Quantum cluster algebra structure on the finite dimensional representations of $U_q(\widehat{sl_2})$

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In this paper, we give a quantum cluster algebra structure on the deformed Grothendieck ring of \mathcal{C}_n , where \mathcal{C}_n is a full subcategory of finite dimensional representations of $U_q(\widehat{sl_2})$ defined in section II.

Keywords: Quantum cluster algebra, Deformed Grothendieck ring

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I. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky³, and they played more and more important role in representation theory and other areas. In 2010, Hernandez and Leclerc¹ posted a conjecture on the monoidal categorification of cluster algebras³ and gave a model of monoidal category for certain cluster algebras. They proved the case n = 1 for the Lie algebras of type $AD^{1,2}$. Almost at the same time, Nakajima¹² used the theory of perverse sheaves and q,t-characters to prove the case n = 1 for the Lie algebras of type ADE. In 2012, Qin⁴ generalized Nakajima's geometric approach and obtained monoidal categorifications of the cluster algebras associated to any acyclic quiver. Recently, Yang and Zheng¹¹ proved the conjecture for the case A_3 , n = 2.

In 2010, Nakajima¹² asked that if there is a quantum cluster algebra structure on the deformed Grothendieck ring of full subcategory of the finite dimensional representation of the quantum loop algebra. In 2012, Qin and Yoshiyuki Kimura⁵ obtained a deformed monoidal categorifications of acyclic quantum cluster algebras with specific coefficients through geometric approach.

In this paper, by the algebraic definition of q, t-characters given by Hernandez⁶, we want to give an answer of Nakajima's problem of $U_q(\widehat{sl_2})$ case. That is , we proved that for any $n \in \mathbb{N}$, there is a quantum cluster algebra structure on the deformed Grothendieck ring of the full subcategory \mathscr{C}_n .

The paper is organized as follows. In Section 2, we present necessary definitions and facts of quantum cluster algebras. In Section 3, we give the main result and the detail of the proof.

II. DEFINITIONS AND NOTATIONS

In this section, we give a simplified introduction to the theory of quantum cluster algebras⁷ and finite dimensional representations of quantum affine algebras^{6,8,9}.

A. Quantum cluster algebra

Definition II.1. Let \widetilde{B} be an $m \times n$ integer matrix with rows labeled by [1,m] and columns labeled by an n-element subset $\mathbf{ex} \subset [1,m]$. Let Λ be a skew-symmetric $m \times m$ integer matrix with rows and columns labeled by [1,m]. We say that a pair (Λ,\widetilde{B}) is compatible if, for every $j \in \mathbf{ex}$ and $i \in [1,m]$, we have

$$\sum_{k=1}^m b_{kj} \lambda_{ki} = \delta_{ij} d_j$$

for some integers $d_j (j \in ex)$.

Fix an index $k \in \mathbf{ex}$ and a sign $\varepsilon \in \{\pm 1\}$, the matrix $\widetilde{B}' = \mu_k(\widetilde{B})$ is defined as

$$\widetilde{B}' = E_{\varepsilon}\widetilde{B}F_{\varepsilon},$$

where E_{ε} is the $m \times m$ matrix with entries

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ max(0, -\varepsilon b_{ik}) & \text{if } i \neq j = k, \end{cases}$$

 F_{ε} is the $n \times n$ matrix with rows and columns labeled by **ex** with entries

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ max(0, \varepsilon b_{ik}) & \text{if } i = k \neq j. \end{cases}$$

Set $\Lambda' = E_{\varepsilon}^T \Lambda E_{\varepsilon}$.

Definition II.2. Let (Λ, \widetilde{B}) be a compatible pair, and $k \in ex$. We say that the compatible pair $(\Lambda', \widetilde{B}')$ is obtained from (Λ, \widetilde{B}) by the mutation in direction k.

Let L be a lattice of rank m with a skew-symmetric bilinear form $\Lambda: L \times L \to \mathbb{Z}$. The based quantum torus associated with L is the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra $T = T(\Lambda)$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$ -basis $\{X^e|e\in L\}$ and the multiplication given by $X^eX^f=q^{\Lambda(e,f)/2}X^{e+f}, X^0=1, (X^e)^{-1}=X^{-e}$. For a skew-field F of T, a toric frame in F is a map $M:\mathbb{Z}^m\to F\setminus\{0\}$ defined as $M(c)=\varphi(X^{\eta(c)})$, where φ is an automorphism of F, and $\eta:\mathbb{Z}^m\to L$ is an isomorphism of lattices.

Definition II.3. A quantum seed is a pair (M, \widetilde{B}) , where M is a toric frame in F; \widetilde{B} is an $m \times n$ integer matrix with rows labeled by [1,m] and columns labeled by an n-element subset $ex \subset [1,m]$; the pair $(\Lambda_M, \widetilde{B})$ is a compatible pair.

Let (M, \widetilde{B}) be a quantum seed, and (M', \widetilde{B}') be the quantum seed obtained by the mutation in direction $k \in \mathbf{ex}$. For $i \in [1, m]$, let $X_i = M(e_i)$ and $X_i' = M'(e_i)$. Then $X_i' = X_i$ for $i \neq k$, and X_k' is given by the following quantum analog of the exchange relation:

$$X_{k}^{'} = M(-e_{k} + \sum_{b_{ik} > 0} b_{ik}e_{i}) + M(-e_{k} + \sum_{b_{ik} < 0} b_{ik}e_{i}).$$

Two quantum seeds are mutation-equivalent if they can be obtained from each other by a sequence of quantum seed mutations. For a quantum seed (M, \widetilde{B}) , we denote by $\widetilde{\mathbf{X}} = (X_1, X_2, \dots, X_m)$ the corresponding "free generating set" in F given by $X_i = M(e_i)$. We call the subset $\mathbf{X} = \{X_j | j \in \mathbf{ex}\} \subset \widetilde{\mathbf{X}}$ the cluster of the quantum seed (M, \widetilde{B}) , and set $\mathbf{C} = \widetilde{\mathbf{X}} - \mathbf{X}$.

Definition II.4. Let S be a mutation-equivalence class of quantum seeds in F. The quantum cluster algebra associated with S is the $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of the ambient skew-field F, generated by the union of clusters of all seeds in S, together with the element of C.

B. Quantum loop algebra

 $U_q(\widehat{sl_2})$ is the corresponding quantum affine algebra with parameter $q \in C^*$ not a root of unity. $\mathscr C$ is the category of finite-dimensional $U_q(\widehat{sl_2})$ -representations of type 1. For $\ell \in \mathbb Z_{\geq 0}$, $\mathscr C_\ell$ be the full subcategory of $\mathscr C$: for any V of $\mathscr C_\ell$, the roots of the Drinfeld polynomials of every composition factor of V belong to $\{q^{-2k} \mid 0 \leq k \leq \ell\}$.

Denote the Grothendieck ring of \mathscr{C}_{ℓ} by R_{ℓ} , then $R_{\ell} = \mathbb{Z}\big[[W_{1,2k}]_{i \in I, 0 \le k \le \ell}\big]$, where $W_{1,2k}$ are Kirillov-Reshetikhin modules with the highest l-weights $m_{1,2k} = Y_{q^{2k}}$.

Theorem II.5. ⁶ In the sl_2 case, the deformed Grothendieck ring $Rep_{t,n}$ is a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra generated by $[W_{1,0}], [W_{1,2}], \cdots, [W_{1,2n}]$ with relations

$$[W_{1,l_1}]*[W_{1,l_2}]=q^r[W_{1,l_2}]*[W_{1,l_1}], if\ l_1\geq l_2\ and\ l_1\neq l_1+2, where\ r=(-1)^{(l_1-l_2)/2}.$$

$$[W_{1,l}]*[W_{1,l-2}]=q^{-1}[W_{1,l-2}]*[W_{1,l}]+(1-q^{-1}).$$

III. MAIN RESULTS

In this section, we construct a quantum cluster algebra structure on the deformed Grothendieck ring of \mathcal{C}_n for the Lie algebra sl_2 . To be easily reading, we give an example for a special case.

The Γ_n is the quiver

$$n+1 \rightarrow n \rightarrow \cdots \rightarrow 1$$

where n + 1 is the frozen point.

Let $\widetilde{B_n}$ is a matrix correspond to Γ_n , i.e

$$\widetilde{B_n} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n+1,n)}$$

Let $\Lambda_n = (\lambda_{ij})_{(n+1,n+1)}$ be the skew matrix defined by

$$\lambda_{ij} = \begin{cases} (-1)^{i+j+1} & \text{if } j > i \text{ and i is odd,} \\ 0 & \text{if } j > i \text{ and i is even,} \\ -\lambda_{ji} & \text{if } j < i. \end{cases}$$

That is

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & -1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{(n+1,n+1)}.$$

It is easy to see $(\Lambda_n, \widetilde{B}_n)$ is compatible.

Let $\mathscr{A}_n = \mathscr{A}(\Gamma_n)$ be the quantum cluster algebra associated with a pair $(\Lambda_n, \widetilde{B}_n)$. The ambient field F of fractions of the quantum torus with generators Y_1, Y_2, \dots, Y_{n+1} satisfying relations $Y_i Y_j = q^{\lambda_{ij}} Y_j Y_i$. For $0 \le i \le n-1$, let $X_{2i} = \mu_{n-i}(Y_{n-i})$, and $X_{2n} = Y_1$.

Proposition III.1. \mathscr{A}_n as a $\mathbb{Z}[q^{\pm 1/2}]$ algebra can be generated by $\{X_{2i} \mid 0 \le i \le n\}$.

Proof. Since quantum clusters are mutation equivalence if and only if the correspondence clusters are mutation equivalence, the number of the quantum cluster variables is equal to the number of the cluster variables. The cluster variable is one to one correspondence to the set $\{[W_{i,j}]|1 \le i \le n+1, 0 \le j \le 2n-2i+2$, where j is even $\}^{10}$.

By the definition of the quantum cluster algebra, we have the following relations:

$$X_{2n-2}X_{2n} = q^{1/2}Y_2 + 1;$$

$$X_{2n-2i}Y_i = q^{-1/2}Y_{i-1} + Y_{i+1}$$
, if *i* is even;
 $X_{2n-2i}Y_i = Y_{i-1} + q^{1/2}Y_{i+1}$, if *i* is odd;

By the above relations, it is easy to see that $Y_i (1 \le i \le n) \in \mathbb{Z}[X_{2i} (0 \le i \le n+1), q^{\pm 1/2}]$. For example, $Y_3 = X_{2n-4}Y_2 + q^{-1/2}Y_1 = q^{-1/2}(X_{2n-4}X_{2n-2}X_{2n} - X_{2n-4} + X_{2n})$.

Set $\mu = \mu_n \mu_{n-1} \cdots \mu_1$, where μ_i is the mutation at the direction i, $(Y_1', Y_2', \cdots, Y_n', Y_{n+1}) = \mu(Y_1, Y_2, \cdots, Y_{n+1})$, $Y_i'' = \mu_i(Y_i')$. By the mentioned above and the T-system, $(Y_1', Y_2', \cdots, Y_n', Y_{n+1})$ is correspondence to $(W_{1,2n-2}, W_{2,2n-4}, \cdots, W_{n,0}, W_{n+1,0})$. So we have $Y_i'' = X_{2n-2-2i} (1 \le i \le n-1)$, $Y_1' = X_{2n-2}$. Similarly, we have $Y_i' (1 \le i \le n) \in \mathbb{Z}[X_{2i} (0 \le i < n), q^{\pm 1/2}]$.

By the induction, we have that any quantum cluster variable belongs to $Z[X_{2i}(0 \le i \le n), q^{\pm 1/2}]$.

Theorem III.2. The map

$$X_i \mapsto [W_{1,i}]$$

extends to a ring isomorphism ι_n from the quantum cluster algebra \mathscr{A}_n to the deformed Grothendieck ring $Rep_{t,n}$ of \mathscr{C}_n .

Proof. First, if i is odd,

$$\begin{split} &X_{2(n-i)}X_{2(n-i-1)}\\ &= (Y_{i-1} + q^{1/2}Y_{i+1})Y_i^{-1}(q^{-1/2}Y_i + Y_{i+2})Y_{i+1}^{-1}\\ &= q^{-1/2}Y_{i-1}Y_{i+1}^{-1} + 1 + Y_{i-1}Y_i^{-1}Y_{i+2}Y_{i+1}^{-1} + q^{-1/2}Y_i^{-1}Y_{i+2}.\\ &X_{2(n-i-1)}X_{2(n-i)}\\ &= (q^{-1/2}Y_i + Y_{i+2})Y_{i+1}^{-1}(Y_{i-1} + q^{1/2}Y_{i+1})Y_i^{-1}\\ &= q^{1/2}Y_{i-1}Y_{i+1}^{-1} + 1 + qY_{i-1}Y_i^{-1}Y_{i+2}Y_{i+1}^{-1} + q^{1/2}Y_i^{-1}Y_{i+2}. \end{split}$$

So we have $X_{2(n-i)}X_{2(n-i-1)} = q^{-1}X_{2(n-i-1)}X_{2(n-i)} + (1-q^{-1})$.

If i is even,

$$\begin{split} &X_{2(n-i)}X_{2(n-i-1)}\\ &= (q^{-1/2}Y_{i-1} + Y_{i+1})Y_i^{-1}(Y_i + q^{1/2}Y_{i+2})Y_{i+1}^{-1}\\ &= q^{-1/2}Y_{i-1}Y_{i+1}^{-1} + 1 + Y_{i-1}Y_i^{-1}Y_{i+2}Y_{i+1}^{-1} + q^{-1/2}Y_i^{-1}Y_{i+2}.\\ &X_{2(n-i-1)}X_{2(n-i)}\\ &= (Y_i + q^{1/2}Y_{i+2})Y_{i+1}^{-1}(q^{-1/2}Y_{i-1} + Y_{i+1})Y_i^{-1}\\ &= q^{1/2}Y_{i-1}Y_{i+1}^{-1} + 1 + qY_{i-1}Y_i^{-1}Y_{i+2}Y_{i+1}^{-1} + q^{1/2}Y_i^{-1}Y_{i+2}. \end{split}$$

So we have $X_{2(n-i)}X_{2(n-i-1)}=q^{-1}X_{2(n-i-1)}X_{2(n-i)}+(1-q^{-1}).$ Secondly,

$$\begin{split} &X_{2(n-i)}X_{2(n-j)}\\ &=\ (Y_{i-1}+q^{1/2}Y_{i+1})Y_i^{-1}(Y_j+q^{1/2}Y_{j+1})Y_j^{-1}\\ &=\ Y_{i-1}Y_i^{-1}Y_{j-1}Y_j^{-1}+q^{1/2}Y_{i+1}Y_i^{-1}Y_{j-1}Y_j^{-1}+q^{1/2}Y_{i-1}Y_i^{-1}Y_{j+1}Y_j^{-1}+qY_{i+1}Y_i^{-1}Y_{j+1}Y_j^{-1}. \end{split}$$

if i and j is odd and i < j;

$$\begin{split} &X_{2(n-j)}X_{2(n-i)}\\ &=Y_{j-1}+q^{1/2}Y_{j+1})Y_{j}^{-1}(Y_{i}+q^{1/2}Y_{i+1})Y_{i}^{-1}\\ &=Y_{j-1}Y_{j}^{-1}Y_{i-1}Y_{i}^{-1}+q^{1/2}Y_{j-1}Y_{j}^{-1}Y_{i+1}Y_{i}^{-1}+q^{1/2}Y_{j+1}Y_{j}^{-1}Y_{i-1}Y_{i}^{-1}+qY_{j+1}Y_{j}^{-1}Y_{i+1}Y_{i}^{-1}\\ &=q^{-1}(Y_{i-1}Y_{i}^{-1}Y_{j-1}Y_{j}^{-1}+q^{1/2}Y_{i+1}Y_{i}^{-1}Y_{j-1}Y_{j}^{-1}+q^{1/2}Y_{i-1}Y_{i}^{-1}Y_{j+1}Y_{j}^{-1}+qY_{i+1}Y_{i}^{-1}Y_{j+1}Y_{j}^{-1})\\ &=q^{-1}X_{2(n-i)}X_{2(n-j)}\\ &=q^{(-1)^{j-i+1}}X_{2(n-i)}X_{2(n-j)}. \end{split}$$

If i is odd,j is even and i + 1 < j;

$$\begin{split} &X_{2(n-j)}X_{2(n-i)}\\ &= (Y_{j-1} + q^{1/2}Y_{j+1})Y_j^{-1}(Y_i + q^{1/2}Y_{i+1})Y_i^{-1}\\ &= Y_{j-1}Y_j^{-1}Y_{i-1}Y_i^{-1} + q^{1/2}Y_{j-1}Y_j^{-1}Y_{i+1}Y_i^{-1} + q^{1/2}Y_{j+1}Y_j^{-1}Y_{i-1}Y_i^{-1} + qY_{j+1}Y_j^{-1}Y_{i+1}Y_i^{-1}\\ &= q(Y_{i-1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i+1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i-1}Y_i^{-1}Y_{j+1}Y_j^{-1} + qY_{i+1}Y_i^{-1}Y_{j+1}Y_j^{-1})\\ &= qX_{2(n-i)}X_{2(n-j)}\\ &= q^{(-1)^{j-i+1}}X_{2(n-i)}X_{2(n-j)} \end{split}$$

If i and j are even, and i < j;

$$\begin{split} &X_{2(n-j)}X_{2(n-i)}\\ &= (Y_{j-1} + q^{1/2}Y_{j+1})Y_j^{-1}(Y_i + q^{1/2}Y_{i+1})Y_i^{-1}\\ &= Y_{j-1}Y_j^{-1}Y_{i-1}Y_i^{-1} + q^{1/2}Y_{j-1}Y_j^{-1}Y_{i+1}Y_i^{-1} + q^{1/2}Y_{j+1}Y_j^{-1}Y_{i-1}Y_i^{-1} + qY_{j+1}Y_j^{-1}Y_{i+1}Y_i^{-1}\\ &= q^{-1}(Y_{i-1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i+1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i-1}Y_i^{-1}Y_{j+1}Y_j^{-1} + qY_{i+1}Y_i^{-1}Y_{j+1}Y_j^{-1})\\ &= q^{-1}X_{2(n-i)}X_{2(n-j)}\\ &= q^{(-1)^{j-i+1}}X_{2(n-i)}X_{2(n-j)} \end{split}$$

If i is even, j is odd, and i + 1 < j;

$$\begin{split} &X_{2(n-j)}X_{2(n-i)}\\ &= (Y_{j-1} + q^{1/2}Y_{j+1})Y_j^{-1}(Y_i + q^{1/2}Y_{i+1})Y_i^{-1}\\ &= Y_{j-1}Y_j^{-1}Y_{i-1}Y_i^{-1} + q^{1/2}Y_{j-1}Y_j^{-1}Y_{i+1}Y_i^{-1} + q^{1/2}Y_{j+1}Y_j^{-1}Y_{i-1}Y_i^{-1} + qY_{j+1}Y_j^{-1}Y_{i+1}Y_i^{-1}\\ &= q(Y_{i-1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i+1}Y_i^{-1}Y_{j-1}Y_j^{-1} + q^{1/2}Y_{i-1}Y_i^{-1}Y_{j+1}Y_j^{-1} + qY_{i+1}Y_i^{-1}Y_{j+1}Y_j^{-1})\\ &= qX_{2(n-i)}X_{2(n-j)}\\ &= q^{(-1)^{j-i+1}}X_{2(n-i)}X_{2(n-j)} \end{split}$$

So we have $X_{2(n-i)}X_{2(n-j)} = q^{j-i}X_{2(n-j)}X_{2(n-i)}$.

Thus, we get the conclusion.

A. Example

The quiver Γ_2 are defined by

$$1 \leftarrow 2 \leftarrow 3$$

where 3 is the frozen point.

$$\widetilde{B}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since

$$\widetilde{B}_2^T \Lambda_2 = \left(egin{array}{ccc} 0 & 1 & 0 \ -1 & 0 & 1 \end{array}
ight) \left(egin{array}{ccc} 0 & -1 & 0 \ 1 & 0 & 0 \ 0 & 0 & 0 \end{array}
ight) = \left(egin{array}{ccc} 1 & 0 & 0 \ 0 & 1 & 0 \end{array}
ight),$$

that is, $(\Lambda_2, \widetilde{B}_2)$ is compatible.

Let $\mathscr{A}_n = \mathscr{A}(\Gamma_2)$ be the quantum cluster algebra associated with a pair $(\Lambda_2, \widetilde{B}_2)$. By the definition of the quantum cluster algebra, we have $X_0 = (q^{-1/2}Y_1 + Y_3)Y_2^{-1}$, $X_2 = Y_1^{-1} + q^{1/2}Y_2Y_1^{-1}$, $X_4 = Y_1$, where (Y_1, Y_2, Y_3) is the initial quantum cluster.

Proposition III.3. \mathscr{A}_2 as a $\mathbb{Z}[q^{\pm 1/2}]$ algebra can be generated by X_0, X_2, X_4 .

Proof. The detailed expressions of quantum cluster variables are shown as follows: Y_1 , Y_2 , Y_3 , $Y_1' = Y_1^{-1} + q^{-1/2}Y_1^{-1}Y_2 = X_2$, $Y_2' = q^{1/2}Y_3Y_1^{-1}Y_2^{-1} + Y_2^{-1} + Y_1^{-1}Y_3$, $Y_1'' = Y_3Y_2^{-1} + q^{-1/2}Y_1Y_2^{-1} = X_0$.

Because the relation $Y_1Y_2 = q^{-1}Y_2Y_1$, we have:

$$Y_1Y_1' = 1 + q^{-1/2}Y_2,$$

 $Y_1''Y_1' = 1 + q^{-1/2}Y_2',$
 $X_0Y_2 = q^{-1/2}Y_1 + Y_3.$

That is:

$$Y_2 = q^{1/2}(Y_1Y_1' - 1) = q^{1/2}(X_4X_2 - 1),$$

$$Y_2' = q^{1/2}(Y_1''Y_1' - 1) = q^{1/2}(X_0X_2 - 1),$$

$$Y_3 = X_0Y_2 - q^{-1/2}X_4.$$

So we get the conclusion.

Theorem III.4. The map

$$X_4 \mapsto [W_{1,4}], \ X_0 \mapsto [W_{1,0}], \ X_2 \mapsto [W_{1,2}]$$

extends to a ring isomorphism ι_2 from the quantum cluster algebra \mathscr{A}_2 to the deformed Grothendieck ring $Rep_{t,2}$ of \mathscr{C}_2 .

Proof. Since

$$Y_1'Y_1''$$

$$= Y_1^{-1}Y_3Y_2^{-1} + q^{-1/2}Y_1^{-1}Y_2Y_3Y_2^{-1} + q^{-1}Y_1^{-1}Y_2Y_1Y_2^{-1} + q^{-1/2}Y_2^{-1}$$

$$= Y_1^{-1}Y_2^{-1}Y_3 + q^{-1/2}Y_1^{-1}Y_3 + q^{-1/2}Y_2^{-1} + 1$$

and

$$\begin{split} &Y_1''Y_1'\\ &= Y_3Y_2^1Y_1^{-1} + q^{-1/2}Y_1Y_2^{-1}Y_1^{-1} + q^{-1/2}Y_3Y_2^{-1}Y_1^{-1}Y_2 + q^{-1}Y_1Y_2^{-1}Y_1^{-1}Y_2\\ &= qY_1^{-1}Y_2^{-1}Y_3 + q^{1/2}Y_1^{-1}Y_3 + q^{1/2}Y_2^{-1} + 1, \end{split}$$

then we have

$$Y_1'Y_1'' = q^{-1}Y_1''Y_1' + (1 - q^{-1}).$$

Since

$$Y_1'Y_1 = 1 + q_{1/2}Y_1^{-1}Y_2Y_1 = 1 + q^{1/2}Y_2$$

and

$$Y_1Y_1' = 1 + q^{-1/2}Y_1,$$

then we have

$$Y_1Y_1' = q^{-1}Y_1'Y_1 + (1 - q^{-1}).$$

Since

$$Y_1Y_1'' = Y_1Y_3Y_2^{-1} + q^{-1/2}Y_1^2Y_2^{-1} = Y_1Y_2^{-1}Y_3 + q^{-1/2}Y_1^2Y_2^{-1}$$

and

$$Y_1''Y_1 = Y_3Y_2^{-1}Y_1 + q^{-1/2}Y_1Y_2^{-1}Y_2 = q^{-1}Y_1Y_2^{-1}Y_3 + q^{-3/2}Y_1^2Y_2^{-1},$$

then we have

$$Y_1Y_1'' = qY_1''Y_1.$$

so we get the conclusion.

REFERENCES

- ¹D. Hernandez, B. Leclerc, Cluster algebras and quantum affine algebras. Duke Math. J. **154** (2010), 265–341.
- ²D. Hernandez, B. Leclerc, Monoidal categorifications of cluster algebras of type A and D. in Symmetries, integrable systems and representations, (K. Iohara, S. Morier-Genoud, B. Rémy, eds.), Springer proceedings in mathematics and statistics **40** (2013), 175–193.
- ³S. Fomin, A. Zelevinsky, Cluster algebras I: Foundations. J. Amer. Math. Soc. **15** (2002), 497–529.
- ⁴F. QIN, Alg'ebres amassees quantiques acycliques. PhD thesis, Universite Paris 7, May 2012.
- ⁵Y. Kimura, F. Qin, Graded quiver varieties, quantum cluster algebras and dual canonical basis.arXiv:1205.2066v2.
- ⁶D. Hernandez, Algebraic approach to q,t-character. Advances in Mathematics, **187**, (2004), 1–52.
- ⁷A. Berenstein, A. Zelevinsky, Quantum cluster algebra. Advances in Mathematics, **195**, (2005), 405–455.
- ⁸V. Chari, A. Pressley, A Guide to Quantum Groups. Cambridge University Press, Cambridge, 1994.
- ⁹V. Chari, A. Pressley, Quantum affine algebras. Comm. Math. Phys. **142** (1991), no. 2, 261–283.

¹⁰B. Leclerc, Quantum loop algebras, quiver variety, and cluster algebra.arXiv:1102.1076v1.

¹¹Y.M. Yang, Z.J. Zheng, Cluster algebra structure on the finite dimensional representations of $U_q(\widehat{A_3})$ for l=2.arXiv:1403.5124.

¹²H. Nakajima, Quiver varieties and cluster algebras.arXiv:0905.0002v5.