

# ANOTHER PROOF OF TWO MODULO 3 CONGRUENCES AND ANOTHER SPT CRANK FOR THE NUMBER OF SMALLEST PARTS IN OVERPARTITIONS WITH EVEN SMALLEST PART

CHRIS JENNINGS-SHAFFER

ABSTRACT. By considering the  $M_2$ -rank of an overpartition as well as a residual crank, we give another combinatorial refinement of the congruences  $\overline{\text{spt}}_2(3n) \equiv \overline{\text{spt}}_2(3n+1) \equiv 0 \pmod{3}$ . Here  $\overline{\text{spt}}_2(n)$  is the total number of occurrences of the smallest parts among the overpartitions of  $n$  where the smallest part is even and not overlined. Our proof depends on Bailey's Lemma and the rank difference formulas of Lovejoy and Osburn for the  $M_2$ -rank of an overpartition. This congruence, along with a modulo 5 congruence, has previously been refined using the rank of an overpartition.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

We recall a partition of a positive integer  $n$  is a non-increasing sequence of positive integers that sum to  $n$ . As an example, the partitions of 4 are 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Similar to this is the idea of an overpartition. An overpartition of  $n$  is a partition of  $n$  where the first occurrence of a part may (or may not) be overlined. For example, the overpartitions of 4 are 4,  $\overline{4}$ , 3 + 1, 3 +  $\overline{1}$ ,  $\overline{3}$  + 1,  $\overline{3}$  +  $\overline{1}$ , 2 + 2,  $\overline{2}$  + 2, 2 + 1 + 1, 2 +  $\overline{1}$  + 1,  $\overline{2}$  + 1 + 1,  $\overline{2}$  +  $\overline{1}$  + 1, 1 + 1 + 1 + 1, and  $\overline{1}$  + 1 + 1 + 1.

We have a weighted count on partitions, and overpartitions, given by counting a partition by the number of times the smallest occurs. We use the convention of not counting overpartitions where the smallest part is overlined. We let  $\text{spt}(n)$  denote the total number of occurrences of the smallest parts among the partitions of  $n$ . We let  $\overline{\text{spt}}(n)$  denote the total number of occurrences of the smallest parts among the overpartitions of  $n$  without smallest part overlined. The function  $\text{spt}(n)$  was introduced by Andrews in [1] and the function  $\overline{\text{spt}}(n)$  was introduced by Bringmann, Lovejoy, and Osburn in [3]. Two restrictions of  $\overline{\text{spt}}(n)$  are  $\overline{\text{spt}}_1(n)$  and  $\overline{\text{spt}}_2(n)$ , where restrict to overpartitions where the smallest part is odd and even respectively. We see  $\text{spt}(4) = 10$ ,  $\overline{\text{spt}}(4) = 13$ ,  $\overline{\text{spt}}_1(4) = 10$ , and  $\overline{\text{spt}}_2(4) = 3$ .

Similar to the work of Andrews, Garvan, and Liang in [2] for  $\text{spt}(n)$ , in [4] Garvan and the author gave combinatorial refinements of congruences satisfied by  $\overline{\text{spt}}(n)$ ,  $\overline{\text{spt}}_1(n)$ , and  $\overline{\text{spt}}_2(n)$ . The idea is to introduce an extra variable into the generating function of each  $\text{spt}$ -function to get a crank type statistic. This statistic can then be shown in certain cases to equally split up the numbers  $\overline{\text{spt}}(n)$ ,  $\overline{\text{spt}}_1(n)$ , and  $\overline{\text{spt}}_2(n)$  based on the residue class of the statistic. We explain this in more detail shortly.

For  $\overline{\text{spt}}_2(n)$  we have the congruences

$$\overline{\text{spt}}_2(3n) \equiv 0 \pmod{3}, \tag{1.1}$$

$$\overline{\text{spt}}_2(3n+1) \equiv 0 \pmod{3}, \tag{1.2}$$

$$\overline{\text{spt}}_2(5n+3) \equiv 0 \pmod{5}. \tag{1.3}$$

In this paper we give another proof of the modulo 3 congruences.

To start, by summing according to the smallest part, we find a generating function for  $\overline{\text{spt}}_2(n)$  to be given by

$$\sum_{n=1}^{\infty} \overline{\text{spt}}_2(n) q^n = \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q)_{\infty}}{(1-q^{2n})^2 (q^{2n+1}; q)_{\infty}}. \tag{1.4}$$

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Here we use the standard product notation,

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (1.5)$$

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n, \quad (1.6)$$

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \quad (1.7)$$

$$(a_1, a_2, \dots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty. \quad (1.8)$$

In [4] we considered the two variable generalization given by

$$\overline{S}_2(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q)_\infty (q^{2n+1}; q)_\infty}{(zq^{2n}; q)_\infty (z^{-1}q^{2n}; q)_\infty}. \quad (1.9)$$

We note setting  $z = 1$  gives the generating function for  $\overline{spt}_2(n)$ . It turns out  $\overline{S}_2(z, q)$  can be expressed in terms of the Dyson rank of an overpartition and a residual crank from [3]. In [6] Lovejoy and Osburn determined difference formulas for the Dyson rank of an overpartition, these formulas are essential to the proofs in [4]. In the same paper, we also used the difference formulas for the  $M_2$ -rank of a partition without repeated odd parts, these formulas were also determined by Lovejoy and Osburn in [7]. Lovejoy and Osburn also found difference formulas for the  $M_2$ -rank of an overpartition [8], but we did not use these formulas in [4].

In [5] the author gave higher order generalizations of  $\overline{spt}(n)$  and  $\overline{spt}_2(n)$  and noted that one could use the  $M_2$ -rank and another residual crank from [3], as used in that paper in working with  $\overline{spt}_2(n)$ , to explain the modulo 3 congruences for  $\overline{spt}_2(n)$ . Here we give the proof. In this paper we instead use

$$S(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q)_\infty (q^{2n+1}; q)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \quad (1.10)$$

$$= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_S(m, n) z^m q^n. \quad (1.11)$$

Again setting  $z = 1$  gives the generating function for  $\overline{spt}_2(n)$ . This is not the same series  $S(z, q)$  used in [4], however we do not want to overly complicate matters with additional notation.

For a positive integer  $t$  we let

$$N_S(k, t, n) = \sum_{m \equiv k \pmod{t}} N_S(m, n). \quad (1.12)$$

We then have

$$\overline{spt}_2(n) = \sum_{m=-\infty}^{\infty} N_S(m, n) = \sum_{k=0}^{t-1} N_S(k, t, n). \quad (1.13)$$

Additionally we see if  $\zeta$  is a  $t^{\text{th}}$  root of unity, then

$$S(\zeta, q) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{t-1} N_S(k, t, n) \zeta^k \right) q^n. \quad (1.14)$$

We consider when  $\zeta = \zeta_3$  is a primitive third root of unity. Here the minimal polynomial for  $\zeta_3$  is  $x^2 + x + 1$ . If  $N_S(0, t, N) + N_S(1, t, N)\zeta_3 + N_S(2, t, N)\zeta_3^2 = 0$  then we must in fact have  $N_S(0, t, N) = N_S(1, t, N) = N_S(2, t, N)$ . That is to say, if the coefficient of  $q^N$  in  $S(\zeta_3, q)$  is zero, then  $\overline{spt}_2(N) = 3 \cdot N_S(0, t, N)$  and so  $\overline{spt}_2(N) \equiv 0 \pmod{3}$ .

Our proof of  $\overline{spt}_2(3n) \equiv \overline{spt}_2(3n+1) \equiv 0$  is to find the 3-dissection of  $S(\zeta_3, q)$  with the  $q^{3n}$  and  $q^{3n+1}$  terms being all zero. This is the same idea that was used in [4], we are just using  $S(z, q)$  rather than  $\overline{S}_2(z, q)$ .

**Theorem 1.1.**

$$S(\zeta_3, q) = A_0(q^3) + qA_1(q^3) + q^2A_2(q^3), \quad (1.15)$$

where

$$A_0(q) = 0, \quad (1.16)$$

$$A_1(q) = 0, \quad (1.17)$$

$$A_2(q) = \frac{(q^6; q^6)_\infty^4}{(q^2; q^2)_\infty (q^3; q^3)_\infty^2} + \frac{2q(-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1 - q^{6n+2}}. \quad (1.18)$$

We prove these formulas by relating  $S(z, q)$  to a certain rank and crank and using dissections of these related functions. We recall the  $M_2$ -rank of an overpartition  $\pi$  is given by

$$M_2\text{-rank} = \left\lfloor \frac{l(\pi)}{2} \right\rfloor - \#(\pi) + \#(\pi_o) - \chi(\pi), \quad (1.19)$$

where  $l(\pi)$  is the largest part of  $\pi$ ,  $\#(\pi)$  is the number of parts of  $\pi$ ,  $\#(\pi_o)$  is the number of non-overlined odd parts, and  $\chi(\pi) = 1$  if the largest part of  $\pi$  is odd and non-overlined and otherwise  $\chi(\pi) = 0$ . The  $M_2$ -rank was introduced by Lovejoy in [8]. We let  $\overline{N2}(m, n)$  denote the number of overpartitions of  $n$  with  $M_2$ -rank  $m$ . Lovejoy found the generating function for  $\overline{N2}$  is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_{2n} q^n}{(zq^2, z^{-1}q^2; q^2)_n} = \frac{(-q; q)_\infty}{(q; q)_\infty} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right). \quad (1.20)$$

We also use a residual crank first defined [3]. We define

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M2}(m, n) z^m q^n = \frac{(-q; q)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty}. \quad (1.21)$$

The interpretation of this is as follows. For an overpartition  $\pi$  of  $n$  we take the crank of the partition  $\frac{\pi_e}{2}$  obtained by taking the subpartition  $\pi_e$ , of the even non-overlined parts of  $\pi$ , and halving each part of  $\pi_e$ . Then  $\overline{M2}(m, n)$  is the number of overpartitions  $\pi$  of  $n$  and such that the partition  $\frac{\pi_e}{2}$  has crank  $m$ . However this interpretation fails when considering overpartitions whose only even non-overlined parts are a single two, as the corresponding interpretation of  $\frac{(q; q)_\infty}{(zq, q/z; q)_\infty}$  as the generating function of the crank for ordinary partitions fails for the partition of 1.

**Theorem 1.2.**

$$S(z, q) = \frac{1}{(1-z)(1-z^{-1})} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\overline{N2}(m, n) - \overline{M2}(m, n)) z^m q^n. \quad (1.22)$$

Using the rank difference formulas derived by Lovejoy and Osburn in [8], we have

**Theorem 1.3.**

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m, n) \zeta_3^m q^n = \overline{N2}_0(q^3) + q\overline{N2}_1(q^3) + q^2\overline{N2}_2(q^3), \quad (1.23)$$

where

$$\overline{N2}_0(q) = \frac{(-q; q)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (-q^3; q^3)_\infty^2}, \quad (1.24)$$

$$\overline{N2}_1(q) = \frac{2(q^3; q^3)_\infty (q^6; q^6)_\infty}{(q; q)_\infty}, \quad (1.25)$$

$$\overline{N2}_2(q) = \frac{4(q^6; q^6)_\infty^4}{(q^2; q^2)_\infty (q^3; q^3)_\infty^2} + \frac{6q(-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1 - q^{6n+2}}. \quad (1.26)$$

**Theorem 1.4.**

$$\frac{(-q; q)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (\zeta_3 q^2, \zeta_3^{-1} q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M2}(m, n) \zeta_3^m q^n = \overline{M2}_0(q^3) + q \overline{M2}_1(q^3) + q^2 \overline{M2}_2(q^3), \quad (1.27)$$

where

$$\overline{M2}_0(q) = \frac{(-q; q)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (-q^3; q^3)_\infty^2}, \quad (1.28)$$

$$\overline{M2}_1(q) = 2 \frac{(q^3; q^3)_\infty (q^6; q^6)_\infty}{(q; q)_\infty}, \quad (1.29)$$

$$\overline{M2}_2(q) = \frac{(q^6; q^6)_\infty^4}{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}. \quad (1.30)$$

We see Theorem 1.1 follows from Theorems 1.2, 1.3, and 1.4, noting  $\frac{1}{(1-\zeta_3)(1-\zeta_3^{-1})} = \frac{1}{3}$ . We give the proofs of Theorems 1.2 and 1.4 in the next section. In Section 3 we give brief combinatorial interpretations of the coefficients  $N_S(m, n)$ , in particular they are non-negative, and in Section 4 we end with a few remarks.

## 2. THE PROOFS

*Proof of Theorem 1.2.* We recall a pair of sequences  $(\alpha_n, \beta_n)$ , is a Bailey pair relative to  $(a, q)$  if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}. \quad (2.1)$$

A limiting case of Bailey's Lemma gives for a Bailey pair  $(\alpha_n, \beta_n)$  that

$$\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left( \frac{aq}{\rho_1 \rho_2} \right)^n \beta_n = \frac{(aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/\rho_1 \rho_2; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n \left( \frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n}{(aq/\rho_1, aq/\rho_2; q)_n}. \quad (2.2)$$

As in [5] the Bailey pair connecting the  $M_2$ -rank of an overpartition and the residual crank is

$$\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^n 2q^{n^2} & n \geq 1 \end{cases}, \quad (2.3)$$

$$\beta_n = \frac{(q; q^2)_n^2}{(q^2; q^2)_{2n}}. \quad (2.4)$$

This is a Bailey pair with respect to  $(1, q^2)$ .

We note that

$$\frac{(q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty (q; q^2)_\infty^2} \cdot \frac{(zq^2, z^{-1}q^2; q^2)_\infty}{(q^2; q^2)_\infty^2} = \frac{(q^2; q^2)_\infty}{(1-z)(1-z^{-1})(q; q^2)_\infty^2} = \frac{(-q; q)_\infty}{(1-z)(1-z^{-1})(q; q)_\infty}. \quad (2.5)$$

With this Bailey pair we have

$$\begin{aligned} S(z, q) &= \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q)_\infty (q^{2n+1}; q)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n} (q^{4n+2}; q^2)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \\ &= \frac{(q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty (q; q^2)_\infty^2} \sum_{n=1}^{\infty} q^{2n} (z, z^{-1}; q^2)_n \beta_n \\ &= \frac{(q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty (q; q^2)_\infty^2} \sum_{n=0}^{\infty} q^{2n} (z, z^{-1}; q^2)_n \beta_n - \frac{(q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty (q; q^2)_\infty^2} \\ &= \frac{(-q; q)_\infty}{(1-z)(1-z^{-1})(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n} (z, z^{-1}; q^2)_n \alpha_n}{(zq^2, z^{-1}q^2; q^2)_n} - \frac{(q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty (q; q^2)_\infty^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-q; q)_\infty}{(1-z)(1-z^{-1})(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}(1-z)(1-z^{-1})\alpha_n}{(1-zq^{2n})(1-z^{-1}q^{2n})} - \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (z, z^{-1}; q^2)_\infty} \\
&= \frac{(-q; q)_\infty}{(1-z)(1-z^{-1})(q; q)_\infty} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right) - \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (z, z^{-1}; q^2)_\infty}.
\end{aligned} \tag{2.6}$$

By equations (1.20) and (1.21) we then have

$$S(z, q) = \frac{1}{(1-z)(1-z^{-1})} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\overline{N2}(m, n) - \overline{M2}(m, n)) z^m q^n. \tag{2.7}$$

□

*Proof of Theorem 1.4.* We begin by noting

$$\frac{(-q; q)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (\zeta_3 q^2, \zeta_3^{-1} q^2; q^2)_\infty} = \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2 (q^6; q^6)_\infty}. \tag{2.8}$$

By Gauss we have

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}. \tag{2.9}$$

By the Jacobi Triple Product Identity we then have

$$\begin{aligned}
\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} &= \frac{1}{2} \sum_{k=0}^2 \sum_{n=-\infty}^{\infty} q^{(3n+k)(3n+k+1)/2} \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+3n)/2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+9n)/2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+15n)/2} \\
&= \frac{1}{2} (-q^6, -q^3, q^9; q^9)_\infty + \frac{1}{2} q (-1, -q^9, q^9; q^9)_\infty + \frac{1}{2} (-q^{-3}, -q^{12}, q^9; q^9)_\infty \\
&= (-q^6, -q^3, q^9; q^9)_\infty + q (-q^9, -q^9, q^9; q^9)_\infty.
\end{aligned} \tag{2.10}$$

Using the above to expand  $\frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2}$  and dividing by  $(q^6; q^6)_\infty$  then gives

$$\overline{M2}_0(q) = \frac{(-q, -q^2, q^3; q^3)_\infty^2}{(q^2; q^2)_\infty}, \tag{2.11}$$

$$\overline{M2}_1(q) = 2 \frac{(-q, -q^2, -q^3, -q^3, q^3, q^3; q^3)_\infty}{(q^2; q^2)_\infty} \tag{2.12}$$

$$\overline{M2}_2(q) = \frac{(-q^3, -q^3, q^3; q^3)_\infty^2}{(q^2; q^2)_\infty}. \tag{2.13}$$

However these products easily reduce to those in the statement of the Theorem 1.4. □

### 3. COMBINATORIAL INTERPRETATIONS

We see  $N_S(m, n)$  can be interpreted in terms of vector partitions. We let  $\overline{V} = \mathcal{D} \times \mathcal{P} \times \mathcal{P} \times \mathcal{D}$ , where  $\mathcal{P}$  denotes the set of all partitions and  $\mathcal{D}$  denotes the set of all partitions into distinct parts. For a partition  $\pi$  we let  $s(\pi)$  denote the smallest part of  $\pi$  (with the convention that the empty partition has smallest part  $\infty$ ),  $\#(\pi)$  the number of parts in  $\pi$ ,  $\#(\pi_e)$  the number of even parts in  $\pi$ , and  $|\pi|$  the sum of the parts of  $\pi$ . For  $\vec{\pi} = (\pi^1, \pi^2, \pi^3, \pi^4) \in \overline{V}$ , we define the weight  $\omega(\vec{\pi}) = (-1)^{\#(\pi^1)-1}$ , the crank( $\vec{\pi}$ ) =  $\#(\pi_e^2) - \#(\pi_e^3)$ , and the norm  $|\vec{\pi}| = |\pi^1| + |\pi^2| + |\pi^3| + |\pi^4|$ . We say  $\vec{\pi}$  is a vector partition of  $n$  if  $|\vec{\pi}| = n$ .

By writing the summands of  $S(z, q)$  as

$$q^{2n} (q^{2n+1}; q)_\infty \cdot \frac{1}{(zq^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty} \cdot \frac{1}{(z^{-1}q^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty} \cdot (q^{2n+1}; q)_\infty, \quad (3.1)$$

we see  $N_S(m, n)$  is the number of vector partitions  $(\pi^1, \pi^2, \pi^3, \pi^4)$  from  $\overline{V}$  of  $n$  with the additional constraints that  $\pi^1$  is non-empty,  $s(\pi^1) \leq s(\pi^2)$ ,  $s(\pi^1) \leq s(\pi^3)$ ,  $s(\pi^1) < s(\pi^4)$ , and  $s(\pi^1)$  is even, but counted with the weight  $\omega$ .

However, this interpretation hides the fact that each  $N_S(m, n)$  is non-negative, as we are counting with a weight that may be negative. We can also interpret  $N_S(m, n)$  in terms of partition pairs. This interpretation makes the non-negativity clear. Using the  $q$ -binomial theorem we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{q^{2n} (q^{4n+2}; q^2)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \sum_{k=0}^{\infty} \frac{(zq^{2n+2}; q^2)_k z^{-k} q^{2nk}}{(q^2; q^2)_k} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{z^{-k} q^{2n+2nk}}{(1-zq^{2n})(zq^{2n+2k+2}; q^2)_\infty (q^2; q^2)_k (q^{2n+1}; q^2)_\infty^2} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \\ & \quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{2n} (q^2; q^2)_n}{(1-zq^{2n})(q^2; q^2)_{n+k} (zq^{2n+2k+2}; q^2)_\infty (q^{2n+1}; q^2)_\infty} \cdot \frac{z^{-k} q^{2nk} (q^2; q^2)_{n+k}}{(q^2; q^2)_k (q^2; q^2)_n (q^{2n+1}; q^2)_\infty} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n}}{(zq^{2n}; q^2)_\infty (q^{2n+1}; q^2)_\infty^2} \\ & \quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{2n}}{(1-zq^{2n})(q^{2n+2}; q^2)_k (zq^{2n+2k+2}; q^2)_\infty (q^{2n+1}; q^2)_\infty} \cdot \frac{z^{-k} q^{2nk} (q^2; q^2)_{n+k}}{(q^2; q^2)_k (q^2; q^2)_n (q^{2n+1}; q^2)_\infty}. \quad (3.2) \end{aligned}$$

We can now give the partition pair interpretation. We let  $PP_2$  denote the set of partition pairs  $(\pi^1, \pi^2)$  such that  $\pi^1$  is non-empty,  $s(\pi^1)$  is even,  $s(\pi^1) \leq s(\pi^2)$ , and the even parts of  $\pi^2$  are at most  $2s(\pi^1)$ . For such a partition pair we let  $k(\pi^1, \pi^2)$  denote the number of even parts of  $\pi^1$  that are either the smallest part or are larger than  $s(\pi_1) + 2\#(\pi_e^2)$ . We note when  $\pi^2$  contains no even parts that  $k(\pi_1, \pi_2)$  reduces to  $\#(\pi_e^1)$ . We define a crank on the elements of  $PP_2$  by

$$c(\pi_1, \pi_2) = k(\pi^1, \pi^2) - \#(\pi_e^2) - 1. \quad (3.3)$$

We claim  $N_S(m, n)$  is also the number of partitions pairs of  $n$  from  $PP_2$  with  $c(\pi^1, \pi^2) = m$ .

For this we note the first series in (3.2) gives the cases when  $\pi^2$  has no even parts. The second series in (3.2) gives the cases when  $\pi^2$  has even parts, since  $\frac{q^{2nk} (q^2; q^2)_{n+k}}{(q^2; q^2)_k (q^2; q^2)_n}$  is the generating function for partitions into even parts with exactly  $k$  parts and each part between  $2n$  and  $4n$  (inclusive).

It may be possible to define a bijection from these partition pairs to marked overpartitions with smallest part even, and through that determine a crank defined on marked overpartitions. However, we do not pursue that here.

#### 4. REMARKS

While  $S(\zeta_3, q)$  can be used to prove  $\overline{\text{spt}}_2(3n) \equiv 0 \pmod{3}$  and  $\overline{\text{spt}}_2(3n+1) \equiv 0 \pmod{3}$ , we cannot use  $S(\zeta_5, q)$  to prove  $\overline{\text{spt}}_2(5n+3) \equiv 0 \pmod{5}$ . In particular we find the coefficient of  $q^8$  in  $S(\zeta_5, q)$  to be  $z^3 + z^2 + 3z + 5 + 3z^{-1} + z^{-2} + z^{-3}$ . That is to say,  $N_S(0, 5, 8) = 5$ ,  $N_S(1, 5, 8) = 3$ ,  $N_S(2, 5, 8) = 2$ ,  $N_S(3, 5, 8) = 2$ , and  $N_S(4, 5, 8) = 3$ .

However  $\overline{\text{spt}}_2(5n+3) \equiv 0 \pmod{5}$  does follow by considering  $\overline{\mathbb{S}}_2(\zeta_5, q)$ . This can be compared with the rank of a partition explaining the congruences for  $p(5n+4)$  and  $p(7n+5)$  but not  $p(11n+6)$ , whereas the crank of an partition does explain all three.

#### REFERENCES

- [1] G. E. Andrews. The number of smallest parts in the partitions of  $n$ . *J. Reine Angew. Math.*, 624:133–142, 2008.
- [2] G. E. Andrews, F. G. Garvan, and J. Liang. Combinatorial interpretations of congruences for the spt-function. *Ramanujan J.*, 29(1-3):321–338, 2012.
- [3] K. Bringmann, J. Lovejoy, and R. Osburn. Rank and crank moments for overpartitions. *J. Number Theory*, 129(7):1758–1772, 2009.
- [4] F. Garvan and C. Jennings-Shaffer. The spt-crank for overpartitions. *ArXiv e-prints*, Nov. 2013.
- [5] C. Jennings-Shaffer. Higher order spt functions for overpartitions, overpartitions with smallest part even, and partitions without repeated odd parts. *ArXiv e-prints*, Feb. 2014.
- [6] J. Lovejoy and R. Osburn. Rank differences for overpartitions. *Q. J. Math.*, 59(2):257–273, 2008.
- [7] J. Lovejoy and R. Osburn.  $M_2$ -rank differences for partitions without repeated odd parts. *J. Théor. Nombres Bordeaux*, 21(2):313–334, 2009.
- [8] J. Lovejoy and R. Osburn.  $M_2$ -rank differences for overpartitions. *Acta Arith.*, 144(2):193–212, 2010.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611, USA  
CJENNINGS@UFL.EDU