$\mathcal{A}$-MANIFOLDS ON A PRINCIPAL TORUS BUNDLE OVER AN $\mathcal{A}$-MANIFOLD BASE

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Abstract. We construct new examples of manifolds with cyclic-parallel Ricci tensor, so called $\mathcal{A}$-manifolds, on a r-torus bundle over a product of almost Hodge A-manifolds.

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1. Introduction

One of the most extensively studied objects in mathematics and physics are Einstein manifolds (see for example [Be]), i.e. manifolds whose Ricci tensor is a constant multiple of the metric tensor. In his work [Gra] A. Gray defined a condition which generalize the concept of an Einsten manifold. This condition states that the Ricci tensor Ric of the Riemannian manifold (M, g) is cyclic parallel, i.e.

$$\nabla_X \mathrm{Ric}(Y, Z) + \nabla_Y \mathrm{Ric}(Z, X) + \nabla_Z \mathrm{Ric}(X, Y) = 0,$$

where ∇ denotes the Levi-Civita connection of the metric g and X, Y, Z are arbitrary vector fields on M. A Riemannian manifold satisfying this condition is called an $\mathcal{A}$-manifold. It is obvious that if the Ricci tensor of (M, g) is parallel, then it satisfies the above condition. On the other hand if Ric is cyclic-parallel, but not parallel then we call (M, g) a strict $\mathcal{A}$-manifold. A. Gray gave in [Gra] first example of such strict $\mathcal{A}$-manifold, which was the sphere S^3 with appropriately defined homogeneous metric. A first example of a non-homogeneous $\mathcal{A}$-manifold was given in [Jel1]. This example is a S^1-bundle over some Kähler-Einstein manifold. This result was generalized in [Jel2] to K-contact manifolds. Namely, over every almost Hodge $\mathcal{A}$-manifold with J-invariant Ricci tensor we can construct a Riemannian metric such that the total space of the bundle is an $\mathcal{A}$-manifold. In the present paper we take a next step in the generalization process and we prove that there exist an $\mathcal{A}$-manifold structure on every r-torus bundle over product of almost Hodge $\mathcal{A}$-manifolds. Our result and that of Jelonek are based on the existence of almost Hodge $\mathcal{A}$-manifolds, which was proven in [Jel3].

2. Conformal Killing tensors

Let (M, g) be any Riemannian manifold. We call a symmetric tensor field of type $(0, 2)$ on M a *conformal Killing tensor* field iff there exists a 1-form P such

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that for any $X \in \Gamma(TM)$

$$\nabla_X K(X,X) = P(X)g(X,X), \tag{1}$$

where ∇ is the Levi-Civita connection of g. The above condition is clearly equivalent to the following

$$\mathcal{C}_{X,Y,Z}\nabla_X K(Y,Z) = \mathcal{C}_{X,Y,Z}P(X)g(Y,Z) \tag{2}$$

for all $X,Y,Z \in \Gamma(TM)$ where $\mathcal{C}_{X,Y,Z}$ denotes the cyclic sum over X,Y,Z. It is easy to prove that the 1-form P is given by

$$P(X) = \frac{1}{n+2}\left(2\mathrm{div}K(X) + d\,\mathrm{tr}\,K(X)\right),$$

where $X \in \Gamma(TM)$ and $\mathrm{div}S$ and $\mathrm{tr}\,S$ are the divergence and trace of the tensor field S with respect to g.

If the 1-form P vanishes, then we call K a *Killing tensor*. Of particular interest in this work is a situation when the Ricci tensor of the metric g is a Killing tensor. We call such a manifold an $\mathcal{A}$*-manifold*. In the more general situation, when the Ricci tensor is a conformal tensor we call (M,g) a $\mathcal{AC}^{\perp}$*-manifold*.

We will use a following easy property of conformal Killing tensors.

Proposition 1. *Suppose that (M,g) is a Riemannian product of (M_i,g_i), $i = 1,2$. Moreover, let K_i be conformal tensors on (M_i,g_i). Then $K = K_1 + K_2$ is a conformal tensor for (M,g).*

A *conformal Killing form* or a *twistor form* is a differential p-form φ on (M,g) satisfying the following equation

$$\nabla_X\varphi = \frac{1}{p+1}X \lrcorner\, d\varphi - \frac{1}{n-p+1}X \wedge \delta\varphi. \tag{3}$$

An extensive description of conformal Killing forms can be found in a series of articles by Semmelmann and Moroianu ([Sem],[S-M]).

It is known that if φ is a co-closed conformal Killing form (also called a *Killing form*) then the $(0,2)$-tensor field K_φ defined by

$$K_\varphi(X,Y) = g(X \lrcorner\, \varphi, Y \lrcorner\, \varphi)$$

is a Killing tensor.

We can prove even more.

Theorem 2. *Let φ and ψ be conformal Killing p-forms. Then the tensor field $K_{\varphi,\psi}$ defined by*

$$K_{\varphi,\psi} = g(X \lrcorner\, \varphi, Y \lrcorner\, \psi) + g(Y \lrcorner\, \varphi, X \lrcorner\, \psi)$$

is a conformal Killing tensor field.

Proof. Let X be any vector field and φ, ψ conformal Killing p-forms. We will check that $K_{\varphi,\psi}$ as defined above satisfies (1).

$$\begin{aligned}\nabla_X K_{\varphi,\psi}(X,X) &= 2X\left(g(X \lrcorner \varphi, X \lrcorner \psi)\right) - 2g(\nabla_X X \lrcorner \varphi, X \lrcorner \psi) \\ &\quad -2g(\nabla_X X \lrcorner \psi, X \lrcorner \varphi) \\ &= 2g(\nabla_X (X \lrcorner \varphi), X \lrcorner \psi) + 2g(X \lrcorner \varphi, \nabla_X (X \lrcorner \psi)) \\ &\quad -2g(\nabla_X X \lrcorner \varphi, X \lrcorner \psi) - 2g(\nabla_X X \lrcorner \psi, X \lrcorner \varphi) \\ &= 2g(X \lrcorner \nabla_X \varphi, X \lrcorner \psi) + 2g(X \lrcorner \varphi, X \lrcorner \nabla_X \psi).\end{aligned}$$

From the fact that φ satisfies (3) we have

$$\begin{aligned}g(X \lrcorner \nabla_X \varphi, X \lrcorner \psi) &= \frac{1}{p+1} g(X \lrcorner (X \lrcorner d\varphi), X \lrcorner \psi) \\ &\quad - \frac{1}{n-p+1} g(X \lrcorner (X \wedge \delta\varphi), X \lrcorner \psi) \\ &= -\frac{1}{n-p+1}\left(g(X,X)g(\delta\varphi, X \lrcorner \psi) - g(X \wedge (X \lrcorner \delta\varphi), X \lrcorner \psi)\right) \\ &= -\frac{1}{n-p+1} g(X,X) g(\delta\varphi, X \lrcorner \psi).\end{aligned}$$

The same is valid for ψ with

$$g(X \lrcorner \nabla_X \psi, X \lrcorner \varphi) = -\frac{1}{n-p+1} g(X,X) g(\delta\psi, X \lrcorner \varphi).$$

Hence we have

$$\nabla_X K_{\varphi,\psi}(X,X) = -\frac{2}{n-p+1} g(X,X)\left(g(\delta\varphi, X \lrcorner \psi) + g(\delta\psi, X \lrcorner \varphi)\right)$$

□

3. Torus bundles

Let (M,h) be a Riemannian manifold and suppose that β_i are closed 2-forms on M for $i = 1,\dots,r$ such that their cohomology classes $[\beta_i]$ are integral. In [Ko] it was proven that to each such cohomology class there corresponds a principal circle bundle $p_i : P_i \to M$ with a connection form θ_i such that

$$d\theta_i = 2\pi p_i^* \beta_i. \tag{4}$$

Taking the Whitney sum of bundles (p_i, P_i, M) we obtain a principal r-torus bundle $p : P \to M$ classified by cohomology classes of β_i, $i = 1,\dots,r$. The connection form θ is a vector valued 1-form with coefficients θ_i, where θ_i are as before. For each connection form θ_i we define a vector field ξ^i by $\theta_i(\xi^i) = 1$. This vector field is just the fundamental vector field for θ_i corresponding to 1 in the Lie algebra of i-th S^1-factor of the bundle (p, P, M).

It is easy to check that the tensor field g given by

$$g = \sum_{i,j=1}^{r} b_{ij}\theta_i \otimes \theta_j + p^*h \tag{5}$$

is a Riemannian metric on P if $[b_{ij}]_{i,j=1}^r$ is some symmetric, positive definite $r \times r$ matrix with real coefficients. This Riemannian metric makes the projection $p : (P,g) \to (M,h)$ a Riemannian submersion (see [ON]).

Lemma 3. *Each vector field ξ^i for $i = 1,\dots,r$ is Killing with respect to the metric g. Moreover, define a tensor field T_i of type $(1,1)$ by $T_iX = \nabla_X\xi^i$ for $X \in \Gamma(TP)$, where ∇ is the Levi-Civita connection of g. Then we have*

$$T_i\xi^j = 0, \quad L_{\xi^i}T_j = 0,$$

for $i \neq j$.

Proof. To prove that ξ^s is a Killing vector field for $s = 1,\dots,r$ observe that

$$L_{\xi^s}g = \sum_{i,j=1}^{r} b_{ij}\left((L_{\xi^s}\theta_i)\otimes\theta_j + \theta_i\otimes(L_{\xi^s}\theta_j)\right).$$

Hence we only have to check that $L_{\xi^s}\theta_i = 0$ for any $i,s = 1,\dots,r$. Using Cartan's magic formula for Lie derivative we have

$$L_{\xi^s}\theta_i = d\left(\theta_i(\xi^s)\right) + \xi^s \lrcorner\, d\theta_i$$

and it is immediate that the first term is zero, since $\theta_i(\xi^s) = \delta_i^s$, where δ_i^s is the Kronecker delta. For the second term we have

$$d\theta_i(\xi^s,X) = \xi^s\left(\theta_i(X)\right) - X\left(\theta_i(\xi^s)\right) - \theta_i\left([\xi^s,X]\right), \tag{6}$$

where X is arbitrary. We will consider two cases, namely when X is a horizontal or vertical vector field. In both cases the first two components vanish, hence we only have to look at the third. In the first case we notice that $[\xi^s,X]$ is a horizontal vector field, since ξ^s is a fundamental vector field on P. This gives us the vanishing of $\xi^s \lrcorner\, d\theta_i$ on horizontal vector fields. When X is vertical we can take it to be just ξ^k and we immediately see that $[\xi^s,\xi^k] = 0$ since the fields ξ^j come from the action of a torus on P.

For the second part of the lemma observe that $g(\xi^i,\xi^j)$ is constant. For any vector field X this gives us

$$0 = Xg(\xi^i,\xi^j) = g(\nabla_X\xi^i,\xi^j) + g(\xi^i,\nabla_X\xi^j) = -g(X,\nabla_{\xi^j}\xi^i) - g(\nabla_{\xi^i}\xi^j,X).$$

Now, since $[\xi^i,\xi^j] = 0$ we have $\nabla_{\xi^i}\xi^j = \nabla_{\xi^j}\xi^i$ which proves that $T_i\xi^j = 0$.

Recall that for any Killing vector field we have

$$L_\xi\nabla_XY = \nabla_{L_\xi X}Y + \nabla_X(L_\xi Y),$$

where X and Y are arbitrary vector fields. In our situation we have

$$(L_{\xi^i}T_j)X = L_{\xi^i}(T_jX) - T_j(L_{\xi^i}X) = \nabla_{[\xi^i,X]}\xi^j + \nabla_X[\xi^i,\xi^j] - \nabla_{[\xi^i,X]}\xi^j = 0,$$

which ends the proof. $\square$

Hence tensor fields T_i are horizontal, i.e. for each i there exists a tensor field $\tilde{T}_i$ on M such that $p_* \circ T_i = \tilde{T}_i \circ p_*$.

We now compute the O'Neill tensors ([ON]) of the Riemannian submersion $p : P \to M$.

Proposition 4. *The O'Neill tensor T is zero. The O'Neill tensor A is given by*

$$A_E F = \sum_{i,j=1}^{r} b^{ij} \left(g(E, T_i F)\xi^j + g(\xi^i, F) T_j E \right), \tag{7}$$

where b^{ij} are the coefficients of the inverse matrix of $[b_{ij}]_{i,j=1}^{r}$ and $E, F \in \Gamma(TP)$.

Observe that from the fact that $\theta_i(\xi^i) = 1$ for $E \in \Gamma(TP)$ we get that

$$g(\xi^i, E) = \sum_{j=1}^{r} b_{ij} \theta_j(E)$$

hence

$$\theta_j(E) = \sum_{i=1}^{r} b^{ji} g(\xi^i, E).$$

Taking the exterior differential we get

$$d\theta_j(E, F) = 2 \sum_{i=1}^{r} b^{ji} g(T_i E, F), \tag{8}$$

where $E, F \in \Gamma(TP)$.

Using formulae from [Be] Chapter 9 and the fact that the fibre of the Riemannian submersion (p, P, M) is totally geodesic and flat, we see that the Ricci tensor on the total space of Riemannian submersion is given by

$$\mathrm{Ric}(U, V) = \sum_{i=1}^{m} g(A_{E_i} U, A_{E_i} V), \tag{9}$$

$$\mathrm{Ric}(X, U) = -\sum_{i=1}^{m} g\left((\nabla_{E_i} A)_{E_i} X, U \right), \tag{10}$$

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_M(X, Y) - 2 \sum_{i=1}^{m} g(A_X E_i, A_Y E_i). \tag{11}$$

Here E_i is an element of the orthonormal basis of the horizontal distribution $\mathcal{H}$, Ric_M is a lift of the Ricci tensor of the base (M, h), X, Y are horizontal vector fields and U, V any vertical vector fields. Using the formula (7) for the O'Neill tensor A we can compute all components of the Ricci tensor Ric. We obtain

$$\mathrm{Ric}(U, V) = \sum_{i=1}^{m} g\left(\sum_{s,t=1}^{r} b^{st} g(\xi^s, U) T_t E_i, \sum_{k,l=1}^{r} b^{kl} g(\xi^k, V) T_l E_i \right), \tag{12}$$

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_M(X, Y) - \frac{1}{2} \sum_{s,t=1}^{r} b^{st} g(T_s X, T_t Y). \tag{13}$$

As for the value of $\mathrm{Ric}(X,U)$ we compute the covariant derivative

$$\begin{aligned}(\nabla_{E_i}A)_{E_i}X &= \nabla_{E_i}\left(\sum_{s,t=1}^{r} b^{st} g\left(E_i, T_s X\right)\xi^t\right) - \sum_{s,t=1}^{r} b^{st} g\left(\nabla_{E_i}E_i, T_s X\right)\xi^t \\ &\quad - \sum_{s,t=1}^{r} b^{st}\left(g\left(E_i, T_s\nabla_{E_i}X\right)\xi^t + g\left(\xi^s, \nabla_{E_i}X\right)T_t E_i\right) \\ &= \sum_{s,t=1}^{r} b^{st} g\left(E_i, \left(\nabla_{E_i}T_s\right)X\right)\xi^t,\end{aligned}$$

where we used the fact that $g(\xi^s, \nabla_{E_i}X) = -g(T_s E_i, X)$ which follows from A_X being anti-symmetric with respect to g for any horizontal vector field X. Now since tensors T_s are anti-symmetric with respect to g so is $\nabla_X T_s$, hence

$$(\nabla_{E_i}A)_{E_i}X = -\sum_{s,t=1}^{r} b^{st} g\left(\left(\nabla_{E_i}T_s\right)E_i, X\right)\xi^t = \sum_{t=1}^{r}\delta d\theta_t(X)\xi^t.$$

As a result we have

$$\mathrm{Ric}(X,U) = \sum_{t=1}^{r}\delta d\theta_t(X) g(\xi^t, U).$$

4. Torus bundle over a product of almost Hodge manifolds

Recall that an almost complex manifold is a pair (M,J) where M is a some differential manifold and J is an endomorphism of TM such that $J^2 = -\mathrm{id}_{TM}$. On such manifolds we can single out particular Riemannian metrics which we call compatible with the almost complex structure J. The compatibility condition for a metric g is $g(JX,JY) = g(X,Y)$. We call such metrics almost Hermitian and the triple (M,g,J) an almost Hermitian manifold. We can define a differential 2-form ω by $\omega(X,Y) = g(JX,Y)$. We call ω a Kähler form of (M,g,J). If the complex structure J is integrable and the Kähler form is closed we call such complex manifold a Kähler manifold. Such manifolds are of no use in this work, since by a result of Sekigawa and Vanhecke ([S-V]) every Kähler $\mathcal{A}$-manifold has parallel Ricci tensor.

There are however manifolds very close to being Kähler which are more suitable for us. Let (M,g,J) be an almost complex manifold with closed Kähler form. We call such manifolds *almost Kähler manifolds*. If the almost complex structure J is not integrable then (M,g,J) is sometimes called a *strictly almost Kähler manifold*. In addition to being closed the Kähler form of an almost Kähler manifold is also coclosed, hence harmonic with respect to g.

In [Jel3] Jelonek constructed a strictly almost Kähler $\mathcal{A}$-manifold with non-parallel Ricci tensor. Moreover the Kähler form of such a manifold has a useful property. It is a constant multiple of some differential 2-form that belongs to an integral cohomology class i.e. a differential form in $H^2(M;\mathbb{Z})$. An almost Kähler manifold whose Kähler form satisfies this condition is called an almost Hodge manifold.

Returning to our construction suppose that (M_i, g_i, J_i), $i = 1, \ldots, n$ are almost Hodge manifolds such that Kähler forms ω_i are constant multiples of 2-forms α_i and their cohomology classes are integral, i.e. $[\alpha_i] \in H^2(M_i; \mathbb{Z})$. Denote by (M, g, J) the product manifold with the product metric and product almost complex structure and let pr_i be the projection on the i-th factor. From our earlier discussion we know that there exists a principal r-torus bundle classified by the forms $\beta_1, \ldots, \beta_r$ given by

$$\beta_j = \sum_{i=1}^{n} a_{ji} pr_i^* \alpha_i,$$

where $[a_{ji}]$ is some $r \times n$ matrix with integer coefficients. By (4) the coefficients θ_j of the connection form of (p, P, M) satisfy

$$d\theta_j = 2\pi p^* \beta_j = 2\pi \sum_{i=1}^{n} a_{ji} p^* \left(pr_i^* \alpha_i \right)$$

for every $j = 1, \ldots, r$. Since α_i's and Kähler forms ω_i of (M_i, g_i, J_i) are connected by $\omega_i = c_i \alpha_i$ for some constants c_i, $i = 1, \ldots, n$ we have

$$d\theta_j = 2\pi \sum_{i=1}^{n} \frac{a_{ji}}{c_i} \omega_i^*, \tag{14}$$

where by ω_i^* we denote the 2-form obtained from lifting ω_i to P. Comparing this with (8) we get a formula for each tensor field $\tilde{T}_i$

$$\tilde{T}_i X = \pi \sum_{j=1}^{r} b_{ij} \sum_{k=1}^{n} \frac{a_{jk}}{c_k} J_k^* X \tag{15}$$

where J_k^* is the almost complex structure tensor of (M_k, g_k, J_k) lifted to the product manifold M.

We will now compute the Ricci tensor of (P, g) using (9)-(11), computations that follows those formulas and above observations. We begin with

$$\begin{aligned} \operatorname{Ric}(U,V) &= \pi^2 \sum_{i=1}^{m} h\left(\sum_{s=1}^{r} g(\xi^s, U) \sum_{k=1}^{n} \frac{a_{sk}}{c_k} J_k^* E_i, \sum_{l=1}^{r} g(\xi^l, U) \sum_{h=1}^{n} \frac{a_{lh}}{c_h} J_h^* E_i \right) \\ &= \pi^2 \sum_{s,l=1}^{r} g(\xi^s, U) g(\xi^l, V) \sum_{i=1}^{m} h\left(\sum_{k=1}^{n} \frac{a_{sk}}{c_k} J_k^* E_i, \sum_{h=1}^{n} \frac{a_{lh}}{c_h} J_h^* E_i \right) \\ &= \pi^2 \sum_{s,l=1}^{r} g(\xi^s, U) g(\xi^l, V) \sum_{i=1}^{m} \sum_{k=1}^{n} g_k\left(\frac{a_{sk}}{c_k} J_k E_i, \frac{a_{lk}}{c_k} J_k E_i \right). \end{aligned} \tag{16}$$

We used the fact that for $k \neq h$ images of J_k and J_h are orthogonal. It is easy to see that

$$\sum_{i=1}^{m} \sum_{k=1}^{n} g_k\left(\frac{a_{sk}}{c_k} J_k E_i, \frac{a_{lk}}{c_k} J_k E_i \right)$$

are constants for each $s, l = 1, \dots, r$. Hence the Ricci tensor of (P, g) on vertical vector fields is a symmetrized product of Killing vector fields.

Next, since the Kähler form of each almost Hodge manifold (M_k, g_k, J_k) is co-closed we see from (14) that

$$\mathrm{Ric}(X, U) = 0 \tag{17}$$

for any horizontal vector field X and vertical vector field U.

The last component of the Ricci tensor of (P, g) is the horizontal one. First observe that Ric_M is the Ricci tensor of the product metric $h = g_1 + \ldots + g_n$ and Ricci tensors Ric_k are J_k-invariant Killing tensors. We have

Theorem 5. *Let K_i be a Killing tensor on (M_i, g_i, J_i) for $i = 1, \dots, n$. Then the lift K^* of $K = K_1 + \ldots + K_n$ to P is a Killing tensor iff each K_i is J_i-invariant.*

Proof. We need to check the cyclic sum condition (2) for different choices of vector fields. It is easy to see that if all three vector fields are vertical then each component of the cyclic sum vanishes, since K^* is non-vanishing only on horizontal vector fields. If only two of the vector fields are vertical then again all components vanish, since $\nabla_{\xi^i}\xi^j = 0$. For three horizontal vector fields we again see that the cyclic sum vanish, since the covariant derivative of K^* with respect to metric g on P is the same as that of K with respect to the product metric h on M. By Proposition 1 K is a Killing tensor for (M, h). The remaining case is when only one vector field is vertical. Let us put $Z = \xi^i$ and X, Y be basic horizontal vector fields. We compute

$$\begin{aligned}\nabla_{\xi^i} K^*(X, Y) &= -K^*(\nabla_{\xi^i} X, Y) - K^*(X, \nabla_{\xi^i} Y) = -K^*(A_X \xi^i, Y) - K^*(X, A_Y \xi^i)\\ &= -K^*(\nabla_X \xi^i, Y) - K^*(X, \nabla_Y \xi^i),\end{aligned}$$

where the before last equality is due to the fact that X and Y are basic (see [ON]) and the last one follows from the definition of the O'Neill tensor A. Next we have

$$\nabla_X K^*(\xi^i, Y) = -K^*(\nabla_X \xi^i, Y).$$

Summing up we have

$$\begin{aligned}\mathcal{C}_{\xi^i, X, Y} \nabla_{\xi^i} K^*(X, Y) &= -2\left(K^*(\nabla_X \xi^i, Y) + K^*(X, \nabla_Y \xi^i)\right)\\ &= -2\left(K(\tilde{T}_i X, Y) + K(X, \tilde{T}_i Y)\right).\end{aligned}$$

Now we use the formula (15) for the tensor $\tilde{T}_i$

$$\mathcal{C}_{\xi^i, X, Y} \nabla_{\xi^i} K^*(X, Y) = -2\pi \sum_{j=1}^{r} b_{ij} \sum_{k=1}^{n} \frac{a_{jk}}{c_k} \left(K(J_k^* X, Y) + K(X, J_k^* Y)\right).$$

Since each J_i projects vector fields on TM_k we see from the definition of K that

$$K(J_k^* X, Y) + K(X, J_k^* Y) = K_k(J_k X, Y) + K_k(X, J_k Y).$$

By J_k-invariance of K_k for $k = 1, \dots, n$ we have completed the proof. □

Remark. It is worth noting, that we cannot lift in that way a conformal Killing tensor with non-vanishing P. In fact taking three vertical vector fields we see that P vanishes on vertical distribution. On the other hand for two vertical vector fields U, V and one horizontal vector field X the left-hand side of (2) vanish and the right-hand side reads $P(X)g(U,V)$, hence P has to vanish also on the horizontal distribution.

Corollary 1. An r-torus bundle with metric defined by (5) can not be an $\mathcal{AC}^\perp$-*manifold.* Especially there are no $\mathcal{AC}^\perp$ structures on K-contact and Sasakian manifolds.

Next we show that the second component of the horizontal part of the Ricci tensor (13) is just a sum of lifts of metrics g_k, $k = 1, \ldots, n$.

$$\sum_{s,t=1}^{r} b^{st} g(T_s X, T_t Y) = \pi^2 \sum_{s,t=1}^{r} h\left(\sum_{j=1}^{r} b_{sj} \sum_{k=1}^{n} \frac{a_{jk}}{c_k} J_k^* X, \sum_{i=1}^{r} b_{ti} \sum_{l=1}^{n} \frac{a_{il}}{c_l} J_l^* Y\right).$$

Since J_k and J_l are orthogonal for different $k, l = 1, \ldots, n$ we obtain

$$\sum_{s,t=1}^{r} b^{st} g(T_s X, T_t Y) = \pi^2 \sum_{s,t=1}^{r} \sum_{k=1}^{n} h\left(\sum_{j=1}^{r} b_{sj} \frac{a_{jk}}{c_k} J_k^* X, \sum_{i=1}^{r} b_{ti} \frac{a_{ik}}{c_k} J_k^* Y\right) \tag{18}$$
$$= \pi^2 \sum_{j,l=1}^{r} b_{jl} \sum_{k=1}^{r} \frac{a_{jk} a_{lk}}{c_k^2} g_k(X,Y).$$

From the above Theorem we infer that, since a Riemannian metric is a Killing tensor and each g_k is J_k-invariant, the tensor field $K(X,Y) = \sum_{s,t=1}^{r} b^{st} g(T_s X, T_t Y)$ is a Killing tensor field.

Now we can prove the following theorem

Theorem 6. *Let P be a r-torus bundle over a Riemannian product (M,h) of almost Hodge $\mathcal{A}$-manifolds (M_k, g_k, J_k), $k = 1, \ldots n$ with metric g defined by* (5)*. Then (P, g) is itself an $\mathcal{A}$-manifold.*

Proof. Since distributions $\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to the Ricci tensor Ric of (P,g) by (17) we can write it as

$$\mathrm{Ric}(E,F) = \pi^2 \sum_{s,l=1}^{r} g(\xi^s, E) g(\xi^l, F) \sum_{i=1}^{m} \sum_{k=1}^{n} g_k\left(\frac{a_{sk}}{c_k} J_k E_i, \frac{a_{lk}}{c_k} J_k E_i\right)$$
$$+\mathrm{Ric}_M(E,F) - \frac{1}{2}\pi^2 \sum_{j,l=1}^{r} b_{jl} \sum_{k=1}^{r} \frac{a_{jk} a_{lk}}{c_k^2} g_k(E,F)$$

using (18) and (16).The first component is a Killing tensor as a symmetrized product of Killing vector fields by Theorem 2. The second and third components are Killing tensors by Theorem 5. Since a sum of Killing tensors with constant coefficients is again a Killing tensor we have proved the theorem. □

Remark. Observe that if at least one of the manifolds (M_k, g_k) has non-parallel Ricci tensor, then the Ricci tensor Ric of (P, g) is also non-parallel with respect to the metric g. Thus we have constructed a large number of strict $\mathcal{A}$-manifolds.

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