Induced 2-Regular Subgraphs in k-Chordal Cubic Graphs

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Abstract

We show that a cubic graph G of order n has an induced 2-regular subgraph of order at least

- $\frac{n-2}{4-\frac{4}{k}}$, if G has no induced cycle of length more than k,
- $\frac{5n+6}{8}$, if G has no induced cycle of length more than 4, and $n>6$, and
- $\left(\frac{1}{4}+\epsilon\right)n$, if the independence number of G is at most $\left(\frac{3}{8}-\epsilon\right)n$.

To show the second result we give a precise structural description of cubic 4-chordal graphs.

Keywords: Induced regular subgraph; induced cycle; independent set; induced matching
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1 Introduction

The problem of finding a largest induced regular subgraph of a given graph goes back to Erdős, Fajtlowicz, and Staton [6]. It follows immediately from Ramsey's theorem [12] that every graph G of order $n(G)$ has an induced regular subgraph of order $\Omega(\log n(G))$. Special cases with fixed regularity such as the independent set problem or the induced matching problem have received a lot of attention. In general, it is NP-hard to find a maximum induced (bipartite) k-regular subgraph of a given graph as shown by Cardoso et al. [2], who also extend the Hoffman upper bound on the independence number to the maximum order of an induced k-regular subgraph. Efficient algorithms for special graph classes [10], exact exponential time algorithms [7], as well as fpt-algorithms [11] for this problem have been studied.

While the components induced by independent sets or induced matchings are clearly of bounded order, there is no upper bound on the order of a component of an induced k-regular subgraph for every k at least 2. Unfortunately, this means that local techniques, which were successfully applied to independent sets and induced matchings, hardly generalize to values of k at least 2. Recently, Henning et al. [8] studied the maximum order $c_{\text{ind}}(G)$ of an induced 2-regular subgraph of a given graph G. They establish NP-hardness of $c_{\text{ind}}(G)$ for graphs of maximum degree 4. For an r-regular graph G, they show

$$c_{\text{ind}}(G) \geq \frac{n(G)}{2(r-1)} + \frac{1}{(r-1)(r-2)},$$

which implies $c_{\text{ind}}(G) \geq \frac{n(G)+2}{4}$ if G is cubic. For a claw-free cubic graph G, they prove the asymptotically best-possible bound $c_{\text{ind}}(G) > 13n(G)/20$. Furthermore, they believe that their general bound can be improved. Specifically, for a cubic graph G, they conjecture $c_{\text{ind}}(G) \geq \frac{n(G)}{2}$, which would be best-possible in view of the graph in Figure 1.

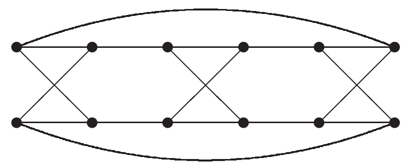

Figure 1: A graph G with $c_{\text{ind}}(G) = \frac{n(G)}{2}$.

In the present paper we study $c_{\text{ind}}(G)$ for cubic graphs that do not have long induced cycles or whose independence number is small.

For an integer k at least 3, a graph G is k-chordal if it does not have an induced cycle of length more than k. Chordal graphs coincide with 3-chordal graphs, and graphs of small chordality were studied in [1, 4, 5, 9]. Note that the components of an induced 2-regular subgraph of a k-chordal graph are of order at most k; that is, imposing k-chordality as an additional hypothesis allows us to apply more local arguments. Our results are an improvement of the bound from [8] for cubic k-chordal graphs as well as a best-possible bound for 4-chordal cubic graphs. In order to prove this last result, we give a precise structural description of 4-chordal cubic graphs.

Before we proceed to our results and their proofs, we would like to mention some further related notions and conjectures.

A set D of vertices of a graph G is a fair dominating set if every vertex in $V(G) \setminus D$ has the same positive number of neighbors in D [3]. This definition implies that, if G is an r-regular graph, then a set D of vertices of G is a fair dominating set of G if and only if $G - D$ is s-regular for some $s < r$. Caro et al. [3] studied bounds on the fair domination number, which is the minimum cardinality of a fair dominating set. Clearly, for an r-regular graph G, the fair domination number of G is equal to $n(G) - \max\{n(H) : H$ is an induced s-regular subgraph of G with $s < r\}$.

Instead of regular induced subgraphs, one might consider induced subgraphs whose components are regular but are allowed to have different degrees. We conjecture that every cubic graph G has an induced subgraph H of order at least $\frac{3}{5}n(G)$ that is the disjoint union of K_1s, K_2s, and induced cycles. The Petersen graphs shows that this is best possible.

2 Results

For a graph G, let $\kappa(G)$ be the number of components of G. Recall that the cyclomatic number $\mu(G)$ of G is $m(G) + \kappa(G) - n(G)$, and that G has a cycle if and only if $\mu(G) > 0$. For a set S of vertices of G, the closed neighborhood $N_G[S]$ of S in G contains S and all neighbors of vertices in S.

Lemma 1 *If G is a connected cubic graph, then G has an induced 2-regular subgraph with* *components* $C_1,\dots,C_t$ *such that*

$$\mu(G-V_{\leq i-1})-\mu(G-V_{\leq i})\leq\begin{cases}2n(C_1) & \text{if } i=1, \text{ and}\\ 2n(C_i)-2 & \text{if } 2\leq i\leq t.\end{cases}$$

where $V_{\leq 0}=\emptyset$ *and* $V_{\leq i}=N_G[V(C_1)\cup\cdots\cup V(C_i)]$ *for each* i *with* $1\leq i\leq t$.

Proof: We construct a sequence $G_0,G_1,\dots,G_t$ of induced subgraphs of G as well as a sequence $C_1,\dots,C_t$ such that, for $i\in\{1,\dots,t\}$, C_i is an induced cycle of G_{i-1}, and G_i arises from G_{i-1} by removing C_i together with its neighbors; that is, $G_i=G_{i-1}-V_i$ where $V_i=N_{G_{i-1}}[V(C_i)]$. Clearly, $c_{\rm ind}(G)\geq\ell_1+\cdots+\ell_t$ where ℓ_i is the order of C_i; that is, $\ell_i=n(C_i)$. Let $n_i=|V_i|$ and let m_i be the number of edges of G_{i-1} that are incident with a vertex in V_i; that is, $n_i=n(G_{i-1})-n(G_i)$ and $m_i=m(G_{i-1})-m(G_i)$. Since G is cubic, we have $n_i\leq 2\ell_i$. Let $\mu_i=\mu(G_{i-1})-\mu(G_i)$.

Let $G_0=G$. Let C_1 be any induced cycle of G. If, for some $i\geq 2$, the graph G_{i-1} has a cycle, then choose C_i as an induced cycle of G_{i-1} such that $\mu_i-\ell_i$ is smallest possible. The sequences terminate as soon as G_i is a forest. It remains to show that $\mu_1\leq 2\ell_1$ and that $\mu_i\leq 2\ell_i-2$ for $2\leq i\leq t$.

Let $i\in\{1,\dots,t\}$.

If V_i is the vertex set of a component of G_{i-1}, then $\kappa(G_i)=\kappa(G_{i-1})-1$, and, since G is cubic,

$$\begin{aligned}\mu_i &= (m(G_{i-1})-m(G_i))+(\kappa(G_{i-1})-\kappa(G_i))-(n(G_{i-1})-n(G_i))\\ &= m_i+1-n_i\\ &\leq \frac{3}{2}n_i+1-n_i\\ &= \frac{1}{2}n_i+1\\ &\leq \ell_i+1\\ &\leq 2\ell_i-2.\end{aligned}$$

Hence, we may assume that V_i is not the vertex set of a component of G_{i-1}, which implies that $\kappa(G_{i-1})-\kappa(G_i)\leq 0$.

Since G is cubic, we have $m_i\leq\ell_i+3(n_i-\ell_i)=3n_i-2\ell_i$. This implies

$$\begin{aligned}\mu_i &\leq m_i-n_i\\ &\leq 3n_i-2\ell_i-n_i\\ &= 2n_i-2\ell_i\\ &\leq 4\ell_i-2\ell_i\\ &= 2\ell_i,\end{aligned}$$

which implies the desired bound for $i=1$. Hence we may assume that $i\geq 2$.

If C_i contains a vertex of degree 2, then $n_i\leq 2\ell_i-1$, and hence

$$\begin{aligned}\mu_i &\leq 2n_i-2\ell_i\\ &\leq 4\ell_i-2-2\ell_i\\ &= 2\ell_i-2.\end{aligned}$$

If m_i is at most $3n_i-2\ell_i-2$, then a similar argument implies $\mu_i\leq 2\ell_i-2$.

In view of the choice of C_i, we may therefore assume that, for every induced cycle C of G_{i-1}, we have that $|V_C|=2n(C)$ where $V_C=N_{G_{i-1}}[V(C)]$, and that there are at least $4n(C)-1$ edges of G_{i-1} that are incident with a vertex in V_C. This implies that V_C contains only vertices that are of degree 3 in G_{i-1}, and that every vertex v in $V_C \setminus C$ has at least one neighbor in $V(G_{i-1}) \setminus V_C$. Since G_{i-1} is not a forest, it has a block B that is distinct from K_2. Since every vertex in B lies on an induced cycle in B, all vertices in B have degree 3 in G_{i-1}. Since $i \geq 2$ and the graph G is connected, this implies that B is not a component of G_{i-1}. Let u be a cutvertex of G_{i-1}. Let C_i be an induced cycle in B that contains u. Let u^- and u^+ be the neighbors of u in C_i. Let v, v^-, and v^+ be the neighbors of u, u^-, and u^+ outside of $V(C_i)$, respectively. Let $G' = G_{i-1} - (V_i \setminus \{v, v^-, v^+\})$. By construction, v does not lie in the same component of G' as v^- or v^+. If v^- and v^+ lie in the same component of G', and P is a shortest v^--v^+-path in G', then $C' = (P \cup C_i) - \{u\}$ is an induced cycle of G_{i-1} with $|N_{G_{i-1}}[V(C')]| < 2n(C')$, which is a contradiction. Hence v, v^-, and v^+ all lie in different components of G'. Since each of these vertices has a neighbor in $V(G_{i-1}) \setminus V_i$, we obtain $\kappa(G_{i-1}) - \kappa(G_i) \leq -2$, and hence

$$\begin{aligned} \mu_i &= m_i + (\kappa(G_{i-1}) - \kappa(G_i)) - n_i \\ &\leq 4\ell_i - 2 - 2\ell_i \\ &= 2\ell_i - 2, \end{aligned}$$

which completes the proof. $\Box$

Theorem 2 *If G is a connected cubic k-chordal graph, then*

$$c_{\text{ind}}(G) \geq \frac{n(G)-2}{4-\frac{4}{k}}.$$

Proof: Let $C_1, \ldots, C_t$ be as in Lemma 1. We use the notation from the proof of Lemma 1. By Lemma 1, we have $\mu_1 \leq 2\ell_1$, and $\mu_i \leq 2\ell_i - 2$ for $2 \leq i \leq t$. Since G is k-chordal, we have $\ell_i \leq k$ for all $i \geq 1$. Therefore,

$$\mu_1 \leq 2\ell_1 = \frac{2\ell_1}{k} + \left(2 - \frac{2}{k}\right)\ell_1 \leq 2 + \left(2 - \frac{2}{k}\right)\ell_1$$

and, for $2 \leq i \leq t$,

$$\mu_i \leq \left(2 - \frac{2}{k}\right)\ell_i.$$

Since G is a connected cubic graph, we have $\mu(G) = \frac{n(G)}{2} + 1$. Since G_t is a forest, we have $\mu(G_t) = 0$. Now

$$\frac{n(G)}{2} + 1 = \mu(G) = \left(\sum_{i=1}^{t} \mu_i\right) + \mu(G_t) = \sum_{i=1}^{t} \mu_i \leq 2 + \sum_{i=1}^{t} \left(2 - \frac{2}{k}\right)\ell_i.$$

This implies

$$c_{\text{ind}}(G) \geq \sum_{i=1}^{t} \ell_i \geq \frac{n(G)-2}{2\left(2-\frac{2}{k}\right)},$$

which completes the proof. $\Box$

It is obvious that the technique used in the proof of Lemma 1 and Theorem 2 can also be applied to r-regular graphs for $r > 3$. Before we proceed to our result on 4-chordal graphs, we show another application of Lemma 1, which relates $c_{\text{ind}}(G)$ to the independence number $\alpha(G)$ of G.

Theorem 3 *Let G be a connected cubic graph. If $\alpha(G) \leq \left(\frac{3}{8} - \epsilon\right) n(G)$ for some $\epsilon > 0$, then*

$$c_{\rm ind}(G) > \left(\frac{1}{4} + \epsilon\right) n(G) - 1.$$

Proof: For a contradiction, we suppose that $c_{\rm ind}(G) \leq \left(\frac{1}{4} + \epsilon\right) n(G) - 1$. Let $C_1, \ldots, C_t$ be as in Lemma 1. We use the notation from the proof of Lemma 1. Since G_t is a forest, $n_i \leq 2\ell_i$, and no vertex of $C_1 \cup \cdots \cup C_t$ is adjacent to a vertex of G_t, we obtain

$$\begin{aligned}
\alpha(G) &\geq \sum_{i=1}^{t} \left(\frac{\ell_i - 1}{2}\right) + \frac{n(G_t)}{2} \\
&= \frac{1}{2}\left(\sum_{i=1}^{t} \ell_i + n(G) - \sum_{i=1}^{t} n_i\right) - \frac{t}{2} \\
&\geq \frac{1}{2}\left(n(G) - \sum_{i=1}^{t} \ell_i\right) - \frac{t}{2} \\
&\geq \frac{1}{2}\left(n(G) - c_{\rm ind}(G)\right) - \frac{t}{2} \\
&> \frac{1}{2}\left(n(G) - \left(\frac{1}{4} + \epsilon\right) n(G)\right) - \frac{t}{2} \\
&\geq \left(\frac{3}{8} - \frac{\epsilon}{2}\right) n(G) - \frac{t}{2}.
\end{aligned}$$

Together with $\alpha(G) \leq \left(\frac{3}{8} - \epsilon\right) n(G)$, this implies $t \geq \epsilon n(G)$. As in the proof of Theorem 2, we obtain

$$\frac{n(G)}{2} + 1 = \sum_{i=1}^{t} \mu_i \leq 2 + \sum_{i=1}^{t} (2\ell_i - 2) = 2 - 2t + 2\sum_{i=1}^{t} \ell_i \leq 2 - 2\epsilon n(G) + 2\sum_{i=1}^{t} \ell_i,$$

which implies the contradiction $c_{\rm ind}(G) \geq \sum\limits_{i=1}^{t} \ell_i \geq \left(\frac{1}{4} + \epsilon\right) n(G) - \frac{1}{2}$. $\Box$

In order to prove our bound for 4-chordal cubic graphs, we describe their structure in detail. Our next result characterizes all non-trivial blocks of a 4-chordal cubic graph. Let K_n, P_n, and C_n, be the complete graph, the path, and the cycle of order n, respectively. Let $K_{n,m}$ be the complete bipartite graph with partite sets of order n and m, respectively. Let $G \Box H$ be the Cartesian product of the graphs G and H.

Figure 2: The graphs D, D', $P_2 \Box K_3$, and $K_{3,3}^-$.

For some integer k at least 2, let B_k be the graph $P_2 \Box P_k$. Note that B_2 is C_4. Let B'_k arise from B_k by adding a new vertex to B_k and joining it to two adjacent vertices of B_k of degree 2. Let B''_k arise from B'_k by adding a new vertex to B'_k and joining it to the two adjacent vertices of B'_k of degree 2. See Figure 3 for an illustration.

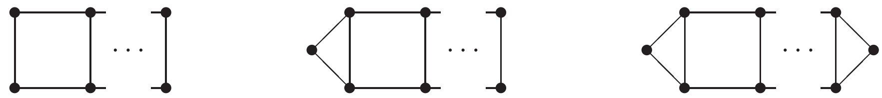

Figure 3: $B_k = P_2 \,\square\, P_k$, B'_k, and B''_k for $k \geq 2$.

Let

$$\begin{aligned}\mathcal{F} &= \{K_3, K_4, D, D', P_2 \,\square\, K_3, K_{2,3}, K_{3,3}, K_{3,3}^-\}, \text{ and} \\ \mathcal{B} &= \{B_k : k \geq 2\} \cup \{B'_k : k \geq 2\} \cup \{B''_k : k \geq 2\}.\end{aligned}$$

Theorem 4 *If G is a 2-connected subcubic 4-chordal graph, then G belongs to $\mathcal{F} \cup \mathcal{B}$.*

Proof: Let G be a 2-connected subcubic 4-chordal graph. Since all graphs in $\mathcal{F}$ are 2-connected, subcubic, and 4-chordal, we may assume that G does not belong to $\mathcal{F}$. We consider different cases.

First, we assume that G contains the diamond D as an induced subgraph. Let a and b be the two vertices of degree 2 in D. Since G is not D, we may assume that a has a neighbor c not in D. Since G is not D', the vertex b is not adjacent to c. Since G is 2-connected, there is a shortest path P between b and c that does not intersect $D - b$. Now P together with a shortest a-b-path in D yields an induced cycle of length more than 4, which is a contradiction. Therefore, we may assume that G is D-free.

Next, we assume that G contains a triangle $T : abca$. If all vertices of T have degree 3 in G, then, since G is D-free and not K_4, the vertices a, b, and c have distinct neighbors, say a', b', and c', outside of T, respectively. Since G is 4-chordal, no two of the vertices a', b', and c' are joined by an induced path of length at least 2 in $G - V(T)$. Since G is 2-connected, every two of the vertices a', b', and c' are joined by an induced path in $G - V(T)$. Hence a', b', and c' induce a K_3, and G is $P_2 \,\square\, K_3$, which is a contradiction. Therefore, we may assume that b has degree 2 in G. Since G is not K_3, we may assume that a has degree 3 in G. Let a_1 be the neighbor of a outside of T. Since G is 2-connected, the graph $G - \{a, b\}$ contains a shortest a_1-c-path P. Since G is 4-chordal and D-free, the path P has order exactly 3. Let c_1 be the unique internal vertex of P. Note that $G[\{a, b, c, a_1, c_1\}]$ is isomorphic to B'_2. If, for some $k \geq 2$, a proper induced subgraph G' of G is isomorphic to B'_k, and a_k and c_k are the two adjacent vertices of degree 2 in G', then we may assume that a_k has a neighbor a_{k+1} outside of G'. Since G is 2-connected, the graph $G - (V(G') \setminus \{c_k\})$ contains a shortest a_{k+1}-c_k-path Q. Since G is 4-chordal, the path Q has order at most 3. If Q has order 2, then G is B''_k. If Q has order 3, then G has an induced subgraph that is isomorphic to B'_{k+1}. By an inductive argument, we obtain that G is B'_k or B''_k for some $k \geq 2$. Therefore, we may assume that G is triangle-free.

Next, we assume that G contains $K_{2,3}$ as an induced subgraph. Let a, b, and c be the three vertices of degree 2 in this $K_{2,3}$. Since G is not $K_{2,3}$, we may assume that a has a neighbor a' outside of $K_{2,3}$. Since G is 2-connected, we may assume, by symmetry between b and c, that P is a shortest a'-b-path in $G - (V(K_{2,3}) \setminus \{b, c\})$. Since G is 4-chordal, the path P has order 2; that is, the vertex a' is adjacent to b. Since G is not $K_{3,3}$, the vertex a' is not adjacent to c. Since G is not $K_{3,3}^-$, we may assume that Q is a shortest a'-c-path in $G - (V(K_{2,3}) \setminus \{c\})$. Since Q is of order at least 3, it is contained in an induced cycle of length at least 5 in G, which is a contradiction. Therefore, we may assume that G is $K_{2,3}$-free.

Since G is 4-chordal, 2-connected, and triangle-free, it contains an induced 4-cycle $C : a_1a_2b_2b_1a_1$. Since C is isomorphic to B_2, we may assume that G is not C. Therefore, we may assume, by symmetry, that a_2 has a neighbor a_3 outside of C. Since G is 2-connected, we may assume that P is a shortest path in $G - a_2$ between a_3 and a vertex in $\{a_1, b_1, b_2\}$. If P is an a_3-b_1-path, then, since G is triangle-free and 4-chordal, the path P has order 2; that is, the vertex a_3 is adjacent to b_1, and G is not $K_{2,3}$-free, which is a contradiction. Hence, we may assume, by symmetry between a_1 and b_2, that P is an a_3-b_2-path. Since G is triangle-free and 4-chordal, the path P has order exactly 3. Let b_3 be the unique internal vertex of P. Since G is triangle-free and $K_{2,3}$-free, the graph $G[\{a_1, a_2, a_3, b_1, b_2, b_3\}]$ is isomorphic to B_3. Applying an inductive argument as above, we obtain that G is B_k for some $k \geq 3$, which completes the proof. $\square$

Let G be a cubic 4-chordal graph. If B is a block of G that is distinct from K_2, then Theorem 4 implies that B belongs to $\mathcal{F} \cup \mathcal{B}$. Furthermore, all edges of G between a vertex in $V(B)$ and a vertex in $V(G) \setminus V(B)$ are bridges. Considering the vertex degrees of the graphs in $\mathcal{F} \cup \mathcal{B}$, this implies that there are at most four edges between $V(B)$ and $V(G) \setminus V(B)$. Therefore, contracting every block of G that is distinct from K_2 to a single vertex, results in a tree of maximum degree 4. Reversing this process leads to the following constructive description of cubic 4-chordal graphs.

Corollary 5 *If G is a connected cubic 4-chordal graph, then G is either 2-connected, in which case G belongs to $\{K_4, K_{3,3}, P_2 \square K_3\}$, or G arises from a tree T of order at least 2 and maximum degree at most 4 by replacing*

- *every endvertex of T with D',*
- *every vertex of T of degree 2 with either D, or $K_{3,3}^-$, or B_k'' for some $k \geq 2$,*
- *some vertices of T of degree 3 with either K_3, or $K_{2,3}$, or B_k' for some $k \geq 2$, and*
- *every vertex of T of degree 4 with B_k for some $k \geq 2$.*

Based on this structural description, we proceed to our second main result.

Theorem 6 *If G is a connected cubic 4-chordal graph that does not belong to the set $\{K_4, K_{3,3}, P_2 \square K_3\}$, then*

$$c_{\rm ind}(G) \geq \frac{5}{8}n(G) + \frac{3}{4}.$$

Furthermore, equality holds if and only if G arises from a tree T of order at least 2 and maximum degree at most 3 by replacing

- *every endvertex of T with D',*
- *every vertex of T of degree 2 with B_3'', and*
- *every vertex of T of degree 3 with K_3.*

Proof: We prove the statement by induction on the order of G. By Corollary 5, the graph G is not 2-connected. Let T be as in the statement of Corollary 5. Since T has at least two endvertices, the order of G is at least 10. If $n(G) = 10$, then G arises from the disjoint union of two copies of D' by connecting the two vertices of degree 2 by a bridge, and $c_{\rm ind}(G) = 7 = \frac{5}{8}n(G) + \frac{3}{4}$. Now let $n(G) > 10$, which implies that T has order at least 3.

First, we assume that G contains B_5 as an induced subgraph. We denote its vertices as in the left of Figure 4.

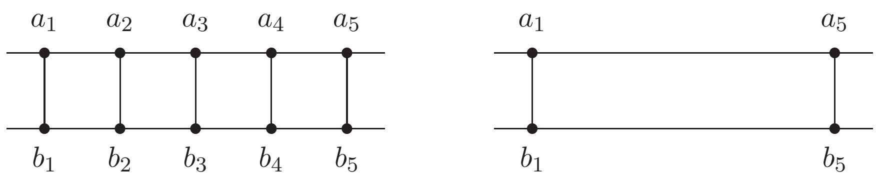

Figure 4: An induced B_5 in G and G''.

Note that $G' = G - \{a_2, a_3, a_4, b_2, b_3, b_4\}$ has exactly two components. Let G'' arise from G' by adding the two edges a_1a_5 and b_1b_5. Clearly, G'' is cubic. Since removing any two edges of $K_{3,3}$ or $P_2 \,\Box\, K_3$ does not disconnect these graphs, G'' does not belong to $\{K_4, K_{3,3}, P_2 \,\Box\, K_3\}$. Note that $a_1a_5b_5b_1a_1$ is the only induced cycle of G'' that is not also a cycle of G. Therefore, G'' is 4-chordal. Let H'' be an induced 2-regular subgraph of G''. If H'' contains the cycle $a_1a_5b_5b_1a_1$, then let $H = (H'' - \{a_1, a_5, b_5, b_1\}) \cup a_1a_2b_2b_1a_1 \cup a_4a_5b_5b_4a_4$. If H'' does not contain the cycle $a_1a_5b_5b_1a_1$, then H'' does not contain any of the two edges a_1a_5 and b_1b_5. Therefore, if $a_1 \in V(H'')$, then $a_5, b_5 \notin V(H'')$, and let $H = H'' \cup a_3a_4b_4b_3a_3$. By symmetry, this implies in all cases that G has an induced 2-regular subgraph H with $n(H) \geq n(H'') + 4$. By induction, we obtain

$$c_{\rm ind}(G) \quad \geq \quad c_{\rm ind}(G'') + 4 \geq \frac{5}{8}(n(G) - 6) + \frac{3}{4} + 4 > \frac{5}{8}n(G) + \frac{3}{4}.$$

Therefore, we may assume that G is B_5-free. By a similar argument, we may assume that G is B_4'-free.

Let $P : uvw \ldots$ be a longest path in T. Note that P has order at least 3, and that all neighbors of v in T that are distinct from w are endvertices of T. Let $U = N_T(v) \setminus \{w\}$. The vertex v in T is either a vertex of G that is of degree 3 or it corresponds to a block B in G according to Corollary 5. In the first case, let $B = K_1$. Every vertex in U corresponds to an induced D' in G that is connected to B by a bridge of G. There is a unique vertex x of G that does not belong to B or to one of the copies of D' that correspond to the vertices in U, such that x has a neighbor y in B. Note that xy is a bridge of G. Let G' be the component of $G - xy$ that contains x, and let B^+ be the component of $G - xy$ that contains y. Let G'' arise from the disjoint union of G' and D' by adding an edge between x and the vertex of degree 2 in D'. Note that G'' is a cubic 4-chordal graph of order less than G that does not belongs to $\{K_4, K_{3,3}, P_2 \,\Box\, K_3\}$. Let H'' be an induced 2-regular subgraph of G'' of order $c_{\rm ind}(G'')$. If $x \in V(H'')$, then H'' contains exactly three vertices of the D' that was added to G'. Therefore, $c_{\rm ind}(G) \geq c_{\rm ind}(G'') - 3 + c_{\rm ind}(B^+ - y)$. Similarly, if $x \notin V(H'')$, then H'' contains exactly four vertices of the D' that was added to G', and hence $c_{\rm ind}(G) \geq c_{\rm ind}(G'') - 4 + c_{\rm ind}(B^+)$. The following table summarizes relevant values for all possibilities for B.

B	$c_{\text{ind}}(B^+)$	$c_{\text{ind}}(B^+ - y)$	$c_{\text{ind}}(G) - c_{\text{ind}}(G'') \geq$	$\frac{5}{8}n(G) - \frac{5}{8}n(G'')$
D	7	6	3	2.5
$K_{3,3}^-$	8	8	4	3.75
B_2''	8	8	4	3.75
B_3''	9	8	5	5
K_1	8	8	4	3.75
K_3	9	8	5	5
$K_{2,3}$	11	10	7	6.25
B_2'	11	10	7	6.25
B_3'	12	12	8	7.5
B_2	13	12	9	8.75
B_3	15	14	11	10
B_4	16	16	12	11.25

Figure 5 illustrates the case $B = B_3''$. In this case, combining two triangles in B with a triangle in D' yields $c_{\text{ind}}(B^+) = 9$. Combining two cycles of length 4, one in B and one in D', yields $c_{\text{ind}}(B^+ - y) = 8$.

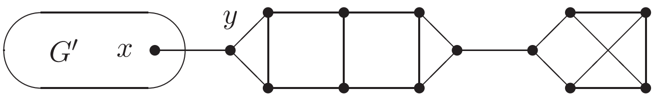

Figure 5: The case $B = B_3''$.

Note that for $B \in \{B_2', B_3'\}$, there are two non-isomorphic configurations for G. Since these lead to the same values, we do not distinguish them within the table.

Since the entries in the second to last column are consistently at least as large as the entries in the last column, the desired bound follows by induction. The statement about the extremal graphs easily follows from the base case of the induction, and the fact that only B_3'' and K_3 lead to equal values within the last two columns of the table. □

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