On matrices with Q^2-scalings

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Abstract

We provide a counterexample to some statements dealing with a sufficient property for the square of a matrix to be a P_0^+-matrix.

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Let us recall the following definitions and notations (see, for example, [1], [2]). If $\mathbf{A}$ is an $n\times n$ matrix, $\mathbf{A}^{(j)}$ $(1 \leq j \leq n)$ denotes its jth compound matrix, i.e. the matrix which consists of all the minors of the jth order of $\mathbf{A}$, numerated lexicographically.

An $n \times n$ matrix $\mathbf{A}$ is called a *Q-matrix* if its sums of principal minors of the same order are all positive (this is equivalent to the following conditions: $\mathrm{Tr}(\mathbf{A}^{(j)}) > 0$ for all $j = 1, \ \ldots, \ n$). An $n \times n$ matrix $\mathbf{A}$ is called a P_0- (P_0^+-) matrix if all its principal minors are nonnegative (respectively, nonnegative with at least one positive principal minor of each order). An $n\times n$ matrix $\mathbf{A}$ is called *anti-sign symmetric* if it satisfies the following conditions:

$$A\begin{pmatrix}\alpha \\ \beta\end{pmatrix} A\begin{pmatrix}\beta \\ \alpha\end{pmatrix} \leq 0$$

for all sets of indices $\alpha, \beta \subset \{1, \ \ldots, \ n\}$, $\alpha \neq \beta$, $|\alpha| = |\beta|$.

The following statement was claimed to be proven in [1] (see [1], p. 115, Proposition 4.4).

Theorem 1 *Let $\mathbf{A}$ be a square matrix. If for every positive diagonal matrix $\mathbf{D}$ the matrix $(\mathbf{DA})^2$ is a Q-matrix then $\mathbf{A}^2$ is a P_0^+-matrix.*

This statement does not hold. Let us consider the following counterexample.

Counterexample. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}. \tag{1}$$

In this case, we have

$$\mathbf{A}^{(2)} = \det(\mathbf{A}) = 7.$$

Multiplying by an arbitrary positive diagonal matrix $\mathbf{D} = \mathrm{diag}\{d_{11}, \ d_{22}\}$, we obtain:

$$\mathbf{DA} = \begin{pmatrix} d_{11} & 2d_{11} \\ -d_{22} & 5d_{22} \end{pmatrix};$$

$$(\mathbf{DA})^2 = \begin{pmatrix} d_{11}^2 - 2d_{11}d_{22} & 2d_{11}^2 + 10d_{11}d_{22} \\ -d_{11}d_{22} - 5d_{22}^2 & -2d_{11}d_{22} + 25d_{22}^2 \end{pmatrix};$$

$$((\mathbf{DA})^2)^{(2)} = \det((\mathbf{DA})^2) = 49d_{11}^2d_{22}^2.$$

It is easy to see that

$$\mathrm{Tr}((\mathbf{DA})^2) = d_{11}^2 - 4d_{11}d_{22} + 25d_{22}^2 = (d_{11} - 2d_{22})^2 + 21d_{22}^2 > 0;$$
$$\det((\mathbf{DA})^2) = 49d_{11}^2 d_{22}^2 > 0$$

for any positive values $d_{11},\ d_{22}$. Thus the matrix $(\mathbf{DA})^2$ is a Q-matrix for every positive diagonal matrix $\mathbf{D}$. However,

$$\mathbf{A}^2 = \begin{pmatrix} -1 & 12 \\ -6 & 23 \end{pmatrix}$$

is not even a P_0-matrix since it has a negative entry on the principal diagonal.

The flaw in the proof is as follows. For a given proper subset α of $\{1, \ldots, n\}$, the authors construct a positive diagonal matrix $\mathbf{D}_\epsilon$:

$$(\mathbf{D}_\epsilon)_{jj} = \begin{cases} 1, & j \in \alpha \\ \epsilon, & j \notin \alpha \end{cases}$$

and claim the following equality for the principal minors: $(\mathbf{D}_0\mathbf{A})^2[\alpha] = \mathbf{A}^2[\alpha]$. However, this is not true. $(\mathbf{D}_0\mathbf{A})^2[\alpha]$ gives the determinant of $(\mathbf{A}_\alpha)^2$ where $\mathbf{A}_\alpha$ is a principal submatrix of $\mathbf{A}$ spanned by rows and columns with the numbers from α, while $\mathbf{A}^2[\alpha]$ gives the determinant of the corresponding submatrix of $\mathbf{A}^2$ (note, that $(\mathbf{A}_\alpha)^2 \neq (\mathbf{A}^2)_\alpha$). For example, if $n = 3$, $\alpha = \{1,\ 2\}$, $\mathbf{A} = \{a_{ij}\}_{i,j=1}^3$, we have $\mathbf{D}_\epsilon = \mathrm{diag}\{1, 1, \epsilon\}$ and

$$\mathbf{D}_0\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, $(\mathbf{D}_0\mathbf{A})^2[1,2] = (a_{11}a_{22} - a_{21}a_{12})^2 = \left(A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right)^2$, which is always positive. However, $\mathbf{A}^2[1,2]$ is equal to $\left(A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right)^2 + A\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} A\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + A\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} A\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and obviously in general case is not equal to $(\mathbf{D}_0\mathbf{A})^2[1,2]$.

The following statements were claimed to be proven in [1] using false Proposition 4.4 (see [1], p. 115, Proposition 4.6 and p. 116, Theorem 4.8).

Theorem 2 *Let* $\mathbf{A}$ *be a* 2×2 *matrix. Then the following are equivalent:*

(i) *For every positive diagonal matrix* $\mathbf{D}$ *the matrix* $(\mathbf{DA})^2$ *is a* Q*-matrix.*

(ii) *The matrix* $\mathbf{A}^2$ *is a* P_0^+*-matrix.*

Theorem 3 *Let* $\mathbf{A}$ *be an anti-sign symmetric matrix. Then the following are equivalent:*

(i) *For every positive diagonal matrix* $\mathbf{D}$ *the matrix* $(\mathbf{DA})^2$ *is a* Q*-matrix.*

(ii) *The matrix* $\mathbf{A}^2$ *is a* P_0^+*-matrix.*

The implication $(i) \Rightarrow (ii)$ is false in both of the statements. An anti-sign symmetric 2×2 matrix $\mathbf{A}$ given by Formula (1) provides the counterexample for both of them. Thus we conclude that Proposition 4.4 fails even in the case of 2×2 matrices.

References

[1] D. Hershkowitz, N. Keller, *Positivity of principal minors, sign symmetry and stability*, Linear Algebra Appl., **364** (2003), 105-124.

[2] A. Pinkus, *Totally positive matrices.* Cambridge University Press, 2010.