

Matched G^k -constructions yield C^k -continuous iso-geometric elements

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Abstract

We show how G^k (geometrically continuous surface) constructions yield C^k iso-geometric elements also at irregular points in a quad-mesh where three or more than four elements come together.

1 Introduction

Change of coordinates, a.k.a. reparameterization, is the concept shared by G^k constructions of manifolds and by iso-geometric analysis and prediction of physical properties. G^k continuity is a notion used to characterize constructions of C^k surfaces by joining two pieces along a common boundary curve so that their derivatives match after reparameterization (see e.g. [GH87, Boe88, Pet02]). Linear combinations of iso-geometric elements serve to approximately compute the solution of a differential equation over a bounded, often geometrically non-trivial, region. Iso-geometric elements are higher-order iso-parametric elements of classical engineering analysis [IZ68]. The term iso-geometric was used in [HCB05] to highlight the case when the region and the image of the elements are spanned by the same space of functions, typically tensor-product splines [dB02]).

To date, the iso-geometric approach has not been investigated at points where more or fewer than four tensor-product elements meet *smoothly*. Such points are called irregular, extraordinary or star points. Since the smooth joining of surface pieces at irregular points is governed by G^k relations, it is natural to apply the concept of geometric continuity to constructing everywhere differentiable iso-geometric elements. This paper shows that when both the region and the image of the iso-geometric elements are from the same space of G^k continuous maps then the iso-geometric elements are C^k . This observation formed the background for the author's presentations in early 2014 [Pet14] and the publication [NKP14].

2 Geometric continuity and Iso-geometric elements

Geometric continuity refers to matching geometric invariants. However for practical constructions, the following parameterization-based definition of matching derivatives after a change of coordinates is widely accepted and equivalent in most relevant cases [Pet02, Sect 3].

Geometric continuity. For $i = 1, 2$, let \square_i be an m -dimensional polytope, for example a unit cube in \mathbb{R}^m . Let $\mathbf{e}_1(s)$, $s \in \mathbb{R}^{m-1}$ parameterize an $m-1$ -dimensional facet of \square_1 with open neighborhood $\mathcal{N}(\mathbf{e}_1) \subsetneq \mathbb{R}^m$ and $\mathbf{e}_2(s)$ parameterize an $m-1$ -dimensional facet of \square_2 such that there exists an invertible C^k reparameterization

$$\rho : \mathcal{N}(\mathbf{e}_1) \rightarrow \mathcal{N}(\mathbf{e}_2), \quad \rho(\mathbf{e}_1) = \mathbf{e}_2, \quad \mathcal{N}(\mathbf{e}_1) \cap \square_1 \rightarrow \mathcal{N}(\mathbf{e}_2) \cap (\mathbb{R}^m \setminus \square_2). \quad (1)$$

Let $\mathbf{x}_1, \mathbf{x}_2 : \square \subsetneq \mathbb{R}^m \rightarrow \mathbb{R}^d$ be two maps whose images join along a common interface $\mathbf{e}(s) := \mathbf{x}_2(\mathbf{e}_2(s)) = \mathbf{x}_2(\rho(\mathbf{e}_1(s))) = \mathbf{x}_1(\mathbf{e}_1(s))$. Denote as the k -jet of a map f at \mathbf{y} as $\mathbf{j}_{\mathbf{y}}^k f$. Then two C^k maps \mathbf{x}_1 and \mathbf{x}_2 join G^k along \mathbf{e} with reparameterization ρ if for every point $\mathbf{e}_1(s)$ of \mathbf{e}_1

$$\mathbf{j}_{\mathbf{e}_1(s)}^k \mathbf{x}_1 = \mathbf{j}_{\mathbf{e}_1(s)}^k (\mathbf{x}_2 \circ \rho), \quad (2)$$

where \circ denotes composition. That is, \mathbf{x}_1 and $\mathbf{x}_2 \circ \rho$ form a C^k function on $\mathcal{N}(\mathbf{e}_1)$.

When $m = 2$, $d = 3$ and \mathbf{x}_i are tensor-product splines then \square is a rectangle and \mathbf{e} is a boundary curve shared by the patches $\mathbf{x}_1(\square)$ and $\mathbf{x}_2(\square)$ (see Fig. 1). Such pairwise G^k constructions are used to construct surfaces where three or more than four tensor-product splines are to be joined smoothly to enclose a point, since placing the rectangular domains directly into \mathbb{R}^2 to form a joint domain from where to map out a neighborhood of the point yields an embedding only if four rectangles meet.

Iso-geometric elements. In the iso-parametric approach to solving partial differential equations, maps \mathbf{x}_i , $i = 1..n$, typically splines mapping into \mathbb{R}^d , $d = 2$ or $d = 3$, parameterize a region or manifold X called *physical domain*. The physical domain is tessellated into pieces $\mathbf{x}_i(\square_i)$,

$$X := \cup_{i=1}^n \mathbf{x}_i(\square_i) \subset \mathbb{R}^d, \quad \mathbf{x}_i : \square_i \subsetneq \mathbb{R}^m \rightarrow \mathbb{R}^d. \quad (3)$$

(That is the interiors of $\mathbf{x}_i(\square_i)$ are disjoint.) In the following we assume that \mathbf{x}_i is injective on their domain, hence \mathbf{x}_i^{-1} is well-defined.

When $m = 2$, i.e. in two variables, the \square_i are for example unit squares. If $m = 2$ and $d = 2$ then X is a region of the xy -plane. If $m = 2$ and $d = 3$, X is a surface embedded in \mathbb{R}^3 .

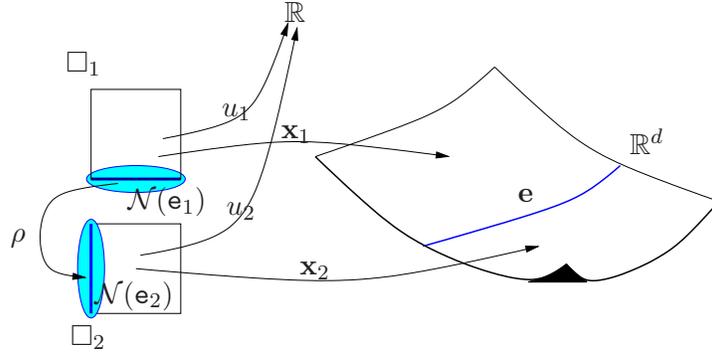


Figure 1: Iso-geometric elements and G^1 continuity

The goal of the iso-geometric approach is to compute functions u_i so that a linear combination of the maps $u_i \circ \mathbf{x}_i^{-1}$ solves a partial differential equation on $\mathbf{x}_i(\square_i)$. Without loss of generality, we choose u_i to be scalar-valued, i.e. $u_i : \square_i \rightarrow \mathbb{R}$. The composition $u_i \circ \mathbf{x}_i^{-1}$ is called *iso-geometric element* if both u_i and the coordinates \mathbf{x}_i are functions from the same space, typically a space of spline functions.

3 Smoothness of the composition

We want to show that G^k constructions yield C^k iso-geometric elements. The proof is inspired by the definition of a smooth function on a manifold.

Lemma 1 (G^k construction yields C^k iso-geometric element) *For $i = 1, 2$, consider C^k maps u_i and \mathbf{x}_i that map \square_i into \mathbb{R}^d , $i = 1, 2$ and \mathbf{x}_i is injective. Let ρ be an invertible C^k reparameterization according to (1) and let \mathbf{x}_1 join G^k with \mathbf{x}_2 according to $\mathbf{j}_{\mathbf{e}_1(s)}^k \mathbf{x}_1 = \mathbf{j}_{\mathbf{e}_1(s)}^k (\mathbf{x}_2 \circ \rho)$ and u_1 join G^k with u_2 according to $\mathbf{j}_{\mathbf{e}_1(s)}^k u_1 = \mathbf{j}_{\mathbf{e}_1(s)}^k (u_2 \circ \rho)$. Then the iso-geometric maps $u_1 \circ \mathbf{x}_1^{-1}$ and $u_2 \circ \mathbf{x}_2^{-1}$ form a C^k function on $\mathbf{x}_1(\square_1) \cup \mathbf{x}_2(\square_2)$.*

Proof Denote by $\mathbf{e}_1(s)$ the pre-image of the point $\mathbf{e}(s)$ under \mathbf{x}_1 and by $\mathbf{e}_2(s)$ the pre-image of the point $\mathbf{e}(s)$ under \mathbf{x}_2 . Below we drop the argument s with the understanding that equalities hold pointwise. By assumption

$$\mathbf{j}_{\mathbf{e}_1}^k (\mathbf{x}_2 \circ \mathbf{x}_2^{-1} \circ \mathbf{x}_1) = \mathbf{j}_{\mathbf{e}_1}^k \mathbf{x}_1 = \mathbf{j}_{\mathbf{e}_1}^k (\mathbf{x}_2 \circ \rho) \quad (4)$$

and hence, by injectivity of \mathbf{x}_2 , $\mathbf{j}_{\mathbf{e}_1}^k \rho = \mathbf{j}_{\mathbf{e}_1}^k (\mathbf{x}_2^{-1} \circ \mathbf{x}_1)$. Then

$$\begin{aligned}
\mathbf{j}_{\mathbf{e}}^k(u_1 \circ \mathbf{x}_1^{-1}) &= \mathbf{j}_{\mathbf{e}_1}^k u_1 \circ \mathbf{j}_{\mathbf{e}}^k \mathbf{x}_1^{-1} = \mathbf{j}_{\mathbf{e}_1}^k (u_2 \circ \rho) \circ \mathbf{j}_{\mathbf{e}}^k (\mathbf{x}_1^{-1} \circ \mathbf{x}_2 \circ \mathbf{x}_2^{-1}) \\
&= \mathbf{j}_{\mathbf{e}_2}^k u_2 \circ \mathbf{j}_{\mathbf{e}_1}^k \rho \circ \mathbf{j}_{\mathbf{e}_2}^k (\mathbf{x}_1^{-1} \circ \mathbf{x}_2) \circ \mathbf{j}_{\mathbf{e}}^k \mathbf{x}_2^{-1} \\
&= \mathbf{j}_{\mathbf{e}_2}^k u_2 \circ \mathbf{j}_{\mathbf{e}_2}^k (\mathbf{x}_2^{-1} \circ \mathbf{x}_1 \circ \mathbf{x}_1^{-1} \circ \mathbf{x}_2) \circ \mathbf{j}_{\mathbf{e}}^k \mathbf{x}_2^{-1} \\
&= \mathbf{j}_{\mathbf{e}}^k (u_2 \circ \mathbf{x}_2^{-1}).
\end{aligned}$$

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In consequence, all G^k surface constructions in the literature can directly be used to solve differential equations not only on surfaces but also on planar regions where $n \neq 4$ pieces come together.

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