability estimate in Theorem 8. □

## 5. Applications to control theory

In this Section, we show several applications of Theorems 1, 2, 4 and 8 in control theory. One can also obtain analogous applications of Theorems 3, 5, 6 and 7.

First of all, we can apply Theorem 1 to get the bang-bang property of the time optimal control problems for the higher order parabolic equations (1.2): let $\Omega$ be a bounded domain with analytic boundary and $\omega \subset \Omega$ a non-empty open set (or a

measurable set with positive measure). Define for each $M>0$ a control constraint set

$$\mathcal{U}_1^M \triangleq \Big\{ f:\Omega\times\mathbb{R}^+\to\mathbb{R} \ \text{ measurable} : |f(x,t)|\le M, \ \text{ a.e. in } \Omega\times\mathbb{R}^+\Big\}.$$

For each $u_0$ in $L^2(\Omega)\setminus\{0\}$, consider the time optimal control problem

$$(TP)_1^M: \qquad T_1^M \triangleq \inf_{\mathcal{U}_1^M}\big\{t>0;\ u(t;u_0,f)=0\big\},$$

where $u(\cdot\,;u_0,f)$ is the solution to the controlled problem

$$\begin{cases} \partial_t u+(-1)^m\Delta^m u=\chi_\omega f, & \text{in } \Omega\times(0,+\infty),\\ u=\nabla u=\cdots=\nabla^{m-1}u=0, & \text{on } \partial\Omega\times(0,+\infty),\\ u(0)=u_0. & \text{in } \Omega. \end{cases}$$

According to Theorem 1 and [33, Theorem 3.3], $T_1^M$ is a positive minimum. A control function $f$ in $\mathcal{U}_1^M$ associated to $T_1^M$ is called an optimal control to this problem. Then, the methods in [3, §5] (See also [35] or [32]), and the fact that standard duality (HUM method) and Theorem 1 imply the null controllability at all times $T>0$ of the system (1.2) with bounded controls acting over measurable sets within $\omega\times(0,T)$, give the following result.

**Corollary 1.** *Problem $(TP)_1^M$ has the bang-bang property: any time optimal control $f$ satisfies, $|f(x,t)|=M$, for a.e. $(x,t)$ in $\omega\times(0,T_1^M)$. Consequently, the problem has a unique time optimal control.*

Theorem 2 implies a weak bang-bang property for the time optimal boundary control problems for the fourth order parabolic equation (1.4): let $\Omega$ be as above and $\Gamma\subset\partial\Omega$ be a non-empty open subset (or a measurable set in $\partial\Omega$ with positive surface measure). Define for each $M>0$ the control constraint set

$$\begin{aligned}\mathcal{U}_2^M \triangleq\Big\{&(g_1,g_2):\partial\Omega\times\mathbb{R}^+\to\mathbb{R}^2 \ \text{ measurable};\\ &\max\big\{|g_1(x,t)|,|g_2(x,t)|\big\}\le M, \ \text{ a.e. } (x,t)\in\Omega\times\mathbb{R}^+\Big\}.\end{aligned}$$

For each $u_0$ in $L^2(\Omega)\setminus\{0\}$ consider the time optimal boundary control problem

$$(TP)_2^M: \quad T_2^M \triangleq \inf_{\mathcal{U}_2^M}\big\{t>0;\ u(t;u_0,g_1,g_2)=0\big\},$$

where $u(\cdot\,;u_0,g_1,g_2)$ denotes the solution to the boundary controlled parabolic equation

$$\begin{cases} \partial_t u+\Delta^2 u=0, & \text{in } \Omega\times(0,T),\\ u=g_1\chi_\Gamma, \ \frac{\partial u}{\partial\nu}=g_2\chi_\Gamma, & \text{on } \partial\Omega\times(0,T),\\ u(0)=u_0, & \text{in } \Omega. \end{cases} \tag{5.1}$$

From Theorem 2 and arguments as those in the proof of [3, Lemma 15], $T_2^M$ is a positive minimum. A control pair of functions $(g_1,g_2)$ associated to $T_2^M$ is called an optimal control to this problem. From Theorem 2 and similar methods to those in [3, §5], give the following non-standard bang-bang property:

**Corollary 2.** *Problem $(TP)_2^M$ has the weak bang-bang property: any time optimal control $(g_1,g_2)$ satisfies that $\max\big\{|g_1(x,t)|,|g_2(x,t)|\big\}=M$, for a.e. $(x,t)$ in $\Gamma\times(0,T_2^M)$.*

To carry out the technical details for Corollary 2 we must first solve (5.1) for $u_0$ in $L^2(\Omega)$ and with lateral Dirichlet data $g_i$, $i=1,2$, in $L^\infty(\partial\Omega\times(0,T))$. For this reason by the solution to

$$\begin{cases} \partial_t u+\Delta^2 u=0, & \text{in } \Omega\times(0,T),\\ u=g_1,\ \frac{\partial u}{\partial \nu}=g_2, & \text{on } \partial\Omega\times(0,T),\\ u(0)=u_0, & \text{in } \Omega, \end{cases} \tag{5.2}$$

with $g_i$, $i=1,2$, in $L^2(\partial\Omega\times(0,T))$ and $u_0$ in $C_0^\infty(\Omega)$, we mean the unique function $u$ over $\Omega\times(0,T)$ such that $v=u-e^{t(-\Delta^2)}u_0$ is the solution defined by transposition [20, p. 209] to

$$\begin{cases} \partial_t v+\Delta^2 v=0, & \text{in } \Omega\times(0,T),\\ v=g_1,\ \frac{\partial v}{\partial \nu}=g_2, & \text{on } \partial\Omega\times(0,T),\\ v(0)=0, & \text{in } \Omega, \end{cases} \tag{5.3}$$

i.e.; the unique $v$ in $L^2(\Omega\times(0,T))$ verifying

$$\int_{\Omega\times(0,T)} v\left(-\partial_t\varphi+\Delta^2\varphi\right)\,dxdt=\int_{\partial\Omega\times(0,T)} g_1\,\tfrac{\partial\Delta\varphi}{\partial\nu}-g_2\,\Delta\varphi\,d\sigma dt,$$

for all $\varphi$ in $C^\infty(\overline{\Omega}\times[0,T])$, with $\varphi(T)\equiv 0$ in $\Omega$ and $\varphi=\nabla\varphi=0$ in $\partial\Omega\times(0,T)$. One can make sense of $v$ because for each $h$ in $C^\infty(\overline{\Omega}\times[0,T))$ there is a unique $\varphi$ in $C^\infty(\overline{\Omega}\times[0,T])$ verifying

$$\begin{cases} -\partial_t\varphi+\Delta^2\varphi=h, & \text{in } \Omega\times(0,T),\\ \varphi=\frac{\partial\varphi}{\partial\nu}=0, & \text{in } \partial\Omega\times(0,T),\\ \varphi(T)=0, & \text{in } \Omega, \end{cases}$$

and

$$\|\varphi\|_{L^\infty((0,T),L^2(\Omega))\cap L^2((0,T),H^4(\Omega)\cap H_0^2(\Omega))}\le N\|h\|_{L^2(\Omega\times(0,T))},$$

with $N=N(\Omega,T)$ [10, p. 140, Theorem 10.2]. The above estimate on $\varphi$, standard traces inequalities [7, p. 258] and duality imply the bound

$$\|v\|_{L^2(\Omega\times(0,T))}\le N\left[\|g_1\|_{L^2(\partial\Omega\times(0,T))}+\|g_2\|_{L^2(\partial\Omega\times(0,T))}\right], \tag{5.4}$$

with $N$ as above. For given $T>0$, $u_0$ in $L^2(\Omega)$ and $\mathcal{J}\subset\partial\Omega\times(0,T)$, a measurable set with positive measure, we may assume that $\mathcal{J}\subset\Omega\times(0,T-2\delta)$ for some small $0<\delta<T/2$. Then, the existence of two bounded boundary control functions $g_i$, $i=1,2$, verifying

$$\|g_1\|_{L^\infty(\mathcal{J})}+\|g_2\|_{L^\infty(\mathcal{J})}\le N\|u_0\|_{L^2(\Omega)},$$

with $N$ the constant in (1.3) for the new set $\mathcal{J}$ and such that the solution $u$ to

$$\begin{cases} \partial_t u+\Delta^2 u=0, & \text{in } \Omega\times(0,T),\\ u=g_1\chi_{\mathcal{J}},\ \frac{\partial u}{\partial\nu}=g_2\chi_{\mathcal{J}}, & \text{on } \partial\Omega\times(0,T),\\ u(0)=u_0, & \text{in } \Omega. \end{cases} \tag{5.5}$$

verifies $u(T)\equiv 0$, can be proved by means of a standard duality argument (Hahn Banach Theorem) based on the observability inequality (1.3) [3, Corollary 1] with the purpose to obtain the existence of two functions $g_i$ in $L^\infty(\mathcal{J})$, $i=1,2$, verifying

$$\int_\Omega u_0\varphi(0)\,dx+\int_{\mathcal{J}} g_1\,\tfrac{\partial\Delta\varphi}{\partial\nu}-g_2\,\Delta\varphi\,d\sigma dt=0, \tag{5.6}$$

for all $\varphi_T$ in $C_0^\infty(\Omega)$ and with $\varphi(t) = e^{(t-T)\Delta^2}\varphi_T$. Recall that the unique weak solution $v$ to (5.3) is in fact in $C^\infty(\overline{\Omega}\times[0,+\infty))$, when $g_i$ are both in $C_0^\infty(\partial\Omega\times[0,+\infty))$; and that there is $N = N(\Omega,\delta)$ such that the estimate

$$\|v\|_{C^{2,1}(\Omega\times[T-\frac{\delta}{2},T])} \leq N\|v\|_{L^2(\Omega\times(0,T))},$$

holds when $\text{supp}(g_i)\subset\partial\Omega\times[0,T-\delta]$, $i=1,2$ [10, p.141]. The latter and (5.4) yield the bound

$$\|v\|_{C^{2,1}(\Omega\times[T-\frac{\delta}{2},T])} \leq N(\Omega,T,\delta)\left[\|g_1\|_{L^2(\partial\Omega\times(0,T))} + \|g_2\|_{L^2(\partial\Omega\times(0,T))}\right], \tag{5.7}$$

when $\text{supp}(g_i)\subset\partial\Omega\times[0,T-\delta]$, $i=1,2$. Finally, letting $u^\epsilon$ denote the $C^\infty(\overline{\Omega}\times[0,+\infty))$ solution to (5.2), when $u_0$ and $g_i$ are replaced respectively by $u_0^\epsilon$ and $g_i^\epsilon$, with $u_0^\epsilon$ in $C_0^\infty(\Omega)$, $g_i^\epsilon$ in $C_0^\infty(\partial\Omega\times(0,T-\delta))$ for $i=1,2$, and in such a way that $u_0^\epsilon$ converges to $u_0$ in $L^2(\Omega)$ and $g_i^\epsilon$ converges to $g_i\chi_\mathcal{J}$ in $L^2(\partial\Omega\times(0,T-\delta))$, with

$$\|g_i^\epsilon\|_{L^\infty(\partial\Omega\times[0,T-\delta])} \leq 2\|g_i\|_{L^\infty(\mathcal{J})},\ \text{for } i=1,2,$$

integration by parts shows that

$$\int_\Omega u^\epsilon(T)\varphi_T\,dx = \int_\Omega u_0^\epsilon\varphi(0)\,dx + \int_\mathcal{J} g_1^\epsilon\,\tfrac{\partial\Delta\varphi}{\partial\nu} - g_2^\epsilon\,\Delta\varphi\,d\sigma dt,$$

when $\varphi = e^{(t-T)\Delta^2}\varphi_T$, $\varphi_T$ is in $C_0^\infty(\Omega)$. Letting then $\epsilon\to 0^+$ together with (5.7) and (5.6) show that the solution $u$ to (5.5) verifies $u\equiv 0$ for $t\geq T$. The proof of Corollary 2 is now standard.

Theorem 4 implies the null controllability of the system (1.5) with controls restricted over $\ell$ different non-empty open sets (or measurable sets of positive measure): assume that $\omega_j\subset\Omega$, $j=1,\dots,\ell$, are non-empty open sets verifying, $\omega_j\cap\omega_k=\emptyset$, for $1\leq j\neq k\leq\ell$. Consider the system

$$\begin{cases}\partial_t\mathbf{u}-\mathbf{L}^*\mathbf{u}=\mathbf{f}, & \text{in } \Omega\times(0,T),\\ \mathbf{u}=0, & \text{on } \partial\Omega\times(0,T),\\ \mathbf{u}(0)=\mathbf{u}_0, & \text{in } \Omega,\end{cases}\qquad \text{with } \mathbf{f}=(\chi_{\omega_1}f_1,\dots,\chi_{\omega_l}f_l), \tag{5.8}$$

$f_\xi$ in $L^\infty(\Omega\times(0,T))$, $\xi=1,\dots,\ell$, are the controls, $\mathbf{u}_0$ in $L^2(\Omega)^\ell$ and $\mathbf{L}$ and its coefficients are as in (1.5). Then, from Theorem 4 and the classical duality argument (cf., e.g., [5] or [3, Corollary 1]) we have

**Corollary 3.** *For each $T>0$ and $\mathbf{u}_0$ in $L^2(\Omega)^\ell$, there are bounded controls $\mathbf{f}=(f_1,\dots,f_\ell)$, with*

$$\|\mathbf{f}\|_{L^\infty(\Omega\times(0,T))}\leq N\|\mathbf{u}_0\|_{L^2(\Omega)^\ell},$$

*such that the solution $\mathbf{u}(\cdot\,;\mathbf{u}_0,\mathbf{f})$ to (5.8) verifies, $\mathbf{u}(T;\mathbf{u}_0,\mathbf{f})=0$. Here, the constant $N=N(T,\Omega,\omega_1,\dots,\omega_\ell)$ is independent of $\mathbf{u}_0$.*

Finally, Theorem 8 implies the bang-bang property of the time optimal controls for some systems of two parabolic equations with only one control force. For this connection we refer the readers to [1], [36] and the references therein: let $T>0$ and $\Omega$ be a as above. Suppose that $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ are real-analytic in $\overline{\Omega}$ and $b(\cdot)\not\equiv 0$. Let $\omega\subset\Omega$ be a non-empty open set (or a measurable set with positive

measure). Consider the controlled parabolic system

$$\begin{cases} \partial_t u - \Delta u + a(x)u + b(x)v = 0, & \text{in } \Omega \times (0,+\infty), \\ \partial_t v - \Delta v + c(x)u + d(x)v = \chi_\omega f, & \text{in } \Omega \times (0,+\infty), \\ u = 0, \;\; v = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0, \;\; v(\cdot,0) = v_0, & \text{in } \Omega, \end{cases} \tag{5.9}$$

where $f$ is a control force taken in the constraint set

$$\mathcal{U}_3^M \triangleq \Big\{ f : \Omega \times \mathbb{R}^+ \to \mathbb{R} \;\text{ measurable} : |f(x,t)| \leq M, \;\; \text{a.e. in } \Omega \times \mathbb{R}^+ \Big\},$$

with $M > 0$. For each $(u_0, v_0)$ in $L^2(\Omega) \times L^2(\Omega) \setminus \{(0,0)\}$, we study the time optimal control problem

$$(TP)_3^M : \qquad T_3^M \triangleq \inf_{\mathcal{U}_3^M} \big\{ t > 0; \; \big(u(t; u_0, v_0, f), v(t; u_0, v_0, f)\big) = (0,0) \big\},$$

where $\big(u(\cdot\,; u_0, v_0, f), v(\,;, u_0, v_0, f)\big)$ is the solution to (5.9) corresponding to the control $f$ and the initial datum $(u_0, v_0)$. Then, the methods in [3, §5] and Theorem 8 give the following consequence.

**Corollary 4.** *The problem $(TP)_3^M$ has the bang-bang property: any time optimal control $f$ satisfies, $|f(x,t)| = M$ for a.e. $(x,t)$ in $\omega \times (0, T_3^M)$. Moreover, it is unique.*

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