

**SINGULARITY OF MACROSCOPIC VARIABLES NEAR
BOUNDARY FOR GASES WITH CUT-OFF
INVERSE-POWER POTENTIAL**

I-KUN CHEN AND CHUN-HSIUNG HSIA

ABSTRACT. In this article, the boundary singularity for stationary solutions of the linearized Boltzmann equation with cut-off inverse power potential is analyzed. In particular, for cut-off hard-potential cases, we establish the asymptotic approximation for the gradient of the moments. Our analysis indicates the logarithmic singularity of the gradient of the moments.

1. INTRODUCTION

The Boltzmann equation is

$$(1.1) \quad \frac{\partial F}{\partial t} + \xi \cdot \frac{\partial F}{\partial t} = Q(F, F),$$

where $F = F(t, x, \xi)$. Q above is the collision operator only involves velocity as follows:

$$(1.2) \quad Q(F, F) = \int \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (F' F'_* - F F_*) B(|V|, \theta) d\theta d\epsilon d\xi_*,$$

where $V = \xi_* - \xi$ and α is a unit vector on a hemisphere parametrized by θ and ϵ such that $\alpha \cdot V = |V| \cos \theta$ and

$$(1.3) \quad F = F(\xi), \quad F_* = F(\xi_*), \quad F' = F(\xi'), \quad F'_* = F(\xi'_*),$$

$$(1.4) \quad \xi' = \xi + (\alpha \cdot V)\alpha,$$

$$(1.5) \quad \xi'_* = \xi_* - (\alpha \cdot V)\alpha.$$

The $B(|V|, \theta) \geq 0$ is called the cross-section. If we consider inverse power force between particles, i.e., $Force = \frac{1}{r^s}$, then the cross-section is in the form

$$(1.6) \quad B(|V|, \theta) = |V|^\gamma \beta(\theta),$$

where $\gamma = \frac{s-5}{s-1}$. The fact $\beta \sim (\frac{\pi}{2} - \theta)^{-\frac{s+1}{s-1}}$ as $\theta \rightarrow \frac{\pi}{2}$, which is not integrable in θ , makes us unable to separate 1.2 into gain and lost parts. To avoid this mathematical difficulty, it was Grad's idea, [13], to consider the cross-section such that

$$(1.7) \quad B(|V|, \theta) \leq C|V|^\gamma \cos \theta \sin \theta.$$

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We will refer these cases as Grad's angular cut-off potential. In particular, in our research we will consider the cases that B as a product of a function of $|V|$ and one of θ , i.e.,

$$(1.8) \quad B(|V|, \theta) = |V|^\gamma \beta(\theta), \quad \beta(\theta) \leq C \cos \theta \sin \theta.$$

To make distinctions, we will refer the cases above, (1.8), as power-law potential with angular cut-off in this paper. We first non-dimensionalize the equation so that the Maxwellian we are interested becomes the standard one:

$$(1.9) \quad w = \frac{1}{\sqrt{\pi}^3} e^{-|\zeta|^2}.$$

We linearize the equation around standard Maxwellian so that

$$(1.10) \quad F = w + w^{\frac{1}{2}} f.$$

We have,

$$(1.11) \quad sh \frac{\partial f}{\partial t} + \zeta \cdot \frac{\partial f}{\partial t} = \frac{1}{\kappa} L(f),$$

where $L(f) = 2w^{-\frac{1}{2}} Q(w, w^{\frac{1}{2}} f)$. Under the assumption of Grad's angular cut-off, the linearized collision operator can be decomposed into a damping multiplicative operator $-\nu$ and a smoothing integral operator K :

$$(1.12) \quad L(\phi)(\zeta) = -\nu(\zeta)\phi(\zeta) + K(\phi)(\zeta).$$

The the following properties of the linearized collision operator were studied by Grad [13] and Caglioli [6]. The collision frequency satisfies the following estimate

$$(1.13) \quad \nu_0(1 + |\zeta|)^\gamma \leq \nu(\zeta) \leq \nu_1(1 + |\zeta|)^\gamma,$$

where $0 < \nu_0 < \nu_1$ and $-2 \leq \gamma \leq 1$ is a parameter from interaction between particles. $\gamma = 1$ is called the hard sphere model; $\gamma = 0$ is called the Maxwellian case. Positive γ 's corresponded to hard potential; negative γ 's correspond to soft-potential.

If we consider power-law potential with angular cut-off, we have further properties :

$$(1.14) \quad \nu(\zeta) \text{ is a function of } |\zeta|.$$

$$(1.15) \quad \|\partial_{\zeta_i} K(f)\|_{L^p} \leq C \|f\|_{L^p}, \quad p \geq 1.$$

In this paper, we restrict our study to the cases $0 < \gamma \leq 1$. We define

$$(1.16) \quad \|f(\zeta)\|_{L_a^\infty} = \sup_{\zeta} (1 + |\zeta|)^a |f(\zeta)|.$$

The integral operator improves the decay:

$$(1.17) \quad \|K(f)\|_{L_{\frac{3}{2}-\gamma}^\infty} \leq C \|f\|_{L^2},$$

$$(1.18) \quad \|K(f)\|_{L_{2+a-\gamma}^\infty} \leq C \|f\|_{L_a^\infty}.$$

We consider the stationary equation:

$$(1.19) \quad \zeta_1 \partial_x f(x, \zeta) = L(f)(\zeta), \text{ for } 0 < x \leq l.$$

The functional space we are considering is as follows:

Definition 1.1.

$$(1.20) \quad L_\zeta^*(\mathbb{R}^3) = \{f, \|f(\zeta)\|_* < \infty\},$$

where

$$(1.21) \quad \|f(\zeta)\|_* = \left(\int f^2(\zeta) \nu(\zeta) d\zeta \right)^{\frac{1}{2}}.$$

Also,

$$(1.22) \quad \|f\| := \sup_{0 \leq x \leq l} \|f\|_*.$$

We say $f \in L_x^\infty([0, l], L_\zeta^*(\mathbb{R}^3))$ is a solution to (1.19) if it satisfies the following integral equation:

$$(1.23) \quad f(x, \zeta) = \begin{cases} e^{-\frac{\nu(\zeta)}{|\zeta_1|}x} f(0, \zeta) + \int_0^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|(x-s)}} K(f)(s, \zeta) ds, & \text{for } \zeta_1 > 0, \\ e^{-\frac{\nu(\zeta)}{|\zeta_1|(l-x)} f(l, \zeta) + \int_x^l \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|(s-x)}} K(f)(s, \zeta) ds, & \text{for } \zeta_1 < 0. \end{cases}$$

Remark 1.2. The solution spaces for both Milne and Kramar's problems given in [11] are in $L_x^\infty([0, l], L_\zeta^*(\mathbb{R}^3))$ if x is restricted to $[0, l]$. ■

The moments are defined as follows:

Definition 1.3. The α moment is defined as

$$(1.24) \quad \sigma_\alpha(x) = \int f(x, \zeta) \phi_\alpha(\zeta) d\zeta,$$

where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_i \text{'s are nonnegative integers,}$$

and

$$(1.25) \quad \phi_\alpha(\zeta) = \zeta^\alpha E^{\frac{1}{2}} = \pi^{-\frac{3}{4}} \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} e^{-\frac{|\zeta|^2}{2}}.$$

We introduce a constant depending on α :

Definition 1.4. Set

$$(1.26) \quad A_\alpha = (2\alpha_1)^{\frac{\alpha_1}{2}} (2\alpha_2)^{\frac{\alpha_2}{2}} (2\alpha_3)^{\frac{\alpha_3}{2}} e^{-\frac{\alpha_1 + \alpha_2 + \alpha_3}{2}},$$

where we follow the convention $0^0 = 1$.

The macroscopic variables are obtained through the moments. For example, $\sigma_{(0,0,0)}$ is the density, $\sigma_{(1,0,0)}$ is the flow velocity in x_1 direction, and $\frac{2}{3}(\sigma_{(2,0,0)} + \sigma_{(0,2,0)} + \sigma_{(0,0,2)}) - \sigma_{(0,0,0)}$ is the temperature. The following inequality will be frequently used later :

$$(1.27) \quad |\phi_\alpha| \leq CA_\alpha e^{-\frac{|\zeta|^2}{4}}.$$

The Main Theorem is as follows

Theorem 1.5. *Suppose $f \in L_x^\infty([0, l], L_\zeta^*(\mathbb{R}^3))$ is a solution to (1.19) with power law potential with angular cut-off with $0 < \gamma \leq 1$ and $\nabla f(0, \zeta) \in L_\zeta^p(\mathbb{R}^{3+})$ for $p > 1$, $f(0, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3+})$, and $f(l, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3-})$. Then, for x small enough,*

$$(1.28) \quad |\partial_x \sigma_\alpha(x)| = -\ln x \int \int \phi_\alpha(0, \zeta_2, \zeta_3) L(f)(0, 0^+, \zeta_2, \zeta_3) d\zeta_2 d\zeta_3 \\ + O(A_\alpha \langle f \rangle'),$$

where

$$(1.29) \quad L(f)(0, 0^+, \zeta_2, \zeta_3) := \lim_{\zeta_1 \rightarrow 0^+} L(f)(0, \zeta_1, \zeta_2, \zeta_3)$$

and

$$(1.30) \quad \langle f \rangle' := 1 + \|f\| + \|f(0, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3+})} + \|f(l, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3-})} + \|\nabla f(0, \zeta)\|_{L_\zeta^p(\mathbb{R}^{3+})}.$$

We first investigate the problem for Grad's angular cut-off potential. We consider the the distribution function for $\zeta_1 > 0$. The case for $\zeta_1 < 0$ can be treated similarly. Differentiating (1.23) for $\zeta_1 > 0$, we have

$$(1.31) \quad \frac{\partial}{\partial x} f(x, \zeta) = -\frac{\nu(\zeta)}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} f(0, \zeta) + \frac{1}{|\zeta_1|} K(f)(x, \zeta) \\ - \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} K(f)(s, \zeta) ds.$$

We observe that the first term is nice in ζ when x is away from zero and has a singularity at $x = 0$. On the other hand, the second and third terms have factors $|\zeta_1|^{-1}$ and $|\zeta_1|^{-2}$ in ζ , which cause a difficulty in our analysis. In order to overcome this difficulty, we reorganize the equation (1.31):

$$(1.32) \quad \frac{\partial}{\partial x} f(x, \zeta) = -\frac{\nu(\zeta)}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} f(0, \zeta) + \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} K(f)(x, \zeta) \\ + \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} (K(f)(x, \zeta) - K(f)(s, \zeta)) ds \\ = \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} L(f)(0, \zeta) + \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} (K(f)(x, \zeta) - K(f)(0, \zeta)) \\ + \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} (K(f)(x, \zeta) - K(f)(s, \zeta)) ds.$$

The contribution from the second and third terms is uniformly bounded. Roughly speaking, we organize the equation in such a way in order to use the "Hölder continuity" of $K(f)$ in x to obtain "differentiability" of f in a very weak sense, which will be explained in detail in Section 2. Also in Section 2, we will deal with the contribution from $\zeta_1 < 0$. In section 3, with further assumption of invers-power potential with angular cut-off and regularity on boundary data, we can extract the singularity from the contribution of the first term on the right hand side of (1.32), which concludes the Theorem 1.5.

2. UPPER BOUND ESTIMATES

As outlined in the introduction, the goal of this section is to prove

Lemma 2.1. *Suppose $f \in L_x^\infty([0, l], L_\zeta^*(\mathbb{R}^3))$ is a solution to (1.19) with Grad's angular cut-off potential with $0 < \gamma \leq 1$ and $f(0, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3+})$ and $f(l, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3-})$.*

Then,

(2.1)

$$|\int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} L(f)(0, \zeta) d\zeta| \leq C(|\ln x| + 1) A_\alpha \langle f \rangle,$$

(2.2)

$$|\int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} (K(f)(x, \zeta) - K(f)(0, \zeta)) d\zeta| \leq C A_\alpha \langle f \rangle,$$

(2.3)

$$|\int_{\zeta_1 > 0} \phi_\alpha \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} (K(f)(x, \zeta) - K(f)(s, \zeta)) ds d\zeta| \leq C A_\alpha \langle f \rangle,$$

where

$$\langle f \rangle := \|f\| + \|f(0, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3+})} + \|f(l, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3-})}.$$

(2.4)

Proof. We observe that f is in fact bounded for all x and ζ if $f(0, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3+})$ and $f(l, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3-})$. For $\zeta_1 > 0$,

$$\begin{aligned} |f(x, \zeta)| &\leq |f(0, \zeta)| + C \|f\|_* \int_0^x \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} ds \\ &\leq \|f(0, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3+})} + C \|f\|_* \nu_0^{-1} \int_0^{\frac{x}{|\zeta_1|}} e^{-z} dz \\ &\leq C (\|f(0, \zeta)\|_{L_\zeta^\infty(\mathbb{R}^{3+})} + \|f\|_*). \end{aligned}$$

(2.5)

A similar inequality holds for $\zeta_1 < 0$. Therefore,

$$\|f\|_{L_{x, \zeta}^\infty} \leq C \langle f \rangle.$$

(2.6)

We observe

$$\begin{aligned}
(2.7) \quad & \left| \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} L(f)(0, \zeta) d\zeta \right| \\
& \leq C A_\alpha \langle f \rangle \left| \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{4}} \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} d\zeta \right| \leq C A_\alpha \langle f \rangle (|\ln x| + 1),
\end{aligned}$$

which concludes (2.1)

We will present the proof for (2.3).

Replacing 0, l by s , x in the (1.23), we can derive

$$\begin{aligned}
(2.8) \quad & K(f)(x, \zeta) - K(f)(s, \zeta) = \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x-s|} - 1) f(s, \zeta_*) d\zeta_* \\
& + \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \int_s^x \frac{1}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x-t|} K(f)(t, \zeta_*) dt d\zeta_* \\
& + \int_{\zeta_{*1} < 0} k(\zeta, \zeta_*) (1 - e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x-s|}) f(x, \zeta_*) d\zeta_* \\
& + \int_{\zeta_{*1} < 0} k(\zeta, \zeta_*) \int_s^x \frac{1}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |s-t|} K(f)(t, \zeta_*) dt d\zeta_* \\
& =: H_1 + H_2 + H_3 + H_4.
\end{aligned}$$

The term H_2 and H_4 have the property to be proved later

$$(2.9) \quad |H_2| \leq C \|f\|_* |x-s|^\beta, \quad |H_4| \leq C \|f\|_* |x-s|^\beta,$$

where $0 < \beta < \frac{\gamma}{2+\gamma}$. We let

$$\begin{aligned}
(2.10) \quad & \int_{\zeta_1 > 0} \int_0^x \phi_\alpha(\zeta) \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} (K(f)(x, \zeta) - K(f)(s, \zeta)) ds d\zeta \\
& = \int_{\zeta_1 > 0} \int_0^x \phi_\alpha(\zeta) \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} (H_1 + H_2 + H_3 + H_4) ds d\zeta \\
& =: B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

Therefore, for $i = 2$ and 4,

$$\begin{aligned}
(2.11) \quad & |B_i| \leq C A_\alpha \|f\|_* \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{4}} \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|} (x-s)} |x-s|^\beta ds d\zeta \\
& \leq C A_\alpha \|f\|_* \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{4}} \nu(\zeta)^{-\beta} |\zeta_1|^{\beta-1} \int_0^{\frac{\nu(\zeta)x}{|\zeta_1|}} z^\beta e^{-z} dz d\zeta \\
& \leq C A_\alpha \|f\|_*,
\end{aligned}$$

where $z = \frac{\nu(\zeta)}{|\zeta_1|} (x-s)$. Estimates for B_1 and B_3 are not so obvious and the analysis is more demanding. We shall present the case for B_1 only and the

case for B_3 can be done similarly. We claim

$$(2.12) \quad |B_1| = \left| \int_{\zeta_1 > 0} \phi_\alpha \int_0^x \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|}|x-s|} - 1) f(s, \zeta_*) d\zeta_* ds d\zeta \right| \leq CA_\alpha \langle f \rangle.$$

Change the order of integration, we have

$$(2.13) \quad |B_1| = \left| \int_0^x \int_{\zeta_{*1} > 0} \left(\int_{\zeta_1 > 0} k(\zeta, \zeta_*) \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} d\zeta \right) (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|}|x-s|} - 1) f(s, \zeta_*) d\zeta_* ds \right|.$$

We observe, for $a \geq 0$

$$(2.14) \quad \left\| \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} \right\|_{L_a^1} \leq C_a A_\alpha |x-s|^{-1} (1 + |\ln|x||),$$

$$(2.15) \quad \left\| \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} \right\|_{L_a^\infty} \leq C_a A_\alpha |x-s|^{-2}.$$

Interpolating the inequalities above, we obtain

$$(2.16) \quad \left\| \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} \right\|_{L_a^p} \leq C_a A_\alpha |x-s|^{-2+\frac{1}{p}} (1 + |\ln|x||)^{\frac{1}{p}},$$

where $1 \leq p \leq \infty$. In particular,

$$(2.17) \quad \left\| \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} \right\|_{L^2} \leq C A_\alpha |x-s|^{-\frac{3}{2}} (1 + |\ln|x||)^{\frac{1}{2}}.$$

Let $h(\theta, \gamma, a) = (\frac{3}{2} - \gamma)\theta + (2 + a - \gamma)(1 - \theta) = 2 + a(1 - \theta) - \gamma - \frac{1}{2}\theta$, where $0 \leq \theta \leq 1$. We have

$$(2.18) \quad \begin{aligned} \|K(f)\|_{L_h^\infty} &= \sup_{\zeta \in \mathbb{R}^3} \left(|f(\zeta)|(1 + |\zeta|)^{\frac{3}{2}-\gamma} \right)^\theta \left(|f(\zeta)|(1 + |\zeta|)^{2+a-\gamma} \right)^{1-\theta} \\ &\leq \|K(f)\|_{L_{\frac{3}{2}-\gamma}^\infty}^\theta \|K(f)\|_{L_{2+a-\gamma}^\infty}^{1-\theta} \leq C \|f\|_{L^2}^\theta \|f\|_{L_a^\infty}^{1-\theta}. \end{aligned}$$

Combining (1.17), (1.18), (2.15), (2.17), and (2.18), we have

$$(2.19) \quad \left\| \int_{\zeta_1 > 0} \phi_\alpha \frac{\nu(\zeta)}{|\zeta_1|^2} e^{-\frac{\nu(\zeta)}{|\zeta_1|}(x-s)} d\zeta_1 \right\|_{L_h^\infty} \leq C C_a^{1-\theta} A_\alpha |x-s|^{-2+\frac{1}{2}\theta} (1 + |\ln|x-s||)^{\frac{1}{2}\theta},$$

Applying (2.19) above with fixed $0 < \theta < 1$ and a large enough, we have

$$\begin{aligned}
(2.20) \quad |B_1| &\leq \left| \int_0^x \int_{\zeta_{*1} > 0} (1 + |\zeta_*|)^{-h(\theta, \gamma, a)} |x - s|^{-(2 - \frac{1}{2}\theta)} (1 + |\ln |x - s||)^{\frac{1}{2}\theta} \right. \\
&\quad \left. (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x - s|} - 1) |f(s, \zeta_*)| d\zeta_* ds \right| \\
&\leq CA_\alpha \left| \int_0^x |x - s|^{-(2 - \frac{1}{2}\theta)} (1 + |\ln |x - s||)^{\frac{1}{2}\theta} \right. \\
&\quad \left. \int_0^{|x - s|} \int_{\zeta_{*1} > 0} \frac{\nu(\zeta_*)}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} t} (1 + |\zeta_*|)^{-h(\theta, \gamma, a)} |f(s, \zeta_*)| d\zeta_* dt ds \right| \\
&\leq CA_\alpha \langle f \rangle \left| \int_0^x |x - s|^{-(1 - \frac{1}{2}\theta)} (1 + |\ln |x - s||)^{1 + \frac{1}{2}\theta} ds \right| \leq CA_\alpha \langle f \rangle.
\end{aligned}$$

The proof for (2.2) is similar and simpler. Replacing s in (2.8) by 0 and denoting these terms as H'_i s, we write

$$\begin{aligned}
(2.21) \quad &\int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} (K(f)(x, \zeta) - K(f)(0, \zeta)) d\zeta \\
&= \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} (H'_1 + H'_2 + H'_3 + H'_4) d\zeta \\
&=: B'_1 + B'_2 + B'_3 + B'_4.
\end{aligned}$$

Using the fact

$$(2.22) \quad |H'_2| \leq C|x|^\beta \|f\|_*, \quad |H'_4| \leq C|x|^\beta \|f\|_*,$$

we have

$$\begin{aligned}
(2.23) \quad |B'_2 + B'_4| &\leq \|f\|_* \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} |x|^\beta d\zeta \\
&\leq C\|f\|_* \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\nu(\zeta) |\zeta_1|^{1 - \beta}} d\zeta \leq C\|f\|_*.
\end{aligned}$$

The treatment for B'_1 and B'_3 is similar, and therefore we present the case for B'_1 only. Changing the order of integration, we have

$$\begin{aligned}
(2.24) \quad B'_1 &= \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} x} - 1) f(0, \zeta_*) d\zeta_* d\zeta \\
&= \int_{\zeta_{*1} > 0} (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} x} - 1) f(0, \zeta_*) \left(\int_{\zeta_1 > 0} k(\zeta, \zeta_*) \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} d\zeta \right) d\zeta.
\end{aligned}$$

Note that

$$(2.25) \quad \left\| \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} \right\|_{L^1} \leq CA_\alpha (1 + |\ln x|),$$

$$(2.26) \quad \left\| \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} \right\|_{L^\infty} \leq C_a A_\alpha \left(\frac{1}{x} \right).$$

By interpolation, we have

$$(2.27) \quad \left\| \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} \right\|_{L^2} \leq C A_\alpha \left(\frac{1}{x} \right)^{\frac{1}{2}} (1 + |\ln x|)^{\frac{1}{2}}.$$

Combining (1.17), (1.18), (2.26), (2.27), and (2.18), we have

$$(2.28) \quad \left\| \int_{\zeta_1 > 0} k(\zeta, \zeta_*) \phi_\alpha \frac{1}{\zeta_1} e^{-\frac{\nu(\zeta)}{|\zeta_1|} x} d\zeta \right\|_{L_h^\infty} \leq C C_a^{1-\theta} A_\alpha \left(\frac{1}{x} \right)^{1-\frac{\theta}{2}} (1 + |\ln x|)^{\frac{1}{2}\theta}.$$

Similar to (2.20), for fixed $0 < \theta < 1$ and a large enough, we have

$$(2.29) \quad \begin{aligned} |B'_1| &\leq C A_\alpha \int_{\zeta_{*1} > 0} (e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} x} - 1) f(0, \zeta_*) (1 + |\zeta_*|)^{-h(\theta, \gamma, a)} \left(\frac{1}{x} \right)^{1-\frac{\theta}{2}} (1 + |\ln x|)^{\frac{1}{2}\theta} d\zeta_* \\ &\leq C A_\alpha \left(\frac{1}{x} \right)^{1-\frac{\theta}{2}} (1 + |\ln x|)^{\frac{1}{2}\theta} \int_0^x \int_{\zeta_{*1} > 0} \frac{\nu(\zeta_*)}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} t} (1 + |\zeta_*|)^{-h(\theta, \gamma, a)} |f(0, \zeta_*)| d\zeta_* dt \\ &\leq C A_\alpha \langle f \rangle x^{\frac{\theta}{2}} (1 + |\ln x|)^{1+\frac{1}{2}\theta} \leq C A_\alpha \langle f \rangle. \end{aligned}$$

We still have to prove (2.9). We will present the proof for H_2 only and the one for H_4 is similar. We will first present the following lemma

Lemma 2.2. *If $f \in L_*^2$ and $\theta \in (\frac{2}{2+\gamma}, 1)$, then*

$$(2.30) \quad \int \frac{1}{|\zeta_{*1}|^{2-2\theta} \nu(\zeta_*)^{2\theta}} |K(f)|^2 d\zeta_* \leq C \|f\|_*^2.$$

The proof Lemma 2.2 follows the idea of the one of Lemma 4.2 in [11]. We will present the proof at the end of this section to make this paper self-contained.

With the Lemma 2.2 above, we have

$$(2.31) \quad \begin{aligned} |H_2| &= \left| \int_s^x \int_{\zeta_{*1} > 0} k(\zeta, \zeta_*) \frac{1}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x-t|} K(f)(t, \zeta_*) d\zeta_* dt \right| \\ &\leq C \left(\int_s^x \left(\int_{\zeta_{*1} > 0} \frac{1}{|\zeta_{*1}|} e^{-\frac{\nu(\zeta_*)}{|\zeta_{*1}|} |x-t|} |K(f)(t, \zeta_*)|^2 d\zeta_* \right)^{\frac{1}{2}} dt \right) \\ &\leq C \left(\int_s^x \frac{1}{|x-t|^\theta} \left(\int_{\zeta_{*1} > 0} \frac{1}{|\zeta_{*1}|^{2-2\theta} \nu(\zeta_*)^{2\theta}} |K(f)(t, \zeta_*)|^2 d\zeta_* \right)^{\frac{1}{2}} dt \right) \\ &\leq C |x-s|^{1-\theta} = C |x-s|^\beta. \end{aligned}$$

□

For $\zeta_1 < 0$, we can yield a similar lemma. Together, we have

Lemma 2.3. *Suppose $f \in L_x^\infty([0, l], L_\zeta^*(\mathbb{R}^3))$ is a solution to (1.19) with Grad's angular cut-off potential with $0 < \gamma \leq 1$ and $f(0, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3+})$ and $f(l, \zeta) \in L_\zeta^\infty(\mathbb{R}^{3-})$. Then,*

$$(2.32) \quad |\partial_x \sigma_\alpha^+(x)| \leq C (|\ln |x|| + 1),$$

$$(2.33) \quad |\partial_x \sigma_\alpha^-(x)| \leq C(|\ln |l-x|| + 1),$$

where

$$(2.34) \quad \sigma_\alpha^+(x) = \int_{\zeta_1 > 0} \phi_\alpha f d\zeta, \quad \sigma_\alpha^-(x) = \int_{\zeta_1 < 0} \phi_\alpha f d\zeta.$$

Proof for Lemma 2.2. We observe

$$(2.35) \quad \begin{aligned} \int |Kf|^2 d\zeta &= \int \left(\int k(\zeta, \zeta_*) f(\zeta_*) d\zeta * \int k(\zeta, \zeta') f(\zeta') d\zeta' \right) d\zeta \\ &\leq C \|f\|_{L^2} \int |f(\zeta_*)| \int k(\zeta, \zeta_*) (1 + |\zeta|)^{-(\frac{3}{2}-\gamma)} d\zeta d\zeta_* \\ &\leq C \|f\|_{L^2} \int |f(\zeta_*)| (1 + |\zeta * |)^{-(\frac{7}{2}-2\gamma)} d\zeta_* \leq C \|f\|_{L^2} \|f\|_* \leq C \|f\|_*^2. \end{aligned}$$

Together with (1.17), we know $|Kf|^2 \in L^\infty \cap L^1$. Interpolating between these two inequalities, we have

$$(2.36) \quad \| |Kf|^2 \|_{L^p} \leq C \|f\|_*^2 \text{ for } 1 \leq p \leq \infty.$$

Therefore, we now only need to prove for some proper $0 < \theta < 1$ and Hölder conjugate of $p, p' \in [1, \infty]$,

$$(2.37) \quad \int \frac{1}{|\zeta_{*1}|^{(2-2\theta)p'} |1 + |\zeta_*||^{2\gamma\theta p'}} d\zeta_* < \infty,$$

which yields the following condition:

$$(2.38) \quad (2 - 2\theta)p' < 1; \quad (2 - 2\theta + 2\gamma\theta)p' > 3.$$

Such p' exists if and only if

$$(2.39) \quad \theta < 1, \quad 3(2 - 2\theta) < 2 - 2\theta + 2\gamma\theta,$$

which concludes Lemma 2.2. \square

3. ASYMPTOTIC FORMULA

In the precious section, we obtain an upper bound for $|\partial_x \sigma_\alpha|$, which diverges to infinity at boundary like a logarithmic function. Through the analysis, we also localize the source of singularity, which is the contribution from the first term on the right hand side in (1.32). In this section, restricted to the inverse-power potential with angular cut-off, the goal is to further single out and factorize the singularity and form an asymptotic formula, i.e.,

Lemma 3.1. *Suppose $f(0, \zeta) \in L_*$ and $\nabla f(0, \zeta) \in L_\zeta^p(\mathbb{R}^{3+})$. Then,*

$$(3.1) \quad \begin{aligned} \frac{\partial}{\partial x} \sigma_{\alpha 1}^+ &:= \int_{\zeta_1 > 0} \phi_\alpha \frac{1}{|\zeta_1|} e^{-\frac{\nu(\zeta_1)}{|\zeta_1|} x} L(f)(0, \zeta) d\zeta \\ &= -\ln x \int \int \phi_\alpha(0, \zeta_2, \zeta_3) L(f)(0, 0^+, \zeta_2, \zeta_3) d\zeta_2 d\zeta_3 + O(A_\alpha \langle g' \rangle), \end{aligned}$$

where

$$(3.2) \quad L(f)(0, 0^+, \zeta_2, \zeta_3) := \lim_{\zeta_1 \rightarrow 0^+} L(f)(0, \zeta_1, \zeta_2, \zeta_3).$$

Proof. If we change to spherical coordinates so that

$$\zeta = (\rho \cos \theta, \rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi),$$

we have

$$(3.3) \quad \frac{\partial}{\partial x} \sigma_{\alpha_1}^+ = \int_0^\infty \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\rho \cos \theta} e^{-\frac{\rho^2}{2}} e^{-\frac{\nu(\rho)}{\rho \cos \theta} x} \bar{F}(\rho, \theta, \phi) \rho^2 \sin \theta d\theta d\phi d\rho,$$

where

$$(3.4) \quad \bar{F}(\rho, \theta, \phi) = \pi^{-\frac{3}{4}} \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} L(f)(0, \zeta) =: g(\zeta).$$

Further letting $z = \cos \theta$, we obtain

$$(3.5) \quad \frac{\partial}{\partial x} \sigma_{\alpha_1}^+ = \int_0^\infty \int_0^{2\pi} \left[\int_0^1 \frac{1}{z} e^{-\frac{\nu(\rho)}{\rho z} x} F(\rho, z, \phi) dz \right] e^{-\frac{\rho^2}{2}} \rho d\phi d\rho,$$

where

$$(3.6) \quad F(\rho, z, \phi) = \bar{F}(\rho, \cos^{-1} z, \phi).$$

Here, we introduce a well-know special function, exponential integral,

$$(3.7) \quad E_1(x) = \int_0^1 \frac{1}{z} e^{-\frac{x}{z}} dz.$$

The $E_1(x)$ has the following properties, [1]:

$$(3.8) \quad E_1(x) = -\gamma - \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k \cdot k!},$$

$$(3.9) \quad \frac{1}{2} e^{-x} \ln\left(1 + \frac{2}{x}\right) \leq E_1(x) \leq e^{-x} \ln\left(1 + \frac{1}{x}\right) \quad \text{for } x > 0.$$

From the properties above, we have

$$(3.10) \quad E_1(x) = -\ln x + O(1), \quad \text{for } 0 < x \leq 1,$$

Let

$$(3.11) \quad H(z, x) = - \int_z^1 \frac{1}{u} e^{-\frac{\nu(\rho)}{\rho u} x} du.$$

Notice that $\frac{\partial}{\partial z} H(z, x) = \frac{1}{z} e^{-\frac{\nu(\rho)}{\rho z} x}$ and $H(0, x) = E_1\left(\frac{\nu(\rho)}{\rho} x\right)$. Performing integration by parts for the inner most integral in (3.5), we obtain

$$(3.12) \quad \begin{aligned} & \int_0^1 \left(\frac{\partial}{\partial z} H(z) \right) F(\rho, z, \phi) dz \\ & = E_1\left(\frac{\nu(\rho)}{\rho} x\right) F(\rho, 0, \phi) - \int_0^1 H(z) \left(\frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz \end{aligned}$$

The first term on the right hand side above is the source of singularity and will be explained in detail later. We will prove the contribution from the second term above is finite. Let

$$(3.13) \quad I := \int_0^\infty \int_0^{2\pi} E_1\left(\frac{\nu(\rho)}{\rho}x\right)F(\rho, 0, \phi)e^{-\frac{\rho^2}{2}}\rho d\phi d\rho.,$$

$$(3.14) \quad II := \int_0^\infty \int_0^{2\pi} \int_0^1 H(z) \left(\frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz e^{-\frac{\rho^2}{2}} \rho d\phi d\rho.$$

Notice that

$$(3.15) \quad |H(z, x)| \leq |\ln z|,$$

$$(3.16) \quad \frac{\partial}{\partial z} F(\rho, z, \phi) = \rho \frac{\partial}{\partial \zeta_1} g + \frac{\rho z}{\sqrt{1-z^2}} \cos \phi \frac{\partial}{\partial \zeta_2} g + \frac{\rho z}{\sqrt{1-z^2}} \sin \phi \frac{\partial}{\partial \zeta_3} g.$$

We have

$$(3.17) \quad \begin{aligned} |II| &\leq \left| \int_0^\infty \int_0^{2\pi} \int_0^1 H(z) \left(\frac{\partial}{\partial z} F(\rho, z, \phi) \right) dz e^{-\frac{\rho^2}{2}} \rho d\phi d\rho \right| \\ &\leq \left| \int_0^\infty \int_0^{2\pi} \left(\int_0^1 C(|\frac{\partial}{\partial \zeta_3} g| + |\frac{\partial}{\partial \zeta_2} g|) dz + \int_0^1 |\ln z| |\frac{\partial}{\partial \zeta_1} g| dz \right) e^{-\frac{\rho^2}{2}} \rho^2 d\phi d\rho \right| \\ &\leq CA_\alpha \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{8}} \left(|L(0, \zeta)| + \left| \frac{\partial}{\partial \zeta_2} L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_3} L(0, \zeta) \right| \right) d\zeta \\ &\quad + A_\alpha \left| \int_0^\infty \int_0^{2\pi} \left(\int_0^1 |\ln z|^q dz \right)^{\frac{1}{q}} \left(\int_0^1 \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p dz \right)^{\frac{1}{p}} e^{-\frac{\rho^2}{4}} \rho^2 d\phi d\rho \right|, \end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let the second term on the right hand side above be II' .

$$(3.18) \quad \begin{aligned} |II'| &\leq \left(\int_0^\infty \int_0^{2\pi} \left(\int_0^1 |\ln z|^q dz \right) e^{\frac{q\rho^2}{8}} \rho^2 d\phi d\rho \right)^{\frac{1}{q}} \left(\int_0^\infty \int_0^{2\pi} \left(\int_0^1 \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p dz \right) e^{\frac{p\rho^2}{8}} \rho^2 d\phi d\rho \right)^{\frac{1}{p}} \\ &\leq C_p A_\alpha \left(\int_{\zeta_1 > 0} \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p e^{-\frac{p|\zeta|^2}{8}} d\zeta \right)^{\frac{1}{p}} \end{aligned}$$

Therefore,

$$(3.19) \quad \begin{aligned} |II| &\leq CA_\alpha \int_{\zeta_1 > 0} e^{-\frac{|\zeta|^2}{8}} \left(|L(0, \zeta)| + \left| \frac{\partial}{\partial \zeta_2} L(0, \zeta) \right| + \left| \frac{\partial}{\partial \zeta_3} L(0, \zeta) \right| \right) d\zeta \\ &\quad + C_p A_\alpha \left(\int_{\zeta_1 > 0} \left| \frac{\partial}{\partial \zeta_1} L(0, \zeta) \right|^p e^{-\frac{p|\zeta|^2}{8}} d\zeta \right)^{\frac{1}{p}}, \end{aligned}$$

where $p > 1$.

Using the assumption and (1.15), we have

$$(3.20) \quad |II| \leq CA_\alpha(\|\nabla f(0, \zeta)\|_{L^p_\zeta(\mathbb{R}^{3+})} + \|f(0, \zeta)\|_*)$$

Finally, we are going to extract the singularity from I . We let

$$(3.21) \quad \rho_0 = \sup\{\rho \mid \frac{\nu(\rho)}{\rho}x > 1\}.$$

We divided the domain of integration of I in (3.13) into two, $0 \leq \rho \leq \rho_0$ and $\rho_0 \leq \rho$, and denoted the integral as I_s and I_l correspondently. Note that if $0 \leq \rho < \rho_0$, then

$$(3.22) \quad \frac{\nu(\rho)}{\rho}x > \frac{c_0}{c_1}.$$

Applying (3.9), we have

$$(3.23) \quad |I_s| \leq CA_\alpha\langle f \rangle.$$

Using the asymptotic formula (3.10), We obtain

$$(3.24) \quad \begin{aligned} I_l &= -\ln(x) \int_{\rho_0}^{\infty} \int_0^{2\pi} F(\rho, 0, \phi) e^{-\frac{\rho^2}{2}} \rho d\phi d\rho + O(A_\alpha\langle f \rangle) \\ &= -\ln x \int \int \phi_\alpha(0, \zeta_2, \zeta_3) L(f)(0, 0^+, \zeta_2, \zeta_3) d\zeta_2 d\zeta_3 + O(A_\alpha\langle f \rangle(1 + \rho_0^2 |\ln x|)). \end{aligned}$$

The remaining task is to estimate ρ_0 . If we assume $x \leq \frac{1}{2c_1}$, then

$$(3.25) \quad 1 = \frac{\nu(\rho_0)}{\rho_0}x \leq c_1 \frac{(1 + \rho_0)^\gamma}{\rho_0}x \leq \frac{(1 + \rho_0)^\gamma}{2\rho_0}.$$

If $\gamma = 1$, we see $\rho_0 \leq 1$. Observe that, for $0 \leq \gamma < 0$, ρ grows faster than the $\frac{1}{2}(1 + \rho)^\gamma$ as $\rho \rightarrow \infty$. Therefore, (3.26) implies $\rho_0 \leq m$ for some $m < \infty$. Therefore, we have

$$(3.26) \quad 1 = \frac{\nu(\rho_0)}{\rho_0}x \leq c_1 \frac{(1 + m)^\gamma}{\rho_0}x.$$

We have

$$(3.27) \quad |\rho_0^2 \ln x| \leq Cx^2 |\ln x| \leq C$$

and conclude the lemma. \square

Finally, combining (2.2) and (2.3) in Lemma 2.1, (2.33) in Lemma 2.3, and Lemma 3.1, we concludes Theorem 1.5.

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DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAI-
WAN

E-mail address: ikun.chen@gmail.com

INSTITUTE OF APPLIED MATHEMATICAL SCIENCES, NATIONAL TAIWAN UNIVERSITY,
TAIPEI 10617, TAIWAN

E-mail address: willhsia@math.ntu.edu.tw