CIRCLE ACTIONS ON UHF-ABSORBING C^*-ALGEBRAS

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Abstract. We study circle actions with the Rokhlin property, in relation to their restrictions to finite subgroups. We construct examples showing the following: the restriction of a circle action with the Rokhlin property (even on a real rank zero C^*-algebra), need not have the Rokhlin property; and even if every restriction of a given circle action has the Rokhlin property, the circle action itself need not have it. As a positive result, we show that the restriction of a circle action with the Rokhlin property to the subgroup $\mathbb{Z}_n$ has the Rokhlin property if the underlying algebra absorbs M_{n^∞}. The condition on the algebra is also necessary in most cases of interest.

Despite the fact that there are no circle actions with the Rokhlin property on UHF-algebras, we construct many such actions on certain UHF-absorbing simple A$\mathbb{T}$-algebras. Additionally, we show that circle actions with the Rokhlin property on $\mathcal{O}_2$-absorbing C^*-algebras are generic, in a suitable sense.

Contents

1. Introduction

The interplay between C^*-algebras and dynamics has a long and rich history. Crossed products have provided some of the most interesting examples of C^*-algebras. Some algebraic properties are preserved under formation of crossed products in great generality. For example, crossed products of type I C^*-algebras by compact groups are type I, and crossed products of nuclear C^*-algebras by amenable groups are nuclear, regardless of the action. On the other hand, for preservation of other (usually stronger) properties, one must assume some kind of freeness condition on the action. This is best seen in the commutative setting, where the Atiyah-Segal completion theorem (specifically as in the statement of Theorem 1.1.1 in [Phi87]) shows how free actions on compact spaces enjoy a number of nice analytic and algebraic properties.

In the noncommutative setting, there are several different notions of freeness

Date: April 6, 2020.

2000 *Mathematics Subject Classification.* Primary 46L55; Secondary 46L35.

Key words and phrases. Rokhlin property, UHF-algebras, Cuntz algebra $\mathcal{O}_2$, Kirchberg algebras.

This material is based upon work supported by the US National Science Foundation through the author's thesis advisor's Grant DMS-1101742. The financial support is gratefully acknowledged.

for actions, and many of them are surveyed in [Phi09]. For finite groups, and in roughly decreasing order of strength, there are: the Rokhlin property (see [Izu04a]), the tracial Rokhlin property (see [Phi11]), pointwise outerness, and hereditary saturation (see [Phi87]), just to mention a few. Among these, the Rokhlin property is the strongest one, and is therefore less common than the other notions of freeness. For example, the Rokhlin property for finite groups implies that the underlying algebra has non-trivial projections, ruling out the existence of such actions on many C^*-algebras of interest, such as the Jiang-Su algebra $\mathcal{Z}$. There are also less obvious K-theoretic obstructions to the Rokhlin property. See Theorem 3.13 in [Izu04a]. On the other hand, the Rokhlin property implies very strong structure preservation results for crossed products (see Theorem 2.3 in [Phi09] for a list of properties that are preserved by Rokhlin actions, and see [HW07], [OP12] and [Phi11] for the proofs of most of them), and it is the hypothesis in most theorems on classification of group actions (see Theorems 3.4 and 3.5 in [Izu04b], and see Theorem 4.7 in [Gar14a]). These have been the main uses of the Rokhlin property: obtaining structure results for the crossed product, and classification of actions.

Besides finite groups, the Rokhlin property has been extensively studied for automorphisms (see [HO84] and [Kis95]) and flows (see [Kis96]). In [HW07], Hirshberg and Winter introduced the Rokhlin property for an action of a second-countable compact group on a unital C^*-algebra, and they proved that absorption of a strongly self-absorbing C^*-algebra and approximate divisibility pass to crossed products by such actions.

It is natural to try to generalize the results on the structure of the crossed product in [OP12] to arbitrary compact groups. This will be done in [Gar14c]. Another natural direction is to explore the classification of Rokhlin actions of compact groups on certain classes of classifiable C^*-algebras, generalizing or at least complementing Izumi's work for finite group actions with the Rokhlin property. As a first step in this direction, we study Rokhlin actions of the circle on C^*-algebras that absorb a UHF-algebra, specifically in relation to their restrictions to finite subgroups. One of our main results, Theorem 3.19, asserts that under suitable assumptions on the UHF-algebra, all such restrictions have the Rokhlin property. This fact, together with Izumi's classification of finite group actions with the Rokhlin property, will be used in subsequent work to classify circle actions on UHF-absorbing C^*-algebras.

Similarly to what happens with finite groups, Rokhlin actions of compact groups are rare, and there are C^*-algebras that do not have any action of any non-trivial compact group with the Rokhlin property, such as the Cuntz algebra $\mathcal{O}_\infty$ or the Jiang-Su algebra $\mathcal{Z}$. In this sense, C^*-algebras that absorb $\mathcal{O}_2$ form a distinguished class since they have many actions with the Rokhlin property. In fact, the set of circle actions with the Rokhlin property on a separable $\mathcal{O}_2$-absorbing C^*-algebra is a dense G_δ-set in the space of all circle actions. See Theorem 2.25.

This paper is organized as follows. We establish the notation that will be used throughout in the rest of the Introduction. In Section 2, we introduce the definition of the Rokhlin property for circle actions on unital C^*-algebra, and derive some of its basic properties which will be frequently used in the later sections. We also provide a number of examples of circle actions on C^*-algebras with the Rokhlin property, mostly on simple C^*-algebras. In contrast, we show in Theorem 2.17 that no direct limit action of the circle on a UHF-algebra can have the Rokhlin property. In Subsection 2.2, we specialize to circle actions on C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_2$, a class of C^*-algebras which is special from the point of view of the Rokhlin property. We show in Theorem 2.25 that circle actions with the Rokhlin property are generic on separable, unital, $\mathcal{O}_2$-absorbing C^*-algebras, a fact that should be contrasted with Theorem 2.17.

Section 3 is devoted to showing the following: if A is a separable, unital C^*-algebra that absorbs the UHF-algebra M_{n^∞}, and if $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action with the Rokhlin property, then the restriction of α to the finite cyclic group $\mathbb{Z}_n \subseteq \mathbb{T}$ has the Rokhlin property. See Theorem 3.19. The condition that A absorb M_{n^∞} is shown to be necessary in most cases of interest; see Theorem 3.20. We also give examples of circle actions with the Rokhlin property such that *no* restriction to any finite cyclic group has the Rokhlin property; see Example 3.8 and Example 3.10. Additionally, Example 3.22 and Example 3.23 show that even if a circle action has the property that *every* restriction to a finite subgroup has the Rokhlin property, the action itself need not have the Rokhlin property, even on Kirchberg algebras satisfying the UCT.

Acknowledgements. The author is grateful to his advisor Chris Phillips for a number of helpful conversations, as well as for feedback on an earlier draft of this work. He also wishes to thank the annonymous referee for a number of suggestions that improved the quality of the present paper.

1.1. **Notation and preliminaries.** We adopt the convention that $\{0\}$ is not a unital C^*-algebra, that is, we require that $1 \neq 0$ in a unital C^*-algebra. We take $\mathbb{N} = \{1, 2, \ldots\}$. For a separable C^*-algebra A, we denote by $\mathrm{Aut}(A)$ the automorphism group of A, which is equipped with the topology of pointwise norm convergence. In this topology, a sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi \in \mathrm{Aut}(A)$ if and only if for every $\varepsilon > 0$ and every compact set $F \subseteq A$, there exists $m \in \mathbb{N}$ such that $\|\varphi_m(a) - \varphi(a)\| < \varepsilon$ for all $a \in F$.

If A is moreover unital, then $\mathcal{U}(A)$ denotes the unitary group of A, and two automorphisms φ and ψ of A are said to be approximately unitarily equivalent if $\varphi \circ \psi^{-1}$ is approximately inner.

For a locally compact group G, an action of G on a C^*-algebra A is always assumed to be a *continuous* group homomorphism from G into $\mathrm{Aut}(A)$, unless otherwise stated. If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A, then we will denote by A^α the fixed-point subalgebra of A under α.

For a C^*-algebra A, we set

$$\begin{aligned}
\ell^\infty(\mathbb{N}, A) &= \left\{(a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} \colon \sup_{n\in\mathbb{N}} \|a_n\| < \infty\right\};\\
c_0(\mathbb{N}, A) &= \left\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \colon \lim_{n\to\infty} \|a_n\| = 0\right\};\\
A_\infty &= \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).
\end{aligned}$$

We identify A with the constant sequences in $\ell^\infty(\mathbb{N}, A)$ and with their image in A_∞. We write $A_\infty \cap A'$ for the central sequence algebra of A, that is, the relative commutant of $A \in A_\infty$. For a bounded sequence $(a_n)_{n\in\mathbb{N}} \in A$, we denote by $\overline{(a_n)}_{n\in\mathbb{N}}$ its image in A_∞. We have

$$A_\infty \cap A' = \left\{\overline{(a_n)}_{n\in\mathbb{N}} \in A_\infty \colon \lim_{n\to\infty} \|a_n a - a a_n\| = 0 \text{ for all } a \in A\right\}.$$

If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A, then there are actions of G on A_∞ and on $A_\infty \cap A'$, both denoted by α_∞. Note that unless the group G is discrete, these actions will not be continuous in general.

Given $n \in \{2, 3, \ldots, \infty\}$, we denote by $\mathcal{O}_n$ the Cuntz algebra with canonical generators $\{s_j\}_{j=1}^n$ satisfying the usual relations (see for example Section 4.2 in [Rør02]).

We denote the circle group by $\mathbb{T}$, and identify it with the set of complex numbers of modulus 1. The finite cyclic group of order n will be denoted by $\mathbb{Z}_n$, and we will usually identify $\mathbb{Z}_n$ with the n-th roots of unity in $\mathbb{T}$, and in this fashion we will regard $\mathbb{Z}_n$ as a subgroup of the circle.

2. The Rokhlin property: rigidity and genericity

We begin this section by recalling the definition of the Rokhlin property for a finite group action on a unital C^*-algebra.

Definition 2.1. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for every $\varepsilon > 0$ and for every finite set $F \subseteq A$ there exist orthogonal projections $e_g \in A$ for $g \in G$ such that

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all g and $h \in G$
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$
(3) $\sum\limits_{g\in G} e_g = 1$.

The definition of the Rokhlin property for finite group actions on C^*-algebras was originally introduced by Izumi in [Izu04a], although a similar notion has been studied by Herman and Jones in [HJ82] for $\mathbb{Z}_2$ actions on UHF-algebras, and by Herman and Ocneanu in [HO84] for integer actions. The Rokhlin property also played a crucial role in the classification of finite group actions on von Neumann algebras.

The following is part of Proposition 2.14 in [Phi09], and we include the proof for the convenience of the reader. This result should be compared with Example 3.8 and Example 3.10.

Proposition 2.2. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $H \subseteq G$ is a subgroup, then $\alpha|_H$ has the Rokhlin property.

Proof. Set $n = \mathrm{card}(G/H)$. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose projections e_g for $g \in G$ as in the definition of the Rokhlin property for F and $\frac{\varepsilon}{n}$. We claim that the projections $f_h = \sum\limits_{\overline{x}\in G/H} e_{hx}$ for $h \in H$, form a family of Rokhlin projections for the action $\alpha|_H$, the finite set F and tolerance ε.

Given h and $k \in H$, we have

$$\begin{aligned} \|\alpha_k(f_h) - f_{kh}\| &= \left\| \sum_{\overline{x}\in G/H} \alpha_k(e_{hx}) - e_{khx} \right\| \\ &\leq \sum_{\overline{x}\in G/H} \|\alpha_k(e_{hx}) - e_{khx}\| \leq \mathrm{card}(G/H)\frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Finally, for $a \in F$ and $h \in H$, we have

$$\|af_h - f_h a\| \leq \sum_{\overline{x}\in G/H} \|ae_{hx} - e_{hx}a\| < \varepsilon.$$

□

Hirshberg and Winter defined the Rokhlin property for an arbitrary action of a compact, second countable group in [HW07]. In the case of the circle, and using semiprojectivity of $C(\mathbb{T})$, one can show that their definition is equivalent to the following.

Definition 2.3. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. Then α is said to have the *Rokhlin property* if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exists a unitary $u \in \mathcal{U}(A)$ such that

(1) $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$ and
(2) $\|ua - au\| < \varepsilon$ for all $a \in F$.

Remark 2.4. In order to check condition (2) in Definition 2.3, it is enough to consider finite subsets of any set of generators of A. It is also immediate to show that if in Definition 2.3 one allows the set F to be compact instead of finite, one obtains an equivalent definition. These easy observations will be used repeatedly and without reference.

We present some basic properties of circle actions with the Rokhlin property, some of which resemble those of free actions on spaces. For instance, the proposition below is the analog of the fact that a diagonal action on a product space is free if one of the factors is free.

Proposition 2.5. Let A be a unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ have the Rokhlin property. If $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ is any action of $\mathbb{T}$ on a unital C^*-algebra, then the tensor product action $\zeta \mapsto \alpha_\zeta \otimes \beta_\zeta$ of $\mathbb{T}$ on $\mathrm{Aut}(A \otimes B)$, for any C^*-tensor product on which it is defined, has the Rokhlin property.

Proof. Let $\varepsilon > 0$ and let $F' \subseteq A$ and $F'' \subseteq B$ be finite subsets of the respective unit balls of A and B. Set

$$F = \{a \otimes b\colon a \in F', b \in F''\},$$

which is a finite subset of $A \otimes B$. Using the Rokhlin property for α, choose a unitary $u \in A$ such that the conditions in Definition 2.3 are satisfied for ε and F'. Set $v = u \otimes 1 \in \mathcal{U}(A \otimes B)$. For $x = a \otimes b \in F$, we have

$$\|vx - xv\| = \|(ua - au) \otimes b\| \leq \|ua - au\|\|b\| < \varepsilon.$$

On the other hand,

$$\|(\alpha \otimes \beta)_\zeta(v) - \zeta v\| = \|(\alpha_\zeta(u) \otimes \beta_\zeta(1)) - \zeta(u \otimes 1)\| = \|\alpha_\zeta(u) - \zeta u\| < \varepsilon,$$

for all $\zeta \in \mathbb{T}$, which finishes the proof. □

We point out that a tensor product action may have the Rokhlin property without any of the tensor factors having the Rokhlin property, even if one of the actions is the trivial action. This is analogous to the fact that a diagonal action on a product space may be free without any of the factors being free, except that such examples with the trivial action as one of the factors do not exist in the commutative setting.

The proposition below is the analog of the fact that an equivariant inverse limit of free actions is free.

Proposition 2.6. If $A = \varinjlim(A_n, \iota_n)$ is a direct limit of C^*-algebras with unital maps, and $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action obtained as the direct limit of actions $\alpha^{(n)}\colon \mathbb{T} \to \mathrm{Aut}(A_n)$, such that $\alpha^{(n)}$ has the Rokhlin property for all n, then α has the Rokhlin property.

Proof. Let $F \subseteq A$ be a finite set, and let $\varepsilon > 0$. Write $F = \{a_1, \ldots, a_N\}$. Since $\bigcup\limits_{n\in\mathbb{N}} \iota_{n,\infty}(A_n)$ is dense in A, there exist $n \in \mathbb{N}$ and $F' = \{b_1, \ldots, b_N\} \subseteq A_n$ such that $\|a_j - \iota_{n,\infty}(b_j)\| < \frac{\varepsilon}{3}$ for $j = 1, \ldots, N$. Since $\alpha^{(n)}$ has the Rokhlin property, there exists a unitary $u \in A_n$ such that $\left\|\alpha^{(n)}_\zeta(u) - \zeta u\right\| < \frac{\varepsilon}{3}$ for all $\zeta \in \mathbb{T}$ and $\|b_j u - u b_j\| < \frac{\varepsilon}{3}$ for all $j = 1, \ldots, N$. Notice that $\iota_{n,\infty}(u)$ is a unitary in A, since the connecting maps are unital. Moreover, if $\zeta \in \mathbb{T}$, then

$$\|\alpha_\zeta(\iota_{n,\infty}(u)) - \zeta \iota_{n,\infty}(u)\| = \left\|\iota_{n,\infty}(\alpha^{(n)}_\zeta(u)) - \iota_{n,\infty}(\zeta u)\right\| < \frac{\varepsilon}{3} < \varepsilon.$$

Finally,

$$\begin{aligned}
&\|\iota_{n,\infty}(u)a_j - a_j\iota_{n,\infty}(u)\| \\
&\quad \leq \|\iota_{n,\infty}(u)a_j - \iota_{n,\infty}(u)\iota_{n,\infty}(b_j)\| + \|\iota_{n,\infty}(u)\iota_{n,\infty}(b_j) - \iota_{n,\infty}(b_j)\iota_{n,\infty}(u)\| \\
&\qquad\quad + \|\iota_{n,\infty}(b_j)\iota_{n,\infty}(u) - a_j\iota_{n,\infty}(u)\| \\
&\quad \leq \|a_j - \iota_{n,\infty}(b_j)\| + \|u b_j - b_j u\| + \|\iota_{n,\infty}(b_j) - a_j\| \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence $\iota_{n,\infty}(u)$ is the desired unitary for F and ε, and thus α has the Rokhlin property. □

We have the following convenient result, which turns out to be crucial in some proofs, in particular in the classification of Rokhlin actions of the circle on Kirchberg algebras; see [Gar14a] and [Gar14b]. In the present work, we will use Proposition 2.7 in the proof of Proposition 2.22.

Proposition 2.7. Let A be a unital C^*-algebra, let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, let $\varepsilon > 0$ and let $F \subseteq A$ be a finite subset. Then there exists a unitary $u \in A$ such that

(1) $\alpha_\zeta(u) = \zeta u$ for all $\zeta \in \mathbb{T}$.
(2) $\|ua - au\| < \varepsilon$ for all $a \in F$.

The definition of the Rokhlin property differs from the conclusion of this proposition in that in condition (1), one only requires $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.

Proof. One can normalize F so that $\|a\| \leq 1$ for all $a \in F$. Set $\varepsilon_0 = \min\left\{\frac{1}{3}, \frac{\varepsilon}{7+2\varepsilon}\right\}$. Choose a unitary $v \in A$ such that conditions (1) and (2) in Definition 2.3 are satisfied for the finite set F with ε_0 in place of ε. Denote by μ the normalized Haar measure on $\mathbb{T}$, and set

$$x = \int_{\mathbb{T}} \overline{\zeta}\alpha_\zeta(v)\, d\mu(\zeta).$$

Then $\|x\| \leq 1$ and $\|x - v\| \leq \varepsilon_0$. One checks that $\|x^*x - 1\| \leq 2\varepsilon_0 < 1$ and that $\alpha_\zeta(x) = \zeta x$ for all $\zeta \in \mathbb{T}$. In particular, x^*x is invertible.

We have

$$\left\|(x^*x)^{-1}\right\| \leq \frac{1}{1 - \|1 - x^*x\|} \leq \frac{1}{1 - 2\varepsilon_0},$$

and thus $\left\|(x^*x)^{-\frac{1}{2}}\right\| \leq \frac{1}{\sqrt{1-2\varepsilon_0}}$.

Set $u = x(x^*x)^{-\frac{1}{2}}$, which is a unitary in A. Using that $\|x\| \leq 1$ at the first step, and that $0 \leq 1 - (x^*x)^{\frac{1}{2}} \leq 1 - x^*x$ at the third step, we get

$$\begin{aligned} \|u - x\| &\leq \left\|(x^*x)^{-\frac{1}{2}} - 1\right\| \\ &\leq \left\|(x^*x)^{-\frac{1}{2}}\right\| \left\|1 - (x^*x)^{\frac{1}{2}}\right\| \\ &\leq \frac{1}{\sqrt{1-2\varepsilon_0}}\|1 - x^*x\| \leq \frac{2\varepsilon_0}{\sqrt{1-2\varepsilon_0}}. \end{aligned}$$

We deduce that

$$\|u - v\| \leq \frac{2\varepsilon_0}{\sqrt{1-2\varepsilon_0}} + \varepsilon_0.$$

For $\zeta \in \mathbb{T}$, we have $\alpha_\zeta(x^*x) = x^*x$ and hence $\alpha_\zeta(u) = \zeta u$, so u satisfies condition (1) of the statement. Finally, for $a \in F$, we have

$$\begin{aligned} \|ua - au\| &\leq \|ua - va\| + \|va - av\| + \|av - au\| \\ &< \|u - v\|\|a\| + \varepsilon_0 + \|a\|\|v - u\| \\ &\leq \frac{4\varepsilon_0}{\sqrt{1-2\varepsilon_0}} + 3\varepsilon_0 < \frac{4\varepsilon_0}{1-2\varepsilon_0} + 3\varepsilon_0 \\ &< \frac{7\varepsilon_0}{1-2\varepsilon_0} < \varepsilon, \end{aligned}$$

as desired. □

Remark 2.8. In the language of [PST14], the proof of Proposition 2.7 shows that the action of $\mathbb{T}$ on $C(\mathbb{T})$ induced by group multiplication, is equivariantly semiprojective. This fact seems not to have been known before.

We now turn to examples of circle actions with the Rokhlin property. As in the finite group case, the Rokhlin property is rare, and it is challenging to construct many examples on simple C^*-algebras. We will give an explicit construction of a family of circle actions with the Rokhlin property on simple A$\mathbb{T}$-algebras, and also on the Cuntz algebra $\mathcal{O}_2$. For more examples on purely inifinite C^*-algebras, see [Gar14b] (the construction of the examples there is not explicit).

Example 2.9. This is an example of a circle action on a simple, unital A$\mathbb{T}$-algebra with the Rokhlin property. For $n \in \mathbb{N}$, set $A_n = C(\mathbb{T}) \otimes M_{n!}$. Consider the action $\alpha^{(n)} \colon \mathbb{T} \to \mathrm{Aut}(A_n)$ given by $\alpha^{(n)}_\zeta(f)(w) = f(\zeta^{-1}w)$ for ζ and $w \in \mathbb{T}$ and for $f \in A_n \cong C(\mathbb{T}, M_{n!})$. In other words, $\alpha^{(n)}$ is the tensor product of the action of left translation of $\mathbb{T}$, with the trivial action on $M_{n!}$. Then $\alpha^{(n)}$ has the Rokhlin property by Proposition 2.5, since the action of left translation of $\mathbb{T}$ on itself trivially has the Rokhlin property.

We construct a direct limit algebra $A = \varinjlim(A_n, \iota_n)$ as follows. Fix a countable dense subset $X = \{x_1, x_2, x_3, \ldots\} \subseteq \mathbb{T}$, and assume that $x_1 = 1$. With $f_x(\zeta) = f(x^{-1}\zeta)$ for $f \in A_n$, for $x \in X$ and for $\zeta \in \mathbb{T}$, define maps $\iota_n \colon A_n \to A_{n+1}$ for

$n \in \mathbb{N}$, by

$$\iota_n(f) = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{x_n} \end{pmatrix}$$

for every $f \in A_n$.Then ι_n is unital and injective, for all $n \in \mathbb{N}$. The limit algebra $A = \varinjlim(A_n, \iota_n)$ is a unital A$\mathbb{T}$-algebra.

It is easy to check that

$$\iota_n \circ \alpha^{(n)}_\zeta = \alpha^{(n+1)}_\zeta \circ \iota_n$$

for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, so that $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{T}$ on A. Then α has the Rokhlin property by Proposition 2.6. Simplicity of A follows from Proposition 2.1 in [DNNP92], since X is assumed to be dense in $\mathbb{T}$.

In the example above, the universal UHF-pattern can be replaced by any other UHF or (simple) AF-pattern, and the resulting C^*-algebra is also a (simple) A$\mathbb{T}$-algebra.

Using the absorption properties of $\mathcal{O}_2$, we can construct an action of the circle on $\mathcal{O}_2$ with the Rokhlin property.

Example 2.10. Let A and α be as in the example above. Then A is a separable, unital, nuclear, simple C^*-algebra. Use Theorem 3.8 in [KP00] to choose an isomorphism $\varphi\colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$, and define an action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ by $\gamma_\zeta = \varphi \circ (\alpha_\zeta \otimes \mathrm{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$ for $\zeta \in \mathbb{T}$. Since α has the Rokhlin property, it follows from Proposition 2.5 that γ has the Rokhlin property as well.

Example 2.11. If A is any unital C^*-algebra such that $A \otimes \mathcal{O}_2 \cong A$, then one can construct a circle action on A with the Rokhlin property by tensoring the trivial action on A with any action on $\mathcal{O}_2$ with the Rokhlin property, such as the one constructed in Example 2.10.

2.1. **Nonexistence of actions with the Rokhlin property.** Our next goal is to prove that UHF-algebras do not admit any direct limit action of the circle with the Rokhlin property; see Theorem 2.17. We begin with an easy lemma which already rules out such actions on matrix algebras; see Corollary 2.13.

Lemma 2.12. Let A be a unital C^*-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then α_ζ is not inner for all $\zeta \in \mathbb{T}$ with $\zeta \neq 1$.

Proof. Let $\zeta \in \mathbb{T} \setminus \{1\}$, and assume that there exists a unitary $v \in A$ such that $\alpha_\zeta = \mathrm{Ad}(v)$. Let $\varepsilon > 0$ satisfy $\varepsilon < \frac{|1-\zeta|}{2}$. Using the Rokhlin property for α, find a unitary $u \in A$ such that $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ and $\|uv - vu\| < \varepsilon$. Then

$$\begin{aligned} \varepsilon &> \|\alpha_\zeta(u) - \zeta(u)\| = \|vuv^* - \zeta u\| \geq |\|u - \zeta u\| - \|vuv^* - u\|| \\ &= |1-\zeta| - \|vu - uv\| > \frac{|1-\zeta|}{2} > \varepsilon, \end{aligned}$$

which is a contradiction. This shows that α_ζ is not inner. ☐

Corollary 2.13. Let $n \in \mathbb{N}$. Then there are no actions of the circle on M_n with the Rokhlin property.

Proof. This is an immediate consequence of Lemma 2.12, since every automorphism of M_n is inner. □

We will generalize the corollary above in Theorem 2.17 below, where we show that there are no direct limit actions of the circle with the Rokhlin property on UHF-algebras. We need a series of preliminary results.

Notation 2.14. Let $n \in \mathbb{N}$. We denote by $\mathcal{U}_n(\mathbb{C})$ the unitary group of M_n. Identify $\mathbb{T}$ with the center $\mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ of $\mathcal{U}_n(\mathbb{C})$ via the map $\zeta \mapsto \mathrm{diag}(\zeta, \ldots, \zeta)$, and denote by $P\mathcal{U}_n(\mathbb{C})$ the quotient group $P\mathcal{U}_n(\mathbb{C}) = \mathcal{U}_n(\mathbb{C})/\mathbb{T}$.

Proposition 2.15. Let $n \in \mathbb{N}$ and let $\gamma \colon \mathbb{T} \to \mathrm{Aut}(M_n)$ be a continuous action. Then there exists a continuous map $v \colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$ such that $\gamma_\zeta = \mathrm{Ad}(v(\zeta))$ for all $\zeta \in \mathbb{T}$.

Proof. Recall that every automorphism of M_n is inner, so that for every $\zeta \in \mathbb{T}$ there exists a unitary $u(\zeta) \in \mathcal{U}_n(\mathbb{C})$ such that $\alpha_\zeta = \mathrm{Ad}(u(\zeta))$. Moreover, $u(\zeta)$ is uniquely determined up to multiplication by elements of $\mathbb{T} = \mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ and hence γ_ζ determines a continuous group homomorphism $u \colon \mathbb{T} \to P\mathcal{U}_n(\mathbb{C})$. Denote by $\rho \colon \mathcal{U}_n(\mathbb{C}) \to P\mathcal{U}_n(\mathbb{C})$ the canonical projection. We want to solve the following lifting problem:

$$\begin{array}{ccc} & & \mathcal{U}_n(\mathbb{C}) \\ & \overset{v}{\nearrow} & \big\downarrow \rho \\ \mathbb{T} & \xrightarrow[u]{} & P\mathcal{U}_n(\mathbb{C}). \end{array}$$

The map u determines an element $[u] \in \pi_1(P\mathcal{U}_n(\mathbb{C}))$ and ρ induces a group homomorphism $\pi_1(\rho) \colon \pi_1(\mathcal{U}_n(\mathbb{C})) \to \pi_1(P\mathcal{U}_n(\mathbb{C}))$. The quotient map $\rho \colon \mathcal{U}_n(\mathbb{C}) \to P\mathcal{U}_n(\mathbb{C})$ is actually a fiber bundle, since $\mathcal{U}_n(\mathbb{C})$ is a manifold and the action of $\mathbb{T}$ on $\mathcal{U}_n(\mathbb{C})$ is free. See the theorem in Section 4.1 of [Pal61]. The long exact sequence in homotopy for this fiber bundle is

$$\cdots \longrightarrow \pi_1(\mathbb{T}) \longrightarrow \pi_1(\mathcal{U}_n(\mathbb{C})) \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow \pi_0(\mathbb{T}).$$

Recall that $\pi_1(\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}$, and that $\pi_0(\mathbb{T}) \cong 0$. The map $\pi_1(\mathbb{T}) \to \pi_1(\mathcal{U}_n(\mathbb{C}))$ is induced by $\zeta \mapsto \mathrm{diag}(\zeta, \ldots, \zeta)$, which on π_1 corresponds to multiplication by n. In other words, the above exact sequence is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow 0,$$

which implies that $\pi_1(P\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}_n$ and that the map $\pi_1(\rho)$ is surjective. It follows that u is homotopic to a map $\widehat{u} \colon \mathbb{T} \to P\mathcal{U}_n(\mathbb{C})$ that is liftable. The homotopy lifting property for fiber bundles implies that u itself is liftable, that is, there exists a continuous map $v \colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$ such that $u(\zeta) = \rho(v(\zeta))$ for all $\zeta \in \mathbb{T}$. (See the paragraph below Theorem 4.41 in [Hat00] for the definition of the homotopy lifting property. Proposition 4.48 in [Hat00] shows that every fiber bundle has this property.) This concludes the proof. □

Lemma 2.16. Let $n \in \mathbb{N}$ and let $v \colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$ be a continuous map. Then for every $u \in \mathcal{U}_n(\mathbb{C})$, there exists $\zeta \in \mathbb{T}$ such that

$$\|v(\zeta)uv(\zeta)^* - \zeta u\| \geq 2.$$

Proof. Assume that there exists $u \in \mathcal{U}_n(\mathbb{C})$ such that $\|v(\zeta)uv(\zeta)^* - \zeta u\| < 2$ for all $\zeta \in \mathbb{T}$. Define $w \in C(\mathbb{T}, M_n)$ by $w(\zeta) = \overline{\zeta} v(\zeta)uv(\zeta)^* u^*$ for all $\zeta \in \mathbb{T}$. Then w is a unitary in $C(\mathbb{T}, M_n)$ and $\left\|w - 1_{C(\mathbb{T},M_n)}\right\| < 2$. It follows that the spectrum of w is not the whole circle, and a standard functional calculus argument using a branch of the logarithm shows that w is an exponential, that is, there is a self-adjoint element $x \in C(\mathbb{T}, M_n)$ with $w = e^{ix}$ (namely $x = \log(w)$). Then the path $t \mapsto e^{itx}$, for $t \in [0, 1]$, defines a homotopy between w and $1_{C(\mathbb{T},M_n)}$. Define now a continuous function $f \colon \mathbb{T} \to \mathbb{T}$ by $f = \det \circ w$. Then f is homotopic to the constant map, and thus its winding number is zero.

On the other hand,

$$f(\zeta) = \det(w(\zeta)) = \det(\overline{\zeta} v(\zeta)uv(\zeta)^* u^*) = \overline{\zeta}^n,$$

so the winding number is actually $-n$. This is a contradiction, and the result follows. □

Theorem 2.17. Assume that $A = \varinjlim(M_{k_n}, \iota_n)$ is an unital UHF-algebra with unital connecting maps. If $\alpha = \varinjlim \alpha^{(n)}$ is a direct limit action of the circle on A, then α does not have the Rokhlin property.

Proof. Assume that α has the Rokhlin property. Take $\varepsilon = 2$ and $F = \emptyset$. A standard approximation argument shows that there exist $n \in \mathbb{N}$ and $u \in \mathcal{U}_{k_n}(\mathbb{C})$ such that

$$\left\|\alpha^{(n)}_\zeta(u) - \zeta u\right\| < 2$$

for all $\zeta \in \mathbb{T}$. By Proposition 2.15, there is a continuous map $v \colon \mathbb{T} \to \mathcal{U}_{k_n}(\mathbb{C})$ such that $\alpha^{(n)}_\zeta = \mathrm{Ad}(v(\zeta))$ for all $\zeta \in \mathbb{T}$. Now, Lemma 2.16 implies that there exist $\zeta_0 \in \mathbb{T}$ such that $\|v(\zeta_0)uv(\zeta_0)^* - \zeta_0 u\| \geq 2$. Therefore, $2 > \left\|\alpha^{(n)}_{\zeta_0}(u) - \zeta_0 u\right\| \geq 2$, which is a contradiction. Thus, α does not have the Rokhlin property. □

2.2. **Genericity on $\mathcal{O}_2$-absorbing algebras.** In this subsection, we specialize to circle actions with the Rokhlin property on $\mathcal{O}_2$-absorbing C^*-algebras. This class of C^*-algebras is special in our context. Indeed, circle actions with the Rokhlin property are generic on separable, unital, $\mathcal{O}_2$-absorbing C^*-algebras; see Theorem 2.25. This fact should be contrasted with Theorem 2.17. Additionally, since $\mathcal{O}_2$ absorbs every UHF-algebra, Theorem 3.19 applies to algebras that absorb $\mathcal{O}_2$, so the restriction of every circle action with the Rokhlin property to a finite subgroup, again has the Rokhlin property.

This subsection is devoted to proving the first of the two results mentioned above. Throughout, A will be a separable, unital C^*-algebra.

Definition 2.18. Given an enumeration $S = \{a_1, a_2, \ldots\}$ of a countable dense subset of the unit ball of A, define metrics on $\mathrm{Aut}(A)$ by

$$\rho^{(0)}_S(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{\|\alpha(a_k) - \beta(a_k)\|}{2^k} \quad \text{and} \quad \rho_S(\alpha, \beta) = \rho^{(0)}_S(\alpha, \beta) + \rho^{(0)}_S(\alpha^{-1}, \beta^{-1}).$$

Denote by $\mathrm{Act}_{\mathbb{T}}(A)$ the set of all circle actions on A. For any enumeration $S = \{a_1, a_2, \ldots\}$ as above, define a metric on $\mathrm{Act}_{\mathbb{T}}(A)$ by

$$\rho_{\mathbb{T},S}(\alpha, \beta) = \max_{\zeta \in \mathbb{T}} \rho_S(\alpha_\zeta, \beta_\zeta).$$

Lemma 2.19. For any S as above, the function $\rho_{\mathbb{T},S}$ is a complete metric on $\mathrm{Act}_{\mathbb{T}}(A)$.

Proof. Let $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Act}_{\mathbb{T}}(A)$, that is, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$, we have $\rho_{\mathbb{T},S}\left(\alpha^{(n)}, \alpha^{(m)}\right) < \varepsilon$. We want to show that there is $\alpha \in \mathrm{Act}_{\mathbb{T}}(A)$ such that $\lim\limits_{n\to\infty} \rho_{\mathbb{T},S}\left(\alpha, \alpha^{(n)}\right) = 0$.

Given $\zeta \in \mathbb{T}$, we have $\rho_S\left(\alpha_\zeta^{(n)}, \alpha_\zeta^{(m)}\right) \leq \rho_{\mathbb{T},S}\left(\alpha^{(n)}, \alpha^{(m)}\right)$, and hence $\left(\alpha_\zeta^{(n)}\right)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{Aut}(A)$. By Lemma 3.2 in [Phi12], the pointwise norm limit of the sequence $\left(\alpha_\zeta^{(n)}\right)_{n\in\mathbb{N}}$ exists, and we denote it by α_ζ. It also follows from Lemma 3.2 in [Phi12] that α_ζ is an automorphism of A, with inverse $\alpha_{\zeta^{-1}}$. Moreover, the map $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ given by $\zeta \mapsto \alpha_\zeta$ is a group homomorphism, since it is the pointwise norm limit of group homomorphisms. It remains to check that it is continuous, and this follows from an $\frac{\varepsilon}{3}$ argument from $\lim\limits_{n\to\infty} \left\|\alpha_\zeta^{(n)}(a_k) - \alpha_\zeta(a_k)\right\| = 0$ for all $k \in \mathbb{N}$, and the fact that $\alpha^{(n)}\colon \mathbb{T} \to \mathrm{Aut}(A)$ is continuous for all $n \in \mathbb{N}$. □

Notation 2.20. Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, let $W_{\mathbb{T}}(F, \varepsilon)$ be the set of all actions $\alpha \in \mathrm{Act}_{\mathbb{T}}(A)$ such that there exists $u \in \mathcal{U}(A)$ with $\|ua - au\| < \varepsilon$ for all $a \in F$ and $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.

It is easy to check that an action $\alpha \in \mathrm{Act}_{\mathbb{T}}(A)$ has the Rokhlin property if and only if $\alpha \in W_{\mathbb{T}}(F, \varepsilon)$ for all finite subsets $F \subseteq A$ and all positive numbers $\varepsilon > 0$.

Lemma 2.21. Let S be a countable dense subset of the unit ball of A, and let $\mathcal{F}$ be the set of all finite subsets of S. Then $\alpha \in \mathrm{Act}_{\mathbb{T}}(A)$ has the Rokhlin property if and only if

$$\alpha \in \bigcap_{F\in\mathcal{F}} \bigcap_{n=1}^{\infty} W_{\mathbb{T}}\left(F, \frac{1}{n}\right).$$

Proof. One just needs to approximate any finite set by scalar multiples of elements in a finite subset of S. We omit the details. □

Using the notation of the lemma above, observe that the family $\mathcal{F}$ is countable.

Proposition 2.22. Let A and $\mathcal{D}$ be unital, separable C^*-algebras, such that there is an action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{D})$ with the Rokhlin property. Suppose that there exists an isomorphism $\varphi\colon A \otimes \mathcal{D} \to A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A. Then for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, the set $W_{\mathbb{T}}(F, \varepsilon)$ is open and dense.

Proof. We first check that $W_{\mathbb{T}}(F, \varepsilon)$ is open. Fix an enumeration $S = \{a_1, a_2, \ldots\}$ of a countable dense subset of the unit ball of A. Let $\alpha \in W_{\mathbb{T}}(F, \varepsilon)$, and choose $u \in \mathcal{U}(A)$ such that $\|ua - au\| < \varepsilon$ for all $a \in F$ and $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. Set

$$\varepsilon_0 = \max_{\zeta\in\mathbb{T}} \|\alpha_\zeta(u) - \zeta u\|,$$

so that $\varepsilon_1 = \varepsilon - \varepsilon_0 > 0$. Choose $k \in \mathbb{N}$ such that $\|a_k - u\| < \frac{\varepsilon_1}{3}$. Now, we claim that if $\alpha' \in \mathrm{Act}_{\mathbb{T}}(A)$ satisfies $\rho_{\mathbb{T},S}(\alpha', \alpha) < \frac{\varepsilon_1}{2^k 3}$, then $\alpha' \in W_{\mathbb{T}}(F, \varepsilon)$. Indeed,

$$\begin{aligned}\|\alpha'_\zeta(u) - \zeta u\| &\leq \|\alpha'_\zeta(u) - \alpha_\zeta(u)\| + \|\alpha'_\zeta(u) - \zeta u\| \\ &\leq \frac{2\varepsilon_1}{3} + \|\alpha'_\zeta(a_k) - \alpha_\zeta(a_k)\| + \varepsilon_0 \\ &\leq \frac{2\varepsilon_1}{3} + 2^k \rho_{\mathbb{T},S}(\alpha, \alpha') + \varepsilon_0 \\ &= \varepsilon_1 + \varepsilon_0 = \varepsilon.\end{aligned}$$

This proves that $W_{\mathbb{T}}(F, \varepsilon)$ is open.

We will now show that $W_{\mathbb{T}}(F, \varepsilon)$ is dense in $\mathrm{Act}_{\mathbb{T}}(A)$. Let α be an arbitrary action in $\mathrm{Act}_{\mathbb{T}}(A)$, let $T \subseteq A$ be a finite set, and let $\delta > 0$. We want to find $\beta \in \mathrm{Act}_{\mathbb{T}}(A)$ such that $\beta \in W_{\mathbb{T}}(F, \varepsilon)$ and $\rho_{\mathbb{T},S}(\alpha, \beta) < \delta$.

Choose $\delta' > 0$ such that $\delta' < \min\{\delta, \varepsilon\}$. Since α is continuous, there is $\delta_0 > 0$ such that whenever $\zeta, \zeta' \in \mathbb{T}$ and $|\zeta - \zeta'| < \delta_0$, then $\|\alpha_\zeta(a) - \alpha_{\zeta'}(a)\| < \frac{\delta'}{4}$ for all $a \in T$. Choose $m \in \mathbb{N}$ and $\zeta_1, \ldots, \zeta_m \in \mathbb{T}$ such that for every $\zeta \in \mathbb{T}$ there is $j \in \mathbb{N}$ with $1 \leq j \leq m$ and such that $|\zeta - \zeta_j| < \delta_0$. Choose $w \in \mathcal{U}(A)$ such that $\|w\varphi(1 \otimes a)w^* - a\| < \frac{\delta'}{2}$ for all $a \in T \cup \bigcup_{j=1}^{m} \alpha_{\zeta_j}(T)$. Set $\psi = \mathrm{Ad}(w) \circ \varphi$ and for $\zeta \in \mathbb{T}$, define an action $\beta \in \mathrm{Act}_{\mathbb{T}}(A)$ by

$$\beta_\zeta = \psi \circ (\gamma_\zeta \otimes \alpha_\zeta) \circ \psi^{-1}.$$

We claim that $\beta \in W_{\mathbb{T}}(F, \varepsilon)$. Choose $w' \in A \otimes \mathcal{D}$ of the form $w' = \sum_{\ell=1}^{r} x_\ell \otimes d_\ell$ for some $d_1, \ldots, d_r \in \mathcal{D}$ and some $x_1, \ldots, x_r \in A$, such that $\|w - w'\| < \frac{\delta}{3}$. Since γ has the Rokhlin property, use Proposition 2.7 to choose $u \in \mathcal{U}(\mathcal{D})$ such that $\gamma_\zeta(u) = \zeta u$ for all $\zeta \in \mathbb{T}$ and $\|ud_\ell - d_\ell u\| < \frac{\varepsilon}{4}$ for all $\ell = 1, \ldots, r$. Then

$$\|(1_A \otimes u)w' - w'(1_A \otimes u)\| < \frac{\delta}{3}$$

and hence $\|(1_A \otimes u)w - w(1_A \otimes u)\| < \delta$. Set $v = \varphi(1_A \otimes u)$. Then

$$\begin{aligned}\|\beta_\zeta(v) - \zeta v\| &= \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*\varphi(1_A \otimes u)w))\right) w^* - \zeta\varphi(1_A \otimes u)\| \\ &\leq \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*\varphi(1_A \otimes u)w))\right) w^* - w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(1_A \otimes u)\right) w^*\| \\ &\quad + \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(1_A \otimes u)\right) w^* - \zeta\varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + \|w\varphi\left(\zeta 1_A \otimes u\right) w^* - \zeta\varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + \frac{\delta'}{2} = \delta' < \varepsilon\end{aligned}$$

for all $\zeta \in \mathbb{T}$, and thus $\|\beta_\zeta(u) - \zeta v\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. On the other hand, given $a \in F$, we have

$$\begin{aligned}\|va - av\| &= \|\varphi(1_A \otimes u)a - a\varphi(1_A \otimes u)\| \\ &\leq \|\varphi(1_A \otimes u)a - \varphi(1_A \otimes u)w\varphi(a \otimes 1_{\mathcal{D}})w^*\| \\ &\quad + \|\varphi(1_A \otimes u)w\varphi(a \otimes 1_{\mathcal{D}})w^* - w\varphi(a \otimes 1_{\mathcal{D}})w^*\varphi(1_A \otimes u)\| \\ &\quad + \|w\varphi(a \otimes 1_{\mathcal{D}})w^*\varphi(1_A \otimes u) - a\varphi(1_A \otimes u)\| \\ &< \frac{\delta'}{2} + 0 + \frac{\delta'}{2} = \delta' < \varepsilon\end{aligned}$$

because $a \otimes 1_{\mathcal{D}}$ and $1_A \otimes u$ commute. This proves the claim.

It remains to prove that $\|\beta_\zeta(a) - \alpha_\zeta(a)\| < \delta$ for all $a \in T$ and all $\zeta \in \mathbb{T}$. For fixed $\zeta \in \mathbb{T}$ and $a \in T$, we have

$$\begin{aligned}\|\beta_\zeta(a) - \alpha_\zeta(a)\| &= \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*aw))\right) - \alpha_\zeta(a)\| \\ &\leq \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(\varphi^{-1}(w^*aw))\right) - w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(a \otimes 1_{\mathcal{D}})\right)\| \\ &\quad + \|w\varphi\left((\alpha_\zeta \otimes \gamma_\zeta)(a \otimes 1_{\mathcal{D}})\right) - \alpha_\zeta(a)\| \\ &< \frac{\delta'}{2} + \|w\varphi(\alpha_\zeta(a) \otimes 1_{\mathcal{D}})w^* - \alpha_\zeta(a)\| \\ &\leq \frac{\delta'}{2} + \|w\varphi(\alpha_\zeta(a) \otimes 1_{\mathcal{D}})w^* - w\varphi(\alpha_{\zeta_j}(a) \otimes 1_{\mathcal{D}})w^*\| \\ &\quad + \|w\varphi(\alpha_{\zeta_j}(a) \otimes 1_{\mathcal{D}})w^* - \alpha_{\zeta_j}(a)\| + \|\alpha_{\zeta_j}(a) - \alpha_\zeta(a)\| \\ &< \frac{\delta'}{2} + \frac{\delta'}{4} + \frac{\delta'}{4} = \delta' < \delta.\end{aligned}$$

This finishes the proof. □

Theorem 2.23. Let A and $\mathcal{D}$ be unital, separable C^*-algebras, such that there is an action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{D})$ with the Rokhlin property. Suppose that there exists an isomorphism $\varphi\colon A \otimes \mathcal{D} \to A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A. Then the set of all circle actions with the Rokhlin property on A is a dense G_δ-set in $\mathrm{Act}_{\mathbb{T}}(A)$.

Proof. By Lemma 2.21, the set of all circle actions on A that have the Rokhlin property is precisely the countable intersection

$$\bigcap_{F \in \mathcal{F}} \bigcap_{n \in \mathbb{N}} W_{\mathbb{T}}\left(F. \frac{1}{n}\right).$$

By Proposition 2.22, each $W_{\mathbb{T}}\left(F, \frac{1}{n}\right)$ is open and dense in $\mathrm{Act}_{\mathbb{T}}(A)$, which is a complete metric space by Lemma 2.19, so the result follows from the Baire Category Theorem. □

Recall that a unital, separable C^*-algebra $\mathcal{D}$ is said to be strongly self-absorbing if it is infinite-dimensional and the map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism. (Strongly self-absorbing C^*-algebras are always nuclear, so there is no ambiguity when talking about tensor products.) The only known examples are the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF-algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ by such UHF-algebras. See [TW07] for more details and results on strongly self-absorbing C^*-algebras.

Remark 2.24. In the context of Theorem 2.23, suppose additionally that $\mathcal{D}$ is a unital, separable strongly self-absorbing C^*-algebra. Then, according to Theorem 7.2.2 in [Rør02], the following are equivalent:

(1) There exists an isomorphism $\varphi\colon A \otimes \mathcal{D} \to A$ such that $a \mapsto \varphi(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to id_A;
(2) There exists some isomorphism $\psi\colon A \otimes \mathcal{D} \to A$.

Theorem 2.25. Let A be a separable unital C^*-algebra such that $A \otimes \mathcal{O}_2 \cong A$. Then the set of all circle actions on A with the Rokhlin property is a dense G_δ-set in $\mathrm{Act}_{\mathbb{T}}(A)$.

Proof. By Example 2.10, there is an action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ with the Rokhlin property. Since A absorbs $\mathcal{O}_2$ tensorially, the hypotheses of Theorem 2.23 are met by Remark 2.24, and the result follows. $\square$

It is a consequence of the theorem above that the Rokhlin property is generic for circle actions on $\mathcal{O}_2$. Nevertheless, we do not know of any such action for which it is possible to describe what the images of the canonical generators of $\mathcal{O}_2$ are. In particular, we do not have a model action on $\mathcal{O}_2$.

3. Restrictions to finite cyclic groups

This section is devoted to proving that for $n \in \mathbb{N}$, the restriction of a circle action with the Rokhlin property on a M_{n^∞}-absorbing C^*-algebra to the finite cyclic group $\mathbb{Z}_n$ again has the Rokhlin property. See Theorem 3.19. This phenomenon cannot be expected to hold in full generality since the Rokhlin property for a circle action does not guarantee the existence of any non-trivial projections. Even more, there are serious K-theoretical obstructions to the Rokhlin property for finite groups. See Example 3.8 and Example 3.10 below.

On the other hand, this result will be used in subsequent work to classify circle actions with the Rokhlin property on unital C^*-algebras that absorb some UHF-algebra of infinite type.

We give a rough outline of what our strategy will be. We will first focus on cyclic group actions which are restrictions of circle actions with the Rokhlin property. These have what we call the "unitary Rokhlin property", which is a weakening of the Rokhlin property of Definition 2.1, that asks for a unitary instead of projections; see Definition 3.5. Dual actions of actions with the unitary Rokhlin property can be completely characterized, and we do so in Proposition 3.13. The relevant notion is that of "strong approximate innerness"; see Definition 3.1. We will later show in (the proof of) Theorem 3.19 that, under a number of assumptions, every strongly approximately inner action of $\mathbb{Z}_n$ is approximately representable, which is the notion dual to the Rokhlin property, as was shown by Izumi in [Izu04a]. The conclusion is then that the original restriction, which a priori had the unitary Rokhlin property, actually has the Rokhlin property.

The following is Definition 3.6 in [Izu04a].

Definition 3.1. Let B be a unital C^*-algebra, and let β be an action of a finite abelian group G on B.

(1) We say that β is *strongly approximately inner* if there exist unitaries $u(g) \in (B^\beta)^\infty$, for $g \in G$, such that
$$\beta_g(b) = u(g)bu(g)^*$$

for $b \in B$ and $g \in G$.

(2) We say that β is *approximately representable* if β is strongly approximately inner and the unitaries $u(g)$ for $g \in G$ as in (1) above, can be chosen to form a representation of $G \in (B^\beta)^\infty$.

Notation 3.2. Let B be a C^*-algebra, let G be a cyclic group (that is, either $\mathbb{Z}$ or $\mathbb{Z}_n$ for some $n \in \mathbb{N}$), and let $\beta\colon G \to \mathrm{Aut}(B)$ be action of G on B. We will usually make a slight abuse of notation and also denote by β the generating automorphism β_1.

If G is a finite cyclic group, we have the following characterization of strong approximate innerness in terms of elements in B, rather than in $(B^\beta)^\infty$.

Lemma 3.3. Let B be a separable, unital C^*-algebra, let $n \in \mathbb{N}$, and let β be an action of $\mathbb{Z}_n$ on B. Then β is strongly approximately inner if and only if for every finite subset $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary $w \in \mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all $b \in F$. Moreover, β is approximately representable if and only if the unitary w above can be chosen so that $w^n = 1$.

Proof. Assume that β is strongly approximately inner. Use a standard perturbation argument to choose a sequence $(u_m)_{m\in\mathbb{N}}$ of unitaries in B^β that represents $u(1) \in (B^\beta)^\infty$. Then $\lim\limits_{m\to\infty} \|\beta(u_m) - u_m\| = 0$, and for $b \in B$, we have $\lim\limits_{m\to\infty} \|\beta(b) - u_m b u_m^*\| = 0$.

Given a finite set $F \subseteq B$ and $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $\|\beta(u_M) - u_M\| < \varepsilon$ and $\|\beta(b) - u_M b u_M^*\| < \varepsilon$ for all $b \in F$, and set $w = u_M$.

Conversely, let $(F_m)_{m\in\mathbb{N}}$ be an increasing sequence of finite subsets of B satisfying $\overline{\bigcup\limits_{m\in\mathbb{N}} F_m} = B$. For $m \in \mathbb{N}$, set $\varepsilon_m = \frac{1}{m}$ and let w_m be as in the statement for ε_m and F_m. Then

$$u = \overline{(w_m)}_{m\in\mathbb{N}} \in (B^\beta)^\infty$$

satisfies $\beta(b) = ubu^*$ for all $b \in F_m$, and hence β is strongly approximately inner.

For the second statement, observe that a unitary of order $n \in (B^\beta)^\infty$ can be lifted to a sequence unitaries of order n in B^β. First, observe that a unitary in $(B^\beta)^\infty$ can always be lifted to a sequence of unitaries in B. Second, a standard functional calculus argument shows that if v is a unitary in B^β such that $\|v^n - 1\|$ is small, then v is close to a unitary $\widetilde{v} \in B^\beta$ such that $\widetilde{v}^n = 1$. We omit the details. $\square$

The following is Lemma 3.8 in [Izu04a].

Lemma 3.4. Let B be a separable unital C^*-algebra, and let β be an action of a finite abelian group G on B.

(1) The action β has the Rokhlin property if and only if the dual action $\widehat{\beta}$ is approximately representable.

(2) The action β is approximately representable if and only if the dual action $\widehat{\beta}$ has the Rokhlin property.

The lemma above should be regarded as the assertion that for finite abelian group actions, the Rokhlin property and approximate representability are dual notions. It is therefore natural to ask what condition on β is equivalent to its dual action being strongly approximately inner, rather than approximately representable. Such a condition will necessarily be weaker than the Rokhlin property. We define the relevant property below.

Definition 3.5. Let B be a unital C^*-algebra, let $n \in \mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action. We say that β has the *unitary Rokhlin property* if for every $\varepsilon > 0$ and for every finite subset $F \subseteq B$, there exists $u \in \mathcal{U}(B)$ such that $\|ub - bu\| < \varepsilon$ for all $b \in F$ and $\left\|\beta_k(u) - e^{2\pi ik/n}u\right\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

Let A be a unital C^*-algebra. Given a continuous action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$, and $n \in \mathbb{N}$, we denote by $\alpha|_n$ the restriction $\alpha|_{\mathbb{Z}_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ of α to

$$\{1, e^{2\pi i/n}, \ldots, e^{2\pi i(n-1)/n}\} \cong \mathbb{Z}_n.$$

Recall that if v is the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ implementing $\alpha|_n$, then the dual action

$$\widehat{\alpha|_n}\colon \mathbb{Z}_n \cong \widehat{\mathbb{Z}_n} \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$$

of $\alpha|_n$ is given by $\left(\widehat{\alpha|_n}\right)_k(a) = a$ for all $a \in A$ and $\left(\widehat{\alpha|_n}\right)_k(v) = e^{2\pi ik/n}v$ for all $k \in \mathbb{Z}_n$.

The following easy lemmas provide us with many examples of cyclic group actions with the unitary Rokhlin property.

Lemma 3.6. If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the Rokhlin property, then $\alpha|_n$ has the unitary Rokhlin property for all $n \in \mathbb{N}$.

Proof. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose a unitary $u \in \mathcal{U}(A)$ such that $\|ua - au\| < \varepsilon$ for all $a \in F$ and $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. If $n \in \mathbb{N}$, then

$$\left\|(\alpha|_n)_k(u) - e^{2\pi ik/n}u\right\| = \left\|\alpha_{e^{2\pi ik/n}}(u) - e^{2\pi ik/n}u\right\| < \varepsilon$$

for all $k \in \mathbb{Z}_n$, as desired. ☐

Lemma 3.7. If $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ has the Rokhlin property, then it has the unitary Rokhlin property.

Proof. Given $\varepsilon > 0$ and a finite subset $F \subseteq B$, choose projections $e_0, \ldots, e_{n-1}$ as in the definition of the Rokhlin property for the tolerance $\frac{\varepsilon}{n}$ and the finite set F, and set $u = \sum\limits_{j=0}^{n-1} e^{-2\pi ij/n}e_j$. Then u is a unitary in B. Moreover, $\|ub - bu\| < \varepsilon$ for all $b \in F$ and

$$\left\|\beta_k(u) - e^{2\pi ik/n}u\right\| = \left\|\sum_{j=0}^{n-1} e^{2\pi ij/n}\beta_k(e_j) - e^{2\pi ik/n}\sum_{j=0}^{n-1} e^{-2\pi ij/n}e_j\right\| < \varepsilon$$

since $\|\beta_k(e_j) - e_{j+k}\| < \frac{\varepsilon}{n}$ for all $j, k \in \mathbb{Z}_n$, and the projections $e_0, \ldots, e_{n-1}$ are pairwise orthogonal. ☐

The converse of the preceding lemma is not in general true, since the unitary Rokhlin property does not ensure the existence of any non-trivial projections on the algebra. We present two examples of how this can fail.

Example 3.8. Consider the action of left translation of $\mathbb{T}$ on $C(\mathbb{T})$. It has the Rokhlin property, so its restriction to any $\mathbb{Z}_n \subseteq \mathbb{T}$ has the unitary Rokhlin property. However, no non-trivial finite group action on $C(\mathbb{T})$ can have the Rokhlin property since $C(\mathbb{T})$ has no non-trivial projections.

Besides merely the lack of projections, there are less obvious K-theoretic obstructions for the restrictions of an action of the circle with the Rokhlin property to have the Rokhlin property. See Example 3.10.

We need a lemma first.

Proposition 3.9. Let G be a connected metric group, let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action (not necessarily with the Rokhlin property). Then $K_*(\alpha_g) = \mathrm{id}_{K_*(A)}$ for all $g \in G$.

Proof. We just prove it for K_0; the proof for K_1 is similar, or follows by replacing (A, α) with $(A \otimes B, \alpha \otimes \mathrm{id}_B)$, where B is any C^*-algebra satisfying the UCT such that $K_0(B) = 0$ and $K_1(B) = \mathbb{Z}$, and using the Künneth formula. (For example, $B = C_0(\mathbb{R})$ will do.)

Denote the metric on G by d. Let $n \in \mathbb{N}$ and let p be a projection in $M_n(A)$. Set $\alpha^{(n)} = \alpha \otimes \mathrm{id}_{M_n}$, the augmentation of α to $M_n(A)$. Since $\alpha^{(n)}$ is continuous, there exists $\delta > 0$ such that $\left\|\alpha_g^{(n)}(p) - \alpha_h^{(n)}(p)\right\| < 1$ whenever g and $h \in G$ satisfy $d(g, h) < \delta$. Since $\alpha_g^{(n)}(p)$ and $\alpha_h^{(n)}(p)$ are projections in $M_n(A)$, it follows that $\alpha_g^{(n)}(p)$ and $\alpha_h^{(n)}(p)$ are homotopic, and hence their classes in $K_0(A)$ agree, that is, $K_0(\alpha_g)([p]_0) = K_0(\alpha_h)([p]_0)$. Denote by e the unit of G. Since g and h satisfying $d(g, h) < \delta$ are arbitrary, and since G is connected, it follows that

$$K_0(\alpha_g)([p]_0) = K_0(\alpha_e)([p]_0) = [p]_0$$

for any $g \in G$. Since p is an arbitrary projection in $A \otimes \mathcal{K}$, it follows that $K_0(\alpha_g) = \mathrm{id}_{K_0(A)}$ for all $g \in G$, as desired. $\square$

Example 3.10. This is an example of a purely infinite simple separable nuclear unital C^*-algebra (in particular, with many projections), and an action of the circle on it satisfying the Rokhlin property, such that no restriction to a finite subgroup of $\mathbb{T}$ has the Rokhlin property.

Let $\{p_n\}_{n\in\mathbb{N}}$ be an enumeration of the prime numbers, and for every $n \in \mathbb{N}$, set $q_n = p_1 \cdots p_n$. Fix a countable dense subset $X = \{x_1, x_2, x_3, \ldots\}$ of $\mathbb{T}$ with $x_1 = 1$. For $x \in X$ and $f \in C(\mathbb{T})$, denote by f_x the function in $C(\mathbb{T})$ given by $f_x(\zeta) = f(x^{-1}\zeta)$ for $\zeta \in \mathbb{T}$. For $n \in \mathbb{N}$, define a unital injective map

$$\iota_n\colon M_{q_n}(C(\mathbb{T})) \to M_{q_{n+1}}(C(\mathbb{T}))$$

by $\iota_n(f) = \mathrm{diag}\left(f_1, f_{x_2}, \ldots, f_{x_{p_n}}\right)$ for $f \in M_{q_n}(C(\mathbb{T}))$. The direct limit $A = \varinjlim(M_{q_n}(C(\mathbb{T})), \iota_n)$ is a unital A$\mathbb{T}$-algebra, and an argument similar to the one exhibited in Example 2.9 shows that A is simple. For $n \in \mathbb{N}$, let $\alpha^{(n)}\colon \mathbb{T} \to \mathrm{Aut}(M_{q_n}(C(\mathbb{T})))$ be the tensor product of the trivial action on M_{q_n} with the action coming from left translation on $C(\mathbb{T})$. Then $\alpha^{(n)}$ has the Rokhlin property by Proposition 2.5. Since $\iota_n \circ \alpha_\zeta^{(n)} = \alpha_\zeta^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, the sequence $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{T}$ on A, which has the Rokhlin property by Proposition 2.6.

Now set $B = A \otimes \mathcal{O}_\infty$ and define $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ by $\beta = \alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$. Then B is a purely infinite, simple, separable, nuclear unital C^*-algebra, and β has the Rokhlin property, again by Proposition 2.5. We claim that for every $m > 1$, the restriction $\beta|_m\colon \mathbb{Z}_m \to \mathrm{Aut}(B)$ does not have the Rokhlin property.

Fix $m > 1$, and assume that $\beta|_m$ has the Rokhlin property. By Proposition 3.9, we have $K_*(\beta_\zeta) = \mathrm{id}_{K_*(B)}$ for all $\zeta \in \mathbb{T}$. By Theorem 3.4 in [Izu04b], it follows that every element of $K_0(B)$ is divisible by m. On the other hand,

$$(K_0(B), [1_B]) \cong (K_0(A), [1_A])$$

$$\cong \left(\left\{ \frac{a}{b} : a \in \mathbb{Z}, b = p_{k_1} \cdots p_{k_n} : n, k_1, \ldots, k_n \in \mathbb{N}, k_j \neq k_\ell \text{ for } j \neq \ell \right\}, 1 \right),$$

where not every element is divisible by m. This is a contradiction.

We will nevertheless show that the restriction of an action of the circle with the Rokhlin property to any finite cyclic subgroup again has the Rokhlin property if the algebra is separable and absorbs the universal UHF-algebra $\mathcal{Q}$. See Corollary 3.21 below.

Lemma 3.11. Let A be a separable unital C^*-algebra, let $n \in \mathbb{N}$ and let $\alpha\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ be an action of $\mathbb{Z}_n$ on A. Regard $\mathbb{Z}_n \subseteq \mathbb{T}$ as the n-th roots of unitry, and let $\gamma\colon \mathbb{Z}_n \to \mathrm{Aut}(C(\mathbb{T}))$ be the restriction of the action by left translation of $\mathbb{T}$ on $C(\mathbb{T})$. Let $\alpha_\infty\colon \mathbb{Z}_n \to \mathrm{Aut}(A_\infty \cap A')$ be the action on $A_\infty \cap A'$ induced by α. Then α has the unitary Rokhlin property if and only if there exists a unital equivariant homomorphism

$$\varphi\colon (C(\mathbb{T}), \gamma) \to (A_\infty \cap A', \alpha_\infty).$$

Proof. Choose an increasing sequence $(F_m)_{m\in\mathbb{N}}$ of finite subsets of A such that $\overline{\bigcup_{m\in\mathbb{N}} F_m} = A$. For each $m \in \mathbb{N}$, there exists a unitary $u_m \in A$ such that

$$\|u_m a - a u_m\| < \frac{1}{m} \quad \text{and} \quad \left\|\alpha_j(u_m) - e^{2\pi i j/n} u_m\right\| < \frac{1}{m}$$

for every $a \in F_m$ and for every $j \in \mathbb{Z}_n$. Denote by $u = \overline{(u_m)_{m\in\mathbb{N}}}$ the image of the sequence of unitaries $(u_m)_{m\in\mathbb{N}}$ in A_∞. Then u belongs to the relative commutant of $A \in A_\infty$. Consider the unital map $\varphi\colon C(\mathbb{T}) \to A_\infty \cap A'$ given by $\varphi(f) = f(u)$ for $f \in C(\mathbb{T})$. One checks that

$$\alpha_j(\varphi(f)) = \varphi(\gamma_{e^{2\pi i j/n}}(f))$$

for all $j \in \mathbb{Z}_n$ and all $f \in C(\mathbb{T})$, so φ is equivariant.

Conversely, assume that there is an equivariant unital homomorphism

$$\varphi\colon C(\mathbb{T}) \to A_\infty \cap A'.$$

Let $z \in C(\mathbb{T})$ be the unitary given by $z(\zeta) = \zeta$ for all $\zeta \in \mathbb{T}$, and let $v = \varphi(z)$. By semiprojectivity of $C(\mathbb{T})$, we can choose a representing sequence $(v_m)_{m\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$ consisting of unitaries. It follows that

$$\lim_{m\to\infty} \left\|\alpha_j(v_m) - e^{2\pi i j/n} v_m\right\| = 0 = \lim_{m\to\infty} \|v_m a - a v_m\|$$

for every $a \in A$, and this is clearly equivalent to α having the unitary Rokhlin property. ☐

The following result is analogous to Proposition 2.7, and so is its proof.

Proposition 3.12. Let B be a separable, unital C^*-algebra, let $n \in \mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action on B. Then β has the unitary Rokhlin property if and only if for every finite set $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary $u \in \mathcal{U}(B)$ such that

(1) $\beta_k(u) = e^{2\pi i k/n} u$ for all $k \in \mathbb{Z}_n$;

(2) $\|ub - bu\| < \varepsilon$ for all $b \in F$.

Similarly to what was pointed out after the statement of Proposition 2.7, the definition of the unitary Rokhlin property differs in that in condition (1), one only requires $\|\beta_k(u) - e^{2\pi ik/n}u\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

Proof. Recall that $(C(\mathbb{T}), \mathbb{T}, \mathtt{Lt})$ is equivariantly semiprojective by Remark 2.8. Since the quotient $\mathbb{T}/\mathbb{Z}_n$ is compact, it follows from Theorem 3.11 in [PST14] that the restriction $(C(\mathbb{T}), \mathbb{Z}_n, \mathtt{Lt})$ is equivariantly semiprojective as well. The result now follows using an argument similar to the one used in the proof of Proposition 2.7. The details are left to the reader. □

Proposition 3.13. Let $n \in \mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action of $\mathbb{Z}_n$ on a unital separable C^*-algebra B.

(1) The action β has the unitary Rokhlin property if and only if its dual action $\widehat{\beta}$ is strongly approximately inner.
(2) The action β is strongly approximately inner if and only if its dual action $\widehat{\beta}$ has the unitary Rokhlin property.

Proof. Part (1). Assume that β has the unitary Rokhlin property. Use Lemma 3.11 to choose a unital equivariant homomorphism $\varphi\colon C(\mathbb{T}) \to B_\infty \cap B'$. Denote by $u \in B_\infty \cap B'$ the image under this homomorphism of the unitary $z \in C(\mathbb{T})$ given by $z(\zeta) = \zeta$ for $\zeta \in \mathbb{T}$, and denote by λ the implementing unitary representation of $\mathbb{Z}_n \in B \rtimes_\beta \mathbb{Z}_n$ for β. In $(B \rtimes_\beta \mathbb{Z}_n)_\infty$, we have $u^*\lambda_j u = e^{2\pi ij/n}\lambda_j$ for all $j \in \mathbb{Z}_n$, and $ub = bu$ (f, $ubu^* = b$) for all $b \in B$. Therefore, if β has the unitary Rokhlin property, then $\widehat{\beta}$ is implemented by u^*, and thus it is strongly approximately inner. The converse follows from the same computation, as we have $(B \rtimes_\beta \mathbb{Z}_n)^{\widehat{\beta}} = B$.

Part (2). Denote by v the canonical unitary in the crossed product, and assume that β is strongly approximately inner. Let $F \subseteq B \rtimes_\beta \mathbb{Z}_n$ be a finite subset, and let $\varepsilon > 0$. Since B and v generate $B \rtimes_\beta \mathbb{Z}_n$, we can assume that there is a finite subset $F' \subseteq B$ such that $F = F' \cup \{v\}$. Choose $w \in \mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all $b \in F'$. Since $\beta(b) = vbv^*$ for every $b \in B$, if we let $u = w^*v$, the first of these conditions is equivalent to $\|vu - uv\| < \varepsilon$, while the second one is equivalent to $\|ub - bu\| < \varepsilon$ for all $b \in F'$. On the other hand, $\widehat{\beta}_k(u) = \widehat{\beta}_k(w^*v) = w^*(e^{2\pi ik/n}v) = e^{2\pi ik/n}u$ for $k \in \mathbb{Z}_n$. Thus, u is the desired unitary, and $\widehat{\beta}$ has the unitary Rokhlin property.

Conversely, assume that $\widehat{\beta}$ has the unitary Rokhlin property. Let $F' \subseteq B$ be a finite subset, and let $\varepsilon > 0$. Set $F = F' \cup \{v\}$. Use Proposition 3.12 to choose u in the unitary group of $A \rtimes_\beta \mathbb{Z}_n$ such that $\|ub - bu\| < \varepsilon$ for all $b \in F$, and $\widehat{\beta}_k(u) = e^{2\pi ik/n}u$ for all $k \in \mathbb{Z}_n$. Set $w = vu^*$. Then $w \in B$ since

$$\widehat{\beta}_k(w) = e^{2\pi ik/n}v\overline{e^{2\pi ik/n}}u^* = vu^* = w$$

for all $k \in \mathbb{Z}_n$ and $(B \rtimes_\beta \mathbb{Z}_n)^{\mathbb{Z}_n} = B$. On the other hand,

$$\|\beta(b) - wbw^*\| = \|vbv^* - vu^*buv^*\| = \|b - u^*bu\| = \|ub - bu\| < \varepsilon,$$

for all $b \in F$. Hence w is an implementing unitary for F' and ε, and β is strongly approximately inner. □

Corollary 3.14. Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, and let $n \in \mathbb{N}$. Then $\widehat{\alpha|_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is strongly approximately inner.

Proof. The restriction $\alpha|_n$ has the unitary Rokhlin property by Lemma 3.6, and by part (1) of Proposition 3.13, its dual $\widehat{\alpha|_n}$ is strongly approximately inner. □

The next ingredient needed is showing that crossed products by restrictions of Rokhlin actions of compact groups preserve absorption of strongly self-absorbing C^*-algebras. For Rokhlin actions, this was shown by Hirshberg and Winter in [HW07]. The more general statement is proved using similar ideas.

Proposition 3.15. Let A be a separable unital C^*-algebra, let G be a compact Hausdorff second countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action satisfying the Rokhlin property. Let H be a closed subgroup of G. If B is a unital separable C^*-algebra which admits a central sequence of unital homomorphisms into A, then B admits a unital homomorphism into the fixed point subalgebra of $\alpha|_H \in A_\infty \cap A'$.

Proof. Notice that $(A_\infty \cap A')^\alpha$ is a subalgebra of $(A_\infty \cap A')^{\alpha|_H}$. The result now follows from Theorem 3.3 in [HW07]. □

Remark 3.16. In the proposition above, if B is moreover assumed to be simple, for example if it is strongly self-absorbing, it follows that the unital homomorphism obtained is actually an embedding, since it is not zero.

Recall the following result by Hirshberg and Winter.

Lemma 3.17. (Lemma 2.3 of [HW07].) Let A and $\mathcal{D}$ be unital separable C^*-algebras. Let G be a compact group and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. If there is a unital homomorphism $\mathcal{D} \to (A_\infty \cap A')^G$, then there is a unital homomorphism

$$\mathcal{D} \to (M(A \rtimes_\alpha G))_\infty \cap (A \rtimes_\alpha G)'.$$

Theorem 3.18. Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra, let A be a $\mathcal{D}$-absorbing, separable unital C^*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then, for every $n \in \mathbb{N}$, the crossed product $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is a unital separable $\mathcal{D}$-absorbing C^*-algebra.

Proof. By Theorem 7.2.2 in [Rør02], there exists a unital embedding of $\mathcal{D}$ into $A_\infty \cap A'$, which is equivalent to the existence of a central sequence of unital embeddings of $\mathcal{D}$ into A. Use Proposition 3.15 to obtain a unital homomorphism of $\mathcal{D}$ into the fixed point subalgebra of $\alpha|_{\mathbb{Z}_n} \in A_\infty \cap A'$. It follows that this homomorphism is actually an embedding, since it is not zero and $\mathcal{D}$ is simple, by Theorem 1.6 in [TW07]. Lemma 2.3 in [HW07] (here reproduced as Lemma 3.17) provides us with a unital embedding of $\mathcal{D}$ into $(A \rtimes_\alpha \mathbb{Z}_n)_\infty \cap (A \rtimes_\alpha \mathbb{Z}_n)'$, which again by Theorem 7.2.2 in [Rør02] is equivalent to $A \rtimes_\alpha \mathbb{Z}_n$ being $\mathcal{D}$-absorbing, since $\mathcal{D}$ is strongly self-absorbing. □

The following is the main theorem of this section.

Theorem 3.19. Let A be a separable unital C^*-algebra, let $n \in \mathbb{N}$ and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Suppose that A absorbs M_{n^∞}. Then $\alpha|_n$ has the Rokhlin property.

Proof. By Lemma 3.4, it is enough to show that $\widehat{\alpha|_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is approximately representable. Recall that by Corollary 3.14, the action $\widehat{\alpha|_n}$ is strongly approximately inner. In view of Lemma 3.10 in [Izu04a], to show that it is in fact approximately representable, it is enough to show that there is a unital map

$$M_n \to \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)',$$

where the relative commutant is taken in $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)_\infty$.

Claim: $\left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = (A_\infty \cap A')^{(\alpha|_n)_\infty}$.

Since $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} = A$, we have

$$\left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = A_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)'$$
$$= \left\{ \overline{(a_m)}_{m\in\mathbb{N}} \in A_\infty \colon \lim_{m\to\infty} \|a_m x - x a_m\| = 0 \text{ for all } x \in A \rtimes_{\alpha|_n} \mathbb{Z}_n \right\}.$$

Let v be the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ that implements $\alpha|_n$ in the crossed product. Then for a bounded sequence $(a_m)_{m\in\mathbb{N}} \in A$, the condition $\lim_{m\to\infty} \|a_m x - x a_m\| = 0$ for all $x \in A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is equivalent to $\lim_{m\to\infty} \|a_m a - a a_m\| = 0$ for all $a \in A$ and $\lim_{m\to\infty} \|a_m v - v a_m\| = 0$. The above set is therefore equal to

$$\left\{ \overline{(a_m)}_{m\in\mathbb{N}} \in A_\infty \colon \begin{array}{c} \lim_{m\to\infty} \|a_m a - a a_m\| = 0 \text{ for all } a \in A \text{ and} \\ \lim_{m\to\infty} \|(\alpha|_n)(a_m) - a_m\| = 0 \end{array} \right\},$$

which is precisely the same as the subset of $A_\infty \cap A'$ that is fixed under the action on $A_\infty \cap A'$ induced by $\alpha|_n$. This proves the claim.

Since A absorbs the UHF-algebra M_{n^∞}, it follows that there is a unital embedding $\iota \colon M_n \to A_\infty \cap A'$. By Proposition 3.15, there is a unital homomorphism $M_n \to (A_\infty \cap A')^{(\alpha|_n)_\infty}$. Using the claim above, we conclude that there is a unital homomorphism

$$M_n \to \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} \right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)'.$$

This homomorphism is necessarily an embedding, since it is not zero. Apply Lemma 3.10 in [Izu04a] to the action $\widehat{\alpha|_n} \colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ to conclude that $\widehat{\alpha|_n}$ is approximately representable, and hence that $\alpha|_n$ has the Rokhlin property, thanks to Lemma 3.4. ☐

The following partial converse to Theorem 3.19 holds. The same result is likely to be true for a larger class of C^*-algebras.

Recall that if A is a stably finite C^*-algebra, then its Murray-von Neumann semigroup $V(A)$ can be naturally identified with the subsemigroup of its Cuntz semigroup $\mathrm{Cu}(A)$ consisting of compact elements. Additionally, if A has real rank zero, then every element in $\mathrm{Cu}(A)$ is the supremum of an increasing sequence in $V(A) \subseteq \mathrm{Cu}(A)$.

Theorem 3.20. Let A belong to one of the following classes of C^*-algebras:

(1) Unital Kirchberg algebras satisfying the UCT;
(2) Simple, nuclear, separable unital C^*-algebras with tracial rank zero satisfying the UCT;
(3) Unital real rank zero direct limits of one-dimensional noncommutative CW-complexes with trivial K_1-group (this includes all AF-algebras).

Let $\alpha\colon \mathbb{T}\to \mathrm{Aut}(A)$ be a continuous action and let $n\in\mathbb{N}$. If $\alpha|_n$ has the Rokhlin property, then A absorbs M_{n^∞}.

Proof. Assume that $\alpha|_n$ has the Rokhlin property. Since α induces the trivial action on K-theory by Proposition 3.9, so does $\alpha|_n$. For algebras in the first two classes, the result follows from Theorem 3.4 in [Izu04b] and Theorem 3.5 in [Izu04b], respectively. For algebras in the third class, it follows that $\alpha|_n$ acts trivially on the Murray-von Neumann semigroup $V(A)$. Since every element in $\mathrm{Cu}(A)$ is the supremum of an increasing sequence in $V(A)$, it follows that $\alpha|_n$ acts trivially on $\mathrm{Cu}(A)$ as well. The result now follows from Theorem 4.2 in [GS15]. □

Denote by $\mathcal{Q}$ the universal UHF-algebra, that is, the unique, up to isomorphism, UHF-algebra with K-theory

$$(K_0(\mathcal{Q}),[1_{\mathcal{Q}}])\cong(\mathbb{Q},1).$$

It is well-known that $\mathcal{Q}\otimes M_{n^\infty}\cong\mathcal{Q}$ for all $n\in\mathbb{N}$, and that $\mathcal{Q}\otimes\mathcal{O}_2\cong\mathcal{O}_2$.

Corollary 3.21. Let A be a separable, $\mathcal{Q}$-absorbing unital C^*-algebra, let $\alpha\colon\mathbb{T}\to \mathrm{Aut}(A)$ be an action with the Rokhlin property and let $n\in\mathbb{N}$. Then $\alpha|_n$ has the Rokhlin property. In particular, restrictions to finite subgroups of circle actions with the Rokhlin property on separable, unital $\mathcal{O}_2$-absorbing C^*-algebras, again have the Rokhlin property.

We finish this work by showing that the Rokhlin property for a circle action cannot in general be determined just by looking at its restrictions to finite subgroups.

Example 3.22. There are a unital C^*-algebra A and a circle action on A such that its restriction to every proper subgroup has the Rokhlin property, but the action itself does not.

Let A be the universal UHF-algebra, that is, $A=\varinjlim(M_{n!},\iota_n)$ where $\iota_n\colon M_{n!}\to M_{(n+1)!}$ is given by $\iota_n(a)=\mathrm{diag}(a,\ldots,a)$ for all $a\in M_{n!}$. For every $n\in\mathbb{N}$, let $\alpha^{(n)}\colon\mathbb{T}\to\mathrm{Aut}(M_{n!})$ be given by

$$\alpha^{(n)}_\zeta=\mathrm{Ad}(\mathrm{diag}(1,\zeta,\ldots,\zeta^{n!-1}))$$

for all $\zeta\in\mathbb{T}$. Then $\iota_n\circ\alpha^{(n)}_\zeta=\alpha^{(n+1)}_\zeta\circ\iota_n$ for all $n\in\mathbb{N}$ and all $\zeta\in\mathbb{T}$, and hence there is a direct limit action $\alpha=\varinjlim\alpha^{(n)}$ of $\mathbb{T}$ on A. This action does not have the Rokhlin property by Theorem 2.17.

On the other hand, we claim that given $m\in\mathbb{N}$, the restriction $\alpha|_m\colon\mathbb{Z}_m\to\mathrm{Aut}(A)$ has the Rokhlin property. So fix $m\in\mathbb{N}$. Then $\alpha|_m$ is the direct limit of the actions $\left(\alpha^{(n)}|_m\right)_{n\in\mathbb{N}}$, whose generating automorphisms are

$$\alpha^{(n)}_{e^{2\pi i/m}}=\mathrm{Ad}(\mathrm{diag}(1,e^{2\pi i/m},\ldots,e^{2\pi i(n!-1)/m})).$$

Let $F\subseteq A$ be a finite subset and let $\varepsilon>0$. Write $F=\{a_1,\ldots,a_N\}$. Since $\bigcup\limits_{n\in\mathbb{N}}M_{n!}$ is dense in A, there are $k\in\mathbb{N}$ and a finite subset $F'=\{b_1,\ldots,b_N\}\subseteq M_{k!}$ such that $\|a_j-b_j\|<\frac{\varepsilon}{2}$ for all $j=1,\ldots,N$.

Let $n\geq\max\{k,m\}$. Then the $\mathbb{Z}_m$-action $\alpha^{(n)}|_m$ on $M_{n!}$ is generated by the automorphism

$$\alpha^{(n)}_{e^{2\pi i/m}}=\mathrm{Ad}(1,e^{2\pi i/m},\ldots,e^{2\pi i(m-1)/m},\ldots,1,e^{2\pi i/m},\ldots,e^{2\pi i(m-1)/m}).$$

(There are $n!/m$ repetitions.) Denote by e_0 the projection

$$1_{M_{(n-1)!}} \otimes \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{m} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{m} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix}$$

in $M_{n!} \subseteq A$, and for $j = 1, \ldots, m-1$, set $e_j = \alpha^{(n)}_{e^{2\pi ij/m}}(e_0) \in A$. One checks that $e_0, \ldots, e_{m-1}$ are orthogonal projections with $\sum\limits_{j=0}^{m-1} e_j = 1$, and moreover that $\alpha^{(n)}_{e^{2\pi i/m}}(e_{m-1}) = e_0$.

By construction, these projections are cyclically permuted by the action $\alpha|_m$ and they sum up to one, so we only need to check that they almost commute with the given finite set. The projections $e_0, \ldots, e_{m-1}$ exactly commute with the elements of F'. Thus, if $k \in \{1, \ldots, N\}$ and $j \in \{0, \ldots, m-1\}$, then

$$\begin{aligned} \|a_k e_j - e_j a_k\| &\leq \|a_k e_j - b_k e_j\| + \|b_k e_j - e_j b_k\| + \|e_j b_k - e_j a_k\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and hence $\alpha|_m$ has the Rokhlin property.

The phenomenon exhibited in the example above is not special to UHF-algebras:

Example 3.23. If A and α are as in Example 3.22, set $B = A \otimes \mathcal{O}_\infty$ and let $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be given by $\beta_\zeta = \alpha_\zeta \otimes \mathrm{id}_{\mathcal{O}_\infty}$ for all $\zeta \in \mathbb{T}$. Then B is a unital Kirchberg algebra satisfying the UCT. We claim that the action β does not have the Rokhlin property. To see this, observe first that the fixed point algebra A^α can be written as an inductive limit $A^\alpha = \varinjlim M_{n!}^{\alpha^{(n)}}$, and that

$$M_{n!}^{\alpha^{(n)}} = \{a \in M_{n!} \colon a \text{ commutes with } \mathrm{diag}(1, \zeta, \ldots, \zeta^{n!-1}) \ \forall\, \zeta \in \mathbb{T}\} = \mathbb{C}^{n!} \subseteq M_{n!},$$

the last embedding being as diagonal matrices. In particular, the fixed point algebra $B^\beta = A^\alpha \otimes \mathcal{O}_\infty$ is purely infinite but not simple, and thus the Rokhlin property for β would contradict Theorem 6.3 in [Gar14a].

On the other hand, for every $m \in \mathbb{N}$, the restriction $\beta|_m \colon \mathbb{Z}_m \to \mathrm{Aut}(B)$ has the Rokhlin property by part (i) of Theorem 2 in [San15], being a tensor product of an action with the Rokhlin property (namely $\alpha|_m$) and another action.

References

[DNNP92] M. Dadarlat, G. Nagy, A. Nemethi, and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. of Math. **153** (1992), no. 2, 267–276.

[Gar14a] E. Gardella, *Classification theorems for circle actions on Kirchberg algebras, I*, preprint, arXiv:1405.2469, 2014.

[Gar14b] ———, *Classification theorems for circle actions on Kirchberg algebras, II*, preprint, arXiv:1406.1208, 2014.

[Gar14c] ———, *Compact group actions with the Rokhlin property and their crossed products*, preprint, arXiv:1408.1946, 2014.

[GS15] E. Gardella and L. Santiago, *Equivariant *-homomorphisms, Rokhlin constraints and equivariant UHF-absorption*, preprint, arXiv:1501.05560, 2015.

[Hat00] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2000.

[HJ82] R. Herman and V. Jones, *Period two automorphisms of uhf c^*-algebras*, J. Funct. Anal. **45** (1982), 169–176.

[HO84] R. Herman and A. Ocneanu, *Stability for Integer Actions on UHF C^*-algebras*, J. Funct. Anal. **59** (1984), 132–144.

[HW07] I. Hirshberg and W. Winter, *Rokhlin actions and self-absorbing C^*-algebras*, Pacific J. of Math. **233** (2007), 125–143.

[Izu04a] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. **122** (2004), no. 2, 233–280.

[Izu04b] ———, *Finite group actions on C^*-algebras with the Rohlin property. II*, Adv. Math. **184** (2004), no. 1, 119–160.

[Kis95] A. Kishimoto, *The Rohlin property for automorphisms of UHF algebras*, J. Reine Angew. Math. **465** (1995), 183–196.

[Kis96] ———, *A Rohlin property for one-parameter automorphism groups*, Commun. Math. Phys. **179** (1996), 599–622.

[KP00] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525** (2000), 17–53.

[OP12] H. Osaka and N. C. Phillips, *Crossed products by finite group actions with the rokhlin property*, Math. Z. **270** (2012), 19–42.

[Pal61] R. Palais, *On the existence of slices for actions of non-compact lie groups*, Ann. of Math. **73** (1961), 295–323.

[Phi87] N. C. Phillips, *Equivariant K-theory and freeness of group actions on C^*-algebras*, Lecture Notes in Mathematics, 1274. Berlin etc.: Springer-Verlag. VIII, 1987.

[Phi09] ———, *Freeness of actions of finite groups on C^*-algebras*, de Jeu, Marcel (ed.) et al., Operator structures and dynamical systems. Satellite conference of the 5th European congress of mathematics, Leiden, Netherlands, July 21–25, 2008. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 503, 217-257, 2009.

[Phi11] ———, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, Amer. J. Math. **133** (2011), no. 3, 581–636. MR 2808327 (2012h:46116)

[Phi12] ———, *The tracial Rokhlin property is generic*, preprint, arXiv:1209.3859, 2012.

[PST14] N. C. Phillips, A. Sørensen, and H. Thiel, *Semiprojectivity with and without group action*. J. Funct. Anal. **268** (2015), 929–973.

[Rør02] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Entropy in operator algebras. Berlin: Springer. Encycl. Math. Sci. 126 (VII), 2002.

[San15] L. Santiago, *Crossed products by actions of finite groups with the Rokhlin property.* Internat. J. Math. **26**, 1550042, (2015).

[TW07] A. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, Trans. Am. Math. Soc. **359** (2007), no. 8, 3999–4029.

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